TWO-SIDED GLUINGS OF TILTED ALGEBRAS

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ABSTRACT. We study the class of algebras A satisfying the property: all but at most finitely many non-isomorphic indecomposable A-modules are such that all their predecessors have projective dimension at most one, or all their successors have injective dimension at most one. Such a class includes the tilted algebras [20], the quasi-tilted algebras [19], the shod algebras [10], the weakly shod [12], and the left and right glued algebras [1].

Let A be an artin algebra. We are interested in studying the representation theory of A, thus in characterizing A by properties of the category mod$A$ of finitely generated right A-modules. One method to achieve this goal is to start from a class of algebras whose representation theory is considered to be sufficiently well-understood, and then to generalize this class to another whose representation theory is close enough to that of the preceding class. Thus, tilted algebras were introduced in [20] as a generalization of hereditary algebras. The class of tilted algebras is now considered to be one of the most useful for the general theory. For instance, it is known that an indecomposable module over an arbitrary algebra which does not lie in an oriented cycle of non-zero non-isomorphisms, is a module over a tilted algebra [28]. It was therefore natural to consider various generalizations of this notion. Thus, over the years, the following classes of algebras were defined and investigated: the quasi-tilted (which generalize the tilted and the canonical algebras of [28]) [19], the shod algebras (which generalize the quasi-tilted) [10], the weakly shod algebras (which generalize the shod and the representation-directed algebras) [11, 12] and the left and the right glued algebras (which generalize the tilted and the representation-finite algebras) [1]. The purpose of the present paper is to introduce a new class of algebras which generalizes all the previous classes.

We define an artin algebra A to be a *laura algebra* if all but at most finitely many non-isomorphic indecomposable A-modules are such that

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all their predecessors have projective dimension at most one, or all their successors have injective dimension at most one. We start by giving various examples and characterizations of laura algebras. We then study the representation theory of laura algebras, and our main theorem (4.6) gives a full description of the Auslander-Reiten quiver of a laura algebra. The class of laura algebras is then characterized in the spirit of [1] as a double gluing of tilted algebras (5.4). Since, in general, laura algebras are representation-infinite, a measure of the complexity of the module category is given by the nilpotency of the infinite radical. We show that, if A is a representation-infinite laura algebra with nilpotent infinite radical, then its nilpotency index lies between 3 and 5, inclusively (6.3).

For further results on laura algebras, we refer the reader to [2, 3].

During the writing of this paper, we have learnt that I. Reiten and A. Skowroński have also independently considered laura algebras, obtaining some of our results here [27].

1. Preliminaries

1.1. **Notations.** Throughout this paper, our algebras are connected artin algebras. For an algebra A, we denote by $\mathrm{mod}A$ its category of all finitely generated right A-modules, and by $\mathrm{ind}A$ a full subcategory of $\mathrm{mod}A$ consisting of one representative from each isomorphism class of indecomposable modules. We denote by $\mathrm{rad}(\mathrm{mod}A)$ the ideal in $\mathrm{mod}A$ generated by all non-isomorphisms between indecomposable modules. The infinite radical $\mathrm{rad}^\infty(\mathrm{mod}A)$ of $\mathrm{mod}A$ is the intersection of all powers $\mathrm{rad}^i(\mathrm{mod}A)$, with $i \geq 1$, of $\mathrm{rad}(\mathrm{mod}A)$. We also denote by $\mathrm{rk}(K_0(A))$ the rank of the Grothendieck group of A, which equals the number of isomorphism classes of simple A-modules. If M is an A-module, we denote by $\mathrm{pd}_A M$ (or $\mathrm{id}_A M$) its projective dimension (or injective dimension, respectively) of M. Also, we denote by $\mathrm{gl.dim}A$ the global dimension of A. An algebra B is called a *full subcategory* of A if there exists an idempotent $e \in A$ such that $B = eAe$. It is called *convex* in A if whenever there exists a sequence $e_i = e_{i_0}, e_{i_1}, \cdots, e_{i_t} = e_j$ of primitive idempotents such that $e_{i_{l+1}} A e_{i_l} \neq 0$ for $0 \leq l < t$, and $ee_i = e_i$, $ee_j = e_j$, then $ee_{i_l} = e_{i_l}$, for all l.

For an algebra A, we denote by $\Gamma(\mathrm{mod}A)$ its Auslander-Reiten quiver, and by $\tau_A = \mathrm{DTr}$, $\tau_A^{-1} = \mathrm{TrD}$ the Auslander-Reiten translations. An indecomposable A-module M is called *right stable* (or *left stable*, or *stable*) provided $\tau_A^n X \neq 0$ for each $n \leq 0$ (or $n \geq 0$, or any n, respectively). If Γ is a connected component of $\Gamma(\mathrm{mod}A)$, we denote by ${}_l\Gamma$ (or ${}_r\Gamma$, or ${}_s\Gamma$) the full subquiver of Γ generated by the left stable (or the right stable, or the stable, respectively) indecomposables in Γ. A component

Γ of $\Gamma(\mathrm{mod}A)$ is called *semiregular* if it does not simultaneously contain a projective module and an injective module, and *non-semiregular* if it does contain simultaneously a projective module and an injective module.

For further definitions or facts needed on $\Gamma(\mathrm{mod}A)$, we refer the reader to [4, 28].

1.2. **Paths.** Given two modules M, N in $\mathrm{ind}A$, a path from M to N of length t in $\mathrm{ind}A$ is a sequence

$$(*) \qquad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} M_{t-1} \xrightarrow{f_t} M_t = N$$

$(t \geq 0)$, where all the M_i lie in $\mathrm{ind}A$, and all the f_i are non-zero morphisms. We write in this case $M \rightsquigarrow N$, and we say that M is a *predecessor* of N, while N is a *successor* of M. Observe that each indecomposable module is a predecessor and a successor of itself. It is sometimes necessary to assume that the f_i are non-isomorphisms, in which case we explicitly say so. The path $(*)$ is called a *path of irreducible morphisms* or a *path in* $\Gamma(\mathrm{mod}A)$ if each of the f_i is irreducible. A path $(*)$ of irreducible morphisms such that $M \cong N$ and $t > 0$ is called a *cycle* in $\Gamma(\mathrm{mod}A)$. A path $(*)$ of irreducible morphisms is called *sectional* if $\tau_A M_{i+1} \ncong M_{i-1}$ for each i such that $0 < i < t$. A *refinement* of the path $(*)$ is a path

$$M = M'_0 \xrightarrow{f'_1} M'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{s-1}} M'_{s-1} \xrightarrow{f'_s} M'_s = N$$

in $\mathrm{ind}A$ with $s \geq t$ together with an order-preserving function $\sigma\colon \{1, \cdots, t-1\} \longrightarrow \{1, \cdots, s-1\}$ such that $M_i \cong M'_{\sigma(i)}$ for each i with $1 \leq i \leq t-1$.

1.3. The following result from [29, 32] will be very useful later on.

LEMMA. *Let A be an artin algebra, M and N be two indecomposable A-modules, and f be a non-zero morphism in $\mathrm{rad}^\infty_A(M, N)$. Then, for each $t \geq 1$,*

(a) *There exists a path in $\mathrm{ind}A$*

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t \xrightarrow{g_t} N$$

where $f_1, \cdots, f_t$ are irreducible morphisms, and $g_t \in \mathrm{rad}^\infty_A(M_t, N)$.

(b) *There exists a path in $\mathrm{ind}A$*

$$M \xrightarrow{g'_t} N_t \xrightarrow{f'_t} N_{t-1} \xrightarrow{f'_{t-1}} \cdots \xrightarrow{f'_2} N_1 \xrightarrow{f'_1} N_0 = N$$

where $f'_1, \cdots, f'_t$ are irreducible morphisms, and $g'_t \in \mathrm{rad}^\infty_A(M, N_t)$.

