

THE LEFT PART AND THE AUSLANDER-REITEN COMPONENTS OF AN ARTIN ALGEBRA

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ABSTRACT. The left part \mathcal{L}_A of the module category of an artin algebra A consists of all indecomposables whose predecessors have projective dimension at most one. In this paper, we study the Auslander-Reiten components of A (and of its left support A_λ) which intersect \mathcal{L}_A and also the class \mathcal{E} of the indecomposable Ext-injectives in the additive subcategory $\text{add}\mathcal{L}_A$ generated by \mathcal{L}_A .

INTRODUCTION

Let A be an artin algebra and $\text{mod}A$ denote the category of finitely generated right A -modules. The class \mathcal{L}_A , called the *left part* of $\text{mod}A$, is the full subcategory of $\text{mod}A$ having as objects all indecomposable modules whose predecessors have projective dimension at most one. This class, introduced in [15], was heavily investigated and applied (see, for instance, the survey [4]).

Our objective in this paper is to study the Auslander-Reiten components of an artin algebra which intersect the left part. Some information on these components was already obtained in [2, 3]. Here we are interested in the components which intersect the class \mathcal{E} of the indecomposable Ext-injectives in the full additive subcategory $\text{add}\mathcal{L}_A$ having as objects the direct sums of modules in \mathcal{L}_A . We start by proving the following theorem.

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THEOREM (A). *Let A be an artin algebra, and Γ be a component of the Auslander-Reiten quiver of A . If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

- (a) *Each τ_A -orbit of $\Gamma \cap \mathcal{L}_A$ intersects \mathcal{E} exactly once.*
- (b) *The number of τ_A -orbits of $\Gamma \cap \mathcal{L}_A$ equals the number of modules in $\Gamma \cap \mathcal{E}$.*
- (c) *$\Gamma \cap \mathcal{L}_A$ contains no module lying on a cycle between modules in Γ .*

If, on the other hand, $\Gamma \cap \mathcal{E} = \emptyset$, then either $\Gamma \in \mathcal{L}_A$ or else $\Gamma \cap \mathcal{L}_A = \emptyset$.

We recall that, by [3] (3.3), the class \mathcal{E} contains only finitely many non-isomorphic modules (hence only finitely many Auslander-Reiten components intersect \mathcal{E}).

As a consequence, we give a complete description of the Auslander-Reiten components lying entirely inside the left part.

We then try to describe the intersection of \mathcal{E} with a component Γ of the Auslander-Reiten quiver $\Gamma(\text{mod}A)$. We find that, in general, $\Gamma \cap \mathcal{E}$ is not a section in Γ (in the sense of [20, 23]) but is very nearly one. This leads us to our second theorem, for which we recall that a component Γ of $\Gamma(\text{mod}A)$ is called *generalised standard* if $\text{rad}_A^\infty(X, Y) = 0$ for all $X, Y \in \Gamma$, see [23].

THEOREM (B). *Let A be an artin algebra and Γ be a component of $\Gamma(\text{mod}A)$ such that all projectives in Γ belong to \mathcal{L}_A . If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

- (a) *$\Gamma \cap \mathcal{E}$ is a section in Γ .*
- (b) *Γ is generalised standard.*
- (c) *$A/\text{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.*

In particular, such a component Γ has only finitely many τ_A -orbits.

The situation is better if we look instead at the intersection of \mathcal{E} with the Auslander-Reiten components of the left support A_λ of A . We recall from [3, 24] that the *left support* A_λ of A is the endomorphism algebra of the direct sum of the indecomposable projective A -modules lying in \mathcal{L}_A . It is shown in [3, 24] that every connected component of A_λ is a quasi-tilted algebra (in the sense of [15]). We prove the following theorem.

THEOREM (C). *Let A be an artin algebra and Γ be a component of the Auslander-Reiten quiver of the left support A_λ of A . If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

- (a) *$\Gamma \cap \mathcal{E}$ is a section in Γ .*

(b) Γ is directed, and generalised standard.

(c) $A_\lambda/\text{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.

We then apply our results to the study of left supported algebras. We recall from [3] that an artin algebra A is *left supported* provided $\text{add}\mathcal{L}_A$ is contravariantly finite in $\text{mod}A$. Several classes of algebras are left supported, such as all representation-finite algebras, and all lura algebras which are not quasi-tilted (see [3, 4]). It is shown in [1] that an artin algebra A is left supported if and only if \mathcal{L}_A consists of all the predecessors of the modules in \mathcal{E} . We give here a proof of this fact which, in contrast to the homological nature of the proof in [1], uses our theorem and the full power of the Auslander-Reiten theory of quasi-tilted algebras. Our proof also yields a new characterisation: an algebra A is left supported if and only if every projective A -module which belongs to \mathcal{L}_A is a predecessor of \mathcal{E} . We end the paper with a short proof of the theorem of D. Smith [25] (3.8) which characterises the left supported quasi-tilted algebras.

Clearly, the dual statements about the right part of the module category, also hold true. Here, we only concern ourselves with the left part, leaving the primal-dual translation to the reader.

We now describe the contents of the paper. After a brief preliminary section 1, the sections 2, 3 and 4 are respectively devoted to the proofs of our theorems (A), (B) and (C). In our final section 5, we consider the applications to left supported algebras.

1. PRELIMINARIES.

1.1. Notation. For a basic and connected artin algebra A , let $\text{mod}A$ denote its category of finitely generated right modules and $\text{ind}A$ a full subcategory consisting of exactly one representative from each isomorphism class of indecomposable modules. We sometimes consider A as a category, with objects a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents, and where $e_i A e_j$ is the set of morphisms from e_i to e_j . An algebra B is a *full subcategory* of A if there is an idempotent $e \in A$, which is a sum of some of the distinguished idempotents e_i , such that $B = eAe$. It is *convex* in A if, for any sequence $e_i = e_{i_0}, e_{i_1}, \dots, e_{i_t} = e_j$ of objects of A such that $e_{i_{l+1}} A e_{i_l} \neq 0$ (with $0 \leq l < t$) and e_{i_l}, e_j objects of B , all e_{i_l} are in B .

