The simple connectedness of a tame weakly shod algebra

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Abstract. We prove that a tame weakly shod algebra A which is not quasi-tilted is simply connected if and only if the orbit graph of its pip-bounded component is a tree, or if and only if its first Hochschild cohomology group $\mathrm{H}^1(A)$ with coefficients in ${}_AA_A$ vanishes. We also show that it is strongly simply connected if and only if the orbit graph of each of its directed components is a tree, or if and only if $\mathrm{H}^1(A) = 0$ and it contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}$, or if and only if it is separated and contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}$.

1. Introduction

Weakly shod algebras were introduced in [**17**], as a generalization of the shod algebras of [**15**], themselves a generalization of the quasi-tilted algebras [**23**]. Since their introduction, they were the subject of many investigations, see, for instance, [**1, 2, 9, 16, 18, 27**]. Here, we study the tame weakly shod algebras from the point of view of simple connectedness.

We recall that, following [**8**], a finite dimensional algebra A over an algebraically closed field k is simply connected if its quiver Q_A has no oriented cycles and, for any presentation $A \cong kQ_A/I$ of A as a bounded quiver algebra, the fundamental group of (Q_A, I) is trivial (see also [**7, 12, 28**]). Simply connected algebras have played an important role in the representation theory of algebras because covering techniques often allow to reduce many problems to the study of simply connected algebras. A well-known result, due to Bongartz and Gabriel [**12**](6.5), states that a representation-finite algebra is simply connected if and only if the orbit graph of its Auslander-Reiten quiver (see (4.1) below for the definition) is a tree. On the other hand, it was shown in [**13**] that a representation-finite algebra A is simply connected if and only if its first Hochschild cohomology group $\mathrm{H}^1(A)$ (with coefficients in the bimodule${}_AA_A$) vanishes. It is natural to ask whether similar results hold for a representation-infinite algebra. In this case, the Auslander-Reiten quiver is no longer connected so one should consider the orbit graph of each of its connected components. However, if one deals with a tilted algebra (in the sense of [**22**]), then much information about the algebra is contained in its connecting component(s),

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namely, the one(s) containing the complete slices. Indeed, it was shown in [**6**] that a tame tilted algebra A is simply connected if and only if the orbit graph of its connected component is a tree and this, by [**21**](1.6) or [**5**](1.4), is equivalent to saying that $\mathrm{H}^1(A) = 0$. This also answered positively (for tilted algebras) Skowroński's question in [**28**](Problem 1) whether it is true that a tame triangular algebra A is simply connected if and only if $\mathrm{H}^1(A) = 0$. Now, the weakly shod algebras which are strict, that is, are not quasi-tilted, have a component resembling the connecting component of a tilted algebra, namely its pip-bounded component [**16**].

In this paper, we introduce a filtration of a strictly weakly shod algebra as an iterated one-point extension by projectives lying in the pip-bounded component. This filtration, which we call maximal filtration, see (3.4), has very nice properties, among which are preservation of the vanishing of the first Hochschild cohomology group, see (3.4), tree type of the orbit graph, see (4.2) and, if the algebra is tame, simple connectedness, see (5.2). Applying these results, we prove our main theorem.

Theorem (A) *Let A be a strictly weakly shod algebra. The following conditions are equivalent:*

(a) $\mathrm{H}^1(A) = 0$.
(b) *The orbit graph of the pip-bounded component of A is a tree.*
If, moreover, A is tame, then the above are further equivalent to:
(c) *A is simply connected.*

Since a similar result was obtained in [**3**] for the tame quasi-tilted algebras, this completely characterizes the simple connectednes of a weakly shod tame algebra. On the other hand, weakly shod algebras have recently been generalized to a larger class, that of the laura algebras [**1, 2, 4, 9, 29, 27**] and we conjecture that the above result holds true as well for (possibly wild) weakly shod or laura algebras which are not quasi-tilted. While we have no counter-example, our proof here, however, fails in these cases.

We then turn to one particular subclass, that of the strongly simply connected algebras, introduced by Skowroński in [**28**]. This subclass seems to be the most accessible and has been the subject of many investigations. Since strongly simply connected tame quasi-tilted algebras have been characterized in [**3**] (see also [**5**]), we seek a criterion for the strong simple connectedness of a strictly weakly shod tame algebra, and we prove the following theorem which generalizes the main result of [**5**]:

Theorem (B) *Let A be a strictly weakly shod tame algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected;*
(b) *the orbit graph of every directed component of $\Gamma(\mathrm{mod}A)$ is a tree;*
(c) $\mathrm{H}^1(A) = 0$ *and A is strongly $\tilde{\mathbb{A}}$-free;*
(d) *A is separated and strongly $\tilde{\mathbb{A}}$-free.*

The paper is organized as follows. The first two sections 2 and 3 are devoted to proving the existence of maximal filtrations in strictly weakly shod algebras and to showing that they preserve the vanishing of the first Hochschild cohomology group. In the following two sections 4 and 5, we prove that the tree type of the orbit graph and simple connectedness are also preserved by maximal filtrations. Finally, sections 6 and 7 are devoted to the proofs of Theorem (A) and (B), respectively.

