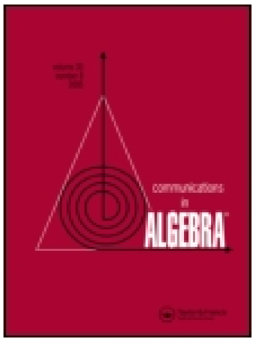

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THE STRONG SIMPLE CONNECTEDNESS OF A TAME TILTED ALGEBRA

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Introduction.

Since its introduction by Happel and Ringel, the class of tilted algebras has attracted a lot of interest in representation theory (see, for instance, [9, 10, 14, 16]). Among other things, it was shown that the Auslander-Reiten quiver $\Gamma(\text{mod}\, A)$ of a tilted algebra A contains postprojective, preinjective and connecting components. If, furthermore, A is representation-finite, then $\Gamma(\text{mod}\, A)$ consists of a unique component, and the orbit graph of $\Gamma(\text{mod}\, A)$ is a tree if and only if A is simply connected (in the sense of [4]), or equivalently, A satisfies the separation condition of [3]. In the representation-infinite case, however, the simple connected algebras are not well-understood. This was the reason for the introduction of a more accessible subclass, that of the strongly simply connected algebras. Let A be a finite dimensional basic and connected algebra over an algebraically closed field k then, as observed in [4], the algebra A can equivalently be considered as a k-category. It is said to be *strongly simply connected* if its ordinary quiver has no oriented cycles and, for each full convex subcategory B of A, the first Hochschild cohomology group $H^1(B)$ with coefficients in the bimodule ${}_BB_B$ vanishes (for equivalent definitions and characterisations, we refer the reader to [17, 2]). If A is representation-finite, then it is strongly simply connected if and only if it is simply connected, or if and only if it satisfies the separation condition.

It was shown in [1] that, if A is a strongly simply connected tilted algebra then the orbit graph of each of the postprojective, the preinjective and the connecting components of $\Gamma(\operatorname{mod} A)$ is a tree. More recently, and more generally, it was shown in [5] that, if A is a (not necessarily tilted) strongly simply connected algebra, then the orbit graph of any convex directing component of $\Gamma(\operatorname{mod} A)$ is a tree. The aim of the present paper is to provide criteria allowing to recognise whether a tame tilted algebra is strongly simply connected or not.

Theorem. *Let A be a tame tilted algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*

(b) *The orbit graph of each of the postprojective, the preinjective and the connecting components of $\Gamma(\operatorname{mod} A)$ is a tree.*

(c) *$H^1(A) = 0$, and A contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}_m$.*

(d) *A satisfies the separation condition and contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}_m$.*

We point out that each of the stated conditions depends only on A, and not on all its full convex subcategories. Also, we notice that, in concrete examples, condition (d) is particularly easy to verify.

The paper is organised as follows. After recalling definitions and proving some preliminary results in section 1, we consider the tame tilted algebras in section 2. Section 3 is devoted to the proof of our theorem above, and section 4 to the particular case of the sincere tame tilted algebras.

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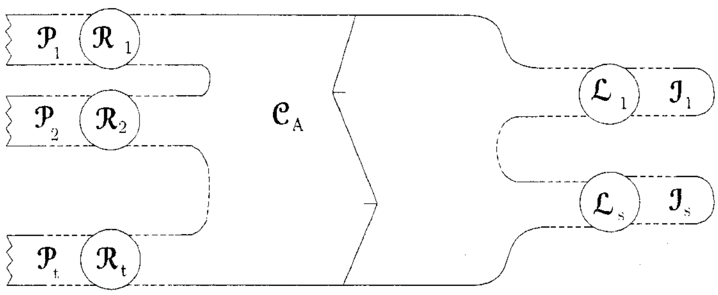
1. Notation and preliminary results.

1.1. Notation. Throughout this paper, k will denote a fixed algebraically closed field. By algebra is meant a basic and connected finite dimensional associative k-algebra with an identity, and by module a finitely generated right module. We sometimes consider an algebra A as a k-catgory, whose object set is denoted by A_0, as in [4]. A full subcategory C of A is called *convex* if, for any path $a_0 \to a_1 \to \cdots \to a_t$ in A, with $a_0, a_t \in C_0$, we have $a_i \in C_0$ for all i. For an algebra A, we denote by $\operatorname{mod} A$ its module category and by P_x (or I_x, or S_x) the indecomposable projective (or injective, or simple, respectively) module corresponding to $x \in A_0$. We use freely and without further reference properties of mod A, the Auslander-Reiten translation $\tau_A = D\mathrm{Tr}$ and $\tau_A^{-1} = \mathrm{Tr}\, D$, and the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of A, as can be found, for instance, on [14].