1.4. The following proposition will be very useful in the sequel.

PROPOSITION. *Let Γ be a component of $\Gamma(modA)$ and $M \in \Gamma$ be an indecomposable module lying in a cycle in Γ.*

(a) *If Γ contains projective modules, then there is a path in $\Gamma(modA)$ from M to a projective.*
(b) *If Γ contains injective modules, then there is a path in $\Gamma(modA)$ from an injective to M.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) Let $M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t = M$ be an oriented cycle in Γ containing M. Suppose first that there exists an index j and an $r \geq 0$ such that $\tau_A^r M_j$ is a projective module. Without loss of generality we may also assume that $\tau_A^l M_i$ is not projective for each $l < r$ and each $i = 0, \cdots, t$. By [15](1.4), there exists a path from M_j to $\tau_A^r M_j$ and so from M to a projective, as required.
Suppose now that each of $M_0, \cdots, M_{t-1}$ is left stable, that is, for each $n \geq 0$ and each $i = 0, \cdots, t-1$, the module $\tau_A^n M_i$ is not projective. Since Γ contains projective modules and it is connected, there exists a walk

$$(*) \qquad N = N_0 - N_1 - \cdots - N_m = P$$

in $\Gamma(\mathrm{mod}A)$ of minimal length, where P is a projective module in Γ and N lies in the τ_A-orbit of M. It follows from the minimality of the length of $(*)$ that each of $N_0, \cdots, N_{m-1}$ is left stable. Therefore, by applying τ_A if necessary, we get a path $N' \longrightarrow \cdots \longrightarrow P$, where $N' = \tau_A^s M$ for some integer s. If $s < 0$, there clearly exists a path $M \rightsquigarrow \tau_A^s M$ in Γ. If $s > 0$, then by [15](1.4), there exists a path $M \rightsquigarrow \tau_A^s M$ in Γ. In both cases, we get a path $M \rightsquigarrow P$, as required. □

1.5. **The subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$.** Following [19], for an algebra A, denote by $\mathcal{L}_A$ and $\mathcal{R}_A$ the following subcategories of ind$A$:

$$\mathcal{L}_A = \{X \in \mathrm{ind}A \colon \mathrm{pd}_A Y \leq 1 \text{ for each predecessor } Y \text{ of } X\}$$

$$\mathcal{R}_A = \{X \in \mathrm{ind}A \colon \mathrm{id}_A Y \leq 1 \text{ for each successor } Y \text{ of } X\}$$

Clearly, $\mathcal{L}_A$ is closed under predecessors, while $\mathcal{R}_A$ is closed under successors. These subcategories played an important role in the study of the quasi-tilted algebras [19, 15], the shod algebras [10] and the weakly shod algebras [12]).

LEMMA. *Let A be an artin algebra.*

(a) *If P is an indecomposable projective A-module, then there are at most finitely many modules $M \in \mathcal{R}_A$ such that there exists a path*

$M \rightsquigarrow P$. Moreover, any such path is refinable to a path of irreducible morphisms, and any such path of irreducible morphisms is sectional.

(b) *If I is an indecomposable injective A-module, then there are only finitely many indecomposable modules $N \in \mathcal{L}_A$ with a path $I \rightsquigarrow N$. Moreover, any such path is refinable to a path of irreducible morphisms, and any such path of irreducible morphisms is sectional.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) Assume that P has infinitely many predecessors in $\mathcal{R}_A$. Then, for each $s \geq 0$, there exists a path in ind$A$

$$(**) \qquad M_s \xrightarrow{f_s} M_{s-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{f_1} M_0 = P$$

where all M_i lie in $\mathcal{R}_A$, and all f_i are non-isomorphisms. We claim that $(**)$ induces another path

$$(*) \qquad N_t \xrightarrow{g_t} N_{t-1} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{g_1} N_0 = P$$

where $t \geq s$, all N_i lie in $\mathcal{R}_A$, and all g_i are irreducible.
Indeed, the non-isomorphism f_1 factors through the right minimal almost split morphism ending with P, so that we have a path $M_1 \xrightarrow{g_1'} N_1 \xrightarrow{g_1} P$ with N_1 indecomposable, g_1 irreducible and $g_1' \neq 0$ (hence N_1 belongs to $\mathcal{R}_A$, becauseM_1 does). Inductively, assume that we have a path

$$M_j \xrightarrow{g_i'} N_i \xrightarrow{g_i} N_{i-1} \longrightarrow \cdots \longrightarrow N_1 \xrightarrow{g_1} P$$

where $i \geq j$, all the N_l are in $\mathcal{R}_A$, all the g_l are irreducible and $g_i' \neq 0$. We have one of two cases. If g_i' is not an isomorphism, then it factors through the right minimal almost split morphism ending with N_i, so that we have a path

$$M_j \xrightarrow{g_{i+1}'} N_{i+1} \xrightarrow{g_{i+1}} N_i \xrightarrow{g_i} \cdots \longrightarrow N_1 \xrightarrow{g_1} P$$

with N_{i+1} indecomposable, g_{i+1} irreducible and $g_{i+1}' \neq 0$ (hence N_{i+1} belongs to $\mathcal{R}_A$, because M_j does). If, on the other hand, g_i' is an isomorphism, then the non-isomorphism $g_i' f_{i+1} \colon M_{j+1} \longrightarrow N_i$ factors through the right minimal almost split morphism ending with N_i, so that we have a path

$$M_{j+1} \xrightarrow{g_{i+1}'} N_{i+1} \xrightarrow{g_{i+1}} N_i \xrightarrow{g_i} \cdots \longrightarrow N_1 \xrightarrow{g_1} P$$

with N_{i+1} indecomposable, g_{i+1} irreducible and $g_{i+1}' \neq 0$. Again, N_{i+1} lies in $\mathcal{R}_A$. This establishes our claim.
We now show that $(*)$ is sectional. If this is not the case, there exists a least j such that $\tau_A N_{j-1} \cong N_{j+1}$ and the subpath $N_j \longrightarrow N_{j+1} \longrightarrow$

$\cdots \longrightarrow N_1 \longrightarrow P$ is sectional. In particular, $\mathrm{Hom}_A(N_{j-1}, P) \neq 0$ by [22, 6], and so $\mathrm{id}_A N_{j+1} \geq 2$, by [28](pg. 74), which contradicts the fact that N_{j+1} lies in $\mathcal{R}_A$.
The sectionality of $(*)$ implies in particular that the N_l are pairwise non-isomorphic [5, 6].
Assume now that $(*)$ is such that $t \geq \mathrm{rk}\ (\mathrm{K}_0(A)) + 1$. By [33], there exist p, q such that $1 \leq p, q, \leq t$ and $\mathrm{Hom}_A(N_p, \tau_A N_q) \neq 0$. Since $\mathrm{Hom}_A(N_q, P) \neq 0$, we have, as before, $\mathrm{id}_A \tau_A N_q \geq 2$, and so $N_p \notin \mathcal{R}_A$, a contradiction which finishes the proof. □

1.6. Corollary. *Let A be an artin algebra.*

(a) *$\mathcal{R}_A$ consists of the modules $M \in indA$ such that, if there exists a path from M to an indecomposable projective module, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.*
(b) *$\mathcal{L}_A$ consists of the modules $N \in indA$ such that, if there exists a path from an indecomposable injective module to N, then this path can be refined to a path of irreducible morphsims, and any such refinement is sectional.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) Assume that $M \in \mathrm{ind}A$ is a module such that, if there exists a path from M to an indecomposable projective, then this path is refinable to a path of irreducible morphisms, and any such refinement is sectional. We claim that M belongs to $\mathcal{R}_A$. If N is a successor of M such that $\mathrm{id}_A N \geq 2$, there exist an indecomposable projective module P and a non-zero morphism $\tau_A^{-1} N \longrightarrow P$. By hypothesis, the composed path $M \rightsquigarrow N \longrightarrow * \longrightarrow \tau_A^{-1} N \longrightarrow P$ is refinable to a path of irreducible morphisms, which is sectional. The ensuing contradiction shows that $\mathrm{id}_A N \leq 1$, and hence our claim. Since the converse follows directly from (1.5), the proof is complete. □

2. Laura algebras: definitions and examples

2.1. We say that a subcategory $\mathcal{C}$ of ind$A$ is *finite* if it contains only finitely many isomorphism classes of indecomposable A-modules. We say that $\mathcal{C}$ is *cofinite* in ind$A$ if all but at most finitely many isomorphism classes of indecomposable A-modules belong to $\mathcal{C}$.

Definition. An artin algebra A is said to be a *laura algebra* provided the union $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in ind$A$.

It follows immediately from this definition that any representation-finite algebra is laura. We now discuss some other classes of laura algebras.

We need to recall a few definitions. An algebra is called *weakly shod* [12] if there exists a positive integer n_0 such that the length of any path from an indecomposable injective module to an indecomposable projective module is bounded by n_0. The class of weakly shod algebras includes the class of *shod algebras* of [10], that is, of the algebras A such that, for any indecomposable A-module M, we have $\mathrm{pd}_A M \leq 1$ or $\mathrm{id}_A M \leq 1$. Since any quasi-tilted algebra [19] (hence, a fortiori, any tilted algebra [20]) is shod, the class of weakly shod algebras contains all the preceding classes. The following reformulation of [12](2.5) shows that all these are examples of laura algebras.

THEOREM. *An artin algebra is weakly shod if and only if it is a laura algebra such that none of the non-semiregular components of the Auslander-Reiten quiver contains oriented cycles.*

We prove in (4.8) below a stronger version of this theorem.

2.2. The class of left and right glued algebras were introduced in [1]. We recall here the definition of right glued algebra, and refer the reader to [1] for the dual definition of *left glued algebra.*

DEFINITION. Let $B_1, \cdots, B_t$ be representation-infinite tilted algebras having complete slices $\Sigma_1, \cdots, \Sigma_t$ respectively, in their preinjective components and no projectives in these components, $B = B_1 \times \cdots \times B_t$ and C be a representation-finite algebra. An algebra A is called a *right gluing of* $B_1, \cdots, B_t$ *by* C *along the slices* $\Sigma_1, \cdots, \Sigma_t$ or, more briefly, to be a *right glued algebra* if $A = C$ or:

(RG1) Each of $B_1, \cdots, B_t$ and C is a full convex subcategory of A, and any object in A belongs to one of these subcategories;

(RG2) No injective A-module is a proper predecessor of the union $\Sigma_1 \cup \cdots \cup \Sigma_t$, considered as embedded in ind$A$; and

(RG3) ind$B$ is cofinite in ind$A$.