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2. Maximal extensions

2.1 **Notations.** Throughout this paper, all algebras are basic and connected finite dimensional algebras over an algebraically closed field k. For an algebra A, we denote by mod$A$ the category of all finitely generated right A-modules, and by ind$A$ a full subcategory of mod$A$ containing exactly one representative from each isomorphism class of indecomposable A-modules.

For an algebra A, we denote by $\Gamma(\text{mod}A)$ its Auslander-Reiten quiver, and by τ_A the Auslander-Reiten translation DTr.

Given $X, Y \in \text{ind}A$, we write $X \rightsquigarrow Y$ in case there exists a path

$$(*) \qquad X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t = Y$$

$(t \geq 0)$, from X to Y in ind$A$, that is, $f_1, \cdots, f_t$ are non-zero non-isomorphisms and $X_0, X_1, \cdots, X_t$ are indecomposable modules. In this case, we say that X is a predecessor of Y and Y is a successor of X. Observe that each indecomposable module is a predecessor and a successor of itself. A path in ind$A$ starting and ending at the same module is called a cycle. An indecomposable module M which lies on no cycle in ind$A$ is called a directing module. When each f_i in the path $(*)$ is an irreducible morphism, we say that $(*)$ is a path of irreducible morphisms or, simply, a path in $\Gamma(\text{mod}A)$.

Following [**23**], we let $\mathcal{L}_A$ denote the full subcategory of ind$A$ consisting of those modules X such that, for any predecessor Y of X, the projective dimension $\text{pd}_A Y$ of Y does not exceed 1. Dually, $\mathcal{R}_A$ is the full subcategory of those modules X such that, for any successor Z of X, the injective dimension $\text{id}_A Z$ of Z does not exceed one.

For the sake of brevity, we refrain from stating the dual of each statement and leave the primal-dual translation to the reader.

2.2 The following lemma is a special case of [**4**](1.5). We prove it for the convenience of the reader.

Lemma. *Let A be an algebra. If P is an indecomposable projective module in $\mathcal{R}_A$, then P is directing.*

Proof: Assume that P is not directing, then there exists a cycle of non-zero non-isomorphisms:

$$P = M_0 \to M_1 \to \cdots \to M_{n-1} \to M_n = P$$

in ind$A$. Since P belongs to $\mathcal{R}_A$, then so does every M_i. It follows from [**1**](1.5) that this cycle may be refined to a cycle of irreducible morphisms and that the latter is sectional. This, however, contradicts [**14**], [**11**].□

2.3 Let $\mathcal{P}(\mathcal{R}_A)$ denote the set of all indecomposable projective modules that lie in $\mathcal{R}_A$. It follows from (2.2) that the successor relation defines a partial order in the set $\mathcal{P}(\mathcal{R}_A)$. Since this set is finite, it certainly contains maximal elements. This leads to the next definition.

Definition. *Let A be an artin algebra, $P = eA$ be a maximal projective in $\mathcal{P}(\mathcal{R}_A)$ for the successor relation, $B = (1-e)A(1-e)$ and $M =$radP. Then the one point extension $A = B[M]$ is said to be a maximal extension.*

The following proof is similar to that of [**18**](2.2)(2.3).

Proposition. *Let $A = B[M]$ be a strictly weakly shod algebra that is a maximal extension. Then for all $i \geq 1$, we have $Ext^i_B(M, M) = 0$.*
Proof: We consider separately the cases $i = 1$ and $i > 1$.

(i=1) Assume $\mathrm{Ext}^1_B(M, M) \neq 0$. Then there exists an indecomposable summand N of M such that $\mathrm{Ext}^1_B(M, N) \neq 0$. Write $M = N \oplus N'$ and let P be the extending projective (that is, $M = \mathrm{rad}_A P$). Then N' is a submodule of P and $L = P/N'$ is indecomposable. Since $\mathrm{Ext}^1_B(M, N) \neq 0$ it follows for [**23**](III.2.2)(a) that $\mathrm{id}_A L \geq 2$. Since P belongs to $\mathcal{P}(\mathcal{R}_A)$ and L is a successor of P we get a contradiction.
($i \geq 2$) Assume now that $\mathrm{Ext}^i_B(M, M) \neq 0$ for some $i \geq 2$. Again, there exists an indecomposable summand N of M such that $\mathrm{Ext}^i_B(M, N) \neq 0$. Applying the functor $\mathrm{Hom}_A(M, -)$ to the short exact sequence

$$0 \to N \to P \to K = P/N \to 0$$

yields an exact sequence:

$$\cdots \to \mathrm{Ext}^{i-1}_A(M, K) \to \mathrm{Ext}^i_A(M, N) \to \mathrm{Ext}^i_A(M, P) \to \cdots$$

