TILTING MODULES OVER SPLIT-BY-NILPOTENT EXTENSIONS

Ibrahim Assem and Nikolaos Marmaridis

Département de mathématiques et d'informatique
Université de Sherbrooke
Sherbrooke, Québec, J1K 2R1, Canada
and
Department of Mathematics
University of Ioannina
45110, Ioannina, Greece

Abstract. Let a finite dimensional algebra R be a split extension of an algebra A by a nilpotent bimodule Q. We give necessary and sufficient conditions for a (partial) tilting module T_A to be such that $T \otimes_A R_R$ is a (partial) tilting module. If this is not the case, but Q_A is generated by the tilting module T_A, then there exists a quotient $\tilde{R}$ of R such that $T \otimes_A \tilde{R}_{\tilde{R}}$ is a tilting module.

Introduction

Let A be a finite dimensional algebra over a commutative field k. By module is meant throughout a finitely generated right A-module. Following [5], we call a module T_A a tilting module if it satisfies the following conditions : If only 1) and 2) are satisfied, then T is said to be a partial tilting module. In this note, we are interested in the problem of extending a (partial) tilting module. More precisely, let A and R be two finite dimensional k-algebras such that there exists a split surjective algebra morphism $R \to A$ whose kernel Q is contained in the radical of R: we then say that R is a split extension of A by the nilpotent bimodule Q, or simply a split-by-nilpotent extension. This notion is easily seen to be equivalent to that of θ-extension, see [7]. It is rather general: indeed, let R be a bound quiver algebra and Q be generated by a family of arrows, then there exists a subalgebra A of R such that R is the split extension of A by Q.

Let thus R be a split extension of A by the nilpotent bimodule Q. The module categories over A and R are related by the classical "extension of scalars" functor $- \otimes_A R_R$ of [3]. We ask under which conditions the image of a tilting A-module under this functor is a tilting R-module. This problem was first solved by Tachikawa and Wakamatsu when R is the trivial extension of A by the dual of its trace ideal [12], then by Miyachi when R is any trivial extension algebra [9]. It was then solved in [8] for θ-extensions. Here, we present a new approach which we believe is more conceptual and allows an easier calculation of examples. Our first result is thus the following theorem.

Theorem A. *Let R be a split extension of A by the nilpotent bimodule Q, and T_A be an A-module. Then $T \otimes_A R$ is a (partial) tilting R-module if and only if T_A is a (partial) tilting A-module,* $\operatorname{Hom}_A(T \otimes_A Q, \tau_A T) = 0$ *and* $\operatorname{Hom}_A(\mathrm{D}({}_AQ), \tau_A T) = 0$.

A (partial) tilting module satisfying the equivalent conditions of the theorem is called extendable. Clearly, T is extendable if and only if $\operatorname{Ext}^1_A(T, T \otimes_A Q) = 0$, $\operatorname{Ext}^1_A(T, \mathrm{D}({}_AQ)) =$

1991 *Mathematics Subject Classification.* 16G20, 16D90.
Key words and phrases. Tilting modules, split-by-nilpotent extensions, trivial extensions.

0. If T is actually a tilting module, this is equivalent to saying that $T \otimes_A Q$ and $\mathrm{D}({}_AQ)$, respectively, are generated by T.

We also show that, it T is an extendable tilting module, the torsion pair induced by $T \otimes_A R$ in $\mathrm{mod}R$ is entirely determined by that induced by T in $\mathrm{mod}A$. Moreover, the endomorphism algebra of $T \otimes_A R$ is itself a split extension of $B = \mathrm{End}\, T_A$ by the nilpotent bimodule $\mathrm{Hom}_A({}_BT, {}_BT \otimes_A Q)$. We next consider the case where T_A is not extendable but Q_A is generated by T_A (this is the case, for instance, when ${}_AQ_A$ is the minimal injective congenerator bimodule ${}_A\,\mathrm{D}\,A_A$). We prove that then T is extendable over a quotient algebra of R, thus generalising [12] (1.6).