Given a full subcategory \mathcal{C} of $\text{mod}A$, we write $M \in \mathcal{C}$ to indicate that M is an object in \mathcal{C} , and we denote by $\text{add}\mathcal{C}$ the full subcategory with objects the direct sums of summands of modules in \mathcal{C} . Given a module M , let $\text{pd}M$ stand for its projective dimension. We also

denote by $\Gamma(\text{mod}A)$ the Auslander-Reiten quiver of A and by $\tau_A = \text{DTr}$, $\tau_A^{-1} = \text{TrD}$ the Auslander-Reiten translations. For further notions or facts needed on $\text{mod}A$ we refer to [7, 22].

1.2. Paths. Let A be an artin algebra and $M, N \in \text{ind}A$. A *path* $M \rightsquigarrow N$ is a sequence

$$(*) \quad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t = N$$

where the f_i are non-zero morphisms and the M_i lie in $\text{ind}A$. We call M a *predecessor* of N and N a *successor* of M . A path from M to M involving at least one non-isomorphism is a *cycle*. An indecomposable module M lying on no cycle is called *directed*. A path $(*)$ is called *sectional* if each f_i is irreducible and $\tau_A M_{i+1} \neq M_{i-1}$ for all i . A *refinement* of $(*)$ is a path

$$M = M'_0 \xrightarrow{f'_1} M'_1 \xrightarrow{f'_2} \cdots \longrightarrow M'_{t-1} \xrightarrow{f'_t} M'_t = N$$

such that there exists an order-preserving injection $\sigma : \{1, \dots, t-1\} \longrightarrow \{1, \dots, s-1\}$ with $M_i = M'_{\sigma(i)}$ for all i . A full subcategory \mathcal{C} of $\text{ind}A$ is *convex* if, for any path $(*)$ with $M, N \in \mathcal{C}$, all the M_i lie in \mathcal{C} .

2. EXT-INJECTIVES IN THE LEFT PART.

2.1. Let A be an artin algebra. The *left part* \mathcal{L}_A of $\text{mod}A$ is the full subcategory of $\text{ind}A$ defined by

$$\mathcal{L}_A = \{M \in \text{ind}A \mid \text{pd} L \leq 1 \text{ for any predecessor } L \text{ of } M\}.$$

An indecomposable module $M \in \mathcal{L}_A$ is called *Ext-projective* (or *Ext-injective*) in $\text{add}\mathcal{L}_A$ if $\text{Ext}_A^1(M, -)|_{\mathcal{L}_A} = 0$ (or $\text{Ext}_A^1(-, M)|_{\mathcal{L}_A} = 0$, respectively), see [9]. While the Ext-projectives in $\text{add}\mathcal{L}_A$ are the projective modules lying in \mathcal{L}_A (see [3] (3.1)), the Ext-injectives are more interesting. Before stating their characterisations we recall that, by [9] (3.7), $M \in \mathcal{L}_A$ is Ext-injective in $\text{add}\mathcal{L}_A$ if and only if $\tau_A^{-1}M \notin \mathcal{L}_A$.

LEMMA [5] (3.2), [3] (3.1). *Let $M \in \mathcal{L}_A$.*

(a) *The following are equivalent :*

- (i) *There exists an indecomposable injective module I such that $\text{Hom}_A(I, M) \neq 0$.*
- (ii) *There exist an indecomposable injective module I and a path $I \rightsquigarrow M$.*
- (iii) *There exist an indecomposable injective module I and a sectional path $I \rightsquigarrow M$.*

(b) The following conditions are equivalent for $M \in \mathcal{L}_A$ which does not satisfy conditions (a):

- (i) There exists an indecomposable projective module $P \notin \mathcal{L}_A$ such that $\text{Hom}_A(P, \tau_A^{-1}M) \neq 0$.
- (ii) There exist an indecomposable projective module $P \notin \mathcal{L}_A$ and a path $P \rightsquigarrow \tau_A^{-1}M$.
- (iii) There exist an indecomposable projective module $P \notin \mathcal{L}_A$ and a sectional path $P \rightsquigarrow \tau_A^{-1}M$.

Letting \mathcal{E}_1 (or \mathcal{E}_2) denote the set of all $M \in \mathcal{L}_A$ verifying (a) (or (b), respectively), and setting $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, then M is Ext-injective in $\text{add}\mathcal{L}_A$ if and only if $M \in \mathcal{E}$. \square

2.2. The following lemma will also be useful.

LEMMA [3] (3.2) (3.4). (a) Any path of irreducible morphisms in \mathcal{E} is sectional.

(b) Let $M \in \mathcal{E}$ and $M \rightsquigarrow N$ with $N \in \mathcal{L}_A$. Then this path can be refined to a sectional path and $N \in \mathcal{E}$. In particular, \mathcal{E} is convex in $\text{ind}A$. \square

2.3. The following immediate corollary will be useful in the proof of our theorem (A).

COROLLARY All modules in \mathcal{E} are directed.

Proof. Assume $M = M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_s = M$ is a cycle in $\text{ind}A$, with $M \in \mathcal{E}$. By (2.2) above, such a cycle can be refined to a sectional cycle with all indecomposables lying in \mathcal{E} . Now compose two copies of this cycle to form a larger cycle in \mathcal{E} of irreducible morphisms. By (2.2), this cycle is also sectional, in contradiction to [11, 12]. \square

2.4. THEOREM (A). Let A be an artin algebra, and Γ be a component of the Auslander-Reiten quiver of A . If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:

- (a) Each τ_A -orbit of $\Gamma \cap \mathcal{L}_A$ intersects \mathcal{E} exactly once.
- (b) The number of τ_A -orbits of $\Gamma \cap \mathcal{L}_A$ equals the number of modules in $\Gamma \cap \mathcal{E}$.
- (c) $\Gamma \cap \mathcal{L}_A$ contains no module lying on a cycle between modules in Γ .