The next result shows that right and left glued algebras are examples of laura algebras.

PROPOSITION. *Let A be a connected algebra. Then*

(a) *A is right glued if and only if $\mathcal{L}_A$ is cofinite in ind$A$.*

(b) *A is left glued if and only if $\mathcal{R}_A$ is cofinite in ind$A$.*

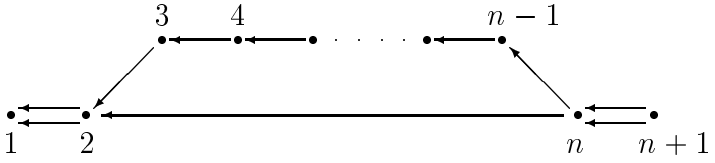
Proof. We only prove (a), since the proof of (b) is dual.

(a) Suppose first that $\mathcal{L}_A$ is cofinite. Then $\mathrm{pd}_A M \leq 1$ for all but at

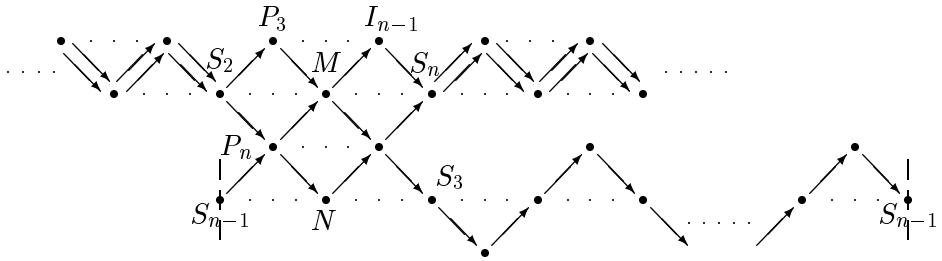
most finitely many isomorphism classes of indecomposable A-modules M. By [1](3.2)(b), A is right glued.
Conversely, assume that A is right glued. Then there are tilted algebras $B_1, \cdots, B_t$ with complete slices $\Sigma_1, \cdots, \Sigma_t$, respectively, and a representation-finite algebra C as in the definition above. Moreover, there are only finitely many isomorphism classes of indecomposable A-modules which are not predecessors of $\Sigma_1 \cup \cdots \cup \Sigma_t$. The result now follows from the facts that $\Sigma_1 \cup \cdots \cup \Sigma_t \subset \mathcal{L}_A$, and $\mathcal{L}_A$ is closed under predecessors. □

2.3. Examples. We now give examples of laura algebras which do not belong to any of the above classes. Let k be a commutative field.

(a) Our first example shows that there are triangular representation-infinite laura algebras of arbitrarily large finite global dimensions. For any $n \geq 4$, let $A = A(n)$ be the radical square zero algebra given by the quiver

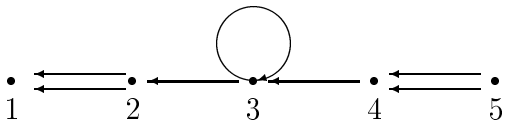


By [17, 16], $\mathrm{pd}_A S_{n+1} = n$ and also $\mathrm{gl.dim} A = n$. Moreover, the Auslander-Reiten quiver $\Gamma(\mathrm{mod} A)$ of A consists of:
(i) the postprojective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subcategory generated by 1 and 2;
(ii) the preinjective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subcategory generated by n and $n+1$;
(iii) a non-semiregular component Γ of the following shape:

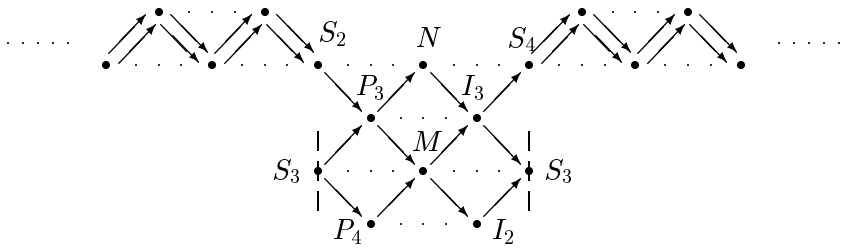


where we identify the two copies of S_{n-1}, along the vertical dotted lines. Here (and in the sequel), we denote by P_i (or I_i or S_i) the indecomposable projective (or injective, or simple, respectively) corresponding to the point i of the quiver. Moreover, the indecomposables M and N are given by $M \cong (P_3 \oplus P_n)/S_2$ and $N \cong P_n/S_{n-1}$.
There are no morphisms from one of the components described in (ii) or from Γ to one of the components described in (i). So, there are no morphisms from injective modules to any of the components described in (i). Therefore, these components are contained in $\mathcal{L}_A$. Also, it is easily seen that the modules in the components of (i) are predecessors of S_2, and $\mathrm{id}_A S_2 > 1$. Therefore, these components lie in $\mathcal{L}_A \setminus \mathcal{R}_A$. Dually, the components described in (ii) are contained in $\mathcal{R}_A \setminus \mathcal{L}_A$. Concerning the component Γ, it is not difficult to see that the modules in Γ which lie in $\mathcal{L}_A$ (or in $\mathcal{R}_A$) are the predecessors of P_3 (or the successors of I_{n-1}, respectively). We then infer that $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in ind$A$ and so, A is laura. It follows from (2.1) and (2.2) that A is neither weakly shod, nor left, nor right glued.

(b) We give an example of a representation-infinite laura algebra of infinite global dimension. Let $A = A(\infty)$ be the radical square zero algebra given by the quiver



We have $\mathrm{pd}_A S_3 = \infty$ and so gl.dim$A = \infty$. Here $\Gamma(\mathrm{mod}A)$ contains a unique non-semiregular component Γ of the following shape



where we identify the two copies of S_3, along the vertical dotted lines. The indecomposables M and N are given by $M \cong (P_3 \oplus P_4)/S_3$ and $N \cong P_3/S_2$. It is not hard to see that those modules in Γ which lie in $\mathcal{L}_A$ (or in $\mathcal{R}_A$) are the predecessors of S_2 (or the successors of S_4,

respectively). As in (a) above, we infer that A is a laura algebra, which is neither weakly shod, nor left, nor right glued.

2.4. We finish this section with the following result which characterizes laura algebras in terms of the number of modules lying in certain paths. A similar result holds true of weakly shod algebras [2](1.4).

THEOREM. *The following statements are equivalent for an algebra A:*

(a) *A is laura.*
(b) *There are only finitely many indecomposable modules M with a path $I \rightsquigarrow M \rightsquigarrow P$ in ind$A$ where I is an injective and P is a projective.*
(c) *There are only finitely many indecomposable modules M with a path $L \rightsquigarrow M \rightsquigarrow N$ in ind$A$ where $L \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$.*

Proof. (a) implies (b). By (1.5), there are at most finitely many indecomposable modules $M \in \mathcal{L}_A \cup \mathcal{R}_A$ such that there exists a path $I \rightsquigarrow M \rightsquigarrow P$ in ind$A$ where I is an injective and P is a projective. Since $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite, the result follows.

(b) implies (a). Let $M \in \text{ind}A$ and suppose $M \notin \mathcal{L}_A \cup \mathcal{R}_A$. Since $M \notin \mathcal{L}_A$, there exists a path $L \rightsquigarrow M$ where $\text{pd}_A L \geq 2$, and so a path

$$I \longrightarrow \tau_A L \longrightarrow * \longrightarrow L \rightsquigarrow M$$

in ind$A$ where I is an injective module. Dually, since $M \notin \mathcal{R}_A$, there exists a path

$$M \rightsquigarrow N \longrightarrow * \longrightarrow \tau_A^{-1} N \longrightarrow P$$

in ind$A$ where P is a projective module. Therefore, for each indecomposable $M \notin \mathcal{L}_A \cup \mathcal{R}_A$, there exists a path $I \rightsquigarrow M \rightsquigarrow P$ with I an injective and P a projective. Therefore, $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite and A is laura.

(b) implies (c). Let $M \in \text{ind}A$ be such that there is a path $L \rightsquigarrow M \rightsquigarrow N$ where $L \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$. As before, there exists a path

$$(*) \qquad I \rightsquigarrow L \rightsquigarrow M \rightsquigarrow N \rightsquigarrow P$$

in ind$A$ where I is an injective module and P is a projective module. Since there are at most finitely many indecomposable modules M lying in paths as $(*)$, the result follows.