Since $i \geq 2$ and $P \in \mathcal{P}(\mathcal{R}_A)$, we have $\mathrm{Ext}^i_A(M, P) = 0$. Thus, $\mathrm{Ext}^i_A(M, N) \neq 0$ implies that $\mathrm{Ext}^{i-1}_A(M, K) \neq 0$. We again consider two cases. If $i \geq 3$, then $\mathrm{Ext}^{i-1}_A(M, K) \neq 0$ implies that $\mathrm{id}_A K \geq 2$ and this contradicts the fact that the indecomposable module K is a successor of $P \in \mathcal{P}(\mathcal{R}_A)$. If $i = 2$, then $\mathrm{Ext}^1_A(M, K) \neq 0$ and the Auslander-Reiten formula imply that there exists an indecomposable summand M' of M such that $\mathrm{Hom}_A(K, \tau_A M') \neq 0$. We thus obtain a path in ind$A$:

$$P \to K \to \tau_A M' \to * \to M' \to P$$

and this contradicts (2.2).□

2.4 The above proposition serves to compare the Hochschild cohomology groups of A and B. For an algebra A, we denote by $\mathrm{H}^i(A)$ its i^{th} Hochschild cohomology group with coefficients in the bimodule ${}_AA_A$ (see [**21, 24**] for details). We recall that Happel has shown [**21**](5.3) that, if $A = B[M]$, then there exists a long exact sequence:

$$\begin{gathered} 0 \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(B) \longrightarrow (\mathrm{End}_A M)/k \longrightarrow \mathrm{H}^1(A) \longrightarrow \\ \longrightarrow \mathrm{H}^1(B) \longrightarrow \mathrm{Ext}^1_B(M, M) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{H}^i(A) \longrightarrow \mathrm{H}^i(B) \longrightarrow \mathrm{Ext}^i_B(M, M) \longrightarrow \cdots \end{gathered}$$

We refer to this sequence in the sequel as Happel's sequence.

Corollary. *Let $A = B[M]$ be an algebra written as maximal extension, then:*

(a) *There exists an exact sequence:*

$$0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to (\mathrm{End}_A M)/k \to \mathrm{H}^1(A) \to \mathrm{H}^1(B) \to 0.$$

(b) *For all* $i \geq 2$, *we have* $\mathrm{H}^i(A) \cong \mathrm{H}^i(B)$.□

2.5 Let x be a source in the quiver Q_A of an algebra A. We recall that x is said to be separating if the number of connected components of the full subquiver Q_A consisting of all points except x equals the number of indecomposables summands of the radical of the indecomposable projective A-module corresponding to x. We also recall that a module N is called a brick if $\mathrm{End}N = k$.

Corollary. *Let* $A = B[M]$ *be an algebra written as maximal extension, then* $\mathrm{H}^1(A) = 0$ *if and only if* $\mathrm{H}^1(B) = 0$, *the extension point is separating, and* M *is a direct sum of bricks.*
Proof: By (2.4)(a), $\mathrm{H}^1(A) = 0$ if and only if $\mathrm{H}^1(B) = 0$, and the sequence:

$$0 \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(B) \longrightarrow (\mathrm{End}_A M)/k \longrightarrow 0$$

is exact. We then apply [**3**](2.2).□

2.6 **Corollary.** *Let* $A = B[M]$ *be an algebra written as maximal extension, then the Hochschild cohomology rings* $\mathrm{H}^*(A)$ *and* $\mathrm{H}^*(B)$ *are isomorphic if and only if* M *is a brick.*
Proof: This follows inmediately from (2.4)(2.5) and [**20**](5.1)(6.2).□

3. Maximal filtrations of weakly shod algebras

3.1 We recall from [**17**] that an algebra is called weakly shod whenever the length of any path from an indecomposable injective to an indecomposable projective is bounded. It is called strictly weakly shod if it is weakly shod but not quasi-tilted.

By [**17**], the Auslander-Reiten quiver of a strictly weakly shod algebra contains a unique faithful pip-bounded component Γ, that is, a non-semiregular component such that there exists an integer n_0 with any path in ind$A$ from an injective module in Γ to a projective module in Γ has length at most n_0. This component is also generalized standard and directed (that is, has no oriented cycles). It follows from [**17**] that all projectives in $\mathcal{R}_A$ actually lie in the pip-bounded component.

Let A be a weakly shod algebra. We let $\mathcal{P}_A^f$ denote the set of all indecomposable projective modules P such that there exist an injective module I and a path $I \rightsquigarrow P$. It follows from [**17**] that the set $\mathcal{P}_A^f$ is also partially ordered by the successor relation, and hence contains maximal elements.

Lemma. *Let* A *be a weakly shod algebra, then* $\mathcal{P}(\mathcal{R}_A) = \phi$ *implies* $\mathcal{P}_A^f = \phi$.
Proof: Assume $\mathcal{P}_A^f \neq \phi$ and let P_1 belong to $\mathcal{P}_A^f$. Then there exist an indecomposable injective I and a path $I \rightsquigarrow P_1$ in ind$A$. In particular, P_1 lies in the pip-bounded component of $\Gamma(\mathrm{mod}A)$. On the other hand, $\mathcal{P}(\mathcal{R}_A) = \phi$ implies $P_1 \notin \mathcal{R}_A$. That is, there exist an indecomposable projective P_2, an indecomposable non-injective M_A and a path in ind$A$:

$$P_1 \rightsquigarrow M \to * \to \tau_A^{-1} M \to P_2.$$

Since P_2 is a successor of P_1, then $P_2 \in \mathcal{P}_A^f$ and then P_2 lies in the pip-bounded component of $\Gamma(\mathrm{mod}A)$. Since, again, $P_2 \notin \mathcal{R}_A$, we may construct inductively an infinite sequence of indecomposable projectives:

$$P_1 \rightsquigarrow P_2 \rightsquigarrow P_3 \rightsquigarrow \cdots$$

all lying in the pip-bounded component of $\Gamma(\text{mod}A)$, in $\mathcal{P}^f_A$ but not in $\mathcal{R}_A$. Since the pip-bounded component is directed, we get a contradiction to the fact that A has only finitely many projectives.□