If, on the other hand, $\Gamma \cap \mathcal{E} = \emptyset$, then either $\Gamma \in \mathcal{L}_A$ or else $\Gamma \cap \mathcal{L}_A = \emptyset$.

Proof. Assume first that $\Gamma \cap \mathcal{E} \neq \emptyset$, that is, the component Γ contains an Ext-injective in $\text{add}\mathcal{L}_A$.

(a) If Γ contains an injective module, then the statement follows from [3] (3.5). We may thus assume that Γ contains no injective. But then $\Gamma \cap \mathcal{E}_1 = \emptyset$, and therefore $\Gamma \cap \mathcal{E}_2 = \Gamma \cap \mathcal{E} \neq \emptyset$. Thus, by (2.1), there exist an indecomposable projective P in Γ such that $P \notin \mathcal{L}_A$, a module $M \in \Gamma \cap \mathcal{E}_2$ and a sectional path $P \rightsquigarrow \tau_A^{-1}M$. Now let $X \in \Gamma \cap \mathcal{L}_A$. Since Γ contains no injective, there exists $s > 0$ such that $\tau_A^{-s}X$ is a successor of P . Hence $\tau_A^{-s}X \notin \mathcal{L}_A$. Since X itself lies in \mathcal{L}_A , there exists $j \geq 0$ such that $\tau_A^{-j}X \in \mathcal{L}_A$ but $\tau_A^{-j-1}X \notin \mathcal{L}_A$, so that $\tau_A^{-j}X$ is Ext-injective in $\text{add } \mathcal{L}_A$. This shows that every τ_A -orbit of $\Gamma \cap \mathcal{L}_A$ intersects \mathcal{E} at least once.

Furthermore, it intersects it only once: if Y and $\tau_A^{-t}Y$ (with $t > 0$) both belong to $\Gamma \cap \mathcal{E}$ then, by (2.2), all the modules on the path

$$Y \rightarrow * \rightarrow \tau_A^{-1}Y \rightarrow \cdots \rightarrow \tau_A^{-t}Y$$

belong to \mathcal{L}_A . In particular, $\tau_A^{-1}Y \in \mathcal{L}_A$ and this contradicts the Ext-injectivity of Y . This completes the proof of (a).

(b) It follows from (a) that the number of τ_A -orbits in $\Gamma \cap \mathcal{L}_A$ does not exceed the cardinality of $\Gamma \cap \mathcal{E}$ (note that by [3] (3.3), the cardinality of \mathcal{E} is finite and does not exceed the rank of the Grothendieck group $K_0(A)$ of A). Since clearly, any element of $\Gamma \cap \mathcal{E}$ belongs to exactly one τ_A -orbit in \mathcal{L}_A , this establishes (b).

(c) Let $(*) \quad M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = M_0$ be a cycle with $M_0 \in \Gamma \cap \mathcal{L}_A$ and all M_i in Γ . Clearly, all M_i belong to $\Gamma \cap \mathcal{L}_A$. By (2.2) and (2.3), none of the M_i belongs to \mathcal{E} and none of the f_i factors through an injective module. Indeed, if f_i factors through the injective I , then some indecomposable summand of I would belong to \mathcal{L}_A and thus M_i would lie in \mathcal{E} , contradicting (2.3). Then the cycle $(*)$ induces a cycle $\tau_A^{-1}M_0 \longrightarrow \tau_A^{-1}M_1 \longrightarrow \cdots \longrightarrow \tau_A^{-1}M_t = \tau_A^{-1}M_0$, and every module in this cycle belongs to $\Gamma \cap \mathcal{L}_A$. We can iterate this procedure and deduce that, for any $m > 0$, the module $\tau_A^{-m}M_0$ lies on a cycle in $\Gamma \cap \mathcal{L}_A$. However, as shown in (a), there exists $s > 0$ such that $\tau_A^{-s}M_0$ does not belong to \mathcal{L}_A , and this contradiction proves (c).

Now assume that the component Γ contains no Ext-injective, that is, $\Gamma \cap \mathcal{E} = \emptyset$. If Γ contains both a module in \mathcal{L}_A and a module which is not in \mathcal{L}_A , then there exists an irreducible morphism $X \longrightarrow Y$ with $X \in \Gamma \cap \mathcal{L}_A$ and $Y \in \Gamma \setminus \mathcal{L}_A$. Since $\Gamma \cap \mathcal{E} = \emptyset$, then $\tau_A^{-1}X \in \mathcal{L}_A$. But this is a contradiction, because $Y \notin \mathcal{L}_A$ and $\text{Hom}_A(Y, \tau_A^{-1}X) \neq 0$. This shows that either $\Gamma \cap \mathcal{L}_A = \emptyset$ or $\Gamma \subseteq \mathcal{L}_A$, as required. \square

We observe that part (c) of the theorem was already proven in [3] (1.5) under the additional hypothesis that Γ contains no injective module.

2.5. COROLLARY [3] (1.6). *Let A be a representation-finite artin algebra. Then \mathcal{L}_A is directed.* \square

2.6. We have a good description of the Auslander-Reiten components which completely lie in \mathcal{L}_A . We need to recall a definition. The endomorphism algebra A_λ of the direct sum of all the projective modules lying in \mathcal{L}_A is called the *left support* of A , see [3, 24]. Clearly, A_λ is (isomorphic to) a full convex subcategory of A , closed under successors, and any A -module lying in \mathcal{L}_A has a natural A_λ -module structure. It is shown in [3] (2.3), [24] (3.1) that A_λ is a product of connected quasi-tilted algebras, and that $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$. The following corollary generalises [3] (5.5).