(c) implies (a). Suppose $\mathcal{L}_A \cup \mathcal{R}_A$ is not cofinite. So, there exists an infinite family $(M_\lambda)_{\lambda \in \Lambda}$ of indecomposable A-modules not lying in $\mathcal{L}_A \cup \mathcal{R}_A$. For each λ, the trivial path $M_\lambda \xrightarrow{id} M_\lambda \xrightarrow{id} M_\lambda$ gives a contradiction to (c). The result is proven. □

3. Quasi-directed components

3.1. The objective of this section is to show that, if A is a laura algebra which is not quasi-tilted, then its Auslander-Reiten quiver $\Gamma(\mathrm{mod}A)$ has a component with some special properties which generalize those of the pip-bounded components of [11].

Definition. Let A be an artin algebra. A component Γ of $\Gamma(\mathrm{mod}A)$ is called *quasi-directed* provided it is generalized standard and at most finitely many modules in Γ lie in oriented cycles.

Remark. Let A be an algebra, and Γ be a quasi-directed component of $\Gamma(\mathrm{mod}A)$. It follows from [32](2.3) that Γ has only finitely many τ_A-orbits.

3.2. Examples. (a) If A is a representation-finite algebra, then $\Gamma(\mathrm{mod}A)$ is clearly quasi-directed.

(b) Let A be a quasi-tilted algebra. It follows from [8, 15] that the quasi-directed components of $\Gamma(\mathrm{mod}A)$ are the postprojective, the preinjective and the connecting components (the latter occurs only in case A is tilted).

(c) Let A be a weakly shod algebra which is not quasi-tilted. It follows from [12] that $\Gamma(\mathrm{mod}A)$ has a unique pip-bounded component Γ, that is, such that there exists a positive integer n_0 such that any path in $\mathrm{ind}A$ from an injective in Γ to a projective in Γ has length at most n_0. Moreover, Γ is faithful, generalized standard and has no oriented cycles. Then, Γ is quasi-directed.

(d) In each of the examples (2.3)(a) and (2.3)(b), the illustrated component Γ is quasi-directed.

(e) We now consider the case of left or right glued algebras. We recall from [7], that, if A is an artin algebra, then a component Γ of $\Gamma(\mathrm{mod}A)$ is called a *π-component* (or an *ι-component*) provided:

(i) All but finitely many modules in Γ lie in the τ_A-orbit of a projective (or of an injective, respectively).
(ii) Only finitely many modules in Γ lie in oriented cycles.

It is shown in [1] that a left (or right) glued algebra has a faithful π-component (or ι-component, respectively). The following lemma says that these are quasi-directed.

Lemma. *Let A be an algebra, and Γ be a component of $\Gamma(\mathrm{mod}A)$.*

(a) *If Γ is a π-component, then Γ is quasi-directed.*

(b) *If* Γ *is an* ι*-component, then* Γ *is quasi-directed.*

Proof. We only prove (a), since the proof of (b) is dual.
(a) It suffices to show that Γ is generalized standard. However, by [7], if M lies in Γ, then it has only finitely many predecessors in ind$A$. In particular, $\mathrm{rad}^\infty(-, M) = 0$ and so, Γ is generalized standard. □

Remark. In fact, the existence of a faithful π-component characterizes left glued algebras. Indeed, assume that A is an algebra such that $\Gamma(\mathrm{mod}A)$ contains a faithful π-component Γ. Then, this π-component is unique: let P_A be an indecomposable projective, the faithfulness of Γ implies the existence of a module M in Γ such that $\mathrm{Hom}_A(P, M) \neq 0$; however, since Γ is a π-component, M has only finitely many predecessors in ind$A$ and therefore P lies in Γ, thus showing that Γ is the unique π-component of $\Gamma(\mathrm{mod}A)$. Applying [1](2.2) and (3.2), we deduce that A is left glued. We have thus shown that an algebra A is left (or right) glued if and only if $\Gamma(\mathrm{mod}A)$ contains a - necessarily unique - faithful π-component (or ι-component, respectively).

3.3. Assume that A is a weakly shod algebra. It follows from [12](1.6) that, if there exists a path in ind$A$ from an indecomposable injective module to an indecomposable projective module, then such a path contains at most finitely many indecomposable modules, and, since it lies in the unique pip-bounded component of $\Gamma(\mathrm{mod}A)$, it is refinable to a path of irreducible morphisms and contains no morphism lying in $\mathrm{rad}^\infty(\mathrm{mod}A)$. We now show that the same statement holds true for laura algebras.

Lemma. *Let* A *be a laura algebra. Any path in ind*A *from an indecomposable injective module to an indecomposable projective module contains at most finitely many modules. Moreover, such a path contains no morphisms lying in* $\mathrm{rad}^\infty(\mathrm{mod}A)$*, and, hence, can be refined to a path of irreducible morphisms.*

Proof. Let I_A, P_A be respectively an indecomposable injective and an indecomposable projective such that there exists a path $I \rightsquigarrow P$ in ind$A$. Such a path is of the form

$$I \rightsquigarrow M' \longrightarrow M \rightsquigarrow N \longrightarrow N' \rightsquigarrow P$$

where M' lies in $\mathcal{L}_A$, N' lies in $\mathcal{R}_A$, while M does not lie in $\mathcal{L}_A$ and N does not lie in $\mathcal{R}_A$, and we make the conventions that, if I does not belong to $\mathcal{L}_A$ (or P does not belong to $\mathcal{R}_A$), then we take $I = M$ (or $P = N$, respectively). By (1.5), the subpaths $I \rightsquigarrow M'$ and $N' \rightsquigarrow P$ can be refined to sectional paths, hence have bounded length. Moreover,

since M does not belong to $\mathcal{L}_A$, and N does not belong to $\mathcal{R}_A$, then no module on the subpath $M \rightsquigarrow N$ lies in $\mathcal{L}_A \cup \mathcal{R}_A$. Since at most finitely many indecomposable A-modules do not belong to $\mathcal{L}_A \cup \mathcal{R}_A$ (because A is laura), this shows that the subpath $M \rightsquigarrow N$ (and hence the path $I \rightsquigarrow P$) contains at most finitely many modules.
We now claim that the subpath $M \rightsquigarrow N$ factors through no morphisms in $\mathrm{rad}^\infty(\mathrm{mod}A)$. Indeed, assume that it factors through the morphism $f \in \mathrm{rad}^\infty_A(L, L')$. Then, for each $t \geq 1$, the given path can be refined to a path in $\mathrm{ind}A$

$$M \rightsquigarrow L = L_0 \xrightarrow{f_1} L_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} L_t = L' \rightsquigarrow N$$

This contradicts the fact that the number of modules of any path $M \rightsquigarrow N$ is bounded. This shows our claim, and hence that no morphism in the path $I \rightsquigarrow P$ lies in $\mathrm{rad}^\infty(\mathrm{mod}A)$. □

3.4. Lemma. *Let A be a laura algebra. Then any non-semiregular component of $\Gamma(\mathrm{mod}A)$ is quasi-directed.*

Proof. Let Γ be a non-semiregular component of $\Gamma(\mathrm{mod}A)$. That Γ has only finitely many modules lying in oriented cycles follows from (1.4) and (3.3). We now have to prove that Γ is generalized standard.
We first show that Γ has only finitely many τ_A-orbits. Assume indeed that this is not the case. Then there exists a connected component Γ' of the right stable part ${}_r\Gamma$ of Γ with infinitely many τ_A-orbits. Moreover, there exists a connected component Γ'' of the left stable part ${}_l\Gamma'$ of Γ' with infinitely many τ_A-orbits. Observe that Γ'' has no oriented cycles (otherwise, it either contains a τ_A-periodic module and so it is a stable tube by [18], or else it has no τ_A-periodic modules, and so stability gives in either case a contradiction to the fact that Γ has at most finitely many modules lying in cycles).
Let now $i \geq 2\ \mathrm{rk}(\mathrm{K}_0(A))$ and M_i be a module in Γ'' such that the least length of a walk from M_i to a non-stable module in Γ is at least i. Let $I = N_0 - N_1 - \cdots - N_r = M_i'$ be a walk of least possible length from an injective I in Γ to a module M_i' in the τ_A-orbit of M_i. The minimality of r implies that $N_1, \cdots, N_r$ are right stable. We deduce, as in the proof of (1.4), a path $I \rightsquigarrow M_i''$ with M_i'' in the τ_A-orbit of M_i. Dually, we construct a path $M_i''' \rightsquigarrow P$, with P a projective in Γ, and M_i''' in the τ_A-orbit of M_i. Applying [15](1.5), we get a path $M_i'' \rightsquigarrow M_i'''$, hence a path $I \rightsquigarrow M_i'' \rightsquigarrow M_i''' \rightsquigarrow P$. This being true for each $i \geq 2\ \mathrm{rk}(\mathrm{K}_0(A))$, we get a contradiction to (3.3). This shows that Γ has only finitely many τ_A-orbits.
We now deduce that Γ is generalized standard. Let $f \in \mathrm{rad}^\infty_A(M, N)$

be a non-zero morphism, with M and N in Γ. Then, for each $s \geq 1$, there exists a path in ind$A$

$$(*) \qquad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_s} M_s$$

of irreducible morphisms, and a non-zero morphism $g_s \in \mathrm{rad}_A^\infty(M_s, N)$, by (1.3). If one of the M_i lies in a cycle, then, by (1.4), there is a path from an injective to such an M_i. If, on the other hand, no M_i lies in a cycle, it follows from the fact that Γ has only finitely many τ_A-orbits that we may assume that $(*)$ crosses the τ_A-orbit of M. Therefore, there exists an s large enough so that there exists an injective I in Γ and a path $I \rightsquigarrow M_s$. We deduce a path $I \rightsquigarrow M_s \xrightarrow{g_s} N$ with $g_s \in \mathrm{rad}_A^\infty(M_s, N)$. Applying the dual argument, we find a projective P in Γ and a path $I \rightsquigarrow M_s \xrightarrow{h} N_s \rightsquigarrow P$ with $h \in \mathrm{rad}_A^\infty(M_s, N_s)$, a contradiction to (3.3). □