1.2. Tilted algebras. Let Δ be a finite connected quiver without oriented cycles, an algebra A is *tilted of type* Δ if there exists a tilting module T over the path algebra $k\Delta$ such that $A = \mathrm{End}\, T_{k\Delta}$. Tilted algebras are characterised by the existence of *complete slices* in a component of their Auslander-Reiten quiver, called *connecting component* [14]. A tilted algebra has at most two connecting components and, if it has two, then it is a concealed algebra, that is, it is the endomorphism algebra of a postprojective (or preinjective) tilting module [15]. The structure of $\Gamma(\mathrm{mod}\, A)$ for a tilted algebra A was given in [8] as follows. If A is not concealed, and $\mathcal{C}_A$ is its unique connecting component, then the *left end algebra* ${}_\infty A$ if A is defined a ${}_\infty A = \mathrm{End}\left(\bigoplus_{P_x \notin \mathcal{C}_A} P_x\right)$. If A is concealed, then ${}_\infty A = A$. We have ${}_\infty A = \prod_{i=1}^{t} A_i$, where each A_i is a tilted algebra having a complete slice in its preinjective component. The *right end algebra* A_∞ is defined dually. Then $\Gamma(\mathrm{mod}\, A)$ has the following shape



It consists of postprojective components $\mathcal{P}_1, \ldots, \mathcal{P}_t$, preinjective components $\mathcal{J}_1, \ldots, \mathcal{J}_s$, the connecting component $\mathcal{C}_A$, families of right stable components $\mathcal{R}_1, \ldots, \mathcal{R}_t$ and families of left stable components $\mathcal{L}_1, \ldots, \mathcal{L}_s$.

We denote by B_i the support algebra of the postprojective component $\mathcal{P}_i$ ($1 \leq i \leq t$), that is, the full convex subcategory of A generated by those $x \in A$, such that $P_x \in \mathcal{P}_i$. In case $\mathcal{C}_A$ is not postprojective, it is easily seen that B_i is the quotient of A_i by those $x \in (A_i)_0$ such that $P_x \in \mathcal{R}_i$.

1.3. Orbit graphs. An indecomposable A-module M is called *directing* if there exists no sequence $M = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t = M$ of non-zero non-isomorphisms between indecomposable A-modules. A component Γ of $\Gamma(\mathrm{mod}\, A)$ is called *directing* if all M in Γ are directing. Moreover, Γ is called *convex* if, in any sequence $M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t$ of non-zero non-isomorphisms between indecomposable A-modules with M_0, M_t in Γ, then all M_i lie in Γ. Examples of convex directing components are the postprojective, preinjective and connecting components of a tilted algebra (see for instance [13]).

Given a component Γ of $\Gamma(\text{mod } A)$, the *orbit graph* $\mathcal{O}(\Gamma)$ is defined as follows: the points of $\mathcal{O}(\Gamma)$ are the τ_A-orbits M^τ of A-modules M in Γ, and there exists an edge M^τ — N^t whenever there is an arrow in Γ of the form $\tau_A^a M \to \tau_A^b N$ or $\tau_A^b N \to \tau_A^a M$ for some $a, b \in \mathbb{Z}$.

Theorem [5]. *Let A be a strongly simply connected algebra and Γ be a convex directing component of $\Gamma(\text{mod } A)$, then $\mathcal{O}(\Gamma)$ is a tree.* □

1.4. Proposition. *Let A be a tilted algebra, and $\mathcal{C}_A$ be a connecting component of $\Gamma(\text{mod } A)$, then $\mathcal{O}(\mathcal{C}_A)$ is a tree if and only if $H^1(A) = 0$.*

Proof: Let Σ be a complete slice in $\mathcal{C}_A$. Then A is tilted of type Σ^{op}. By [7, (4.2)], we have $H^1(A) = H^1(k\Sigma^{op})$. By [7, (1.6)], we deduce that $H^1(A) = 0$ if and only if Σ is a tree. □

The results of [1] are in fact a direct consequence of the above proposition. They may be presented (and generalised) as follows:

Corollary. *Let A be a tilted algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*

(b) *For every full convex subcategory B of A, the graph $\mathcal{O}(\mathcal{C}_B)$ is a tree.*

(c) *For every full convex subcategory B of A, and every directing component Γ of $\Gamma(\text{mod } B)$, the graph $\mathcal{O}(\Gamma)$ is a tree.*

