THE LEFT AND THE RIGHT PARTS OF A MODULE CATEGORY

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Dedicated to the memory of Sheila Brenner.

Abstract. In this paper, we study, for an artin algebra, the class $\mathcal{L}_A$ (and $\mathcal{R}_A$) which is a full subcategory of the category mod$A$ of finitely generated A-modules, and which consists of all indecomposable A-modules whose predecessors (and successors) have projective dimension (and injective dimension, respectively) at most one. We consider quotient algebras of A, which contain the information on these classes, then define and characterize those algebras for which the class is $\mathcal{L}_A$ is contravariantly finite (and $\mathcal{R}_A$ is covariantly finite, respectively).

While defining the class of quasi-tilted algebras in [17], Happel, Reiten and Smalø have introduced two classes of modules which turned out to be very useful in the representation theory of algebras. Let A be an artin algebra, and mod$A$ denote the category of finitely generated right A-modules, then the class $\mathcal{L}_A$ (or $\mathcal{R}_A$) is the full subcategory of mod$A$ consisting of all indecomposable A-modules whose predecessors (or successors) have projective dimension (or injective dimension, respectively) at most one. These classes, respectively called the left part and the right part of the module category mod$A$, were heavily investigated and applied, see, for instance [2, 17, 13, 14, 15, 3, 5, 27, 22, 12, 20].

Our objective in this paper is to give a reasonably good description of these classes. Following Skowroński [27] (3.1), we consider, for an arbitrary artin algebra A, an algebra A_λ, which we call the *left support of* A, and such that $\mathcal{L}_A$ embeds nicely inside modA_λ. It turns out that the left support algebra is always a direct product of quasi-tilted algebras.

Returning to our original aim, and motivated by the belief that a class which behaves well is likely to afford a good description, we consider those algebras A such that the full subcategory add$\mathcal{L}_A$ of mod$A$

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having as objects the direct sums of modules in $\mathcal{L}_A$ is contravariantly finite (in the sense of Auslander and Smalø [9]) and call such algebras *left supported*. *Right supported algebras* are defined dually by requiring that add$\mathcal{R}_A$ be covariantly finite. Contravariantly and covariantly finite subcategories have been very useful in the representation theory of algebras, see, for instance, [8, 9, 6]. This paper is mainly devoted to the study of (left) supported algebras.

As we shall see, there exist many classes of algebras which are left supported such as, for instance, the laura algebras of [2, 3, 5, 22, 27] as well as many classes of tilted algebras.

We now state our results. We start with an arbitrary artin algebra A, and describe explicitly a module E, which we prove to be the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-injectives in the subcategory add$\mathcal{L}_A$. If we add to this module the direct sum of a complete set of representatives of the isomorphism classes of the indecomposable projectives not lying in $\mathcal{L}_A$, we get a partial tilting module $T = E \oplus F$. This leads us to our first main theorem.

Theorem (A). *Let A be an artin algebra. The following conditions are equivalent:*

- (a) *A is left supported.*
- (b) *add$\mathcal{L}_A$ coincides with the class Cogen$E$ of the A-modules cogenerated by E.*
- (c) *$T = E \oplus F$ is a tilting module.*

The module T above generalizes the canonical tilting module constructed for shod algebras in [12] and thus our theorem can be read as a generalization of the results in [12].

We next return to our original aim, namely to give a good description of the modules and the irreducible morphisms in $\mathcal{L}_A$, and achieve this as follows: let A_λ be the left support of A. Then E has a natural A_λ-module structure, and we have.

Theorem (B). *Let A be an artin algebra. Then A is left supported if and only if each connected component of A_λ is a tilted algebra and the restriction of E to this component is a slice module.*

This paper is organized as follows. After a brief preliminary section, we look at the left support of an artin algebra in Section 2, then we study the indecomposable Ext-injectives in $\mathcal{L}_A$ in Section 3. The next two sections are respectively devoted to proving our two theorems.