Theorem B. *Let R be the split extension of A by the nilpotent bimodule Q, and T_A be a tilting module which generates Q_A. Let $\tilde{R}$ be the split extension of A by the nilpotent bimodule $\tilde{Q} = Q/Q'$ where $Q' = \{q \in Q \mid Tq = 0\}$. Then $T \otimes_A \tilde{R}$ is a tilting $\tilde{R}$-module.*

Clearly, the results dual to the ones above, for cotilting modules, hold. We leave to the reader their straightforward formulations.

The authors were informed that some results of this note were generalised in [6] to the case of tilting modules of large projective dimension, in [10] to the case of tilting complexes, and in [4] to the case of $*$-modules.

We use freely and without further reference properties of the module categories and Auslander-Reiten sequences as can be found, for instance, in [2,11]. For an algebra C, we denote by τ_C the Auslander-Reiten translation DTr_C in $\mathrm{mod}\, C$. For tilting theory, we refer the reader to [1]. The paper is organised as follows: in section (1), we survey those properties of split extensions that will be needed, our two theorems will be proved respectively in sections (2) and (3), while section (4) is devoted to examples.

1. Extension and restriction of scalars

1.1. Let A and R be two finite dimensional algebras over a commutative field k, such that R is a split extension of A by the nilpotent bimodule Q, that is, we have a split short exact sequence of abelian groups

$$0 \to Q \xrightarrow{\iota} R \xrightarrow{\pi} A \to 0$$

where $\iota : q \mapsto (0, q)$ is the inclusion of Q as a two-sided ideal of $R = A \oplus Q$, and the projection (algebra) morphism $\pi : (a, q) \mapsto a$ has as section the inclusion morphism $\sigma : a \mapsto (a, 0)$ Thus, the k-vector space $R = A \oplus Q$ has the multiplication

$$(a_1, q_1)(a_2, q_2) = (a_1a_2, a_1q_2 + q_1a_2 + q_1q_2)$$

(where $a_1, a_2 \in A$ and $q_1, q_2 \in Q$) .

Associated with π and σ are the tensor product functors

$$- \otimes_R A_A : \mathrm{mod}\, R \to \mathrm{mod}\, A \quad \text{and} \quad - \otimes_A R_R : \mathrm{mod}\, A \to \mathrm{mod}\, R$$

(see [3]). We clearly have an isomorphism of functors

$$(- \otimes_A R_R) \otimes_R A_A \cong 1_{\mathrm{mod}\, A}.$$

For our purposes, another expression of the functor $- \otimes_R A_A$ will be useful. Let X_R be an R-module. The R-module X/XQ is annihilated by Q and hence has a canonical A-module structure. This yields a functor $\mathrm{mod}\, R \to \mathrm{mod}\, A$ which is additive and right exact (because it is a cokernel functor). Hence, by Watts' theorem, it is functorially isomorphic to $- \otimes_R (R/RQ)_A \cong - \otimes_R A_A$.

Lemma. *For any R-module X, the canonical R-linear epimorphism $p_X : X \to X/XQ$ is minimal.*

Proof. By Nakayama's lemma, the canonical epimorphism $f : X \to X/X \operatorname{rad} R$ is minimal. Since $Q \subseteq \operatorname{rad} R$, we have a canonical morphism $g : X/XQ \to X/X \operatorname{rad} R$ such that $f = g \circ p_X$. In particular, g is an epimorphism. Now let $h : W \to X$ be such that $p_X \circ h$ is an epimorphism. Then so is $g \circ p_X \circ h = f \circ h$. Since f is minimal, h is an epimorphism. □

1.2. **Lemma.** *Let M be an A-module. There exists a bijective correspondence between the isomorphism classes of indecomposable projective summands of M in* $\operatorname{mod} A$*, and the isomorphism classes of indecomposable summands of $M_A \otimes_A R_R$ in* $\operatorname{mod} R$*, given by $N \mapsto N \otimes_A R_R$.*