COROLLARY. *Let A be a representation-infinite artin algebra, and Γ be a component of $\Gamma(\text{mod}A)$ lying entirely in \mathcal{L}_A . Then Γ is one of the following: a postprojective component, a regular component (directed, stable tube or of type $\mathbb{Z}A_\infty$), a semiregular tube without injectives, or a ray extension of $\mathbb{Z}A_\infty$.*

Proof. Indeed, the component Γ lies entirely in $\text{mod}A_\lambda$ and thus is a component of $\Gamma(\text{mod}A_\lambda)$. Since $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$, then Γ is a component of $\Gamma(\text{mod}A_\lambda)$ lying in the left part \mathcal{L}_{A_λ} . The statement then follows from the well-known description of the Auslander-Reiten components of quasi-tilted algebras, as in [13, 18]. \square

3. EXT-INJECTIVES AS SECTIONS IN $\Gamma(\text{mod}A)$.

3.1. We recall the following notion from [20, 23]. Let A be an artin algebra and Γ be a component of $\Gamma(\text{mod}A)$. A full connected subquiver Σ of Γ is called a *section* if it satisfies the following conditions:

- (S₁) Σ contains no oriented cycle.
- (S₂) Σ intersects each τ_A -orbit of Γ exactly once.
- (S₃) Σ is convex in Γ .
- (S₄) If $X \rightarrow Y$ is an arrow in Γ with $X \in \Sigma$, then $Y \in \Sigma$ or $\tau_A Y \in \Sigma$.
- (S₅) If $X \rightarrow Y$ is an arrow in Γ with $Y \in \Sigma$, then $X \in \Sigma$ or $\tau_A^{-1} X \in \Sigma$.

As we show next, the intersection of \mathcal{E} with a component of $\Gamma(\text{mod}A)$ satisfies several of these conditions (but generally not all).

PROPOSITION. *Assume Γ is a component of $\Gamma(\text{mod}A)$ which intersects \mathcal{E} . Then Γ satisfies (S₁), (S₃), (S₅) above, and the following conditions*

- (S'₂) $\Gamma \cap \mathcal{E}$ intersects each τ_A -orbit of Γ at most once.

(S'_4) If $X \rightarrow Y$ is an arrow in Γ with $X \in \Gamma \cap \mathcal{E}$ and Y non-projective, then $Y \in \mathcal{E}$ or $\tau_A Y \in \mathcal{E}$.

Proof. (S_1) follows from Theorem (A) (c).

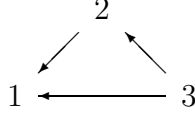
(S'_2) follows from Theorem (A) (a).

(S_3) follows from (2.2).

(S'_4) If $Y \in \mathcal{L}_A$, then $X \in \mathcal{E}$ and (2.2) imply $Y \in \mathcal{E}$. Otherwise, since Y is non-projective, there exists an arrow $\tau_A Y \rightarrow X$. Since $X \in \mathcal{E}$, then $\tau_A Y \in \mathcal{L}_A$. Since $Y = \tau_A^{-1}(\tau_A Y) \notin \mathcal{L}_A$, we get $\tau_A Y \in \mathcal{E}$.

(S_5) If X is injective then, since it lies in \mathcal{L}_A (because it precedes Y), it belongs to \mathcal{E} . So assume it is not and consider the arrow $Y \rightarrow \tau_A^{-1} X$. If $\tau_A^{-1} X \notin \mathcal{L}_A$ then, again, $X \in \mathcal{E}$ while, if $\tau_A^{-1} X \in \mathcal{L}_A$, then $Y \in \mathcal{E}$ and (2.2) imply $\tau_A^{-1} X \in \mathcal{E}$. \square

3.2. EXAMPLE. Let k be a field and A be the radical square zero k -algebra given by the quiver



Here, A is representation finite and \mathcal{E} consists of the two indecomposable projectives P_1 and P_2 corresponding to the points 1 and 2, respectively. Clearly, $\mathcal{E} = \{P_1, P_2\}$ is not a section in $\Gamma(\text{mod} A)$: indeed, there is an arrow $P_1 \rightarrow P_3$ with $P_3 \notin \mathcal{E}$ and, moreover, \mathcal{E} does not intersect each τ_A -orbit of $\Gamma(\text{mod} A)$.

3.3. We are now in a position to prove our second main theorem.

THEOREM (B). *Let A be an artin algebra and Γ be a component of $\Gamma(\text{mod} A)$ such that all projectives in Γ belong to \mathcal{L}_A . If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

(a) $\Gamma \cap \mathcal{E}$ is a section in Γ .

(b) Γ is generalised standard.

(c) $A/\text{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.

Proof.

(a) We start by observing that, if $X \rightarrow P$ is an arrow in Γ , with $X \in \mathcal{E}$ and P projective then, by hypothesis, $P \in \mathcal{L}_A$. Thus, (2.2) implies $P \in \mathcal{E}$. This shows that (S_4) is satisfied. In view of the lemma, it suffices to show that $\Gamma \cap \mathcal{E}$ cuts each τ_A -orbit of Γ .

We claim that if $M \in \mathcal{E}$ and $N \in \Gamma$ lie in two neighbouring orbits, then \mathcal{E} intersects the τ_A -orbit of N . This claim and induction imply the

statement. We assume that \mathcal{E} does not intersect the orbit of N and try to reach a contradiction. There exist $n \in \mathbb{Z}$ and an arrow $\tau_A^n M \rightarrow X$ or $X \rightarrow \tau_A^n M$, with X in the τ_A -orbit of N , where we may suppose, without loss of generality, that $|n|$ is minimal.

Suppose first that $n < 0$. If there exists an arrow $X \rightarrow \tau_A^n M$ then there exists an arrow $\tau_A^{n+1} M \rightarrow X$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $\tau_A^n M \rightarrow X$, then there is a path in Γ of the form $M \rightarrow * \rightarrow \tau_A^{-1} M \rightsquigarrow X$. Since $M \in \mathcal{E}$ then $\tau_A^{-1} M \notin \mathcal{L}_A$. Hence $X \notin \mathcal{L}_A$. In particular, X is not projective, so there exists an arrow $\tau_A^{n+1} M \rightarrow \tau_A X$, contrary to the minimality of $|n|$.

