COMMUNICATIONS IN ALGEBRA, 10(19), 2121-2139 (1982)

TILTED ALGEBRAS OF TYPE $\mathbb{A}_n$

Ibrahim Assem

Department of Mathematics and Statistics
Carleton University, Ottawa, Ontario
K1S 5B6, Canada

Let A be a finite-dimensional algebra (associative, with an identity) over a perfect field k. All our A-modules will be finite-dimensional right A-modules. Following [9], we shall call a module T_A a tilting module if the following properties are satisfied:

(T1) pd $T_A \leq 1$,

(T2) $\mathrm{Ext}_A^1(T,T) = 0$,

(T3) there is a short exact sequence

$0 \to A_A \to T'_A \to T''_A \to 0$ with T', T'' direct sums of summands of T_A .

If A is hereditary, the endomorphism ring $B = \mathrm{End}\, T_A$ of a tilting module is called a tilted algebra [9]. The aim of this paper is to characterize those tilted algebras which are of type $\mathbb{A}_n$ (that is, which are endomorphism rings of tilting modules over tensor algebras $T(\Sigma)$, where the valued graph of the species Σ is $\mathbb{A}_n$). In fact, we shall prove:

Theorem: A connected basic finite-dimensional algebra over a perfect field k is tilted of type $\mathbb{A}_n$ if and only if it is given by a k-species $\Sigma = (F_i, {}_iM_j)_{i,j \in I}$ with relation ideal $R = \bigoplus_{i,j} {}_iR_j$ such that:

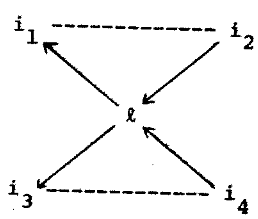
(α_1) The graph G of Σ is a tree.

(α_2) For each vertex i of G, $F_i = E$, where E is a skew field, finite-dimensional over k, and, for each arrow $i \to j$, ${}_iM_j = {}_EE_E$.

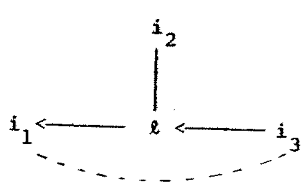
(α_3) All relations are zero-relations of length two.

(α_4) Each vertex has at most four neighbours.

(α_5) If a vertex ℓ of G has four neighbours then we have a full connected subgraph of G of the form:



(α_6) If a vertex ℓ of G has three neighbours, then we have a full connected subgraph of G of the form:



(α_7) There is no full connected subgraph of the form G_t:

$$i_1 \longleftarrow i_2 \longleftarrow i_3 \text{——} i_4 \text{——} i_5 \text{——} \cdots\, i_{t-2} \longleftarrow i_{t-1} \longleftarrow i_t$$

where $t \geq 4$, and there is no other relation between i_2 and i_{t-1}.

(Here, dotted lines denote zero-relations, and non-oriented edges can be oriented arbitrarily).

The particular case of a lower-triangular matrix ring was already considered in [10]. In [2], the generalized tilted algebras of type $\mathbb{A}_n$ (now called iterated tilted) were defined and classified, namely, these are the algebras satisfying the above conditions (α_1) through (α_6). The necessity part of the present result was already proved by the author in [1].

1. <u>Preliminaries</u>:

(1.1) <u>Definition</u>: Let $\Sigma = (F_i, {}_iM_j)_{i,j \in I}$ be a k-species (cf. [6] or [7]), and $T(\Sigma)$ its tensor algebra. An ideal R of $T(\Sigma)$ is called a *relation ideal* whenever it is contained in $\mathrm{rad}^2\, T(\Sigma)$. The factor algebra $A = T(\Sigma)/R$ is then said to be given by the *bounden k-species* (Σ,R).

For a perfect field k, every finite-dimensional basic k-algebra is given by a bounden species [1].

Given a pair $(i,j) \in I \times I$, the relation ideal R defines an $F_i - F_j$ subbimodule ${}_iR_j = F_i R F_j$ of ${}_i\tilde{M}_j = F_i T(\Sigma) F_j$. Each ${}_iR_j$ is called a *relation* on Σ. Clearly, $R = \bigoplus_{i,j} {}_iR_j$. A relation ${}_iR_j$ is of *length* ℓ if ${}_iR_j \subseteq \mathrm{rad}^{\ell}\, T(\Sigma)$ but ${}_iR_j \not\subseteq$

$\mathrm{rad}^{\ell+1}T(\Sigma)$. It is called a *zero-relation* if ${}_iR_j = {}_i\tilde{M}_j$.