Proof: By [6, (III.6.5)], any full convex subcategory of a tilted algebra is itself tilted. The equivalence of (a) and (b) follows directly from the above proposition. Since (c) implies (b) trivially, we show that (b) implies (c). Let B be a full convex subcategory of A, and Γ be a directing component of $\Gamma(\text{mod } B)$. We let $B' = B/\text{Ann }\Gamma$, where $\text{Ann }\Gamma = \{b \in B \mid Mb = 0 \text{ for all } M \in \Gamma\}$ is an ideal of B. It is easily shown that B' is a full convex subcategory of A, hence it is tilted, and Γ is a connecting component of $\Gamma(\text{mod } B')$ (see, for example, [10, (2.4)] [18, (3.1)]). The result follows. □

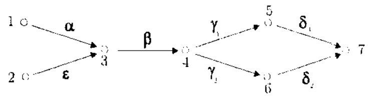
1.5. To compute the Hochschild cohomology groups, the following is useful. Let A be a one-point extension of an algebra B by a B-module M, that is, $A = B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$ with the usual matrix operations. Then B is a full convex subcategory of A. It is shown in [7, (5.3)] that there is an exact sequence

$$0 \to H^0(A) \to H^0(B) \to \text{End } M/k \to H^1(A) \to H^1(B) \to$$
$$\to \text{Ext}_B^1(M, M) \to H^2(A) \to \cdots$$

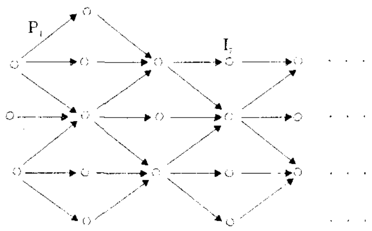
Lemma. *Let $A = B[M]$ be tilted of type Δ, where Δ is a tree. Then B is tilted of tree type if and only if $\text{Ext}_B^1(M, M) = 0$.*

Proof: If Δ is a tree, then $H^i(A) = H^i(k\Delta) = 0$ for all $i \geq 1$, by [7, (1.6)]. The type Σ of B is a tree if and only if $0 = H^1(k\Sigma) \cong H^1(B) \cong \mathrm{Ext}^1_B(M, M)$. $\square$

Examples: (a) Let A be given by the quiver

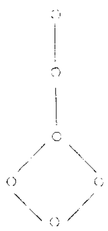
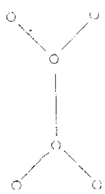


bound by $\alpha\beta\gamma_1\delta_1 = 0$ and $\beta\gamma_1\delta_1 = \beta\gamma_2\delta_2$. Then A is a tilted algebra whose connecting component $\mathcal{C}_A$ contains the full subquiver



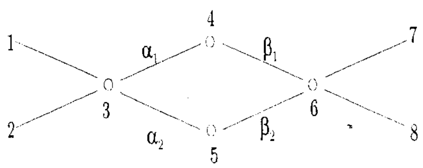
Thus the orbit graph of $\mathcal{C}_A$ is the tree

In particular, $H^1(A) = 0$. Also, $\Gamma(\mathrm{mod}\, A)$ contains a unique postprojective component $\mathcal{P}$, which is that of the algebra $B = A/\langle e_1 \rangle$, and a unique preinjective component $\mathcal{J}$, which is that of $B' = A/\langle e_7 \rangle$. Hence

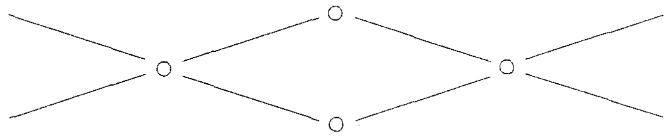
$\mathcal{O}(\mathcal{P})$  and $\mathcal{O}(\mathcal{J})$ 

Clearly, A is not strongly simply connected. Moreover, $H^1(B) \neq 0$.

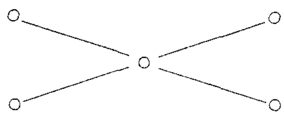
(b) Let A be given by the quiver



bound by $\alpha_1\beta_1 = 0$, $\alpha_2\beta_2 = 0$. Then A is tilted and $\mathcal{O}(\mathcal{C}_A)$ is the graph



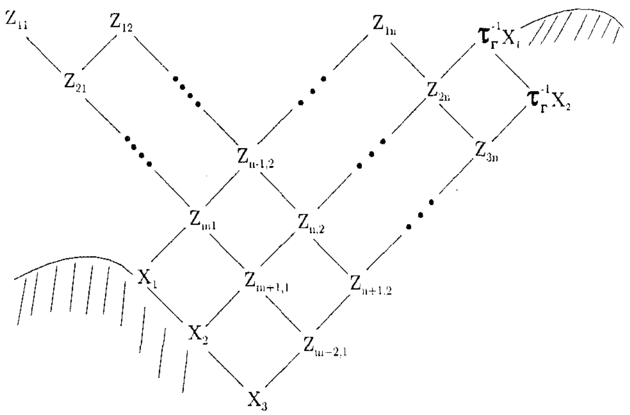
In particular, $H^1(A) \neq 0$. On the other hand, the postprojective and preinjective components of $\Gamma(\mathrm{mod}\, A)$ have tree orbit graphs, namely, the orbit graph of each is