Suppose now that $n > 0$. If there exists an arrow $\tau_A^n M \rightarrow X$, then there exists an arrow $X \rightarrow \tau_A^{n-1} M$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $X \rightarrow \tau_A^n M$, then there is a path in Γ of the form $X \rightarrow \tau_A^n M \rightsquigarrow M$. Hence $X \in \mathcal{L}_A$. In particular, X is not injective (otherwise, $X \in \mathcal{E}$, a contradiction). Hence there exists an arrow $\tau_A^{-1} X \rightarrow \tau_A^{n-1} M$, contrary to the minimality of $|n|$.

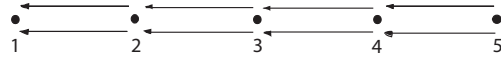
We have thus shown that necessarily $n = 0$, that is, there is an arrow $M \rightarrow X$ or $X \rightarrow M$. If $M \rightarrow X$ then, by (S_4) , $X \in \mathcal{E}$ or $\tau_A X \in \mathcal{E}$, in any case a contradiction. If $X \rightarrow M$, then (3.1) yields $X \in \mathcal{E}$ or $\tau_A^{-1} X \in \mathcal{E}$, again a contradiction in any case. This completes the proof of (a).

(b) By [23], Theorem 2, it suffices to show that for any $X, Y \in \Gamma \cap \mathcal{E}$, we have $\text{Hom}_A(X, \tau_A Y) = 0$. But $Y \in \mathcal{E}$ implies $\text{pd} Y \leq 1$. Therefore the Ext-injectivity of X in $\text{add} \mathcal{L}_A$ implies that

$$\text{Hom}_A(X, \tau_A Y) \simeq \text{D Ext}_A^1(Y, X) = 0.$$

(c) This follows directly from [20] (2.2). □

3.4. EXAMPLE. Let k be a field and A be the radical square zero algebra given by the quiver



Let Γ be the component containing the injective I_1 corresponding to the point 1. Clearly, $I_1 \in \mathcal{E}$, so that $\Gamma \cap \mathcal{E} \neq \emptyset$. On the other hand, the only projective lying in Γ is P_3 , and it belongs to \mathcal{L}_A . Thus, the hypotheses of the theorem apply here. Note that $A/\text{Ann}(\Gamma \cap \mathcal{E})$ is equal to the left support A_λ of A , that is, the full convex subcategory with objects $\{1, 2, 3\}$.

4. EXT-INJECTIVES AND THE LEFT SUPPORT

4.1. In this section we study the intersection of \mathcal{E} with the components of the Auslander-Reiten quiver of the left support A_λ of the artin algebra A .

We observe first that if Y is an A_λ -module and $\tau_A Y \in \mathcal{L}_A$ then $\tau_A Y = \tau_{A_\lambda} Y$. In particular, Y is not projective in $\text{mod} A_\lambda$. Indeed, since $\text{mod} A_\lambda$ is closed under extensions in $\text{mod} A$, then the inclusion $\mathcal{L}_A \subseteq \text{ind} A_\lambda$ implies that the almost split sequence in $\text{mod} A$ ending at Y is entirely contained in $\text{mod} A_\lambda$ (See also [7], p. 187). Similarly, if $\tau_A^{-1} Y \in \mathcal{L}_A$, then $\tau_A^{-1} Y = \tau_{A_\lambda}^{-1} Y$, and Y is not an injective A_λ -module.

LEMMA. *If an indecomposable injective A_λ -module I is a predecessor of \mathcal{E} , then $I \in \mathcal{E}$.*

Proof. This is clear if I is an indecomposable injective A -module. So assume it is not. Since I precedes \mathcal{E} , then $I \in \mathcal{L}_A$. By the above observation we obtain that $\tau_A^{-1} I \notin \mathcal{L}_A$, because I is A_λ -injective. This proves that $I \in \mathcal{E}$, as desired. \square

4.2. The following is an easy consequence of (3.1) and the results in [3].

LEMMA. *Let $E = \bigoplus_{X \in \mathcal{E}} X$. Then E is a convex partial tilting A_λ -module. In particular, $|\mathcal{E}| \leq \text{rk } K_0(A_\lambda)$.*

Proof. Indeed, since $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$ (see [3], (2.1)), $\mathcal{E} \subseteq \mathcal{L}_A$ implies $\text{pd}_{A_\lambda} E \leq 1$. Since $\text{Ext}_A^1(E, E) = 0$ and A_λ is a full convex subcategory of A , we also have $\text{Ext}_{A_\lambda}^1(E, E) = 0$. Finally, the convexity of E in $\text{ind} A_\lambda$ follows from its convexity in $\text{ind} A$ (see (2.2)). \square

4.3. THEOREM C. *Let A be an artin algebra and Γ be a component of the Auslander-Reiten quiver of the left support A_λ of A . If $\Gamma \cap \mathcal{E} \neq \emptyset$, then:*

- (a) $\Gamma \cap \mathcal{E}$ is a section in Γ .
- (b) Γ is directed, and generalised standard.
- (c) $A/\text{Ann}(\Gamma \cap \mathcal{E})$ is a tilted algebra having Γ as a connecting component and $\Gamma \cap \mathcal{E}$ as a complete slice.

Proof. (a) In order to show that $\Gamma \cap \mathcal{E}$ is a section in Γ , we just have to check the conditions of the definition in (3.1). Clearly, (S_1) follows from (2.3) and the observation that any cycle in $\text{ind} A_\lambda$ induces one in $\text{ind} A$. Also, (S_3) follows from (4.2). We start by proving (S_4) and (S_5) .