A representation of Σ (cf. [6]) is said to be *bound by* R if the associated $T(\Sigma)$-module is annihilated by the ideal R . Thus, we have an equivalence between the category $\mathrm{mod}\ (T(\Sigma)/R)$ of (right, finite-dimensional) $T(\Sigma)/R$-modules, and the category of all representations of Σ bound by R .

Let G_Σ be the valued graph of the species Σ [6], and $A = T(\Sigma)/R$. We shall use small letters $i,j,\ldots$ to denote the vertices. The simple A-module corresponding to the vertex i of G_Σ will be denoted by $S(i)$, and its projective cover (respectively, its injective hull) by $P(i)$ (respectively, $I(i)$). We shall use freely properties of the Auslander-Reiten sequences of $\mathrm{mod}\ A$ and of the Auslander-Reiten graph Γ_A of A such as can be found in [3] or [8].

(1.2) We now summarize briefly the results of [2]:

Definition: A finite-dimensional k-algebra B is *iterated tilted* if:

1) there exists a sequence of algebras $A_0, A_1, \ldots A_m = B$ with A_0 hereditary,

2) there exists a sequence of tilting modules $T^{(i)}_{A_i}$ $(0 \le i \le m-1)$ such that $\mathrm{End}\ T^{(i)}_{A_i} = A_{i+1}$ and, for every indecomposable A_{i+1}-module M , we have either $M \otimes_{A_{i+1}} T^{(i)} = 0$, or $\mathrm{Tor}_1^{A_{i+1}}(M, T^{(i)}) = 0$.

B is said to be *of type* Δ , for a given valued graph Δ , if A_0 is the tensor algebra of a species whose valued graph is Δ.

We have the following theorem:

Theorem: Let k be a perfect field. A connected basic finite-dimensional k-algebra is iterated tilted of type $\mathbb{A}_n$ if and only if it is given by a k-species $\Sigma = (F_i, {}_iM_j)_{i,j\in I}$ with relation ideal $R = \bigoplus_{i,j} {}_iR_j$ such that:

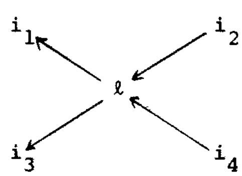
(α_1) The graph G of Σ is a tree.

(α_2) For each vertex i of G, $F_i = E$, where E is a skew field, finite dimensional over k, and, for each arrow $i \to j$, ${}_iM_j = {}_EE_E$.

(α_3) All relations are zero-relations of length two.

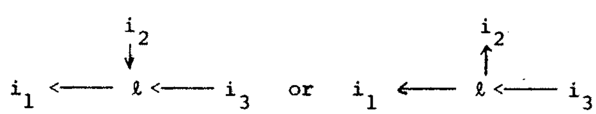
(α_4) Each vertex has at most four neighbours.

(α_5) If a vertex ℓ has four neighbours, then G has a full connected subgraph of the form



with the zero-relations ${}_{i_2}M_\ell \otimes {}_\ell M_{i_1}$ and ${}_{i_4}M_\ell \otimes {}_\ell M_{i_3}$.

(α_6) If a vertex ℓ has three neighbours, then G has a full connected subgraph of one of the forms:



with the zero-relation ${}_{i_3}M_\ell \otimes {}_\ell M_{i_1}$.

Let A be an iterated tilted algebra of type $\mathbf{A}_n$, then:

(1.2.1) The Auslander-Reiten graph Γ_A of A has no oriented cycles, thus every indecomposable A-module can be written in the form $\tau^{-r}P$, with $r \geq 0$ and P_A indecomposable projective (or $\tau^s I$, with $s \geq 0$ and I_A indecomposable injective), where τ denotes the Auslander-Reiten translation $\tau = DTr$. Also, modules are uniquely determined by their dimension-vectors [9].

(1.2.2) There are at most two arrows in Γ_A with a prescribed source or target. If M, N are two indecomposable A-modules, then:

$$|\mathrm{Irr}\,(M,N)_{\mathrm{End}\,M}| \leq 1 ,$$

and

$$|{}_{\mathrm{End}\,N}\mathrm{Irr}(M,N)| \leq 1 ,$$

where Irr(M,N) is the bimodule of irreducible maps [11].