Proof. We first show that if N_A is indecomposable in $\operatorname{mod} A$, then $N \otimes_A R_R$ is indecomposable in $\operatorname{mod} R$. Indeed, assume that $N \otimes_A R_R \cong X_1 \oplus X_2$. Then $N \cong N \otimes_A R_R \otimes_R A_A \cong (X_1 \otimes_R A) \oplus (X_2 \otimes_R A)$. If N_A is indecomposable, then $X_1 \otimes_R A = 0$ or $X_2 \otimes_R A = 0$, say the former. Since $p_{X_1} : X_1 \to X_1/X_1Q$ is minimal, and $X_1/X_1Q \cong X_1 \otimes_R A = 0$, then $X_1 = 0$. Thus $N \otimes_A R$ is indecomposable.

To complete the proof it suffices to observe that $N_1 \cong N_2$ if and only if $N_1 \otimes_A R_R \cong N_2 \otimes_A R_R$. □

Remark. Applying the lemma to A_A and $A_A \otimes_A R_R \cong R_R$, there exists a bijective correspondence between the isomorphism classes of indecomposable projective A-modules and the isomorphism classes of indecomposable R-modules, given by $P \mapsto P \otimes_A R_R$.

1.3. **Lemma.** *If $f : P_A \to M_A$ is a projective cover in* $\operatorname{mod} A$*, then $f \otimes 1_R : P \otimes_A R_R \to M \otimes_A R_R$ is a projective cover in* $\operatorname{mod} R$*.*

Proof. Considering P and M as R-modules, we have a commutative diagram of R-modules and R-linear epimorphisms

$$\begin{array}{ccc} P \otimes_A R & \xrightarrow{f\otimes 1_R} & M \otimes_A R \\ \downarrow{\scriptstyle p_{P\otimes R}} & & \downarrow{\scriptstyle p_{M\otimes R}} \\ P & \xrightarrow{f} & M \end{array}$$

Since $P \otimes_A R$ is projective, we must only show that the epimorphism $f \otimes 1_R$ is minimal. Assume that $h : X \to P \otimes_A R$ is such that $(f \otimes 1_R) \circ h$ is an epimorphism. Then so is $f \circ p_{P\otimes R} \circ h = p_{M\otimes R} \circ (f \otimes 1_R) \circ h$. Since f and $p_{P\otimes R}$ are minimal, h is an epimorphism. □

Corollary. *If $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ is a projective presentation, then so is*

$$P_1 \otimes_A R \xrightarrow{f_1\otimes 1_R} P_0 \otimes_A R \xrightarrow{f_0\otimes 1_R} M \otimes_A R \longrightarrow 0.$$

Further, if the first is minimal, so is the second.

Proof. The first statement is obvious. Assume that the first presentation is minimal. By the lemma, $f_0 \otimes 1_R$ is a projective cover. Also, since $f_1 : P_1 \to f_1(P_1)$ is a projective cover, then so is $f_1 \otimes 1_R : P_1 \otimes_A R \to f_1(P_1) \otimes_A R$. Since $f_1(P_1) \otimes_A R_R \cong (f_1 \otimes 1_R)(P_1 \otimes_A R) \cong \operatorname{Ker}(f_0 \otimes 1_R)$, we are done. □

2. Extension of (partial) tilting modules

2.1. **Lemma.** *For an A-module M, we have*

$$\tau_R (M \otimes_A R) \cong \operatorname{Hom}_A ({}_R R_A, \tau_A M) .$$

Proof. (see [9]). Let $P_1 \to P_0 \to M \to 0$ be a minimal projective presentation of M_A. By (1.3) and a well-known functorial isomorphism, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} \operatorname{Hom}_R(P_0 \otimes_A R, R) & \longrightarrow & \operatorname{Hom}_R(P_1 \otimes_A R, R) & \longrightarrow & \operatorname{Tr}_R(M \otimes_A R) & \longrightarrow & 0 \\ \downarrow f & & \downarrow g & & \downarrow h & & \\ R \otimes_A \operatorname{Hom}_A(P_0, A) & \longrightarrow & R \otimes_A \operatorname{Hom}_A(P_1, A) & \longrightarrow & R \otimes_A \operatorname{Tr}_A M & \longrightarrow & 0 \end{array}$$