2. Tame tilted algebras.

2.1. We start with some considerations that are well-known to specialists, and are included for the convenience of the reader. Let (Γ, τ) be a translation quiver. A point x in Γ is a *ray vertex* if there is an infinite sectional path $x = x_1 \to x_2 \to \cdots \to x_i \to \cdots$ on Γ, which is called a *ray* starting at x, such that, for any $i > 0$, the path $x = x_1 \to x_2 \to \cdots \to x_i$ is the only sectional path of length i starting at x.

Let x be a ray vertex in Γ, and $n > 0$. We define a translation quiver $\Gamma(x, n)$ as follows. The points of $\Gamma(x, n)$ are those of Γ, and additional points z_{ij}, with $i \geq 1$, $1 < j \leq n$. The arrows of $\Gamma(x, n)$ are those of Γ except those starting at x_i other than $x_i \to x_{i+1}$ (for $i \geq 1$), and additional arrows as in the following figure



The translation of $\Gamma(x, n)$ is defined in the obvious way. If the process is carried out inductively, then $\Gamma(x_1, n_1) \ldots (x_s, n_s)$ is said to be obtained from Γ by a sequence of *ray insertions* [14]. We define dually corays and coray insertions.

Lemma. *Let Γ' be a standard component of $\Gamma(\mathrm{mod}\, A)$ such that, as translation quiver, $\Gamma' = \Gamma(x, n)$. Consider the indecomposable projective P_s corresponding to* *the point* z_{n1} *in* Γ'. *Assume that, for all indecomposable projectives* P_s *not in* Γ', *we have* $\operatorname{Hom}_A(P_x, P_x) = 0$. *Then:*

(a) *s is a source in the quiver of A.*

(b) *Letting $B = A/\langle e_s \rangle$, we have that Γ is a standard component of $\Gamma(\operatorname{mod} B)$. Furthermore, the module corresponding to a point y in Γ is the module $i(y)$ in Γ', where $i: \Gamma \to \Gamma'$ is the obvious embedding.*

Proof: The proof is straightforward, we just indicate the main steps. We use the notation above, and denote by $M(y)$ the A-module corresponding to the point y in Γ'. Since (a) follows from the assumption on P_s, and the standardness of Γ', we show (b).

Clearly, any irreducible morphism $f: M \to N$ in Γ' such that neither M nor N lies in the support of $\operatorname{Hom}_A(P_s, -)|_\Gamma$, remains irreducible in $\operatorname{mod} B$. We thus have to show that the sequences

$$0 \to M(x_1) \xrightarrow{f_1} M(x_2) \xrightarrow{g_1} M(\tau^{-1}x_1) \longrightarrow 0 \text{ and}$$

$$0 \longrightarrow M(x_i) \xrightarrow{\begin{bmatrix} f_i \\ g_i \end{bmatrix}} M(x_{i+1}) \oplus M(\tau_\Gamma^{-1} x_{i-1}) \xrightarrow{[g_{i+1}, f'_{i-1}]} M(\tau_\Gamma^{-1} x_i) \longrightarrow 0 \text{ for } i > 2$$

are almost split in $\operatorname{mod} B$, where $f_i: M(x_i) \to M(x_{i+1})$ and $f'_i: M(\tau_\Gamma^{-1} x_i) \to M(\tau_\Gamma^{-1} x_{i+1})$, for $i \geq 1$ are the irreducible morphisms in $\operatorname{mod} A$, as given in Γ', and $g_{i+1}: M(x_{i+1}) \to M(\tau_\Gamma^{-1} x_i)$ is the composition of the $n+1$ irreducible morphisms in Γ' on the path $M(x_{i+1}) \to M(z_{n+i-1,1}) \to \cdots \to M(z_{i+1,n}) \to M(\tau_\Gamma^{-1} x_i)$ (whenever x_i is non-injective, otherwise the non-zero morphism on the left of the sequence is left almost split).