Also, if P_A is projective, with indecomposable radical R, then there is at most one arrow of Γ_A of target [R]. Dually, if I_A is injective with I/soc I indecomposable, then there is at most one arrow of Γ_A of source [I/soc I].

(1.2.3) Given an indecomposable A-module M and a vertex i of G , we have $\dim M_i \leq 1$. Thus, any two paths with the same extremities in Γ_A define the same map.

(1.2.4) Recall that a path $X_0 \to X_1 \to \dots \to X_m$ in the Auslander-Reiten graph Γ_A is *sectional* provided $X_{i-1} \not\cong \tau X_{i+1}$

for $1 \leq i \leq m-1$. Also, a connected subgraph S of Γ_A is a *subsection* if each path in S is sectional [4]. Then, for every indecomposable projective P_A, the set of those indecomposables M such that $\mathrm{Hom}_A(M,P) \neq 0$ is the set of vertices lying on the two maximal sectional paths of Γ_A ending at $[P]$.

(1.3) Let now B be a finite-dimensional k-algebra. A *complete slice* S [9] is a set of indecomposable B-modules with the following properties:

(S1) There is no chain of irreducible maps $S_o \to S_1 \to \ldots \to S_m \to S_o$ with all the $S_i \in S$.

(S2) If $X_o \to X_1 \to \ldots \to X_m$ is a chain of irreducible maps between indecomposable modules, and $X_o, X_m \in S$, then $X_i \in S$ for all $0 \leq i \leq m$.

(S3) Given any indecomposable module X, S contains precisely one module from the orbit $\{\tau^z X \mid z \in \mathbb{Z}\}$.

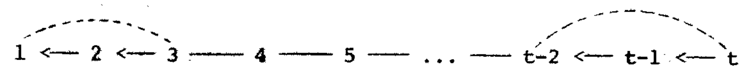
Then we have the following:

Theorem [9]: A finite-dimensional k-algebra B of finite representation type is tilted if and only if its Auslander-Reiten graph Γ_B contains a complete slice.

2. Proof of the theorem:

Observe that, by Theorem (1.2), the wanted result can be restated as:

Theorem: An iterated tilted algebra of type $\mathbf{A}_n$ is a tilted algebra if and only if its bounden graph contains no full connected subgraph of the form G_t :



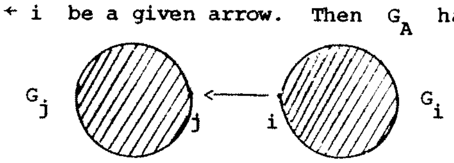
with the indicated zero-relations, where $t \geq 4$, all edges between 3 and t-2 can be oriented arbitrarily, and there are no other zero-relations between 2 and t-1 .

(2.1) <u>Proof of the necessity</u>: Let A be an iterated tilted algebra of type $\mathbf{A}_n$, and G_A be its graph. We shall suppose that G_A contains a G_t as above as a full connected subgraph and show that this implies that Γ_A contains no complete slice.

Let us first introduce some notation. For a vertex i of G_A , we define the set of successors of i , Σ_i , to be the set of those vertices j of G_A such that there exists a non-zero path of non-negative length from i to j . Dually, the set Π_i of predecessors of i will be the set of those vertices h of G_A such that there exists a non-zero path of non-negative length from h to i .

Thus, P(i) is the representation defined by the fact that its support Supp P(i) (that is, the set of those vertices j of G_A such that $P(i)_j \neq 0$) is equal to Σ_i. Similarly, I(i) is given by Supp I(i) = Π_i .

Let $j \leftarrow i$ be a given arrow. Then G_A has the form:



we define $P_j(i)$ by Supp $P_j(i) = \Sigma_i \cap \Sigma_j$. Clearly, $P_j(i)$ is an indecomposable submodule of rad $P(i)$. Dually, one can define $I^i(j)$ by Supp $I^i(j) = \Pi_i \cap \Pi_j$, this is an indecomposable image of $I(j)/\mathrm{soc}\, I(j)$.