3.5. Proposition. *Let A be a laura algebra which is not quasi-tilted. Then $\Gamma(\mathrm{mod}A)$ has a non-semiregular quasi-directed component.*

Proof. Since A is not quasi-tilted, it follows from [19](II.1.14) that there exists an indecomposable projective A-module P not lying in $\mathcal{L}_A$. This means that there is an indecomposable module M such that $\mathrm{pd}_A M \geq 2$ which is a predecessor of P. Consequently, there exist an indecomposable injective A-module I and a path in ind$A$

$$I \longrightarrow \tau_A M \longrightarrow * \longrightarrow M \rightsquigarrow P$$

By (3.3), this path can be refined to a path of irreducible morphisms and therefore I and P belong to the same component Γ of $\Gamma(\mathrm{mod}A)$, which is thus non-semiregular. By (3.4), Γ is quasi-directed. □

4. Left and right end algebras

4.1. Our objective now is to give a complete description of the Auslander-Reiten quiver of a laura algebra. We show that, if the algebra is not quasi-tilted, then it has a unique non-semiregular quasi-directed faithful component while the other components are components of (direct product of) tilted algebras which we call the left and the right end algebras of the given laura algebra. The use of this term comes from the fact that they generalize the left and the right end algebras of a tilted algebra, as defined in [23].

Throughout this section, we let A be a laura algebra which is not quasi-tilted, and we let Γ be a non-semiregular component of $\Gamma(\mathrm{mod}A)$. Such a component exists by (3.5).

Lemma. *Let A and Γ be as above.*

(a) *Assume that I_A is an indecomposable injective module such that there exists a path $I \rightsquigarrow M$ with $M \in \Gamma$, then I belongs to Γ.*
(b) *Assume that P_A is an indecomposable projective module such that there exists a path $N \rightsquigarrow P$ with $N \in \Gamma$, then P belongs to Γ.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) Suppose there exists a path $I \rightsquigarrow M$ in ind$A$, with $M \in \Gamma$ and I an indecomposable injective not in Γ. Clearly, such a path factors through a morphism in $\mathrm{rad}^\infty(\mathrm{mod}A)$. Then, by (1.3), there exists, for each $t \geq 0$, a path in ind$A$

$$(\xi_t) \qquad M_t \xrightarrow{f_t} M_{t-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{f_1} M_0 = M$$

of irreducible morphisms, and a path $I \rightsquigarrow M_t$, which factors through a morphism in $\mathrm{rad}^\infty(\mathrm{mod}A)$. Since Γ has only finitely many τ_A-orbits, we may assume that the paths (ξ_t) cross arbitrarily many times the τ_A-orbit of M. In particular, M is left stable. Let now $M' = N_0 - N_1 - \cdots - N_s = P$ be a walk of least length between an indecomposable projective module P in Γ and a module M' in the τ_A-orbit of M. It follows from the minimality of s and the fact that M is left stable that $N_0, \cdots, N_{s-1}$ are also left stable. Applying τ_A if necessary, we get a path $M'' \rightsquigarrow P$ with M'' in the τ_A-orbit of M. Replacing, if necessary, M and M'' by other modules in the same τ_A-orbit, we get a path from I to P passing through a morphism in $\mathrm{rad}^\infty(\mathrm{mod}A)$, a contradiction to (3.3). □

4.2. In the sequel, we use the following notation: if $\mathcal{C}$ and $\mathcal{D}$ are two classes of A-modules, then $\mathrm{Hom}_A(\mathcal{C}, \mathcal{D}) = 0$ (or $\mathrm{Hom}_A(\mathcal{C}, \mathcal{D}) \neq 0$) means that there exists no non-zero morphism (or that there exists a non-zero morphism, respectively) from a module in $\mathcal{C}$ to a module in $\mathcal{D}$. With this notation, we have the following lemma.

LEMMA. *Let $M \in indA$ be a module not in Γ.*
(a) *If $Hom_A(M, \Gamma) \neq 0$, then M belongs to $\mathcal{L}_A \setminus \mathcal{R}_A$.*
(b) *If $Hom_A(\Gamma, M) \neq 0$, then M belongs to $\mathcal{R}_A \setminus \mathcal{L}_A$.*
(c) *Either $Hom_A(\Gamma, M) = 0$, or $Hom_A(M, \Gamma) = 0$.*

Proof. (a) Suppose there is a non-zero morphism $f\colon M \longrightarrow N$ with $N \in \Gamma$. Clearly, $f \in \mathrm{rad}^\infty(\mathrm{mod}A)$ and so the left stable part of Γ is infinite. By (1.3), there exists, for each $t \geq 1$, a path in ind$A$

$$(*) \qquad M \xrightarrow{g_t} M_t \xrightarrow{f_t} M_{t-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{f_1} M_0 = N$$

where $f_1, \cdots, f_t$ are irreducible morphsims. Since Γ is non-semiregular and has only finitely many τ_A-orbits, there exists an indecomposable projective P, a direct summand L of its radical and $t_0 \geq 1$ such that

there is a path $M_{t_0} \rightsquigarrow \tau_A L$. Since $\mathrm{id}_A \tau_A L \geq 2$, we deduce that M does not belong to $\mathcal{R}_A$. Now, assume that M does not belong to $\mathcal{L}_A$. Then M has a predecessor M' such that $\mathrm{pd}_A M' \geq 2$. Hence, there exists an indecomposable injective I and a path $I \longrightarrow \tau_A M' \longrightarrow * \longrightarrow M' \rightsquigarrow M$ in $\mathrm{ind}A$. This gives a path

$$I \rightsquigarrow M \xrightarrow{g} M_{t_0} \rightsquigarrow \tau_A L \rightsquigarrow P$$

which factors through a morphism in $\mathrm{rad}^\infty(\mathrm{mod}A)$, a contradiction to (3.3).
(b) The proof is dual to that of (a).
(c) It follows directly from (a) and (b). □

4.3. Assume now that A is representation-infinite. Then the left stable part ${}_l\Gamma$ or the right stable part ${}_r\Gamma$ of Γ is infinite. Suppose ${}_l\Gamma$ is infinite. Since Γ has only finitely many τ_A-orbits, then, clearly ${}_l\Gamma$ has only finitely many non-trivial components (that is, containing more than one point). We choose, for each such left stable component, a maximal subsection, and denote these by ${}_1\Sigma, \cdots, {}_s\Sigma$. For each i, with $1 \leq i \leq s$, we denote by ${}_\infty A_i$ the full subcategory of A generated by the support of (all the A-modules lying on) ${}_i\Sigma$. We define the *left end algebra* ${}_\infty A$ of A by ${}_\infty A = {}_\infty A_1 \times \cdots \times {}_\infty A_s$.

We define dually the *right end algebra* A_∞ of A.

Clearly, these notions generalize those introduced for tilted algebras in [23].

LEMMA. *With the above notations,*

(a) *For each i, ${}_\infty A_i$ is a tilted algebra having ${}_i\Sigma$ as a complete slice.*
(b) *If P, P' are indecomposable projective A-modules such that $Hom_A(P, P') \neq 0$, and P' is a projective ${}_\infty A_i$-module, then P is also a projective ${}_\infty A_i$-module. In particular, for each i, ${}_\infty A_i$ is a full convex subcategory of A.*

Proof. (a) It follows from the definition of ${}_\infty A_i$ that the direct sum M of all the indecomposable A-modules lying in ${}_i\Sigma$ is a faithful ${}_\infty A_i$-module. Since Γ is generalized standard (3.4), we have $\mathrm{Hom}_{{}_\infty A_i}(U, \tau_{{}_\infty A_i} V) = 0$ for any two indecomposable summands U and V of M. Applying [26, 31], we infer that ${}_\infty A_i$ is tilted, having ${}_i\Sigma$ as a complete slice.
(b) Since M is a faithful ${}_\infty A_i$-module, there exist $m > 0$ and a monomorphism $P \longrightarrow M^{(m)}$. The second statement follows. □

4.4. LEMMA. *With the above notations.*

(a) *If $P \in indA$ is a projective module which is not an ${}_\infty A$-module, then P lies in Γ.*