where f and g are isomorphisms. So, the induced morphism h is also an isomorphism, and

$$\begin{aligned} \tau_R\left(M \otimes_A R\right) &\cong \mathrm{D}\left(R \otimes_A \operatorname{Tr}_A M\right) \\ &\cong \operatorname{Hom}_A\left({}_R R_A, \operatorname{Hom}_k\left(\operatorname{Tr}_A M, k\right)\right) \cong \operatorname{Hom}_A\left({}_R R_A, \tau_A M\right). \end{aligned}$$

□

2.2. **Lemma.** *For an A-module M, we have* $\operatorname{pd}\left(M \otimes_A R_R\right) \leq 1$ *if and only if* $\operatorname{pd} M_A \leq 1$ *and* $\operatorname{Hom}_A\left(\mathrm{D}\left({}_A Q\right), \tau_A M\right) = 0$.

Proof. We have $\operatorname{pd}\left(M \otimes_A R_R\right) \leq 1$ if and only if $\operatorname{Hom}_R\left(\mathrm{D} R, \tau_R\left(M \otimes_A R\right)\right) = 0$ and similarly $\operatorname{pd} M_A \leq 1$ if and only if $\operatorname{Hom}_A\left(\mathrm{D} A, \tau_A M\right) = 0$ (see, for instance, (2.4) (1)). The statement follows from the sequence of vector space isomorphisms:

$$\begin{aligned} &\operatorname{Hom}_R\left(\mathrm{D} R, \tau_R\left(M \otimes_A R\right)\right) \cong \operatorname{Hom}_R\left(\mathrm{D} R, \operatorname{Hom}_A\left({}_R R_A, \tau_A M\right)\right) \cong \\ &\operatorname{Hom}_A\left(\mathrm{D} R \otimes_R R_A, \tau_A M\right) \cong \operatorname{Hom}_A\left(\mathrm{D}\left({}_A A\right) \oplus \mathrm{D}\left({}_A Q\right), \tau_A M\right) \cong \\ &\operatorname{Hom}_A\left(\mathrm{D} A, \tau_A M\right) \oplus \operatorname{Hom}_A\left(\mathrm{D}\left({}_A Q\right), \tau_A M\right). \end{aligned}$$

□

2.3 *Proof of Theorem (A).* By (1.2), the number of non-isomorphic indecomposable summands of T_A equals the number of non-isomorphic indecomposable summands of $T \otimes_A R_R$. Also, the Grothendieck groups $K_0(A)$ of A and $K_0(R)$ of R have equal ranks. Consequently, it suffices to prove the statement for partial tilting modules. Let T be an A-module, we have vector space isomorphisms:

$$\begin{aligned} &\operatorname{Hom}_R(T \otimes_A R, \tau_R(T \otimes_A R)) \cong \operatorname{Hom}_R(T \otimes_A R, \operatorname{Hom}_A({}_R R_A, \tau_A T)) \cong \\ &\operatorname{Hom}_A(T \otimes_A R \otimes_R R_A, \tau_A T) \cong \operatorname{Hom}_A(T \otimes_A R_A, \tau_A T) \cong \\ &\operatorname{Hom}_A(T \otimes_A (A \oplus Q)_A, \tau_A T) \cong \operatorname{Hom}_A(T, \tau_A T) \oplus \operatorname{Hom}_A(T \otimes_A Q, \tau_A T). \end{aligned}$$

Let now T_A be a partial tilting A-module. Since $\operatorname{pd} T_A \leq 1$, we have $\operatorname{Hom}_A(T, \tau_A T) \cong \mathrm{D}\operatorname{Ext}^1_A(T, T) = 0$. Further, if $\operatorname{Hom}_A(T \otimes_A Q, \tau_A T) = 0$, we have $\operatorname{Hom}_R(T \otimes_A R, \tau_R(T \otimes_A R)) = 0$ and consequently

$$\operatorname{Ext}^1_R(T \otimes_A R, T \otimes_A R) \cong \mathrm{D}\,\overline{\operatorname{Hom}}_R(T \otimes_A T, \tau_R(T \otimes_A R)) = 0.$$

Finally, $\operatorname{Hom}_A(\mathrm{D}\, Q, \tau_A T) = 0$ implies, by (2.2), that $\operatorname{pd}(T \otimes_A R)_R \leq 1$. This shows the sufficiency.