3.2 **Lemma.** *Let A be a strictly weakly shod algebra, and $A = B[M]$ be a maximal extension. Then, if $\mathcal{P}(\mathcal{R}_B) = \phi$:*

(a) *$\mathcal{P}^f_A$ has at most one element, and this element is the unique extending projective P, and*
(b) *every connected component of B is tilted.*

Proof: (a) Assume there exists $Q \in \mathcal{P}^f_A$, such that $Q \ncong P$. Hence Q is a projective indecomposable $B-$module. Since $\mathcal{P}(\mathcal{R}_B) = \phi$, there exists another projective $Q_1 \in \text{ind}B$ and a path $Q = Q_0 \rightsquigarrow Q_1$ in ind$B$. Since, again, $\mathcal{P}(\mathcal{R}_B) = \phi$, we get a projective Q_2 and a path $Q = Q_0 \rightsquigarrow Q_1 \rightsquigarrow Q_2$. Inductively, we reach a cycle of projectives in the algebra B (because there are only finitely many non-isomorphic projective indecomposable $B-$modules). Embedding everything in mod$A$, we obtain a path:

$$I \rightsquigarrow Q = Q_0 \rightsquigarrow Q_1 \rightsquigarrow Q_2 \rightsquigarrow \cdots$$

with I injective (because $Q \in \mathcal{P}^f_A$), and a cycle of projectives. This is a contradiction to A being weakly shod.
(b) Assume first that $P \in \mathcal{P}^f_A$. Now A weakly shod implies B weakly shod, [**17**] or [**2**]. Then, by Lemma (3.2), $\mathcal{P}(\mathcal{R}_B) = \phi$ implies $\mathcal{P}^f_B = \phi$. On the other hand, the maximality of the extending projective P in $\mathcal{R}_A$ implies clearly its maximality in $\mathcal{P}^f_A$. Hence the result follows from [**17**](4.8). Now, if $P \notin \mathcal{P}^f_A$ we have $\mathcal{P}^f_A = \phi$. Let Q be an indecomposable projective $A-$module. Then Q has no injective predecessor, which implies that every predecessor of Q has projective dimension at most one. In particular, $Q \in \mathcal{L}_A$. Therefore, by [**23**], A is quasi-tilted. This contradicts the fact that A is strict.□

3.3 **Proposition.** *Let A be a strictly weakly shod algebra. Then there exist a sequence of algebras $A_0, A_1, \cdots, A_m = A$ with A_0 a product of tilted algebras, and a sequence of modules $M_{iA_{i-1}}$ such that $A_i = A_{i-1}[M_i]$ is a maximal extension, for each i with $0 < i \leq m$.*
Proof: This follows immediately from the preceding discussion and induction.□

3.4 We use in the sequel the following notation. Let A be a strictly weakly shod algebra and

$$B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$$

be a filtration of A as iterated maximal extensions with B tilted, as in the above proposition. For each i with $0 < i \leq m$, we let M_i be the A_{i-1} module such that $A_i = A_{i-1}[M_i]$, P_i be the extending projective A_i-module (thus, $M_i = \text{rad}_A P_i$ and P_i is maximal in $\mathcal{P}(\mathcal{R}_{A_i})$) and x_i be the extension point associated to P_i. Such a filtration will be called a *maximal filtration* of A. Note that, while B is a tilted algebra, all A_i, with $i > 0$, are strictly weakly shod. Indeed, it follows from [**2**] that each A_i is weakly shod, and from [**19**] (5.2) that it is strict.

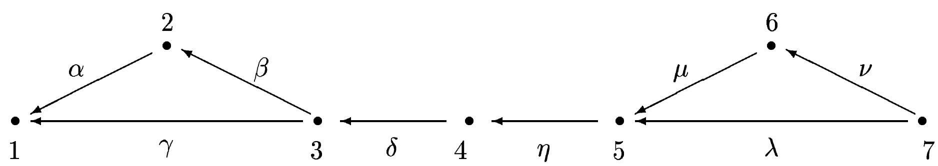
Corollary. *Let $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$, be a maximal filtration of A. Then $\mathrm{H}^1(A) = 0$ if and only if:*

(a) $\mathrm{H}^1(A_0) = 0$, *and*

(b) *each x_i is separating.*

Proof: The pip-bounded component of a strictly weakly shod algebra C contains all projectives in $\mathcal{P}(\mathcal{R}_C)$. Moreover, since it is generalized standard and directed, all its indecomposables are bricks. Therefore the statement follows from (2.5) and an obvious descending induction.□

3.5 **Example.** Let $A = kQ/I$ be given by the quiver:



bound by $\beta\delta = 0$, $\delta\eta = 0$ and $\eta\lambda = 0$.