(i) $[I(1)]$ lies on the left of the sectional path P from $[P(t-1)]$ to $[P(t)]$:

We start by constructing a sequence of non-zero maps defining an oriented path from $[I(1)]$ to $[P(t)]$. Recall that, by the argument above, we have an epimorphism $I(1) \to I^2(1)$ and a monomorphism $P_{t-1}(t) \to P(t)$. We shall now construct a representation M such that:

a) M is indecomposable,

b) $I^2(1)$ is a subrepresentation of M ,

c) $P_{t-1}(t)$ is an epimorphic image of M .

Indeed, let us put $M_j = E$ (where E is the skew field defined in (α_2)) if and only if j satisfies one of the following three conditions:

1) $2 \leq j \leq t-1$,

2) $j \in \Pi_1 \cap \Pi_2$,

3) $j \in \Sigma_t \cap \Sigma_{t-1}$.

This, together with the obvious maps between the coordinate vector spaces, defines clearly the wanted representation M . We thus have an oriented path in Γ_A :

$$[I(1)] \to [I^2(1)] \to [M] \to [P_{t-1}(t)] \to [P(t)] \ .$$

Let us remark that $M_{t-1} \neq 0$, and $I(1)_{t-1} = 0$ imply that $M \ncong I(1)$. Finally, $\mathrm{Hom}_A(P(t-1), P(t)) \neq 0$ shows that there is a unique oriented path $\mathcal{P}$ in Γ_A from $[P(t-1)]$ to $[P(t)]$ which moreover is sectional. Consequently, the fact that the map $P(t-1) \to P(t)$ factors through M implies that $[M]$ lies on $\mathcal{P}$ and $[I(1)]$ on its left.

(ii) Γ_A contains no complete slice:

If $\mathcal{S}$ is a complete slice in Γ_A , it cannot contain modules lying on the left of $\mathcal{P}$. For, let $[N] \in \mathcal{S}$ lie on the left of $\mathcal{P}$, we thus have an oriented path in Γ_A : $[N] \to \ldots \to [L]$ with $[L] \in \mathcal{P}$. We may assume that $[L]$ is chosen such that, for every $[L']$ between $[N]$ and $[L]$ on this path, we have $[L'] \notin \mathcal{P}$.

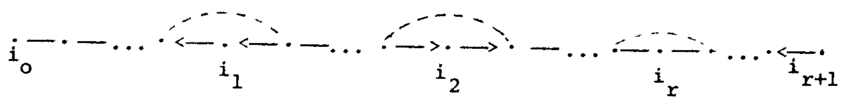
Since $\mathcal{S}$ is a complete slice, it must contain a module of the form $\tau^{-m}P(t)$ (with $m \geq 0$). We thus have a chain of irreducible maps:

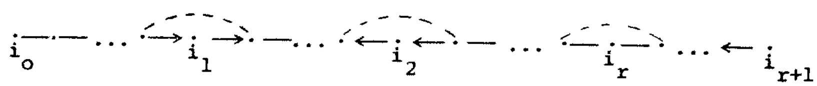
$$N \to \ldots \to L_{-1} \to L \to L_1 \to \ldots \to P(t) \to \ldots \to \tau^{-m} P(t) .$$

Since $[N]$ and $[\tau^{-m}P(t)]$ belong to $\mathcal{S}$, all the intermediate modules are on $\mathcal{S}$. In particular, we necessarily have $m = 0$. Now $[L] \in \mathcal{P}$ and $[P(t)] \in \mathcal{P}$, therefore $[L_1] \in \mathcal{P}$. However, we also have $[L_{-1}] \notin \mathcal{P}$, hence, by the property (1.2.2) of Γ_A, $L_{-1} = \tau L_1$, and this contradicts the fact that $\mathcal{S}$ is a complete slice.