Conversely, let $T \otimes_A R$ be a partial tilting R-module. By (2.2) we have $\operatorname{pd} T_A \leq 1$ and $\operatorname{Hom}_A(\mathrm{D}\, Q, \tau_A T) = 0$. Also,

$$\operatorname{Hom}_R(T \otimes_A R, \tau_R(T \otimes_A R)) \cong \mathrm{D}\operatorname{Ext}^1_R(T \otimes_A R, T \otimes_A R) = 0$$

yields $\operatorname{Hom}_R(T \otimes_A Q, \tau_A T) = 0$ and $\operatorname{Hom}_A(T, \tau_A T) = 0$. Since the latter equality implies $\operatorname{Ext}^1_A(T, T) = 0$, this completes the proof. □

2.4. We say that a (partial) tilting A-module T is **extendable** whenever it satisfies the two conditions $\operatorname{Hom}_A(T\otimes_A Q, \tau_A T) = 0$ and $\operatorname{Hom}_A(\operatorname{D} Q, \tau_A T) = 0$ or, equivalently, $\operatorname{Ext}^1_A(T, T\otimes_A Q) = 0$ and $\operatorname{Ext}^1_A(T, \operatorname{D} Q) = 0$). As a first application, we determine the torsion pair associated to an extended tilting module. Recall that a tilting module T determines a torsion pair, also called torsion theory, $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$, where $\mathcal{T}(T)$ (or $\mathcal{F}(T)$) is the full subcategory of those modules M such that $\operatorname{Ext}^1_A(T, M) = 0$ (or such that $\operatorname{Hom}_A(T, M) = 0$, respectively), see [1], [5]). If T_A is extendable it turns out that the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ entirely determines the torsion pair $(\mathcal{T}(T\otimes_A R), \mathcal{F}(T\otimes_A R))$ in $\operatorname{mod} R$.

Corollary. *For an R-module X, denote by X_A the underlying A-module. Let T_A be an extendable tilting A-module, then*

$$X_R \in \mathcal{T}(T\otimes_A R) \text{ if and only if } X_A \in \mathcal{T}(T)$$

and

$$X_R \in \mathcal{F}(T\otimes_A R) \text{ if and only if } X_A \in \mathcal{F}(T).$$

Proof. (see [9]). This follows from the isomorphisms

$$\operatorname{Hom}_R(X, \tau_R(T\otimes_A R)) \cong \operatorname{Hom}_R(X, \operatorname{Hom}_A({}_R R_A, \tau_A T)) \cong \operatorname{Hom}_A(X\otimes_R R_A, \tau_A T),$$
$$\operatorname{Hom}_R(T\otimes_A R, X) \cong \operatorname{Hom}_A(T, \operatorname{Hom}_R({}_A R_R, X))$$

and the observation that $-\otimes_R R_A$ and $\operatorname{Hom}_R({}_A R_R, -)$ are two expressions for the forgetful functor $\operatorname{mod} R \to \operatorname{mod} A$. □

2.5. **Proposition.** *If T_A is an extendable tilting module, then $E = \operatorname{End}(T\otimes_A R)$ is the split extension of $B = \operatorname{End} T_A$ by the nilpotent bimodule ${}_B W_B = \operatorname{Hom}_A({}_B T_A, {}_B T_A \otimes_A Q_A)$.*

