THE SIMPLE CONNECTEDNESS OF A TAME TILTED ALGEBRA

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Introduction

Let k be an algebraically closed field. By algebra A, we mean a finite dimensional associative k-algebra, which we assume moreover to be basic and connected. For such an algebra A, there exists a (uniquely determined) connected quiver Q_A, and (at least) a surjective algebra morphism ν from the path algebra kQ_A of Q_A into A, whose kernel is denoted by I_ν, see [6]. The algebra A is called triangular if Q_A contains no oriented cycles. For each pair (Q_A, I_ν), called a presentation of A, we can define the fundamental group $\pi_1(Q_A, I_\nu)$, see [10]. A triangular algebra A is called simply connected if, for every presentation (Q_A, I_ν) of A, the group $\pi_1(Q_A, I_\nu)$ is trivial [4], or, equivalently, if and only if A admits no proper Galois coverings (see [10]). As is well-known, covering techniques allow to reduce many problems of the representation theory of algebras to problems about simply connected algebras (see, for instance, [6,10]).

Let A be an algebra, T_A be a tilting module (in the sense of [8]) and $B = \operatorname{End} T_A$. It has long been conjectured that, if A is simply connected, then so is B. This is known to be the case if A is representation-finite [5]. In this paper, we consider the case where A is hereditary, thus is the path algebra of a quiver Q. Then B is called tilted of type Q. Since a hereditary algebra is simply connected if and only if its quiver is a tree, the above conjecture reduces to say that a tilted algebra B of type Q is simply connected if and only if Q is a tree. This is known to be the case if the underlying graph of Q is euclidean [4], or if B is tame and contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}_m$, see [2]. Our objective in this paper is to show that the latter conjecture holds true if B is tame. In the proof, we shall make an essential use of the first cohomology space $H^1(B)$ (of the algebra B with coefficients in the bimodule ${}_BB_B$) whose vanishing is known to be related to the simple connectedness of B. Indeed, it has been shown that a representation-directed algebra A is simply connected if and only if $H^1(A) = 0$, see [7] (5.5). Similarly, if A is the Auslander algebra of a representation-finite algebra over a field k of characteristic zero, then A is simply connected if and only if $H^1(A) = 0$, see [1]. Moreover, for a tilted algebra B of type Q, it follows easily from [7] (see also [2] (1.4)) that Q is a tree if and only if $H^1(B) = 0$. We are now able to state our main result, which answers positively for tilted algebras the first problem of [11].

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Theorem. *Let B be a tame tilted algebra of type Q. Then B is simply connected if and only if Q is a tree.*

Since the underlying graph of Q coincides with the orbit graph of the connecting component of the Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, B)$ of B, the reader may compare this result with that in [2], where it is shown that, if B is tame and tilted, then B is strongly simply connected if and only if the orbit graph of each of the directing components of $\Gamma(\mathrm{mod}\, B)$ is a tree.

1. Preliminaries

1.1. Throughout this paper, an algebra A is equivalently considered as a k-category, whose object set is denoted by A_0, as in [6]. A full subcategory C of A is called **convex** if, for any path $a_0 \to a_1 \to \cdots \to a_t$ in A, with $a_0, a_t \in C_0$, we have $a_i \in C_0$ for all i. For an algebra A, we denote by $\mathrm{mod}\, A$ the category of the finitely generated right A-modules, and by P_x the indecomposable projective module corresponding to $x \in A_0$. It is well-known that, if $A = kQ/I$ is a bound quiver algebra, then $\mathrm{mod}\, A$ is equivalent to the category of all representations of Q bound by I. We thus identify a module M with the corresponding representation $(M(x), M(\alpha))$, see [6]. The **support** $\mathrm{Supp}\, M$ of an A-module M is the full subcategory of A generated by those $x \in A_0$ such that $M(x) \neq 0$.

1.2. For the fundamental group and simple connectedness, we refer the reader to [3]. We shall need in particular the following result. Let $B = C[M]$ be a one-point extension algebra, and x be the extension point. Denote by $\approx$ the smallest equivalence relation on the set of all arrows with source x such that $\alpha \approx \beta$ whenever there exists a minimal relation $\sum \lambda_i w_i$ starting at x, with $w_1 = \alpha v_1$ and $w_2 = \beta v_2$. Given a presentation (Q_B, I_ν) of B, let $t(\nu)$ be the number of equivalence classes $[\beta_1]_\nu, \dots [\beta_{t(\nu)}]_\nu$ of arrows with source x. For each i, with $1 \leq i \leq t(\nu)$, let $\ell(i)$ be the number of tuples of paths $(v_1, v_2, \dots v_{2s-1}, v_{2s})$ in Q such that there are minimal relations $\lambda_{11}\alpha_1 v_1 + \lambda_{12}\alpha_2 v_2 + \sum_{j\geq 3} \lambda_{1j} u_{1j}$ from x to y_1 (say), $\dots$ $\lambda_{s1}\alpha_s v_{2s-1} + \lambda_{s2}\alpha_{s+1} v_{2s} + \sum_{j\geq 3} \lambda_{sj} u_{sj}$ from x to y_s (say), with $\alpha_1 = \alpha_{s+1}$ and $\alpha_1, \dots \alpha_s$ distinct arrows in $[\beta_i]_\nu$.