Since the exactness of the sequences is easily shown, we must prove that any morphism $f: M(x_i) \to N$ in $\operatorname{mod} B$, which is not a section, factors through the middle term of the above sequence. If $i = 1$, there exists a morphism $[h', h'']: M(x_2) \oplus P_s \to N$ such that $f = h' f_1 + h''_j$, where $j: M(x_1) \to P_s$ is the inclusion. However, $\operatorname{Hom}_A(P_s, N) = 0$ yields $h'' = 0$ and $f = h' f_1$. If $i > 1$, there exists a morphism $[h'_i, h''_i]: M(x_{i+1}) \oplus M(x_{n+i-1,1}) \to N$ such that $h = h'_i f_i + h''_i g_{i1}$, where $g_{i1}: M(x_i) \to M(z_{n+i-1,1})$ is the obvious irreducible morphism. Factorising successively h''_i through the modules $M(z_{n+i-2,2}), \ldots, M(z_{i,n})$, $M(\tau_\Gamma^{-1} x_i)$ yields the result. □

2.2. By [9], the connected components of the Auslander-Reiten quiver of a tilted algebra are either directing, stable of type $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$ (for some $n > 0$) or are obtained by ray or coray insertions from components of type $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$. If in particular, A is a tame tilted algebra, then the components of $\Gamma(\operatorname{mod} A)$ are either directing, stable tubes, ray tubes or coray tubes. We need the following lemma and its dual, for which we use the notation of (1.2).

Lemma. *Let A be a tame tilted algebra which is not concealed such and that $\mathcal{C}_A$ is not postprojective, then, for each $1 \leq i \leq t$,*

(a) *A_i is a tilted algebra of Euclidean type.*

(b) *B_i is a tame concealed algebra, and A_i is a domestic tubular extension of B_i.*

Proof: Let $\mathcal{P}_i$ be a postprojective component of $\Gamma(\text{mod}\, A)$. Since $\mathcal{P}_i \neq \mathcal{C}_A$, there are no injective A-modules in $\mathcal{R}_i$ or $\mathcal{P}_i$. Hence by [8, (4.2)], $A_i = \text{End}\, T_H$, where H is a tame hereditary algebra and T is a tilting H-module without preinjective direct summands. By [14, (4.9)], A_i is a domestic tubular algebra, that is, a tilted algebra of Euclidean type. The algebra B_i is obtained from A_i by deleting those $x \in (A_i)_0$ such that $P_x \in \mathcal{R}_i$. Since $\mathcal{R}_i$ is a standard orthogonal tubular family [14], we apply repeatedly (2.1) to get that the tubes in $\Gamma(\text{mod}\, B_i)$ are stable. Hence all injective B_i-modules lie in the preinjective component. Thus $\Gamma(\text{mod}\, B_i)$ has two components with complete slices: B_i is concealed. □

2.3. Proposition. *Let A be a tame tilted algebra which is not concealed. The following conditions are equivalent:*

(a) *The postprojective and preinjective components of $\Gamma(\text{mod}\, A)$ which are different from $\mathcal{C}_A$ have tree orbit graphs.*

(b) *The support algebra of any postprojective or preinjective component which is different from $\mathcal{C}_A$ is tame concealed of type $\tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$.*

(c) *A contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}_m$.*

Proof: The equivalence of (a) and (b) follows from (2.2) and its dual. That (c) implies (b) follows from the fact that the support algebra of a postprojective, or a preinjective component, is a full convex subcategory of A. In order to show that (b) implies (c), let C be a full convex subcategory of A which is hereditary of type $\tilde{\mathbb{A}}_m$. Let $(M_\lambda)_\lambda$ be an infinite family of pairwise non-isomorphic indecomposable C-modules with the same dimension-vector. Observe that the M_λ are necessarily sincere C-modules. By (1.2), the family $(M_\lambda)_\lambda$ lies in one of $\mathcal{R}_1, \ldots, \mathcal{R}_t$ or $\mathcal{L}_1, \ldots, \mathcal{L}_s$. Assume that $M_\lambda \in \mathcal{R}_1$ for all λ. We may assume that all the M_λ lie in stable tubes. For any $x \in C_0$, we have $\text{Hom}_A(P_x, M_\lambda) \neq 0$. Hence $P_x \in \mathcal{P}_1$ and C is contained in the support algebra B_1 of $\mathcal{P}_1$. Since B_1 is concealed, then $B_1 = C$, a contradiction. □

2.4. Proposition. *Let A be a tame tilted algebra and assume that $A = B[M]$ and that the postprojective and connecting components of $\Gamma(\text{mod}\, A)$ have tree orbit graphs. Then $H^1(B) = 0$.*

Proof: We assume that $H^1(B) \neq 0$ to obtain a contradiction. By (1.4), we have $H^1(A) = 0$. Hence (1.5) implies that $\text{Ext}^1_B(M, M) \neq 0$.

We use the description of $\Gamma(\text{mod}\, B)$ and the notation of (1.2). We claim that there exist some $1 \le i \le t$ such that M is a regular homogeneous B_i-module.