Proof. The vector space isomorphisms

$$E \cong \operatorname{Hom}_A(T, \operatorname{Hom}_R({}_A R_R, T\otimes_A R_R)) \cong \operatorname{Hom}_A(T, T\otimes_A R_A) \cong$$
$$\operatorname{Hom}_A(T, T) \oplus \operatorname{Hom}_A(T, T\otimes_A Q).$$

yield a split short exact sequence

$$0 \to W \to E \xrightarrow{\phi} B \to 0$$

where $\phi : E \to B$ is an algebra morphism, and the ideal structure of W is induced from its canonical $B-B$-bimodule structure. There remains to show that W is nilpotent. Now, the multiplication in W is induced from that in E, and, for any $w \in W$, the image of w is contained in $T\otimes_A Q$. Since Q is nilpotent, there exists $s \geq 0$ such that, for any sequence $w_1, w_2, \dots w_s$ of elements of W, we have $\operatorname{Im}(w_1 w_2 \dots w_s) \subseteq T\otimes_A Q^s = 0$. Thus $W^s = 0$. □

Remark. This proof shows that the degree of nilpotency of W does not exceed that of Q. In particular, if R is the trivial extension of A by Q, then E is the trivial extension of B by W.

3. Extension to a quotient of the split extension

3.1. Let T_A be a tilting module which generates Q_A. The module $T\otimes_A R$ is generally not faithful (see example 4.2). This leads to consider its annihilator.

Lemma. $\operatorname{Ann}_R(T \otimes_A R) = \{(0,q) \in A \oplus Q \mid Tq = 0\}$.

Proof. Let $(x, \sum y_i \otimes q_i) \in T \otimes_A R \cong T \oplus (T \otimes_A Q)$ and $(a,q) \in R \cong A \oplus Q$. Then

$$\left(x, \sum y_i \otimes q_i\right)(a,q) = \left(xa, \sum y_i \otimes q_i a + xq + \sum y_i \otimes q_i q\right).$$

If $a = 0$ and $Tq = 0$, this product equals $\sum y_i \otimes q_i q$. Since Q is generated by T, then $Tq = 0$ also implies $q_i q = 0$. Consequently, $(0,q) \in \operatorname{Ann}_R(T \otimes_A R)$. Conversely, if $(a,q) \in \operatorname{Ann}_R(T \otimes_A R)$, then $a = 0$ (since T is faithful). Letting all q_i equal zero, we get $xq = 0$ for any $x \in R$. Therefore $Tq = 0$. $\square$

This shows that $\operatorname{Ann}_R(T \otimes_A R)$ can be identified with the subbimodule $Q' = \{q \in Q \mid Tq = 0\}$ of Q. We then set

$$\tilde{R} = (A \oplus Q)/\operatorname{Ann}_R(T \otimes_A R) \cong A \oplus \tilde{Q}$$

where $\tilde{Q} = Q/Q'$ with the induced multiplication. Since Q is nilpotent in R, so is $\tilde{Q}$ in $\tilde{R}$. Hence $\tilde{R}$ is a split extension of A by the nilpotent bimodule $\tilde{Q}$.

3.2 *Proof of theorem (B).* Since Q_A is generated by T, so is $\tilde{Q} = Q/Q'$. Hence $\operatorname{Hom}_A(\tilde{Q}, \tau_A T) \cong \operatorname{D}\operatorname{Ext}^1_A(T, \tilde{Q}) = 0$ and consequently

$$\operatorname{Hom}_A\left(T \otimes_A \tilde{Q}, \tau_A T\right) \cong \operatorname{Hom}_A\left(T, \operatorname{Hom}_A\left(\tilde{Q}, \tau_A T\right)\right) = 0$$

Next we claim that $\operatorname{Hom}_A\left(\operatorname{D}\left({}_A\tilde{Q}\right), \tau_A T\right) = 0$, or equivalently that $\operatorname{D}\tilde{Q}$ is generated by T. By definition, $T \otimes_A \tilde{R}$ is a faithful $\tilde{R}$-module. Hence there exists an $\tilde{R}$-linear epimorphism $(T \otimes_A \tilde{R})^{(n)} \to \operatorname{D}\tilde{R}$, which induces an A-linear epimorphism $[(T \otimes A) \oplus (T \otimes \tilde{Q})]^{(n)} \to \operatorname{D}A \oplus \operatorname{D}\tilde{Q}$. Since $T \otimes_A \tilde{Q}$ is generated by T, we have a sequence of A-linear epimorphisms

$$T^{(n)} \oplus T^{(m)} \to T^{(n)} \oplus (T \otimes \tilde{Q})^{(n)} \to \operatorname{D}A \oplus \operatorname{D}\tilde{Q} \to \operatorname{D}\tilde{Q}.$$