(2.2) The following lemma is needed for the proof of the sufficiency:

<u>Lemma</u>: Let A' be an iterated tilted algebra of type $\mathbb{A}_n$ given by a graph G_A' of one of the forms:

(I) 

(II) 

(where dotted lines represent zero-relations) such that:

(i) there is no zero-relation between i_j and i_{j+1} $(0 \leq j \leq r)$,

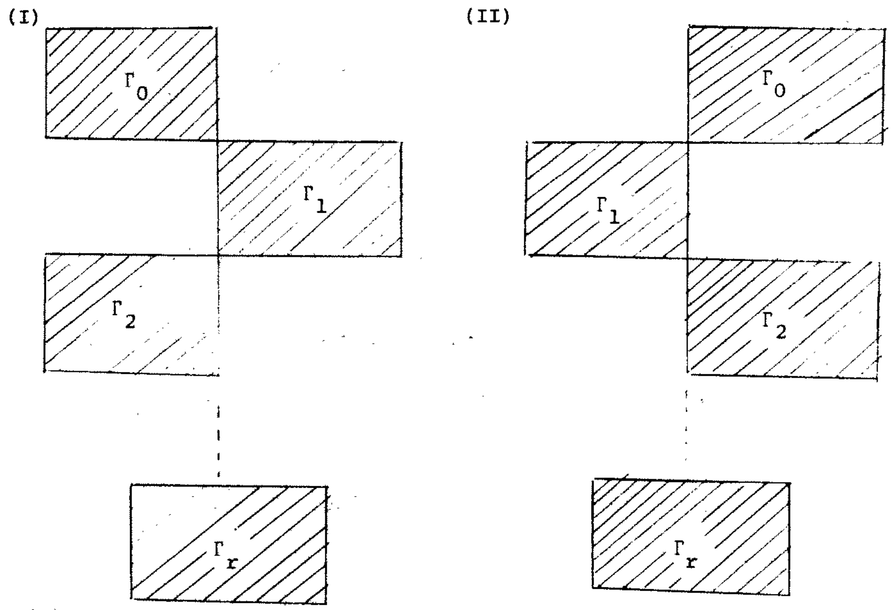
(ii) non-oriented arrows, and the last zero-relation can be oriented arbitrarily,

(iii) no two consecutive zero-relations are oriented in the same direction.

Then, either $\mathrm{Hom}_{A'}(I(i_0), P(i_{r+1})) \neq 0$, or there is no oriented path in $\Gamma_{A'}$ from $[I(i_0)]$ to $[P(i_{r+1})]$.

<u>Proof</u>: We can write $G_A' = \bigcup_{j=0}^{r} G_j$, where G_j is the full connected subgraph of G_A' given by i_j ·——·——·..·——·——·i_{j+1}. Thus, the graph of G_j is $\mathbb{A}_n$ for an appropriate n , and its tensor algebra A_j is hereditary. The existence of the zero-relations implies that an indecomposable (bound) representation of G_A' (thus an indecomposable A'-module) is in fact an indecomposable representation of G_j for some j (thus an A_j-module). Conversely, any indecomposable A_j-module is indecomposable as an A'-module.

We claim that $\Gamma_{A'}$ has the following shape:



where $\Gamma_j = \Gamma_{A_j}$, and $\Gamma_j \cap \Gamma_{j+1} = \{[S(i_{j+1})]\}$.

We shall only prove (I), since the proof of (II) is similar. Let L be an indecomposable A_o-module, and M an indecomposable A_1-module. If $\mathrm{Hom}_{A'}(L,M) \neq 0$, then $\mathrm{Supp}\, L \cap \mathrm{Supp}\, M \neq \phi$. Now $\mathrm{Supp}\, L \subseteq G_0$, $\mathrm{Supp}\, M \subseteq G_1$ and $G_0 \cap G_1 = \{i_1\}$, hence $\mathrm{Supp}\, L \cap \mathrm{Supp}\, M = \{i_1\}$. Observe that $S(i_1)$ is injective as an A_o-module, and projective as an A_1-module. Therefore $L_{i_1} \neq 0$ and $M_{i_1} \neq 0$ imply that $\mathrm{Hom}_{A'}(L, S(i_1)) \neq 0$ and $\mathrm{Hom}_{A'}(S(i_1),M) \neq 0$. So any map from L to M must factor through $S(i_1)$. Also, $\mathrm{Hom}_{A'}(\bar{M},\bar{L}) = 0$ for any indecomposable A_o-module $\bar{L}$, and A_1-module $\bar{M}$. For, again, the existence of such a map implies that $\mathrm{Supp}\, \bar{L} \cap \mathrm{Supp}\, \bar{M} = \{i_1\}$, hence $\mathrm{Hom}_{A'}(\bar{L},S(i_1)) \neq 0$ and $\mathrm{Hom}_{A'}(S(i_1),\bar{M}) \neq 0$.