First, we observe that $t \ge 1$ and that M belongs to one of the families $\mathcal{R}_i$, $1 \le i \le t$. To show this, assume that $M \in \mathcal{L}_j$. Then $\mathcal{C}_B$ remains a component in $\Gamma(\text{mod}\, A)$ and it is not connecting in $\Gamma(\text{mod}\, A)$. Since A is a tilted algebra, $\mathcal{C}_B$ is postprojective in $\Gamma(\text{mod}\, B)$. The hypothesis implies that $\mathcal{C}_B$ has a tree orbit graph, hence $H^1(B) = 0$, a contradiction. Hence, we may assume that $M \in \mathcal{R}_1$.

We now show that M is a B_1-module. Since M is not directing as a B-module (and hence as an A-module), the component Γ of $\Gamma(\text{mod}\, A)$ where M lies is a ray tube (by (2.2) and the fact that, if P is the projective A-module such that $M = \text{rad}\, P$, then P lies in Γ). By (2.1), there are exactly two arrows of source M in Γ, namely

$M \to P$ and $M \to N$ (say). By (2.2) again, $M \to N$ is the unique arrow of source M on $\Gamma(\operatorname{mod} B)$. Therefore M does not lie on rays starting at projective modules in $\mathcal{R}_1$, that is, M is a B_1-module. The same argument shows that M lies on the mouth of a tube in $\Gamma(\operatorname{mod} B_1)$. Since $\operatorname{Ext}_B^1(M, M) \neq 0$, that tube is actually homogeneous. This completes the proof of our claim.

By (2.3) and our hypothesis, B_1 is concealed of type $\tilde{\mathbb{D}}_n$ or $\tilde{\mathbb{E}}_p$ ($p = 6, 7, 8$). On the other hand, $B_1[M]$ is tame, because it is a full convex subcategory of the tame algebra A. This implies that the tubular type of $B_1[M]$ is $(2, 2, 2, 2)$ (that is, the tubular type of B_1 is $(2, 2, 2)$) and thus $B_1[M]$ is a tubular algebra. This contradicts the fact that, since it is a full convex subcategory of the tilted algebra A, $B_1[M]$ is itself tilted [6, (III.6.5)]. □

3. Proof of the theorem.

Since the four conditions in our theorem are clearly equivalent for tame concealed algebras, we may assume that our algebra is not concealed.

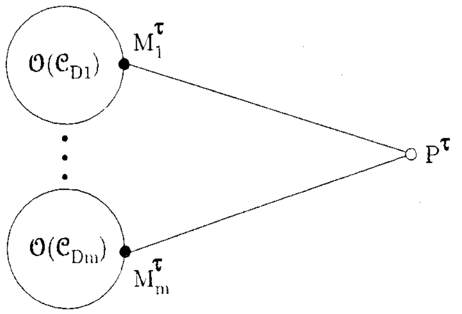
3.1. We first show the equivalence of the first three conditions. That (a) implies (b) follows from (1.3). The equivalence of (b) and (c) follows from (1.4) and (2.3). There remains to show that (b) implies (a). Let B be a connected full convex subcategory of A. We prove by descending induction on the number $|B_0|$ of objects of B that all the directing components of $\Gamma(\operatorname{mod} B)$ have tree orbit graphs. The strong simple connectedness of A then follows from (1.4).

If $|B_0| = |A_0|$, the desired property is granted by hypothesis. Clearly, it is enough to consider the case where $A = B[M]$ and show the corresponding property for B. By (2.4), $H^1(B) = 0$ and (1.4) yields that the connecting component of $\Gamma(\operatorname{mod} B)$ has a tree orbit graph. Since we have already shown that (b) implies (c), A contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}_m$. Hence neither does B. Since B is tame tilted, (2.3) implies that the postprojective and the preinjective components of $\Gamma(\operatorname{mod} B)$ have tree orbit graphs. □

3.2. We now show the equivalence of the first three conditions with the last one. Clearly, (a) implies (d). We show that (d) implies (b). Again, we use the description of $\Gamma(\operatorname{mod} A)$ and the notation of (1.2). By hypothesis and (2.3), the postprojective and the preinjective components distinct from $\mathcal{C}_A$ have tree orbit graphs. We now prove that $\mathcal{O}(\mathcal{C}_A)$ is a tree.