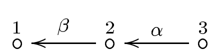
Hence $\operatorname{D}\tilde{Q}$ is generated by T. Applying theorem (A), we see that T_A is extendable to a tilting $\tilde{R}$-module. $\square$

Remark. If follows from the proof and (2.5) that $\tilde{E} = \operatorname{End}(T \otimes_A \tilde{R})$ is the split extension of $B = \operatorname{End} T$ by the nilpotent bimodule ${}_B W_B = \operatorname{Hom}_A(T, T \otimes_A \tilde{Q})$.

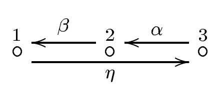
4. Examples

In the following examples, algebras are given by their bound quivers, indecomposables by their Loewy series, and the idempotent corresponding to a point i in the quiver of the algebra is denoted by e_i.

4.1. Let A be a hereditary algebra given by the quiver



and R be given by the quiver



bound by $\beta\eta\alpha\beta\eta = 0$. Then R is the split extension of A by the nilpotent bimodule Q generated by η. A k-basis of Q is the set

$$\{\eta, \eta\alpha, \eta\alpha\beta, \eta\alpha\beta\eta, \eta\alpha\beta\eta\alpha, \eta\alpha\beta\eta\alpha\beta, \beta\eta, \beta\eta\alpha, \beta\eta\alpha\beta, \alpha\beta\eta, \alpha\beta\eta\alpha, \alpha\beta\eta\alpha\beta\}$$

so that

$$Q_A = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}^{(4)} \quad \text{and} \quad D({}_AQ) = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}^{(3)} \oplus (3)^{(3)}.$$

Consider in $\operatorname{mod} A$ the tilting module $T = (1) \oplus (3) \oplus \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Then $B = \operatorname{End} T$ is given by the quiver

$$\overset{1}{\circ} \xleftarrow{\ \mu\ } \overset{2}{\circ} \xleftarrow{\ \lambda\ } \overset{3}{\circ}$$

bound by $\lambda\mu = 0$. First, $\mathrm{D}({}_AQ)$ is clearly generated by T. To compute $T \otimes_A Q$, we consider the projective resolution of T_A

$$0 \to e_2A \to e_1A \oplus (e_3A)^{(2)} \to (3) \oplus (1) \oplus \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \to 0.$$

Applying $- \otimes_A Q$, we get

$$e_2Q \to e_1Q \oplus (e_3Q)^{(2)} \to T \otimes_A Q \to 0.$$

Now, $e_2Q \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $e_1Q \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $e_3Q \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}^{(2)}$, so that $T \otimes_A Q \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}^{(2)}$ which is generated by T. Thus T is extendable.

To compute $T \otimes_A R$, we apply $- \otimes_A R$ to each of the projective resolutions for the summands of T, obtaining $e_1R \cong (1) \otimes_A R, e_3R \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \otimes_A R$ and an exact sequence

$$e_2R \to e_3R \to (3) \otimes_A R \to 0.$$

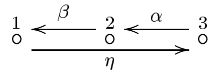
Therefore, $T \otimes_A R = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 2 \\ 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} \oplus (3).$

Then $E = \operatorname{End}(T \otimes_A R)$ is given by the quiver

$$\overset{1}{\circ} \xleftarrow{\ \mu\ } \overset{2}{\circ} \;\underset{\xrightarrow[\ \nu\ ]{}}{\xleftarrow{\ \lambda\ }}\; \overset{3}{\circ}$$

bound by $\lambda\mu = 0, \nu\lambda\nu\lambda = 0$. Thus E is the split extension of B by the bimodule generated by ν.