(b) *If $I \in indA$ is an injective module which is not an A_∞-module, then I lies in Γ.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) The existence of P implies that $A \neq_\infty A$. Since A is connected, there is a sequence of indecomposable projective modules $P' = P_0$, $P_1, \cdots, P_t = P$ such that P' is a projective ${}_\infty A$-module and, for each $i = 1, \cdots, t$, we have either $\mathrm{Hom}_A(P_{i-1}, P_i) \neq 0$ or $\mathrm{Hom}_A(P_i, P_{i-1}) \neq 0$. Without loss of generality, we may assume that, for $i > 0$, P_i is not a projective ${}_\infty A$-module. In particular, it follows from (4.3)(b) that $\mathrm{Hom}_A(P_1, P) = 0$. Therefore, $\mathrm{Hom}_A(P, P_1) \neq 0$. Hence, there exists an index j such that $\mathrm{Hom}_A({}_j\Sigma, P_1) \neq 0$ (because ${}_j\Sigma$ is a complete slice in $\mathrm{mod}_\infty A_i$, and P_i is not an ${}_\infty A$-module). Applying (4.1), we infer that P_1 belongs to Γ. Now, if $\mathrm{Hom}_A(P_1, P_2) \neq 0$, then, again by (4.1), P_2 belongs to Γ. Assume that $\mathrm{Hom}_A(P_2, P_1) \neq 0$. If P_2 does not belong to Γ, then any morphism $P_2 \longrightarrow P_1$ would factor through the union ${}_1\Sigma \cup \cdots \cup_s \Sigma$, and so P_2 would be an ${}_\infty A$-module, a contradiction. Therefore, P_2 lies in Γ. Proceeding inductively in this fashion, we infer that P lies in Γ, as required. □

4.5. We have shown that an indecomposable projective (or injective) A-module either lies in Γ or is a projective ${}_\infty A$-module (or an injective A_∞-module, respectively). We now show that the endomorphism algebra of the projectives in Γ having the property that the corresponding injectives lie also in Γ forms a full convex subcategory of A.

COROLLARY. *Let P denote the direct sum of all indecomposable projective A-modules P_x which lie in Γ and such that the corresponding indecomposable injective I_x also lies in Γ. Then $C = EndP$ is a full convex subcategory of A.*

Proof. This follows from the fact that the class of projectives in ${}_\infty A$ is closed under projective predecessors and, dually, the class of injectives in A_∞ is closed under injective successors, by (4.3)(b). □

4.6. We are now ready to show the main result of this section.

THEOREM. *Let A be a laura algebra which is not quasi-tilted. Then $\Gamma(modA)$ has a unique non-semiregular component Γ which is quasi-directed and faithful.*
Further, if Γ' is a component of $\Gamma(modA)$ distinct from Γ, then Γ' is a semiregular component satisfying exactly one of the following conditions:

(i) Γ' *is a component of* $\Gamma(mod_\infty A)$ *such that* $Hom_A(\Gamma', \Gamma) \neq 0$ *and lying in* $\mathcal{L}_A \setminus \mathcal{R}_A$.
(ii) Γ' *is a component of* $\Gamma(modA_\infty)$ *such that* $Hom_A(\Gamma, \Gamma') \neq 0$ *and lying in* $\mathcal{R}_A \setminus \mathcal{L}_A$.

Proof. By (3.5), $\Gamma(\mathrm{mod}A)$ has a non-semiregular quasi-directed component Γ. If ${}_\infty A = 0$, then by (4.4), all the projectives lie in Γ. Moreover, $\mathrm{rad}^\infty_A(-, M) = 0$ for all $M \in \Gamma$. Therefore, Γ is a π-component containing all the projective modules. By [1](2.2), A is a left glued algebra. Dually, if $A_\infty = 0$, then A is a right glued algebra. In these two cases, the required result follows from [1](3.5). We may thus assume that $A_\infty \neq 0$ and ${}_\infty A \neq 0$. This means that the right and the left stable parts of Γ are infinite.
By (4.3)(a), any indecomposable projective ${}_\infty A$-module can be embedded in a direct sum of modules in Γ. Since the remaining projectives lie in Γ, we infer that Γ is faithful.
Let now Γ' be a component of $\Gamma(\mathrm{mod}A)$ distinct from Γ, and M be a module in Γ'. We claim that, if M is not an ${}_\infty A$-module, then M belongs to $\mathcal{R}_A \setminus \mathcal{L}_A$ and, dually, if M is not an A_∞-module, then M belongs to $\mathcal{L}_A \setminus \mathcal{R}_A$. Indeed, assume that M is not an ${}_\infty A$-module. Using (4.4), we infer that there exists a projective P in Γ such that $\mathrm{Hom}_A(P, M) \neq 0$. By (4.2), M belongs to $\mathcal{R}_A \setminus \mathcal{L}_A$. This establishes our claim.
This fact entails several consequences.

(a) Every indecomposable in Γ' is an ${}_\infty A$-module or an A_∞-module. Indeed, if M is neither an ${}_\infty A$-module nor an A_∞-module, then it belongs to both $\mathcal{R}_A \setminus \mathcal{L}_A$ and $\mathcal{L}_A \setminus \mathcal{R}_A$, an absurdity.

(b) Γ' is either a component of $\Gamma(\mathrm{mod}_\infty A)$ or a component of $\Gamma(\mathrm{mod}A_\infty)$. Indeed, assume that Γ' contains at the same time an A_∞-module L and an ${}_\infty A$-module N. Since Γ' is connected, we may assume that there exists an irreducible morphism $L \longrightarrow N$ or $N \longrightarrow L$. Since L is an A_∞-module, it lies in $\mathcal{R}_A \setminus \mathcal{L}_A$ and similarly N lies in $\mathcal{L}_A \setminus \mathcal{R}_A$. Since $\mathcal{L}_A$ is closed under predecessors and $\mathcal{R}_A$ is closed under successors, both $L \longrightarrow N$ or $N \longrightarrow L$ lead to contradictions.

(c) Assume Γ' is a component of $\Gamma(\mathrm{mod}_\infty A)$, then it lies entirely inside $\mathcal{L}_A \setminus \mathcal{R}_A$. Moreover, we have $\mathrm{Hom}_A(\Gamma', \Gamma) \neq 0$, because any indecomposable in Γ' embeds into a direct sum of modules in Γ. Further, Γ' is semiregular without injectives (since any injective in Γ' would embed into a direct sum of modules in Γ). Dually, if Γ' is a component of $\Gamma(\mathrm{mod}A_\infty)$, then it lies entirely inside $\mathcal{R}_A \setminus \mathcal{L}_A$, satisfies $\mathrm{Hom}_A(\Gamma, \Gamma') \neq 0$, and is semiregular without projective modules.

Since the above arguments show at the same time that Γ is the unique non-semiregular component of $\Gamma(\mathrm{mod}A)$, the proof is complete. □

REMARK. We have shown in the course of the proof that, if A is a laura algebra which is not quasi-tilted, then A is left (or right) glued if and only if ${}_\infty A = 0$ (or $A_\infty = 0$, respectively).

4.7. COROLLARY. *Let A be a laura algebra which is not quasi-tilted. Then*

(a) *$ind_\infty A \cup ind A_\infty$ is cofinite in $indA$.*
(b) *$\mathcal{L}_A \cap \mathcal{R}_A$ is finite and lies in the unique non-semiregular component of $\Gamma(modA)$.*

Proof. (a) All indecomposable A-modules which are neither ${}_\infty A$-modules nor A_∞-modules lie in Γ, by the proof of (4.6) and, further, at most finitely many indecomposable modules in Γ which are neither ${}_\infty A$-modules nor A_∞-modules.

(b) By (4.6), the indecomposables not in Γ lie in $\mathcal{L}_A \setminus \mathcal{R}_A$ or in $\mathcal{R}_A \setminus \mathcal{L}_A$. Finally, at most finitely many indecomposables in Γ lie neither in $\mathcal{L}_A \setminus \mathcal{R}_A$ nor in $\mathcal{R}_A \setminus \mathcal{L}_A$. □

4.8. COROLLARY. *Let A be a laura algebra which is not quasi-tilted. Then A is weakly shod if and only if the unique non-semiregular component of $\Gamma(modA)$ contains no oriented cycles.*

4.9. **The Auslander-Reiten quiver of a laura algebra.** We are now able to describe the shapes of the components of the Auslander-Reiten quiver of a laura algebra A which is not quasi-tilted. By (4.6), $\Gamma(\mathrm{mod}A)$ has a unique non-semiregular quasi-directed and faithful component Γ. Also, if Γ' is a component of $\Gamma(\mathrm{mod}A)$ distinct from Γ, then it is a component of a tilted algebra (which is itself a connected factor of ${}_\infty A$ or A_∞). Using the well-known description of the Auslander-Reiten quiver of tilted algebras [25], we deduce the possible shapes of the components of $\Gamma(\mathrm{mod}A)$.

(a) A unique and faithful non-semiregular and quasi-directed component.
(b) Postprojective component(s) (those of $\Gamma(\mathrm{mod}_\infty A)$).
(c) Preinjective component(s) (those of $\Gamma(\mathrm{mod}A_\infty)$).
(d) Stable tubes.
(e) Components of type $\mathbf{ZA}_\infty$.
(f) Components obtained from tubes or from components of type $\mathbf{ZA}_\infty$ by finitely many ray inserctions of by finitely many coray insertions.