In this case we have $A = A_3 = A_2[M_3]$ where $M_3 = \mathrm{rad}_{A_4} P_7$. As second step we get $A_2 = A_1[M_2]$ with $M_2 = \mathrm{rad}_{A_2} P_6 (= \mathrm{rad}_{A_4} P_6)$. Finally we have $A_1 = A_0[M_1]$ where $M_1 = \mathrm{rad}_{A_1} P_5 (= \mathrm{rad}_{A_4} P_5)$, and $B = A_0$ is a tilted algebra. Then we get a filtration:

$$B = A_0 \subset A_1 \subset A_2 \subset A_3 = A,$$

because P_7 is maximal in $\mathcal{P}(\mathcal{R}_{A_3})$, P_6 is maximal in $\mathcal{P}(\mathcal{R}_{A_2})$ and P_5 is maximal in $\mathcal{P}(\mathcal{R}_{A_1})$.

4. The orbit graph

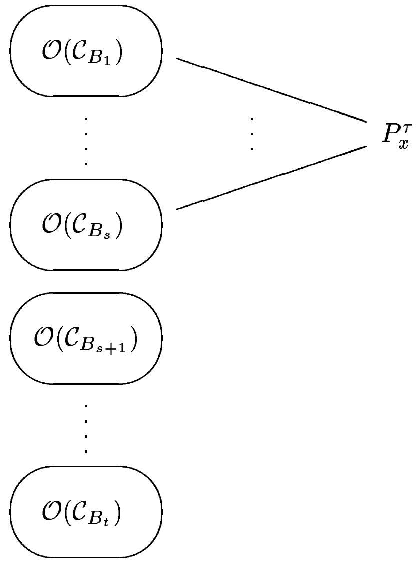
4.1 We recall that, if Γ is a locally finite, directed and connected translation quiver, then the orbit graph $\mathcal{O}(\Gamma)$ of Γ (see [**12**](4.2)) is defined as follows. The vertices of Γ are the τ-orbits x^τ of the points x in Γ, and the edges of Γ between x^τ and y^τ are the σ-orbits of arrows $\tau^n x \to \tau^m y$ or $\tau^m y \to \tau^n x$ for some $n, m \in \mathbb{Z}$.

We now discuss the orbit graph of the pip-bounded component of a weakly shod algebra. Let A be a strictly weakly shod algebra, not necessarily connected. We say that A is of *tree type* whenever, for each connected component C of A, the orbit graph of the pip-bounded component of Γ_C is a tree. We then have the following lemma.

Lemma. *Let $A = B[M]$ be a (not necessarily connected) strictly weakly shod algebra, which is a maximal extension. Then A is of tree type if and only if B is of tree type and the extension point is separating.*

Proof: Sufficiency. Let x denote the extension point, let $B = B_1 \times \cdots \times B_t$ where each of the B_i is connected. Since x is separating, there exists $s \leq t$ such that $\mathrm{rad} P_x = M = M_1 \oplus \cdots \oplus M_s$ where (changing the order if necessary) for each i, M_i is an indecomposable B_i-module. Moreover, it follows from [**17**](5.3)(5.4) that each M_i is an indecomposable lying in the pip-bounded component of the weakly shod algebra B_i. Then the orbit graph of the pip-bounded component $\mathcal{O}(\mathcal{C}_A)$ of A

is obtained from the orbit graphs of the pip-bounded components $\mathcal{O}(\mathcal{C}_{B_i})$ of the B_i as follows:



where we have added to $\mathcal{O}(\mathcal{C}_B)$ the point P_x^τ and at most one edge between P_x^τ and each $\mathcal{O}(\mathcal{C}_{B_i})$, with $1 \leq i \leq s$. Since B is of tree type, this shows that A is of tree type as well.
Necessity. Assume that A is of tree type. We show that the extension point x is separating. Assume that this is not the case, let $B = B_1 \times \cdots \times B_t$ where each of the B_i is connected, and $\text{rad}P_x = M = M_1 \oplus \cdots \oplus M_s$ where for each i, M_i is an indecomposable B-module. Since x is not separating, there exist i, j with $i \neq j$ such that both M_i, M_j lie in the same connected component of B, say B_l. Thus there exists a walk:

$$M_i^\tau \text{---} \cdots \text{---} M_j^\tau$$

in $\mathcal{O}(\mathcal{C}_{B_l})$ (since both M_i and M_j lie in the pip-bounded component $\mathcal{C}_{B_l}$, by [**17**](5.3)(5.4)). But then the edges P_x^τ —M_i^τ and P_x^τ —M_j^τ would give a non trivial cycle in $\mathcal{O}(\mathcal{C}_A)$, a contradiction. It is now clear that $\mathcal{O}(\mathcal{C}_B)$ is a (disjoint union of) tree(s).□

4.2 **Corollary.** *Let $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$, be a maximal filtration of A. Then A is of tree type if and only if:*

(a) *each A_i is of tree type, and*
(b) *each x_i is separating.*

Proof: This follows from (4.1) and induction.□

5. Simple connectedness

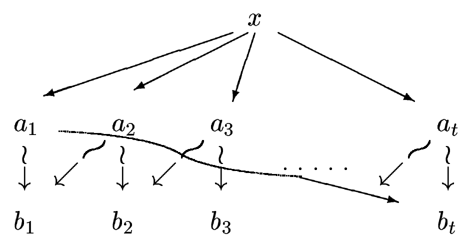
5.1 An algebra A is called simply connected provided its ordinary quiver Q_A has no oriented cycles and, for any presentation of A as a bound quiver algebra $A \cong kQ_A/I$, the fundamental group $\pi_1(Q_A, I)$ is trivial (see [**7, 28**] for details). By [**7**](2.6), if A is simply connected, then every source x in Q_A is separating.