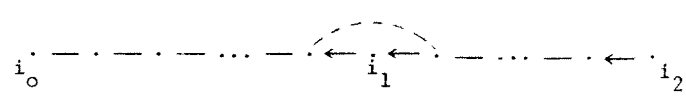
Therefore, we would have an oriented cycle $[\bar{L}] \to [S(i_1)] \to [\bar{M}] \to [\bar{L}]$ in $\Gamma_{A'}$, and this is impossible.

Next, let M be an indecomposable A_1-module, and N an indecomposable A_2-module. If $\mathrm{Hom}_{A'}(M,N) \neq 0$, then $\mathrm{Supp}\, M \cap \mathrm{Supp}\, N = \{i_2\}$, hence $\mathrm{Hom}_{A'}(N,S(i_2)) \neq 0$, and $\mathrm{Hom}_{A'}(S(i_2),M) \neq 0$. We thus have an oriented cycle in $\Gamma_{A'}$: $[M] \to [N] \to [S(i_2)] \to [M]$, a contradiction which shows that $\mathrm{Hom}_{A'}(M,N) = 0$. By the same argument, a non-zero map from an indecomposable A_2-module $\bar{N}$ to an indecomposable A_1-module $\bar{M}$ must factor through $S(i_2)$.

An obvious induction completes the proof of our claim.

Let us now suppose that there is an oriented path in $\Gamma_{A'}$ from $[I(i_o)]$ to $[P(i_{r+1})]$. It is clear that $r \geq 1$, for, if not, A' is hereditary and there is no such path. Also, G'_A cannot be of type (II), for, $I(i_o)$ is an A_o-module, and the previous argument shows that there is no oriented path from an A_o-module to an A_j-module, for $j \geq 1$. Similarly, if G'_A is of type (I) and $r \geq 2$, the previous argument shows that an oriented path from $[I(i_o)]$ to $[P(i_{r+1})]$ must factor through an A_1-module and an A_2-module, and this is impossible.

Therefore there only remains to consider the case of an iterated tilted algebra given by a graph of the form:



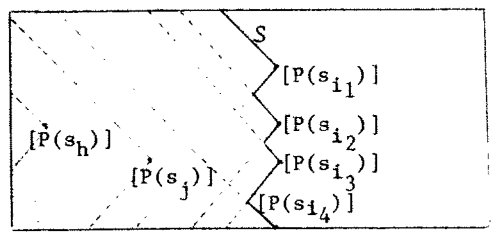
Let us define the indecomposable A'-modules L and M to be the unique ones such that Supp $L = G_o$, Supp $M = G_1$. Now $L_{i_o} \neq 0$, $L_{i_1} \neq 0$ imply that we have non-zero maps $L \to I(i_o)$, $L \to S(i_1)$ and the second map is clearly surjective. Similarly, we have non-zero maps $P(i_2) \to M$ and $S(i_1) \to M$, and the second map is injective. Thus $\mathrm{Hom}_{A'}(L,M) \neq 0$. On the other hand, the oriented path from $[I(i_o)]$ to $[P(i_2)]$ must factor through $[S(i_1)]$, and we have thus a composite path in $\Gamma_{A'}$, given by:

$$[L] \to [I(i_o)] \to \dots \to [S(i_1)] \to \dots \to [P(i_2)] \to [M]$$

and $\mathrm{Hom}_{A'}(L,M) \neq 0$ implies that the composition of the above maps is non-zero. In particular $\mathrm{Hom}_{A'}(I(i_o), P(i_2)) \neq 0$, and this completes the proof of the lemma.

(2.3) <u>Proof of the sufficiency</u>: We shall construct a complete slice in Γ_A. We already know that Γ_A has no oriented cycles. Let $s_1, \dots s_t$ be the sources of G_A , and $P(s_1), \dots P(s_t)$ the corresponding indecomposable projective A-modules.

Let $\mathcal{Q}$ be the full connected subgraph of Γ_A consisting of the vertices $[M]$ such that there is an oriented path $[M] \to \dots \to [P(s_i)]$ for some source s_i , and $\mathcal{S}$ be the right border of $\mathcal{Q}$: that is to say, $\mathcal{S}$ is the full connected subgraph of $\mathcal{Q}$ consisting of those vertices $[M]$ such that, whenever there is an oriented path from $[M]$ to $[P(s_i)]$, for some i , then such a path is sectional. Thus $\mathcal{S}$ is, by definition, a subsection in Γ_A :



We claim that S is a complete slice in Γ_A. Let us observe that, by construction, any vertex [M] of Γ_A which is not on S is either on the left or on the right of S (in other words, if $[M] \notin S$, then either there exists an oriented path from [M] to S or from S to [M]).