Moreover, the components of $\Gamma(\mathrm{mod}_\infty A)$ (or $\Gamma(\mathrm{mod}A_\infty)$) which are fully embedded in $\Gamma(\mathrm{mod}A)$ are semiregular without injective (or projective, respectively) modules and are contained in $\mathcal{L}_A \setminus \mathcal{R}_A$ (or in $\mathcal{R}_A \setminus \mathcal{L}_A$, respectively).

Thus, $\Gamma(\mathrm{mod}A)$ has a shape similar to that of the Auslander-Reiten quiver of a tilted algebra which is not concealed [23](4.1) (we stress, however, that, in general, the non-semiregular component Γ may contain cycles and, even, if it does not, is generally not a connecting component).

4.10. The above results yield an explicit description of the classes $\mathcal{L}_A$ and $\mathcal{R}_A$. Assume that A is a laura algebra which is not quasi-tilted, and let Γ denote the faithful non-semiregular quasi-directed component of $\Gamma(\mathrm{mod}A)$. Then, Γ contains at the same time an injective and a projective. Following [21], we say that a primitive idempotent $e \in A$ is a *strong sink* if the corresponding indecomposable injective I_e is such that there is no non-trivial path from another indecomposable injective to I_e. We consider the full connected subquiver Σ_- of Γ consisting of the modules M such that there exist a strong sink e and a path $I_e \rightsquigarrow M$, and, moreover, any such path is sectional. Then, by definition, Σ_- is a maximal subsection of Σ, called the *left extremal subsection* of Γ. We construct, dually, the *right extremal subsection* Σ_+ of Γ.

COROLLARY. *Let A be a weakly shod algebra which is not quasi-tilted.*

(a) *$\mathcal{L}_A$ consists of all the predecessors of Σ_-, and its support is a tilted algebra, having ${}_\infty A$ as a full convex subcategory.*
(b) *$\mathcal{R}_A$ consists of all the successors of Σ_+, and its support is a tilted algebra, having A_∞ as a full convex subcategory.*

Proof. We only prove (a) since the proof of (b) is dual.
(a) The first statement follows easily from (1.5), the above description and the definition of strong sink. Let B denote the support algebra of Σ_-. The direct sum M of the indecomposable A-modules lying in Σ_- is a faithful B-module. Since Γ is generalized standard, we have $\mathrm{Hom}_A(U, \tau_B V) = 0$ for any two indecomposable summands U and V of M. Applying [26, 31], we get that B is tilted, having Σ_- as a complete slice. The last statement follows from (4.3). □

5. Two sided gluings of tilted algebras

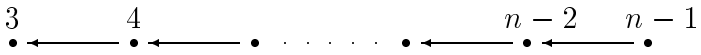
5.1. The results of Section 4 show that a laura algebra which is not quasi-tilted can be seen as a two-sided gluing of tilted algebras. The aim of this section is to formalize this idea and to characterize the laura algebras in this way.

DEFINITION. Let ${}_1B, \cdots, {}_sB$ be representation-infinite tilted algebras having complete slices ${}_1\Sigma, \cdots, {}_s\Sigma$ in components not containing projective modules, let $B_1, \cdots, B_r$ be representation-infinite tilted algebras having complete slices $\Sigma_1, \cdots, \Sigma_r$ in components not containing injective modules, and let C be a representation-finite algebra. Write ${}_\infty B = {}_1 B \times \cdots \times {}_s B$ and $B_\infty = B_1 \times \cdots \times B_r$. We say that an algebra A is a *two sided gluing of* ${}_1B, \cdots, {}_sB, B_1, \cdots, B_r$ *by* C *along the slices* ${}_1\Sigma, \cdots, {}_s\Sigma, \Sigma_1, \cdots, \Sigma_r$ (or simply a *double glued algebra*) provided $A = C$ or:

(a) Each of ${}_1B, \cdots, {}_sB, B_1, \cdots, B_r$ and C is a full convex subcategory of A and any primitive idempotent in A belongs to one of these subcategories;
(b) $\mathrm{ind}_\infty B \cup \mathrm{ind} B_\infty$ is cofinite in $\mathrm{ind} A$.
(c) Each ${}_i\Sigma$ is fully embedded in $\Gamma(\mathrm{mod} A)$ and no injective A-module is a proper predecessor of ${}_1\Sigma \cup \cdots \cup {}_s\Sigma$, considered as embedded in $\mathrm{ind} A$ and, dually, each Σ_j is fully embedded in $\Gamma(\mathrm{mod} A)$ and no projective A-module is a proper successor of $\Sigma_1 \cup \cdots \cup \Sigma_r$, considered as embedded in $\mathrm{ind} A$.

5.2. EXAMPLES. (a) Assume ${}_\infty B = 0$, then A is left glued. Conversely, any left glued algebra is of this form. Dually, an algebra A is right glued if and only if it is double glued with $B_\infty = 0$.

(b) Examples (2.3) (a) and (b) show double glued algebras. In the example (2.3)(a), ${}_\infty B$ and B_∞ are two copies of the Kronecker algebra, while C is the radical square zero algebra given by the following quiver



In the example (2.3)(b), ${}_\infty B$ and B_∞ are again two copies of the Kronecker algebra, while C is the radical square zero algebra given by the quiver

5.3. REMARKS. (a) Let A be a double glued algebra. Since C is an arbitrary representation-finite algebra, a component of $\Gamma(\mathrm{mod} A)$ containing modules not in $\mathrm{ind}_\infty B \cup \mathrm{ind} B_\infty$ may contain periodic modules

and oriented cycles. It is actually a faithful non-semiregular quasi-directed component. As we see below, it is unique.

(b) Let A be a double glued algebra. It is not difficult to see that there are no non-zero morphisms from a projective in B_∞ to one in $C \times_\infty B$, nor from one in C to one in ${}_\infty B$. In particular, A may be written in matrix form

$$A \cong \begin{pmatrix} {}_\infty B & 0 & 0 \\ M_1 & C & 0 \\ M_2 & M_3 & B_\infty \end{pmatrix}$$

where M_1, M_2, M_3 are appropriate bimodules. Consequently, A may be obtained from C by a sequence of one-point extensions and co-extensions.

(b) It is easy to see that A is representation-equivalent to ${}_\infty B \times B_\infty$, so that A is tame if and only if so is each of ${}_1B, \cdots, {}_s B, B_1, \cdots, B_r$.

5.4. The main theorem of this section is the following.

THEOREM. *Let A be an algebra which is not quasi-tilted. Then A is laura if and only if A is double glued.*

Proof. Suppose that A is a laura algebra which is not quasi-tilted, and let C be as in (4.5). Then it follows easily from (4.3), (4.5) and (4.7) that A is a two-sided gluing of ${}_\infty A$, A_∞ by C along the slices considered in (4.3).
Conversely, assume that A is a double glued algebra, and assume the notations in the definition (5.1) above. By hypothesis, each of the slices ${}_i\Sigma$ (with $1 \leq i \leq s$) and Σ_j (with $1 \leq j \leq r$) is fully emdedded in ind$A$. Let ${}_\infty\Sigma =_1 \Sigma \cup \cdots \cup_s \Sigma$ and $\Sigma_\infty = \Sigma_1 \cup \cdots \cup \Sigma_r$. All predecessors of ${}_\infty\Sigma$ lie in ind$_\infty B$. Moreover, each indecomposable module in ind$_\infty B$ which precedes ${}_\infty\Sigma$ lies in $\mathcal{L}_A$, because no injective A-module is a proper predecessor of ${}_\infty\Sigma$. Dually, all successors of Σ_∞ lie in indB_∞ and each indecomposable module in indB_∞ which is a successor of Σ_∞ lies in $\mathcal{R}_A$. Therefore, $\mathcal{L}_A \cup \mathcal{R}_A$ is contained in ind$_\infty$ $A \cup$ indA_∞. Consequently, A is a laura algebra. □

6. The infinite radical of a laura algebra

6.1. The study of the Auslander-Reiten quiver $\Gamma(\text{mod}A)$ of an algebra A gives important informations on the category mod$A$. However, the morphisms in rad$^\infty$(mod$A$) are not represented there, and so it is important to study also this ideal to understand the complexity of mod$A$. Of particular interest is the study of when rad$^\infty$(mod$A$) is nilpotent. This has been considered, for instance, in [24, 13, 14, 30, 9]. In this

section, we use the description of laura algebras given in Section 5 to study these algebras such that $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ is nilpotent.

Let A be a representation-infinite algebra. If there exists a positive integer η_A such that $(\mathrm{rad}^{\infty}(\mathrm{mod}A))^{\eta_A} = 0$ but $(\mathrm{rad}^{\infty}(\mathrm{mod}A))^{\eta_A - 1} \neq 0$, then we say that $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ is *nilpotent of index* η_A. Otherwise, we just write $\eta_A = \infty$. It follows from [13] that A is representation-finite if and only if $(\mathrm{rad}^{\infty}(\mathrm{mod}A))^2 = 0$ and so, if A is representation-infinite, then $\eta_A \geq 3$. Also, by [30], one can find algebras A with finite but arbitrarily large nilpotency index.