Theorem [3] (2.4)**.** *Let $B = C[M]$ be a one-point extension of a connected algebra C. Let (Q_B, I_ν) be a presentation of B whose restriction to C is (Q_C, I'_ν). For any abelian group Z, there is an exact sequence of abelian groups*

$$0 \to Z^{t(\nu)-1} \to \mathrm{Hom}(\pi_1(Q_B, I_\nu), Z) \to \mathrm{Hom}(\pi_1(Q_C, I'_\nu), Z) \to \prod_{i=1}^{t(\nu)} Z^{\ell(i)}. \qquad \square$$

1.3. For tilted algebras, we refer the reader to [8,9]. The following result is a particular case of the main theorem in [4].

Theorem. *Let B be a tilted algebra of euclidean type Q. Then B is simply connected if and only if Q is a tree.* $\square$

1.4. Let A be an algebra, and Γ be a component of its Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, A)$. The **orbit graph** $\mathcal{O}(\Gamma)$ of Γ has as its points the τ-orbits M^τ of the indecomposable A-modules M in Γ, and there exists an edge $M^\tau — N^\tau$ whenever there exist $a, b \in \mathbb{Z}$ and an irreducible morphism $\tau^a M \to \tau^b N$, or $\tau^b N \to \tau^a M$; moreover, the number of such edges equals the dimension of the space of irreducible morphisms $\mathrm{Irr}(\tau^a M, \tau^b N)$, or $\mathrm{Irr}(\tau^b N, \tau^a M)$, respectively.

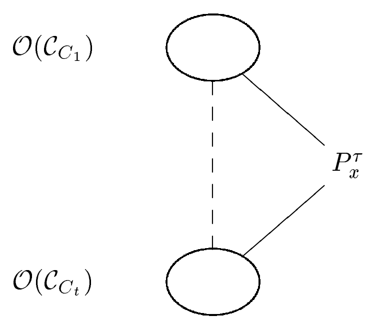
If A is representation-finite, then, by [6] (4.2), A is simply connected if and only if $\mathcal{O}(\Gamma \operatorname{mod} A))$ is a tree. Also, if A is tilted of type Q, then the underlying graph of Q equals the orbit graph of a connecting component of $\Gamma(\operatorname{mod} A)$.

2. Proof of the Theorem

We may clearly assume that B is representation-infinite and not concealed.

We first show the sufficiency. Suppose that the type of B is a tree. As observed before, this means that $H^1(B) = 0$. We must show that B is simply connected. Assume that this is not the case, and that B is a counterexample such that the number of objects of B_0 is minimal. Then there is at least one projective in the connecting component $\mathcal{C}_B$ of B : indeed, if this is not the case, then B is of euclidean type, hence is simply connected by (1.3), a contradiction. Let thus P_x be a projective in $\mathcal{C}_B$. We may assume that x is a source in the quiver of B, and so we can write $B = C[M]$, where $C = C_1 \times \cdots \times C_t$, with each C_i connected. Since $H^1(B) = 0$ then, by [11] (3.2), the source x is separating. Also, for each i, the orbit graph $\mathcal{O}(\mathcal{C}_{C_i})$ is a subgraph of $\mathcal{O}(\mathcal{C}_B)$, hence is a tree. By our minimality assumption, each C_i is simply connected. Applying [11] (3.2) (see also [3] (2.5)), we infer that B is simply connected, a contradiction.

We now show the necessity. Assume that B is simply connected. We must show that $H^1(B) = 0$. If the connecting component $\mathcal{C}_B$ contains no projectives, then B is of euclidean type and we are done by (1.3). If it does, let P_x be a projective in $\mathcal{C}_B$ which is maximal with respect to the order induced by the arrows. In particular, x is a source, so we can write $B = C[M]$ with $M = \operatorname{rad} P_x$ and $C = C_1 \times \cdots \times C_t$, where each C_i is connected. By [3] (2.6), x is separating. It is then easy to see that $\mathcal{O}(\mathcal{C}_B)$ is obtained by gluing together the orbit graphs of the connecting components of the C_i as follows.

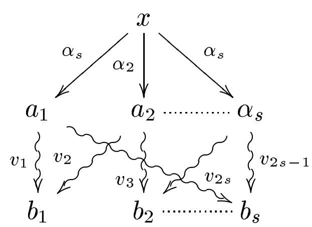


Therefore, we may assume that $t = 1$, that is, C is connected, and hence M is indecomposable.