(S₄) Assume $X \rightarrow Y$ is an arrow in Γ , with $X \in \mathcal{E}$. If $Y \in \mathcal{L}_A$, then (2.2) implies $Y \in \mathcal{E}$. Assume $Y \notin \mathcal{L}_A$. Then, in particular, Y is not a projective A_λ -module. Since Y is an A_λ -module, it is not a projective A -module either, so there is an irreducible morphism $\tau_A Y \rightarrow X$ in $\text{mod}A$. Then $\tau_A Y$ precedes $X \in \mathcal{E}$ and therefore lies in \mathcal{L}_A . Thus, as we observed in (4.1), $\tau_A Y = \tau_{A_\lambda} Y$. Since $\tau_A^{-1}(\tau_{A_\lambda} Y) = Y \notin \mathcal{L}_A$, we conclude that $\tau_{A_\lambda} Y \in \mathcal{E}$, as required.

(S₅) Assume $X \rightarrow Y$ is an arrow in Γ , with $Y \in \mathcal{E}$. If $X \notin \mathcal{E}$, then $\tau_A^{-1} X \in \mathcal{L}_A$ and, again by the observation in 4.1, we know that X is not an injective A_λ -module. Hence $\tau_{A_\lambda}^{-1} X = \tau_A^{-1} X \in \mathcal{L}_A$. Since there is an arrow $Y \rightarrow \tau_{A_\lambda}^{-1} X$, we conclude that $\tau_{A_\lambda}^{-1} X \in \mathcal{E}$, as required.

There remains to prove (S₂), that is, that \mathcal{E} intersects each orbit of Γ exactly once. We use the same technique as in the proof of Theorem (B). Clearly, the situation is different and so the arguments vary slightly.

We start by proving that \mathcal{E} intersects each orbit of Γ at least once. We claim that if $M \in \mathcal{E}$ and $N \in \Gamma$ lie in two neighbouring orbits, then \mathcal{E} intersects the τ_{A_λ} -orbit of N . This claim and induction imply the statement. We assume that \mathcal{E} does not intersect the orbit of N and try to reach a contradiction. There exist $n \in \mathbb{Z}$ and an arrow $\tau_{A_\lambda}^n M \rightarrow X$ or $X \rightarrow \tau_{A_\lambda}^n M$, with X in the τ_{A_λ} -orbit of N , where we may suppose, without loss of generality, that $|n|$ is minimal.

Suppose first that $n < 0$. If there exists an arrow $X \rightarrow \tau_{A_\lambda}^n M$ then there exists an arrow $\tau_{A_\lambda}^{n+1} M \rightarrow X$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $\tau_{A_\lambda}^n M \rightarrow X$, then there is a path in Γ of the form $M \rightarrow * \rightarrow \tau_{A_\lambda}^{-1} M \rightsquigarrow X$. Now, $M \in \mathcal{E}$ implies $\tau_A^{-1} M \notin \mathcal{L}_A$. By [7] p. 186, there exists an epimorphism $\tau_A^{-1} M \rightarrow \tau_{A_\lambda}^{-1} M$. Hence $\tau_{A_\lambda}^{-1} M \notin \mathcal{L}_A$ and so $X \notin \mathcal{L}_A$. In particular, X is not a projective A_λ -module, so there exists an arrow $\tau_{A_\lambda}^{n+1} M \rightarrow \tau_{A_\lambda} X$, contrary to the minimality of $|n|$.

Suppose now that $n > 0$. If there exists an arrow $\tau_{A_\lambda}^n M \rightarrow X$, then there exists an arrow $X \rightarrow \tau_{A_\lambda}^{n-1} M$, a contradiction to the minimality of $|n|$. If, on the other hand, there exists an arrow $X \rightarrow \tau_{A_\lambda}^n M$, then there is a path in Γ of the form $X \rightarrow \tau_{A_\lambda}^n M \rightsquigarrow M$, hence X is a predecessor of \mathcal{E} . Since $X \notin \mathcal{E}$, by hypothesis, then we know by (4.1) that X is not injective in $\text{mod}A_\lambda$. Hence there exists an arrow $\tau_{A_\lambda}^{-1} X \rightarrow \tau_{A_\lambda}^{n-1} M$, contrary to the minimality of $|n|$.

This shows that necessarily $n = 0$, that is, there is an arrow $M \rightarrow X$ or $X \rightarrow M$. If $M \rightarrow X$, then (S₄) yields $X \in \mathcal{E}$ or $\tau_{A_\lambda} X \in \mathcal{E}$, in any case a contradiction. If $X \rightarrow M$, then (S₅) yields $X \in \mathcal{E}$ or $\tau_{A_\lambda}^{-1} X \in \mathcal{E}$, again a contradiction in any case.

We proved that \mathcal{E} intersects each τ_{A_λ} -orbit of Γ . Suppose now that $M \in \mathcal{E}$ and $\tau_{A_\lambda}^{-t}M \in \mathcal{E}$ with $t > 0$. Then the epimorphism $\tau_A^{-1}M \rightarrow \tau_{A_\lambda}^{-1}M$ yields a path $\tau_A^{-1}M \rightarrow \tau_{A_\lambda}^{-1}M \rightsquigarrow \tau_{A_\lambda}^{-t}M$, so that $\tau_A^{-1}M \in \mathcal{L}_A$. This is a contradiction because $M \in \mathcal{E}$. Thus (b) is proven.

(b) Since, by [13], directed components of quasi-tilted algebras are postprojective, preinjective or connecting, thus always generalised standard (see [20, 23]), it suffices to show that Γ is directed. If this is not the case then, by [18] (4.3), Γ is a stable tube, of type $\mathbb{Z}A_\infty$ or obtained from one of these by finitely many ray or coray insertions.

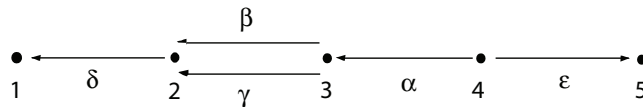
We first notice that by (2.3), any $E_0 \in \Gamma \cap \mathcal{E}$ is directed in $\text{ind}A$, hence in $\text{ind}A_\lambda$. In particular, Γ is neither a stable tube, nor of type $\mathbb{Z}A_\infty$. Therefore Γ is obtained from one of these by ray or coray insertions.