Our purpose here is to show that if A is a representation-infinite laura algebra, then $\eta_A = 3, 4, 5$ or ∞. A similar result has been proven for tilted algebras in [9].

6.2. The following proposition characterizes the infinite radical of the module category of a quasi-tilted algebra.

PROPOSITION. [34](Corollary B) *Let A be a quasi-tilted algebra. Then the following conditions are equivalent:*

(a) *A is domestic.*
(b) *A is tame and no full convex subcategory of A is a tubular algebra.*
(c) *rad^{∞} $(modA)$ is nilpotent.*
(d) *$(rad^{\infty}$ $(modA))^5 = 0$.*

6.3. We now generalize the above result to laura algebras as follows.

THEOREM. *Let A be a representation-infinite laura algebra. The following conditions are equivalent:*

(a) *A is domestic.*
(b) *A is tame and no full convex subcategory of A is a tubular algebra.*
(c) *rad^{∞} $(modA)$ is nilpotent.*

Furthermore, if this is the case, and η_A is the nilpotency index of $rad^{\infty}(modA)$, then, $3 \leq \eta_A \leq 5$. Moreover, $\eta_A = 3$ if one of the following holds:

(i) *A is tilted and one of ${}_{\infty}A$ or A_{∞} is zero.*
(ii) *A is not quasi-tilted and one of ${}_{\infty}A$ or A_{∞} is zero.*

Proof. If A is quasi-tilted, then the equivalence of (a), (b) and (c) follows from (6.2). Moreover, if A is tilted such that one of ${}_{\infty}A$ or A_{∞} is zero, then $\eta_A = 3$ by [9].
We may then assume that A is not quasi-tilted.
We first assume that (c) holds. By (3.5) and the results of section 4, there exists a faithful non-semiregular quasi-directed component Γ. Moreover, at least one of ${}_{\infty}A$ or A_{∞} is non-zero. Suppose that ${}_{\infty}A$ is

non-zero. By construction, ${}_\infty A$ is a product of tilted algebras whose connecting components contains no projective modules. On the other hand, since $\mathrm{rad}^\infty(\mathrm{mod}A)$ is nilpotent, we get from [24] that ${}_\infty A$ is tame. We then infer from [23] that ${}_\infty A$ is a product of tilted algebras of Euclidean type. Dually, if A_∞ is non-zero, then it is the product of tilted algebras of Euclidean type. Since both ${}_\infty A$ and A_∞ are domestic, so is A. This shows (a). Since, clearly, (a) implies (b), we just have to show that (b) implies (c). Note that, by [2](3.4), if A is a representation-infinite laura algebra which is not quasi-tilted, then it contains no full subcategory which is tubular, therefore assuming (b) reduces to assuming that A is tame, and this implies that both ${}_\infty A$ and A_∞ are tame, thus each of them is a product of tilted algebras of Euclidean type. We then consider 3 cases: (1) ${}_\infty A = 0$ and $A_\infty \neq 0$; (2) ${}_\infty A \neq 0$ and $A_\infty = 0$; (3) ${}_\infty A \neq 0$ and $A_\infty \neq 0$.

Case 1: ${}_\infty A = 0$ and $A_\infty \neq 0$.
In this case, A is a left glued algebra and Γ is a π-component of $\Gamma(\mathrm{mod}A)$. Moreover, Γ contains injective modules since, otherwise, by [1](2.8), it would be a connecting component. By construction, $(\mathrm{rad}^\infty(\ \mathrm{mod}_i A_\infty))^3 = 0$, for each $i = 1, \cdots, s$ (in the notation of (4.3)). Observe also that $\mathrm{ind}(A_\infty)$ is cofinite in $\mathrm{ind}A$, and that all the indecomposable A-modules which are not A_∞-modules belong to Γ. If now $(\mathrm{rad}^\infty(\mathrm{mod}A)\ )^3 \neq 0$, then there is a path in $\mathrm{ind}A$

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4$$

with the f_i in $\mathrm{rad}^\infty(\mathrm{mod}A)$ and such that $f_3 f_2 f_1 \neq 0$. Observe first that M_4 does not belong to Γ: indeed, by [1], we have that $\mathrm{rad}^\infty_A(-, N) = 0$ for each $N \in \Gamma$. Hence, M_4 is an indecomposable ${}_i A_\infty$-module for some i. Since $f_3 f_2$ is a non-zero morphism in $(\mathrm{rad}^\infty(\mathrm{mod}A)\ )^2$ then, by the description of the Auslander-Reiten quiver of a tilted algebra of Euclidean type, we infer that M_3 is a regular ${}_i A_\infty$-module and M_2 is either a postprojective ${}_i A_\infty$-module or a module in $\mathrm{ind}A \setminus \mathrm{ind}A_\infty$. In both cases, M_2 lies in Γ and hence $\mathrm{rad}^\infty_A(-, M_2) = 0$. This, however, contradicts our assumption on f_1. Therefore, in this case, $\mathrm{rad}^\infty(\mathrm{mod}A)$ is nilpotent of index $\eta_A = 3$.

Case 2: ${}_\infty A \neq 0$ and $A_\infty = 0$.
This case is dual to the first one, and we leave to the reader the details of the proof.

Case 3: ${}_\infty A \neq 0$ and $A_\infty \neq 0$.
By [14](2.1), we have

$$(\mathrm{rad}^\infty(\mathrm{mod}_\infty A_i))^3 = 0 = (\mathrm{rad}^\infty(\mathrm{mod}_j A_\infty))^3$$

for all i, j, $1 \leq i \leq t$, $i \leq j \leq s$. Moreover, it is easily seen that $\mathrm{Hom}_A(M, N) = 0$, in the following cases:

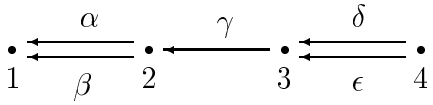
(i) $M \in \mathrm{ind}_\infty A_i \setminus \Gamma$ and $N \in \mathrm{ind}_\infty A_j \setminus \Gamma$, with $i \neq j$.
(i') $M \in \mathrm{ind}_i A_\infty \setminus \Gamma$ and $N \in \mathrm{ind}_j A_\infty \setminus \Gamma$, with $i \neq j$.
(ii) $M \in \mathrm{ind}_i A_\infty$ and $N \in \mathrm{ind}_\infty A_j \setminus \Gamma$, for all i and j.
(ii') $M \in \mathrm{ind}_i A_\infty \setminus \Gamma$ and $N \in \mathrm{ind}_\infty A_j$, for all i and j.
(iii) $M \in \Gamma$ and $N \in \mathrm{ind}_\infty A_j \setminus \Gamma$, for all j.
(iii') $M \in \mathrm{ind}_i A_\infty \setminus \Gamma$ and $N \in \Gamma$, for all i.

Suppose now that $(\mathrm{rad}^\infty(\mathrm{mod}A))^5 \neq 0$. Then there exists a path in $\mathrm{ind}A$

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \xrightarrow{f_4} M_5 \xrightarrow{f_5} M_6$$

with $f_i \in \mathrm{rad}^\infty(\mathrm{mod}A)$, for $i = 1, \cdots, 5$, and $f_5 \cdots f_1 \neq 0$. Using the above observations, it is not difficult to see that if $M_j \in \mathrm{ind}_\infty A_i \setminus \Gamma$, for some i, then $j \leq 2$ and, dually, if $M_j \in \mathrm{ind}_i A_\infty \setminus \Gamma$, for some i, then $j \geq 5$. Therefore, M_3 and M_4 both belong to Γ, and $\mathrm{rad}^\infty_A(M_3, M_4) \neq 0$, which is a contradiction, because Γ is generalized standard. Therefore, $\mathrm{rad}^\infty(\mathrm{mod}A)$ is nilpotent and $\eta_A \leq 5$. This completes the proof. □

6.4. Example. While it is easy to find examples of laura algebras with $\eta_A = 3$ or $\eta_A = 5$, we now give an example of an algebra having $\eta_A = 4$. Let A be given by the quiver



bound by $\alpha\gamma = 0$, $\gamma\delta = 0$, and $\gamma\epsilon = 0$. Then it is easily seen that A is a strict shod algebra. Moreover, any postprojective ${}_\infty A$-module M (or preinjective ${}_\infty A$-module N) has support the full convex subcategory of A generated by $\{1, 2\}$ (or $\{3, 4\}$, respectively). Therefore $\mathrm{Hom}_A(M, N) = 0$. This clearly implies $(\mathrm{rad}^\infty(\mathrm{mod}A))^4 = 0$. On the other hand, $(\mathrm{rad}^\infty(\mathrm{mod}A))^3 \neq 0$, as is seen from the morphisms

$$P_3 \longrightarrow S_3 \longrightarrow U \longrightarrow I_3$$

where U is the uniserial module of length two with socle S_3 and top S_4.

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