1. Preliminaries

1.1. **Notation.** Throughout this paper, all our algebras are artin algebras. For an algebra A, we denote by mod$A$ its category of finitely generated right A-modules. For a full subcategory $\mathcal{C}$ of mod$A$, we denote by ind$\mathcal{C}$ a full subcategory of mod$A$ having as objects a full set of representatives of the isomorphism classes of the indecomposable objects in $\mathcal{C}$, and we abbreviate ind(mod$A$) as ind$A$. Also, we denote by add$\mathcal{C}$ the full subcategory of mod$A$ having as objects the direct sums of indecomposable summands of objects in $\mathcal{C}$, and, if M is a module, we abbreviate add$\{M\}$ as add$M$. We denote by $\mathrm{rk}(K_0(A))$ the rank of the Grothendieck group of A, which equals the number of isomorphism classes of simple A-modules. If M is an A-module, we denote by $\mathrm{pd}_A M$ (or $\mathrm{id}_A M$) its projective dimension (or injective dimension, respectively). Also, we denote by gl.dim.A the global dimension of A. An algebra B is called a *full subcategory of* A if there exists an idempotent $e \in A$ such that $B = eAe$. It is called convex in A if, whenever there exists a sequence $e_i = e_{i_0}, e_{i_1}, \cdots, e_{i_t} = e_j$ of primitive orthogonal idempotents such that $e_{i_{l+1}} A e_{i_l} \neq 0$ for $0 \leq l < t$, and $ee_i = e_i, ee_j = e_j$, then $ee_{i_l} = e_{i_l}$, for each l. Finally, A is called *triangular* if there exists no sequence of primitive orthogonal idempotents $e_{i_0}, e_{i_1}, \cdots, e_{i_t} = e_{i_0}$ such that $e_{i_{l+1}} A e_{i_l} \neq 0$ for $0 \leq l < t$.

For further definitions or facts needed on mod$A$, its Auslander-Reiten quiver $\Gamma(\mathrm{mod}A)$, and its Auslander-Reiten translations $\tau_A = \mathrm{DTr}$ and $\tau_A^{-1} = \mathrm{TrD}$, we refer the reader to [7, 23]. For tilting theory, we refer to [1].

1.2. **Paths.** Given two modules M, N in ind$A$, a *path from M to N of length t* in ind$A$ is a sequence

$$(*) \qquad M = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t = N$$

($t \geq 0$) where all M_i lie in ind$A$, and all f_i are non-zero. We write in this case $M \rightsquigarrow N$, and say that M is a *predecessor of* N, or that N is a *successor of* M. The path $(*)$ is called a *path of irreducible morphims* if each f_i is irreducible. A path $(*)$ is called a *cycle* if $M \cong N$, at least one of the f_i is not an isomorphism, and $t > 0$. An indecomposable module M is called *directed* provided it lies on no cycle in ind$A$, and a full subcategory $\mathcal{C}$ of mod$A$ is called *directed* if each object in $\mathcal{C}$ is directed. A full subcategory $\mathcal{C}$ of mod$A$ is called *convex* if, for any path $(*)$ from M to N in ind$A$, where M and N lie in $\mathcal{C}$, all the M_i lie in $\mathcal{C}$ as well. A path $(*)$ of irreducible morphisms is called *sectional* if

$\tau_A M_{i+1} \neq M_{i-1}$ for all $1 < i < t$. A *refinement of* $(*)$ is a path

$$M = M'_0 \xrightarrow{f'_1} M'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{s-1}} M'_{s-1} \xrightarrow{f'_s} M'_s = N$$

in ind$A$ with $s \geq t$ such that there exists an order-preserving function $\sigma\colon \{1, \cdots, t-1\} \longrightarrow \{1, \cdots, s-1\}$ such that $M_i \cong M'_{\sigma(i)}$ for each i with $1 \leq i \leq t-1$. We need the following result.