By [7] (5.3), we have an exact sequence

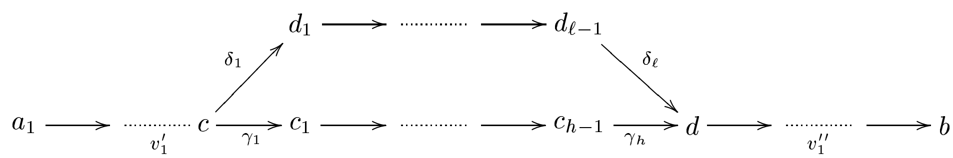
$$\cdots \to \operatorname{End} M/k \to H^1(B) \to H^1(C) \to \operatorname{Ext}^1_C(M,M) \to \cdots$$

Since M lies in $\mathcal{C}_B$, we have $\operatorname{End} M = k$ and $\operatorname{Ext}^1_C(M,M) = 0$. Hence $H^1(B) \cong H^1(C)$. If C is simply connected, then $H^1(C) = 0$ by induction and we are done. Otherwise, let (Q_B, I_ν) be a presentation of B. Since C is not simply connected, we may assume that the restriction (Q_C, I'_ν) of (Q_B, I_ν) to C is such that $\pi_1(Q_C, I'_\nu) \neq 1$. Since $\pi_1(Q_B, I_\nu) = 1$, applying (1.2) yields that $\ell(1) \neq 0$. That is, there are distinct arrows $\alpha_1 : x \to a_1, \ldots \alpha_s : x \to a_s$ and paths $v_{2i-1} : a_i \to b_i, v_{2i} : a_{i+1} \to b_i$ (where $1 \leq i \leq s$ and $a_{s+1} = a_1$) such that there exist minimal relations $\lambda_{11}\alpha_1 v_1 + \lambda_{12}\alpha_2 v_2 + \sum_{j\geq 3} \lambda_{1j}u_{1j}, \ldots, \lambda_{s1}\alpha_s v_{2s-1} + \lambda_{s2}\alpha_1 v_{2s} + \sum_{j\geq 3}\lambda_{sj}u_{sj}$.



Let D be the convex hull in B of the points $a_1, \ldots a_s, b_1, \ldots b_s$. We claim that $D = \operatorname{Supp} M$, and that D is hereditary of type $\tilde{\mathbb{A}}_m$ for some $m \geq 1$.

We first notice that D is tame and tilted, because it is a full convex subcategory of the tame tilted algebra B. This implies in particular that, for each i, v_{2i-1} (or v_{2i}) is the unique path in D from a_i to b_i (or a_{i+1} to b_i, respectively), where $1 \leq i \leq s$ and $a_{s+1} = a_1$. Assume that this is not the case, and that the convex hull of v_1 (say) contains a subcategory of the form



where $v_1 = v_1'\gamma_1 \ldots \gamma_h v_1''$. Assume first that $v_1'\delta_1 \ldots \delta_\ell v_1'' \neq 0$ in B. Since B is tame, there is a minimal relation linking this path with v_1. Let N be the indecomposable D-module of support the full subcategory generated by all points except $c_1, \ldots c_{h-1}, d_1, \ldots d_{\ell-1}$ and such that $N(y) = k$ for all $y \in (\operatorname{Supp} N)_0$. Then it is easily seen that $\operatorname{pd} N_D \geq 2$ and $\operatorname{id} N_D \geq 2$, showing that D is not tilted, a contradiction. On the other hand, if $v_1'\delta_1 \ldots \delta_\ell v_1'' = 0$, then the tameness of D implies that $v_1'\delta_1 = 0$ and $\delta_\ell v_1'' = 0$. Then, let L be the indecomposable D-module of support $\{d_1, \ldots d_{\ell-1}\}$ such that $L(d_i) = k$ for all i, we have $\operatorname{pd} L_D \geq 2$ and $\operatorname{id} L_D \geq 2$, again a contradiction.

The above remark implies that D is a full subcategory of $\operatorname{Supp} M$. Also, the tameness of B implies that, if there is an arrow $y \to z$, with $y \in D_0$ and $z \notin D_0$, then $z \notin (\operatorname{Supp} M)_0$ and, further, $\operatorname{Supp} P_x$ contains no non-zero path from x to some $y \notin D_0$. This completes the proof of our claim.

Now, we have $\dim_k M(y) \leq 1$ for all $y \in D_0$: for, if this is not the case, and $y \in D_0$ is such that $\dim_k M(y) \geq 2$, then the tame algebra D would contain a wild full subcategory of the form

$$b_i \longrightarrow x \rightrightarrows b_j$$

and this is a contradiction. Therefore, M is a simple homogeneous D-module. Hence $\operatorname{Ext}^1_D(M, M) \neq 0$, which implies that $\operatorname{Ext}^1_B(M, M) \neq 0$, a contradiction. □

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