Since A is tame, it follows from [16] that $\mathcal{C}_A$ is not regular. Then an arbitrary complete slice Σ contains a module which is not left stable or a module which is not right stable. Up to duality, we may assume that Σ contains a module which is not left stable. Applying repeatedly τ_A yields a complete slice Σ' containing a projective module P_a such that, for any projective P_x on Σ', we have $\operatorname{Hom}_A(P_a, P_x) = 0$. This implies that, for any projective A-module $P_y \not\cong P_a$, we have $\operatorname{Hom}_A(P_a, P_y) = 0$: indeed, if $\operatorname{Hom}_A(P_a, P_y) \neq 0$, the sincerity of Σ' yields a module M on Σ' such that $\operatorname{Hom}_A(P_y, M) \neq 0$, and the convexity of Σ' implies that $P_y \in \Sigma'$, a contradiction.

Let $\operatorname{rad} P_a = M_1 \oplus \cdots \oplus M_m$, where $M_1, \ldots, M_m$ are indecomposable. By hypothesis, A satisfies the separation condition, thus, since a is a source in the quiver of A, we have $A/\langle e_a \rangle = D_1 \times \cdots \times D_m$ such that M_i is an indecomposable D_i-module, for each $1 \leq i \leq m$. Clearly, D_i is a tilted algebra and by [13], M_i belongs to the connecting component $\mathcal{C}_{D_i}$. Then $\mathcal{O}(\mathcal{C}_A)$ is obtained as the graph



Each D_i, with $1 \leq i \leq m$, is a tame tilted algebra satisfying (d) (because $D_1 \times \cdots \times D_m$ is obtained from A by deleting the source a). By induction, $\mathcal{O}(\mathcal{C}_{D_i})$ is a tree and hence so is $\mathcal{O}(\mathcal{C}_A)$. This completes the proof. □

4. Sincere tame tilted algebras.

4.1. Let A be a tame tilted algebra with postprojective components $\mathcal{P}_1, \ldots, \mathcal{P}_t$ and preinjective components $\mathcal{I}_1, \ldots, \mathcal{I}_s$. If $\mathcal{C}_A$ is neither postprojective nor preinjective, then $\mu_A = s + t$ is the number of one-parameters for A; if $\mathcal{C}_A$ is postprojective then $t = 1$ and $\mu_A = s$ while if $\mathcal{C}_A$ is preinjective, then $s = 1$ and $\mu_A = t$.

In case A is sincere (by which we mean that A has a sincere and directing indecomposable module), it was shown in [11] that $\mu_A \leq 2$. If $\mu_A = 2$, then $\mathcal{C}_A$ is neither postprojective nor preinjective and $s = t = 1$. The classification of sincere tame tilted algebras A with $\mu_A = 2$ and $|A_0| \geq 20$ was given in [12]. We deduce some consequences of these results.

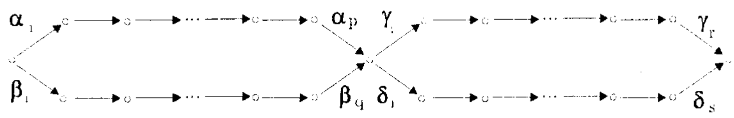
Proposition. *Let A be a sincere tame tilted algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*

(b) *Each of the postprojective and the preinjective component of $\Gamma(\text{mod}\, A)$ has a tree as orbit graph.*

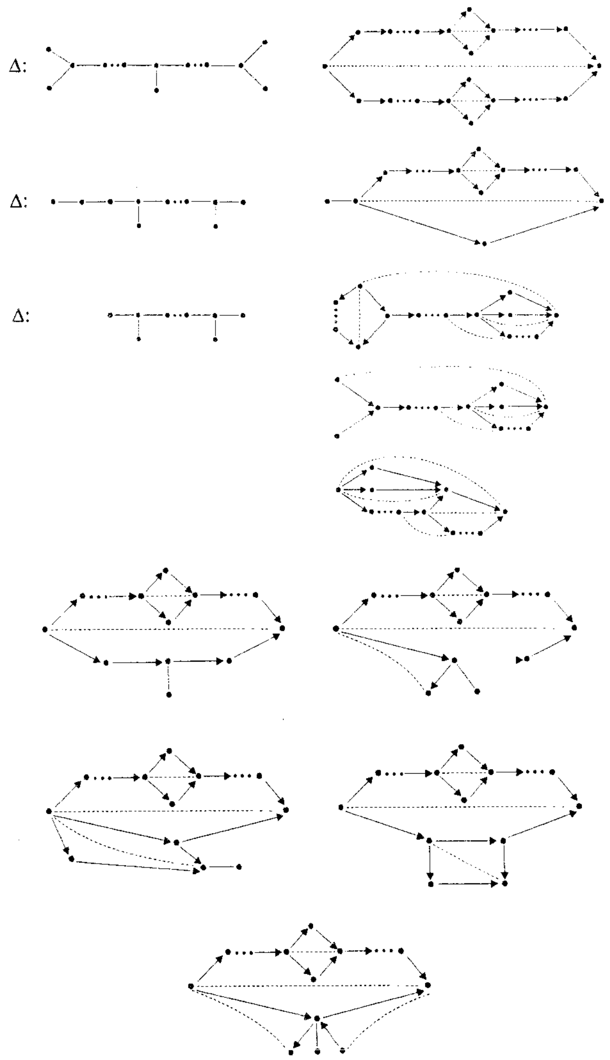
(c) *A has no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}_m$.*