Recall moreover that, following [**12**], we may consider $A \cong kQ_A/I$ as a k-category having as object class A_0 the set of points in Q_A and, for any $x, y \in A_0$, having as morphism set $A(x,y)$ the linear combination of paths from x to y in Q_A modulo the ideal I.

Lemma. *Let $A = B[M]$ be a strictly weakly shod algebra, which is a maximal extension. Assume that A is tame and simply connected. Then:*

(a) *B is simply connected, and*
(b) *the extension point is separating.*

Proof: Since (b) follows directly from the simple connectedness of A and [**7**](2.6), there remains to prove (a). Assume B is not simply connected. By [**6**](2.3), there exist an idempotent $e = e^2 \in A$, and a full convex subcategory $C = eAe$ of A which is the convex hull of the following (weak) crown (with $2 \le t \le 4$) topped by the extension point x:



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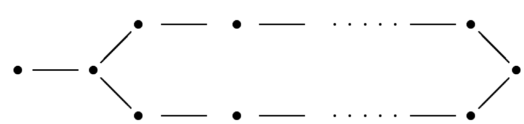
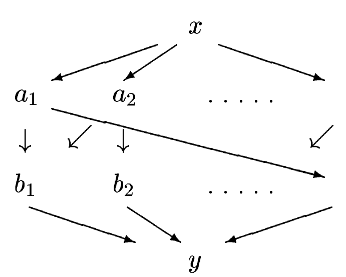
Let $N = Me$. Thus N is the radical of the indecomposable projective C−module P'_x corresponding to x. We claim that N is a submodule of the indecomposable projective A−module P_x corresponding to x.

Firstly, we claim that, if there exists a non-zero path from x to y in A passing through points of C, then $y \in C_0$. Assume that this is not the case. A straightforward analysis shows that either A contains a (not necessarily full) wild subcategory with underlying graph of the form:



which contradicts tameness, or else A contains a full subcategory E of the form:



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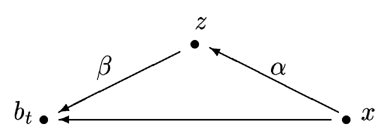
In this latter case, since $A(x,y) \neq 0$ by hypothesis and E is tame, then $\dim_k E(x,y) = 1$. Since E is full then $\dim_k A(x,y) = 1$. Now, the full subcategory E' of E consisting of all objects except x is a tilted algebra which is a one-point coextension of a hereditary algebra of type $\tilde{\mathbb{A}}$ by a simple homogeneous S.

Moreover, the radical N' of P_x in mod$E$ is the indecomposable injective E'-module corresponding to the vertex y, hence lies in the tube of $\Gamma(\text{mod}E')$ containing S. Therefore $E = E'[N']$ contains a projective-injective in a tube. In particular, by [**17**](3.7), E is not weakly shod, a contradiction to [**2**]. This establishes our claim.

Secondly, we claim that every non-zero path from x to the crown passes through one of the a_i. Assume that there exists a non-zero path $w: \quad x \rightsquigarrow z$, with $z \in C_0$. We can assume that z lies between a_t and b_t. Note that we have $\dim_k A(x,z) = 1$, otherwise the full subcategory generated by x, a_t, z, b_{t-1} is wild. We now show that all paths $x \rightsquigarrow z \rightsquigarrow b_t$ are non-zero. If this is not the case then there exists (inside the weak crown) a non-zero path $x \rightsquigarrow a_t \rightsquigarrow b_t$. Therefore, the subcategory generated by x, z, b_t is a split extension of the algebra given by the quiver:



bound by $\alpha\beta = 0$, and the latter is a representation-finite algebra which is not representation-directed, hence not weakly shod. This contradicts [**9**](Theorem A). Therefore all such paths $x \rightsquigarrow z \rightsquigarrow b_t$ are not zero. We infer that for each path

$$x \rightsquigarrow a_t \rightsquigarrow z \rightsquigarrow b_t$$

there exists a minimal relation involving the corresponding subpath $w'; \quad x \rightsquigarrow a_t \rightsquigarrow z$ with the given path $w: \quad x \rightsquigarrow z$. In particular, w contains at least three vertices $w: \ x \rightsquigarrow z' \rightsquigarrow z$, and the induced path $z' \rightsquigarrow z \rightsquigarrow b_t$ is non-zero (indeed, if it were zero, then we obtain a contradiction to the statement that all paths $x \rightsquigarrow z \rightsquigarrow b_t$ are non-zero). Then the subcategory generated by z, z', a_i, b_i, with $1 \leq i \leq t$, is wild, a contradiction. This completes the proof of our second claim.