4.2. Let A, T_A and B be as in (4.1), and R be given by the quiver



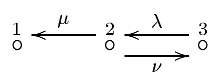
bound by $\alpha\beta\eta = 0$. Then R is the split extension of A by the nilpotent bimodule Q generated by η. A k-basis of Q is the set $\{\eta, \eta\alpha, \eta\alpha\beta, \beta\eta, \beta\eta\alpha, \beta\eta\alpha\beta\}$ so that

$$Q_A = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}^{(2)} \quad \text{and} \quad \mathrm{D}({}_AQ) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}^{(3)}.$$

Now, clearly, $\mathrm{Hom}_A(\mathrm{D}({}_AQ), \tau_A T) \neq 0$ so that T is not extendable. On the other hand, Q_A is generated by T. A k-basis for $Q' = \{q \in Q | Tq = 0\}$ is easily found to be $\{\beta\eta, \beta\eta\alpha, \beta\eta\alpha\beta\}$ (that is, Q' is the subbimodule generated by $\beta\eta$). Therefore a k-basis of $\tilde{Q} = Q/Q'$ is the set $\{\eta, \eta\alpha, \eta\alpha\beta\}$. The split extension $\tilde{R}$ of A by the nilpotent bimodule $\tilde{Q}$ is given by the same quiver as above bound by $\beta\eta = 0$. To compute $T \otimes_A \tilde{R}$ we apply $- \otimes_A \tilde{R}$ to each of the projective resolutions for the summands of T_A, obtaining $e_1\tilde{R} \cong (1) \otimes_A \tilde{R}, e_3\tilde{R} \cong \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \otimes_A \tilde{R}$ and an exact sequence

$$e_2\tilde{R} \to e_3\tilde{R} \to (3) \otimes_A \tilde{R} \to 0$$

Therefore $T \otimes_A \tilde{R} = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \oplus (3)$ and $\tilde{E} = \mathrm{End}(T \otimes_A \tilde{R})$ is given by the quiver



bound by $\lambda\mu = 0, \nu\lambda = 0$.

Acknowledgments

The first author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada.

References

1. Assem, I., *Tilting theory - an introduction, Topics in Algebra*, Banach Center Publication, vol. 26, PWN, Warsaw, 1990, p. 127–180.
2. Auslander, M., Reiten, I. and Smalø, S.O., *Representation theory of Artin algebras*, Cambridge University Press, 1995.
3. Cartan, H. and Eilenberg, S., *Homological Algebra*, Princeton University Press, Princeton, N.J, 1956.
4. Fuller, K., *$*$-Modules over split extensions* (1996), preprint.
5. Happel, D. and Ringel, C.M., *Tilted algebras*, Trans, Amer. Math. Soc. **2** (1982), no. 274, 399–443.
6. Hoshino, M., *Private communication.*
7. Marmaridis, N., *On extensions of abelian categories with applications to ring theory*, J. Algebra **156** (1993), 50–64.
8. Marmaridis, N., *On extensions of finite dimensional algebras and tilting modules*, Canadian Mathematical Society Proceedings **11** (1991), 213–230.
9. Miyachi, J.-I., *Tilting modules over trivial extensions of artin algebras*, Comm. Algebra **13** (1985), no. 6, 1319–1326.

10. Miyachi, J.-I., *Extensions of rings and tilting complexes*, J. Pure Appl. Algebra **2** (1995), no. 105, 183–194.
11. Ringel, C.M., *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics (1984), no. 1099, Springer, Berlin.
12. Tachikawa, H. and Wakamatsu, T., *Extensions of tilting functors and QF-3 algebras*, J. Algebra (1986), no. 103, 662–676.

Département de mathématiques et d'informatique, Faculté des sciences, Université de Sherbrooke, Sherbrooke, (Québec) J1K 2R2, Canada
E-mail address: ibrahim.assem@dmi.usherb.ca

Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
E-mail address: nmarmar@cc.uoi.gr