(i) No indecomposable projective A-module lies on the right of S:

Let P(i) be an indecomposable projective. If i is a source, it is clear by the construction of S that [P(i)] does not lie on the right of S. If i is not a source, there exists a source s_i and an oriented path in G_A (unique, since G_A is a tree) from s_i to i which gives an oriented path from [P(i)] to $[P(s_i)]$. Thus, [P(i)] does not lie on the right of S.

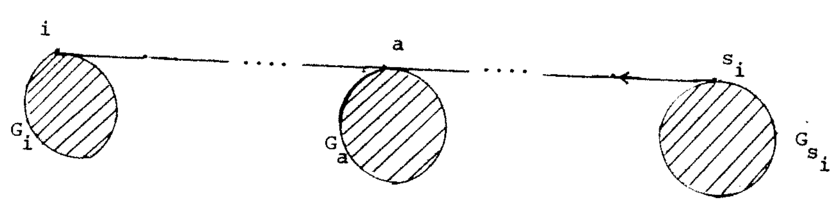
(ii) No indecomposable injective A-module lies on the left of S:

Let us assume that the indecomposable injective [I(i)] lies on the left of S. Thus there is an oriented path from [I(i)] to $[P(s_i)]$ for some source s_i. G_A is a tree, hence there is a unique (non-oriented) path joining the vertices i and s_i and defining a full connected subgraph G_A' of G_A which is of one of the types (I) or (II) of Lemma (2.2). Let A' be the algebra given by the graph G_A', and let $I'(i)$ and $P'(j)$ denote respectively the indecomposable injective A'-module corresponding to the vertex i, and the indecomposable projective A'-module corresponding to the vertex s_i. It is known [12] that there is a full, faithful and exact embedding $\phi : \text{mod } A' \to \text{mod } A$ which is the unique such that $\rho\phi = 1$, where ρ is the restriction functor. Thus $\phi I'(i)_i \neq 0$, and hence there exists a non-zero map $\phi I'(i) \to I(i)$. Similarly, we have a non-zero map $P(s_i) \to \phi P'(s_i)$. Thus, the existence of an oriented path from $[I(i)]$ to $[P(s_i)]$ implies the existence of an oriented path in Γ_A from $[\phi I'(i)]$ to $[\phi P'(s_i)]$ given by a sequence of non-zero maps:

$$\phi I'(i) \to I(i) \to M_1 \to M_2 \to \dots \to M_m \to P(s_i) \to \phi P'(s_i) .$$

We claim that, by applying the restriction functor ρ, this gives an oriented path in $\Gamma_{A'}$ from $[I'(i)]$ to $[P'(s_i)]$. Indeed, since $\rho\phi = 1$, this is the case of $\text{Supp } M_j \cap G_A' \neq \phi$ for all $1 \leq j \leq m$.

Let us denote by G_a the branch of the tree G_A attached at the vertex a of G_A':