Now, the two claims above imply that N is a submodule of P_x, as required.

Next, let D be the full subcategory of C generated by all objects except x. Then $C = D[N]$. Furthermore, D has a full subcategory H (generated by the objects a_i, b_i) which is hereditary of type $\tilde{\mathbb{A}}$, and by [**26**](1.2), there exists a full and faithful embedding mod$H$ $\hookrightarrow$mod$D$, which has as its image a subcategory closed under extensions, and is left inverse to the restriction functor. Now, N is the image under this embedding of a simple homogeneous $H-$module. Consequently, $\text{Ext}^1_D(N,N) \neq 0$. By Auslander-Reiten's formula, this implies that $\text{Hom}_D(N, \tau_D N) \neq 0$. It follows from [**25**](2.1) that $\tau_C N = (\text{Hom}_D(N,\tau_D N), \tau_D N, ev)$ where ev is the evaluation morphism. Since $\text{Hom}_D(N,\tau_D N) \neq 0$, this implies the existence in ind$C$ of a non-zero morphism $P'_x \to \tau_C N$, from which we deduce the path $P'_x \to \tau_C N \to * \to N$ in ind$C$ (hence in ind$A$).

Clearly, $\text{Hom}_A(P_x, P'_x) \neq 0$ and we have seen that there is an injection $N \hookrightarrow P_x$. Thus the above path induces a path

$$P_x \to P'_x \to \tau_C N \to * \to N \to P_x$$

in ind$A$, and this contradicts (2.2).□

5.2 **Corollary.** *Let $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$, be a maximal filtration of A. If A is tame then A is simply connected if and only if:*

(a) *each A_i is simply connected, and*
(b) *each x_i is separating.*

Proof: This obviously follows from (5.1), [**7**](2.5) and induction.□

6. Simple connected tame weakly shod algebras

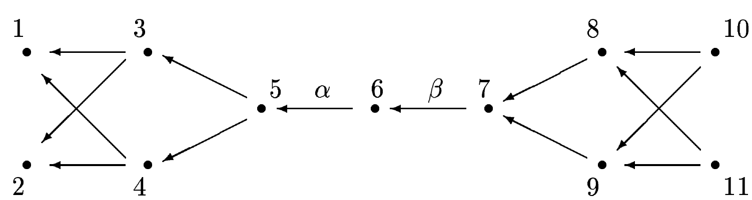
6.1 **Proof of Theorem (A):** We first show that (a) is equivalent to (b). Let $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$ be a maximal filtration of A. Applying (3.4), we see that $\mathrm{H}^1(A) = 0$ if and only if $\mathrm{H}^1(B) = 0$ and each x_i is separating. Since B is a direct product of tilted algebras, $\mathrm{H}^1(B) = 0$ if and only if it is of tree type (see [**21**](1.6)). The statement then follows from (4.2). Assume now that A is tame. If A is simply connected, then, using the same notation as above, it follows from (5.2) that each A_i is simply connected and each x_i is separating. In particular, B is simply connected. Since B is a direct product of tilted tame algebras, this implies, by [**6**], that $\mathrm{H}^1(B) = 0$. Applying (4.2) yields $\mathrm{H}^1(A) = 0$. Conversely, if $\mathrm{H}^1(A) = 0$, then $\mathrm{H}^1(B) = 0$ and each x_i is separating. Since B is a direct product of tilted tame algebras, it follows from [**6**] that each connected component of B is simply connected. Since each x_i is separating, it follows from [**7**](2.5) and induction that A is simply connected.□

6.2 **Corollary** *Let A be a tame strictly weakly shod algebra, and let $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$, be a maximal filtration of A. The following conditions are equivalent:*

(a) $\mathrm{H}^1(A) = 0$;
(b) $\mathrm{H}^1(A_i) = 0$, *for each* i;
(c) *A is simply connected;*
(d) *A_i is simply connected, for each i;*
(e) *A is of tree type;*
(f) *A_i is of tree type, for all i.* □

6.3 **Examples.**

(1) Let $A = kQ_A/I$ be given by the quiver:



bound by $\alpha\beta = 0$ and all commutativity relations.
We get the maximal filtration:

$$B = A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4 \subset A_5 = A,$$

with $A_5 = A_4[\mathrm{rad}P_{11}]$; $A_4 = A_3[\mathrm{rad}P_{10}]$; $A_3 = A_2[\mathrm{rad}P_9]$; $A_2 = A_1[\mathrm{rad}P_8]$, and $A_1 = A_0[\mathrm{rad}P_7]$. All extensions are maximal and $B = A_0$ is tilted.

(2) Consider A as in the Example (3.5). In this case, A is a weakly shod algebra with a maximal filtration. On the other hand, A is tame but it is not simply connected.

7. The strong simple connectedness of a tame weakly shod algebra

7.1 We recall the following definitions. An algebra A is strongly simply connected provided each full convex subcategory of A is simply connected. It is called strongly $\tilde{\mathbb{A}}$-free if it contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}$. Finally, it is called separated if each point y in Q_A (not necessarily a source) is separating as a source in the full convex subcategory of A with objects all points of Q_A except those points z such that there exists a path $z \rightsquigarrow y$ of length at least one in Q_A.