Proof: By (1.3), (a) implies (b). The equivalence of (b) and (c) follows from (2.3) if $\mu_A = 2$, and is clear if $\mu_A = 1$. We show that (b) implies (a). By our theorem, it suffices to show that $\mathcal{O}(\mathcal{C}_A)$ is a tree. Assume that this is not the case. By hypothesis, $\mathcal{C}_A$ is neither postprojective nor preinjective and hence $\mu_A = 2$. By [12, (1.4)], $\mathcal{O}(\mathcal{C}_A)$ is of type $\tilde{\mathbb{A}}_m$ and A is given by the quiver



bound by $\alpha_p\gamma_1 = 0, \beta_q\delta_1 = 0, \alpha_1 \dots \alpha_p\delta_1 \dots \delta_s = \beta_1 \dots \beta_q\gamma_1 \dots \gamma_r$. It is straightforward to check that the postprojective component of $\Gamma(\text{mod}\, A)$ has an orbit graph of type $\tilde{\mathbb{A}}_{r+s}$, a contradiction. □

4.2. Proposition. *Let A be a sincere tame concealed algebra. Assume that A is strongly simply connected and has at least 20 vertices. Then A or A^{op} belong to one of the following families of algebras. Moreover A is tilted of type Δ. A dotted line $x - - - y$ indicates that the sum of all paths from x to y is zero.*



Proof: Straightforward checking of the list given in [13]. □

References

[1] Assem, I. and Liu, S.: *Strongly simply connected tilted algebras*, Ann. Sci. Math. Québec 21 (1997) 13–22.

[2] Assem, I. and Liu, S.: *Strongly simply connected algebras*, to appear.

[3] Bautista, R., Larrión, F. and Salmerón, L.: *On simply connected algebras*, J. London Math. Soc. 27 No. 2 (1983) 212–220.

[4] Bongartz, K. and Gabriel, P.: *Covering spaces in Representation Theory*, Invent. Math. 65 No. 3 (1981/82) 331–378.

[5] Gastaminza, S., de la Peña, J. A., Platzeck, M. I., Redondo M. J. and Trepode, S.: *Algebras with vanishing Hochschild cohomology*, J. Algebra 212, 1-16 (1999).

[6] Happel, D.: *Triangulated categories in the Representation Theory of finite dimensional algebras*, London Math. Soc. Lecture Note Series 119, Cambridge Univ. Press (1988).

[7] Happel, D.: *Hochschild cohomology of finite dimensional algebras*, Séminaire M.-P. Malliavin, Lecture Notes in Math. 1404, Springer (1989) 108–126, Berlin-Heidelberg-New York.

[8] Kerner, O.: *Tilting wild algebras*, J. London Math. Soc. Math. Soc. 39 (1989) 29–47.

[9] Liu, S.: *The connected components of the Auslander-Reiten quiver of a tilted algebra*, J. Algebra 161 (1993) 505–523.

[10] Liu, S.: *Tilted algebras and generalized standard components*, Arch. Math. 61 (1993), 12–19.

[11] de la Peña, J. A.: *Tame algebras with sincere directing modules*, J. Algebra 161 (1993) 171–185.

[12] de la Peña, J. A.: *The families of two-parametric tame algebras with directing modules*, Can. Math. Soc. Conf. Proc. 14 (1993) 361–392.

[13] de la Peña, J. A. and Takane, M.: *Constructing the directing components of an algebra*, Colloq. Math. 74 (1997) 29–46.

[14] Ringel, C. M.: *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics 1099, Springer (1984).

[15] Ringel, C. M.: *Representation Theory of algebras*, in: Representations of algebras, (Durham, 1985), London Math. Soc. Lecture Note Series 116, Cambridge University Press (1986) 7–79.

[16] Ringel, C. M.: *The regular components of the Auslander-Reiten quiver of a tilted algebra*, Chin. Ann. of Math. 913, No. 1 (1988) 1–18.

[17] Skowroński, A.: *Simply connected algebras and Hochschild cohomologies*, Can. Math. Soc. Conf. Proc. 14 (1993) 431–447.

[18] Skowroński, A.: *Generalized standard Auslander-Reiten components without oriented cycles*, Osaka J. Math. 30 (1993), 515–527.

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