Assume first that Γ is an inserted tube or component of type $\mathbb{Z}A_\infty$, and let $E_0 \in \Gamma \cap \mathcal{E}$. We claim that $E_0 \in \mathcal{E}_2$. Indeed, if this is not the case, then there exists an injective A -module I such that $\text{Hom}_A(I, E_0) \neq 0$, by (2.1). However, $I \in \mathcal{L}_A$ implies that I is an A_λ -module, so that I is an injective A_λ -module. But this is impossible because no injective A_λ -module precedes an inserted tube or component of type $\mathbb{Z}A_\infty$. This establishes our claim. Thus, there exists an indecomposable projective module $P \notin \mathcal{L}_A$ such that $\text{Hom}_A(P, \tau_A^{-1}E_0) \neq 0$, by (2.1). On the other hand, $\tau_{A_\lambda}^{-1}E_0 \in \Gamma$, therefore there exist a non-directed projective $P' \in \Gamma$ and a path $\tau_{A_\lambda}^{-1}E_0 \rightsquigarrow P'$ in Γ . This is clear if Γ is an inserted tube, and follows from [10, 17] if Γ is an inserted component of type $\mathbb{Z}A_\infty$. Hence there exists a path $P \rightarrow \tau_A^{-1}E_0 \rightarrow \tau_{A_\lambda}^{-1}E_0 \rightsquigarrow P'$ in $\text{ind}A$. Since $P \notin \mathcal{L}_A$, then $P' \notin \mathcal{L}_A$. However, $P' \in \Gamma$, hence P' is a projective A -module lying in \mathcal{L}_A , a contradiction.

Assume next that Γ is a co-inserted tube or component of type $\mathbb{Z}A_\infty$, and let $E_0 \in \Gamma \cap \mathcal{E}$. Then, among the predecessors of E_0 lies a cycle $M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_t = M_0$, with all $M_i \in \Gamma$. Since all M_i precede E_0 and, by hypothesis, $E_0 \in \mathcal{E} \subseteq \mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$, then this cycle lies in \mathcal{L}_{A_λ} . This contradicts Theorem (A) (c) (also [3] (1.5) (b)).

(c) This follows directly from [20] (2.2). \square

4.4. EXAMPLE. It is important to underline that, while the components of $\Gamma(\text{mod } A_\lambda)$ which cut \mathcal{E} are directed, the same does not hold for the components of $\Gamma(\text{mod } A)$. Indeed, let k be a field and A be given by the quiver



bound by $\beta\delta = 0$, $\gamma\delta = 0$ and $\alpha\beta = 0$. The simple projective S_5 corresponding to the point 5 lies in an inserted tube Γ , it is a directed module and also an Ext-injective in $\text{add}\mathcal{L}_A$, that is, $S_5 \in \Gamma \cap \mathcal{E}$. On the other hand, Γ is clearly not directed. Observe that we have an irreducible morphism $S_5 \rightarrow P_4$ and $P_4 \notin \mathcal{L}_A$ (compare with (3.1)).

4.5. LEMMA. *Let Γ be a component of $\Gamma(\text{mod}A_\lambda)$.*

- (a) *If Γ is a non-connecting postprojective component, then $\Gamma \cap \mathcal{E} = \emptyset$.*
- (b) *If Γ is a non-connecting preinjective component, then $\Gamma \cap \mathcal{E} = \emptyset$.*
- (c) *If Γ intersects \mathcal{E} , then Γ is connecting.*
- (d) *If a connected component B of A_λ is not tilted, then $\text{mod}B \cap \mathcal{E} = \emptyset$.*

Proof. (a) Assume that Γ is a non-connecting postprojective component of $\Gamma(\text{mod}A_\lambda)$ such that $\Gamma \cap \mathcal{E} \neq \emptyset$. Let B be the (unique) connected component of A_λ such that Γ consists of B -modules. We claim that Γ does not contain every indecomposable projective B -module. Indeed, if this is not the case, then the number of τ_B -orbits in Γ coincides with $\text{rk } K_0(B)$. By Theorem (C) (a), \mathcal{E} intersects each τ_B -orbit of Γ exactly once. Hence $\Gamma \cap \mathcal{E}$ has $\text{rk } K_0(B)$ elements. From this and (4.2) we deduce that $E_0 = \bigoplus_{X \in \Gamma \cap \mathcal{E}} X$ is a convex tilting B -module. By [6], (2.1), $\Gamma \cap \mathcal{E}$ is a complete slice in $\text{mod}B$. But this is a contradiction, because Γ was assumed to be non-connecting. This establishes our claim.

Now, let $Q \notin \Gamma$ be an indecomposable projective B -module. Since B is a connected algebra, there exists a walk of projective B -modules $P = P_0 - P_1 - \dots - P_s = Q$, with $P \in \Gamma$. Thus there exists i such that $P_i \in \Gamma$ and $P_{i+1} \notin \Gamma$. Since Γ does not receive morphisms from other components of $\Gamma(\text{mod}B)$, then $\text{Hom}_B(P_i, P_{i+1}) \neq 0$. By [21] (2.1) there exists, for each $s > 0$, a path

$$P_i = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \dots \xrightarrow{f_s} M_s = L \xrightarrow{f} P_{i+1}$$

with f_i irreducible. Since s is as large as we want, and \mathcal{E} intersects each τ_B -orbit of Γ , we may choose s so that L is a proper successor of $\Gamma \cap \mathcal{E}$. On the other hand, P_{i+1} is a projective B -module, hence a projective A -module lying in \mathcal{L}_A . Thus $L \in \mathcal{L}_A$. Since L is a successor of \mathcal{E} , by (2.2), $L \in \mathcal{E}$, a contradiction which proves (a).