Proof of Theorem (B): (a) clearly implies (d).
(d) implies (c). Indeed, if A is separated, then, by [**28**](2.3), A is simply connected. Then Theorem (A) gives $\mathrm{H}^1(A) = 0$ and we are done.
(c) implies (b). Indeed, if $\mathrm{H}^1(A) = 0$, then, by Theorem (A), the orbit graph of the pip-bounded component is a tree. There only remains to consider the orbit graphs of the postprojective and the preinjective components. Assume Γ to be a postprojective component of $\Gamma(\mathrm{mod}A)$. Then Γ is also a postprojective component of the Auslander-Reiten quiver of a tilted algebra B which is a full convex subcategory of A. Moreover, B is tame because A is tame, and strongly $\tilde{\mathbb{A}}$-free, because A is. We then consider two cases. If B is not concealed, then the result follows at once from [**5**](2.3). On the other hand, if B is concealed, then B is the support algebra of Γ, and being strongly $\tilde{\mathbb{A}}$-free and tilted of euclidean type, must be of type $\tilde{\mathbb{D}}$ or $\tilde{\mathbb{E}}$. Then $\mathcal{O}(\Gamma)$ is a tree.
(b) implies (a). We show that A is strongly simply connected by showing that, for each full convex subcategory A' of A, we have $\mathrm{H}^1(A') = 0$. This is true if $A' = A$, by Theorem (A). Indeed, we recall that, since A is weakly shod, then it is triangular and by an obvious induction on $|A_0|$, we may assume that A is a one-point extension of A' (up to duality). Assume thus $A = A'[X]$. We claim that $\mathrm{H}^1(A') = 0$.

We consider two cases. Assume first that the projective indecomposable A-module P_x corresponding to the extension point x (that is, such that $\mathrm{rad}P_x = X$) is maximal in $\mathcal{R}_A$. By (2.3), we have $\mathrm{Ext}^1_{A'}(X,X) = 0$. Since it is known that $\mathrm{H}^1(A) = 0$, Happel's sequence (2.1)

$$\cdots \to \mathrm{H}^1(A) \to \mathrm{H}^1(A') \to \mathrm{Ext}^1_{A'}(X,X) \to \cdots$$

yields $\mathrm{H}^1(A') = 0$. Assume now that P_x is not maximal in $\mathcal{R}_A$. Then we can write $A = A_1[M_1]$, a maximal extension. Clearly, then, P_x is a projective indecomposable A_1-module. If P_x is maximal in $\mathcal{R}_{A_1}$, then $\mathrm{Ext}^1_{A_1}(X,X) = 0$. Hence $\mathrm{Ext}^1_A(X,X) = 0$ (because A_1 is a full convex subcategory of A), so Happel's sequence gives again $\mathrm{H}^1(A') = 0$. We continue in this fashion by induction on the terms of a filtration $B = A_m \subset \cdots \subset A_1 \subset A_0 = A$ as in (3.3). If, for any i such that $i < m$, P_x is a maximal projective in $\mathcal{R}_{A_i}$, then $\mathrm{H}^1(A') = 0$, and, if not, then it is an indecomposable projective A_{i+1}-module. Thus, by induction, we may assume that P_x is an indecomposable projective B-module. Since A is tame, then so is B. Moreover, since $\mathrm{H}^1(A) = 0$ then, by (3.4), we have $\mathrm{H}^1(B) = 0$. Since B is tilted, then it is of tree type. Moreover, X is a B-module. We wish to show that $\mathrm{Ext}^1_B(X,X) = 0$.

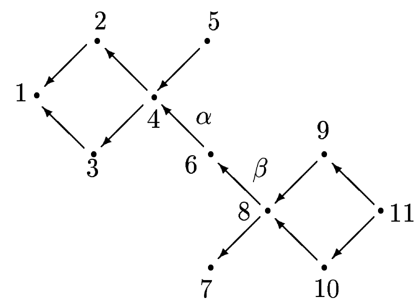
Now, we can write B in the form $B = B'[X]$. Since B is of tree type, the orbit graph of the connecting component of the Auslander-Reiten quiver of (each of the connected components of) B is a tree. On the other hand, the orbit graph of each of the postprojective components of A is a tree, by hypothesis. Then, by [**5**](2.4), $\mathrm{H}^1(B') = 0$. On the other hand $\mathrm{H}^2(B) = 0$, because B is a tilted algebra [**21**](4.3). Hence Happel's sequence

$$\cdots \to \mathrm{H}^1(B') \to \mathrm{Ext}^1_B(X,X) \to \mathrm{H}^2(B) \to \cdots$$

yields $\mathrm{Ext}^1_B(X,X) = 0$. As before, this yields $\mathrm{Ext}^1_A(X,X) = 0$ and hence $\mathrm{H}^1(A') = 0$. □

Examples.

(1) Let $A = kQ_A/I$ be given by the quiver:



bound by $\alpha\beta = 0$ and all possible commutativity relations. Then A satisfies the equivalent conditions of the Theorem (B).

(2) Consider A as in the Example (6.3). In this case, A is a weakly shod algebra with a maximal filtration. On the other hand, A is tame simply connected but not strongly simply connected.

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