Suppose that $\operatorname{Supp} M_j \cap G'_A = \phi$ for some $1 \leq j \leq m$, then, since $\operatorname{Supp} I(i) \cap G'_A \neq \phi$ and $\operatorname{Supp} P(s_i) \cap G'_A \neq \phi$, there exist $t_1 < t_2$ such that all M_t $(t_1 \leq t < t_2)$ have their supports not intersecting G'_A, while both M_{t_1-1} and M_{t_2} have their supports intersecting G'_A. Now, since M_{t_1} is indecomposable, and its support does not intersect G'_A, there exists a vertex a in G'_A such that $\operatorname{Supp} M_{t_1} \subseteq G_a$. For the same reason M_t $(t_1 \leq t < t_2)$ has its support contained in the same G_a. However, $\operatorname{Hom}_A(M_{t_1-1}, M_{t_1}) \neq 0$ and $\operatorname{Hom}_A(M_{t_2-1}, M_{t_2}) \neq 0$ imply that $\operatorname{Supp} M_{t_1-1} \cap \operatorname{Supp} M_{t_1} \neq \phi$ and that $\operatorname{Supp} M_{t_2-1} \cap \operatorname{Supp} M_{t_2} \neq \phi$. Therefore $a \in \operatorname{Supp} M_{t_1-1}$ and $a \in \operatorname{Supp} M_{t_2}$, which imply the existence of non-zero maps $f_1 : P(a) \to M_{t_1-1}$ and $f_2 : P(a) \to M_{t_2}$. But all paths in Γ_A give rise to the same map, hence f_2 is equal to the composition of f_1 with the map $M_{t_1-1} \to M_{t_1} \to \dots \to M_{t_2-1} \to M_{t_2}$. However, $a \notin \operatorname{Supp} M_{t_1}$ so $\operatorname{Hom}_A(P(a), M_{t_1}) = 0$ and the composition of f_1 with the map $M_{t_1-1} \to M_{t_1}$ is zero. This contradicts the fact that $f_2 \neq 0$. We have thus proved the existence of an oriented path in $\Gamma_{A'}$ from $[I'(i)]$ to $[P'(s_i)]$.

By Lemma (2.2), this implies that $\operatorname{Hom}_{A'}(I'(i), P'(s_i)) \neq 0$ and hence that $\operatorname{Hom}_A(\phi I'(i), \phi P'(s_i)) \neq 0$, which yields that $\operatorname{Hom}_A(I(i), P(s_i)) \neq 0$. Thus $[I(i)]$ lies on one of the sectional paths on $[P(s_i)]$ (for any source s_i such that there is an oriented path from $[I(i)]$ to $[P(s_i)]$). But this means precisely that $[I(i)] \in \mathcal{S}$, and hence cannot lie on the left of $\mathcal{S}$.

(iii) S is a complete slice:

(i) and (ii) imply that S contains at least one indecomposable from each τ-orbit. In fact, S contains at most one (and hence exactly one), since it is a subsection of Γ_A. Finally, conditions (S1) and (S2) for a complete slice are trivially satisfied.

Note: After the completion of this paper, the author learned that K. Bongartz and P. Gabriel had also characterized the tilted algebras of type $\mathbf{A}_n$ in [5]. However, their motivations and methods are quite different from the ones used here.

ACKNOWLEDGEMENTS

I would like to express my thanks to O. Roldán for the discussions we held on the subject. I am also indebted to V. Dlab and C. M. Ringel for their helpful remarks made during the preparation of this article.

REFERENCES

[1] I. ASSEM: Iterated tilted algebras, Ph.D. thesis, Carleton University, July 1981.

[2] I. ASSEM and D. HAPPEL: Generalized tilted algebras of type $\mathbf{A}_n$, Comm. Algebra 9 (1981), 2101-2125; Erratum, Comm. Algebra 10 (1982),1475.

[3] M. AUSLANDER and I. REITEN: Representation theory of Artin algebras III and IV, Comm. Algebra 3 (1975), 239-294 and 5 (1977), 443-518.

[4] R. BAUTISTA: Sections in Auslander-Reiten quivers, Proc. ICRA II 1979, Springer Lecture Notes No. 832 (1980), 74-96.

[5] K. BONGARTZ and P. GABRIEL: Covering spaces in Representation theory, preprint.

[6] V. DLAB: Representations of valued graphs. Séminaire de Mathématiques Supérieures, 1979, Presses de l'Université de Montréal (1980).

[7] P. GABRIEL: Indecomposable representations II, Symposia Math. Ist. Naz. Alta. Mat. 11 (1973), 81-104.

[8] P. GABRIEL: Auslander-Reiten sequences and representation-finite algebras, Proc. ICRA II 1979, Springer Lecture Notes No. 831 (1980), 1-71.

[9] D. HAPPEL and C.M. RINGEL: Tilted Algebras. To appear in Trans. Amer. Math. Soc.

[10] D. HAPPEL and C.M. RINGEL: Construction of tilted algebras. Preprint.

[11] C.M. RINGEL: Report on the Brauer-Thrall conjectures, Proc. ICRA II, 1979, Springer Lecture Notes No. 831 (1980), 104-136.

[12] C.M. RINGEL: Tame algebras, Proc. ICRA II, Springer Lecture Notes No. 831 (1980), 137-287.

Received: September 1981