(b) Assume that Γ is a non-connecting preinjective component of $\Gamma(\text{mod}A_\lambda)$ such that $\Gamma \cap \mathcal{E} \neq \emptyset$. Using the same reasoning as in (a), there exist $M \in \Gamma$, which is a proper predecessor of $\Gamma \cap \mathcal{E}$, and an indecomposable injective A_λ -module $I \notin \Gamma$ such that $\text{Hom}_{A_\lambda}(I, M) \neq 0$. Since I precedes \mathcal{E} then, by (4.1), $I \in \mathcal{E}$. The convexity of \mathcal{E} yields the contradiction $M \in \mathcal{E}$. This establishes (b).

(c) It is shown in [3, 24] that every connected component of A_λ is quasi-tilted. By Theorem (C), \mathcal{E} intersects only directed components of $\Gamma(\text{mod}A_\lambda)$. Furthermore, directed components of quasi-tilted algebras are necessarily postprojective, preinjective or connecting. Now the result follows from (a) and (b).

(d) Is a consequence of (c). \square

4.6. PROPOSITION *Let B be a connected component of the left support A_λ , such that $\text{mod}B \cap \mathcal{E} \neq \emptyset$. Then:*

(a) *B is a tilted algebra and $\text{mod}B \cap \mathcal{E}$ is a complete slice in $\text{mod}B$.*

(b) *Let Γ be a component of $\Gamma(\text{mod}A_\lambda)$ such that $\Gamma \cap \mathcal{E} \neq \emptyset$. Then $B = A_\lambda / \text{Ann}(\Gamma \cap \mathcal{E})$.*

Proof. (a) Let Γ be a component of $\Gamma(\text{mod}A_\lambda)$ such that $\Gamma \cap \mathcal{E} \neq \emptyset$. By (c) of the previous lemma, we know that Γ is a connecting component. Since, on the other hand, \mathcal{E} intersects each τ_B -orbit of Γ exactly once (by Theorem (C) (a)), we have $|\Gamma \cap \mathcal{E}| = \text{rk}K_0(B)$. But by (4.2), $|\Gamma(\text{mod}B) \cap \mathcal{E}| \leq \text{rk}K_0(B)$. Hence $\Gamma \cap \mathcal{E} = \Gamma(\text{mod}B) \cap \mathcal{E}$ and the direct sum of the modules in $\Gamma(\text{mod}B) \cap \mathcal{E}$ is a convex tilting B -module. The result then follows from [6](2.1).

(b) We need to prove that $\text{Ann}_{A_\lambda}(\Gamma \cap \mathcal{E})$ is the product of the connected components of A_λ different from B . Clearly $\text{Ann}_{A_\lambda}(\Gamma \cap \mathcal{E})$ contains this product. To prove the other inclusion it is enough to see that $\text{Ann}_B(\Gamma \cap \mathcal{E}) = 0$. But this is again a consequence of (a), since $\Gamma \cap \mathcal{E}$ is faithful in $\text{mod}B$. \square

5. LEFT SUPPORTED ALGEBRAS.

5.1. An artin algebra A is *left supported* if $\text{add}\mathcal{L}_A$ is contravariantly finite in $\text{mod}A$, in the sense of [8]. It is shown in [3] (5.1) that an artin algebra A is left supported if and only if each connected component of A_λ is tilted and the restriction of \mathcal{E} to this component is a complete slice. Several other characterisations of left supported algebras are given in [1, 3]. In particular, it is shown in [1] that A is left supported if and only if $\mathcal{L}_A = \text{Pred}\mathcal{E}$, where $\text{Pred}\mathcal{E}$ denotes the full subcategory of $\text{ind}A$ having as objects all the $M \in \text{ind}A$ such that there exists $E_0 \in \mathcal{E}$ and a path $M \rightsquigarrow E_0$. Our objective in this section is to give another proof of this theorem, using the results above. Our proof also yields a new characterisation of left supported algebras.

THEOREM. *Let A be an artin algebra. Then the following conditions are equivalent:*

(a) *A is left supported.*

(b) $\mathcal{L}_A = \text{Pred } \mathcal{E}$.

(c) *Every projective A -module which belongs to \mathcal{L}_A is a predecessor of \mathcal{E} .*

Proof. (a) implies (b). Assume that A is left supported. By [3](4.2), \mathcal{L}_A is cogenerated by the direct sum of the modules in \mathcal{E} . In particular, $\mathcal{L}_A \subseteq \text{Pred } \mathcal{E}$. Since the reverse inclusion is obvious, this completes the proof of (a) implies (b).

Clearly (b) implies (c). To prove that (c) implies (a) we assume that every projective A -module which belongs to \mathcal{L}_A is a predecessor of \mathcal{E} . Let B be a connected component of A_λ and P be an indecomposable projective B -module. Since $P \in \mathcal{L}_A$, there exist $E_0 \in \mathcal{E}$ and a path $P \rightsquigarrow E_0$ in \mathcal{L}_A , hence in $\text{mod } B$. Therefore, $\text{mod } B \cap \mathcal{E} \neq \emptyset$. By (4.6), B is a tilted algebra and $\text{mod } B \cap \mathcal{E}$ is a complete slice in $\text{mod } B$. Hence A is left supported. \square

5.2. We end this paper with a short proof of a result by D. Smith.

THEOREM.([25] (3.8)) *Let A be a quasi-tilted algebra. Then A is left supported if and only if A is tilted having a complete slice containing an injective module.*

Proof. Since A is quasi-tilted, then $A = A_\lambda$. Assume that A is left supported. Then $\mathcal{L}_A = \text{Pred } \mathcal{E}$. By (5.1), A is tilted and \mathcal{E} is a complete slice in $\Gamma(\text{mod } A)$. Furthermore, since A is quasi-tilted, then all projective A -modules lie in \mathcal{L}_A , so that $\mathcal{E}_2 = \emptyset$ and $\mathcal{E} = \mathcal{E}_1$. Thus \mathcal{E} must contain an injective module.

Conversely, if A has a complete slice containing an injective, then there exists a complete slice Σ having all its sources injective. By (2.1), $\Sigma \subseteq \mathcal{E}$. Since $|\Sigma| = \text{rk } K_0(A)$, it follows from [3] (3.3) that A is left supported. \square

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