Proposition. [2](1.4) *Let Γ be a component of Γ(modA) and M be an indecomposable module lying in a cycle in Γ.*

(a) *If Γ contains projective modules, then there is a path of irreducible morphisms from M to a projective.*
(b) *If Γ contains injective modules, then there is a path of irreducible morphisms from an injective to M.*

1.3. **The subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$.** For an algebra A, we denote by $\mathcal{L}_A$ and $\mathcal{R}_A$ the following subcategories of ind$A$:

$$\mathcal{L}_A = \{X \in \text{ind}A\colon \text{pd}_A Y \leq 1 \text{ for each predecessor } Y \text{ of } X\}$$

$$\mathcal{R}_A = \{X \in \text{ind}A\colon \text{id}_A Z \leq 1 \text{ for each successor } Z \text{ of } X\}$$

Clearly, $\mathcal{L}_A$ is closed under predecessors, while $\mathcal{R}_A$ is closed under successors. These subcategories played an important rôle in the study of the quasi-tilted algebras [17], the shod algebras [13], the weakly shod algebras [14, 15] and the laura algebras [2, 3, 5, 27, 22].

Lemma. [2](1.5) *Let A be an artin algebra.*

(a) *If P is an indecomposable projective A-module, then there are at most finitely many modules $M \in \mathcal{R}_A$ such that there exists a path $M \rightsquigarrow P$. Moreover, any such path is refinable to a path of irreducible morphisms, and any such path of irreducible morphisms is sectional.*
(b) *If I is an indecomposable injective A-module, then there are at most finitely many modules $N \in \mathcal{L}_A$ such that there exists a path $I \rightsquigarrow N$. Moreover, any such path is refinable to a path of irreducible morphisms, and any such path of irreducible morphisms is sectional.*

1.4. Corollary. [2](1.6) *Let A be an artin algebra.*

(a) *$\mathcal{R}_A$ consists of the modules $M \in$ indA such that, if there exists a path from M to an indecomposable projective module, then this path can be refined to a path of irreducible morphisms, and any such path of irreducible morphisms is sectional.*

(b) $\mathcal{L}_A$ *consists of the modules* $N \in indA$ *such that, if there exists a path from an indecomposable injective module to* N*, then this path can be refined to a path of irreducible morphisms, and any such path of irreducible morphisms is sectional.*

1.5. Lemma. *Let* A *be an artin algebra, and* Γ *be a component of* $\Gamma(modA)$*.*

(a) *If* Γ *contains projective modules, then* $\mathcal{R}_A \cap \Gamma$ *is directed.*
(b) *If* Γ *contains injective modules, then* $\mathcal{L}_A \cap \Gamma$ *is directed.*

Proof. (a) Assume $M \in \mathcal{R}_A \cap \Gamma$, and that $M = M_0 \longrightarrow \cdots \longrightarrow M_t = M$ is a cycle. By (1.2), there exists a path $M = N_0 \longrightarrow \cdots \longrightarrow N_s = P$, where P is projective. By (1.3)(a), the composed path $M = M_0 \longrightarrow \cdots \longrightarrow M_t = M = N_0 \longrightarrow \cdots \longrightarrow N_s = P$ is refinable to a sectional path of irreducible morphisms. But this contradicts the non-sectionality of cycles [10, 11, 18]. □

1.6. Corollary. *Let* A *be a representation-finite artin algebra. Then* $\mathcal{L}_A$ *and* $\mathcal{R}_A$ *are directed.*

2. The left and right support algebras.

2.1. Proposition. *Let* $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ *be an artin algebra written in triangular matrix form. Then* $\mathcal{L}_A \subseteq \mathcal{L}_B$ *if and only if, for each primitive idempotent* $e_c \in C$*, the corresponding projective* A*-module* P_c *does not lie in* $\mathcal{L}_A$*.*

Proof. Sufficiency. We first observe that $\mathcal{L}_A \subseteq \text{ind}B$. Indeed, if $X \in \mathcal{L}_A$ and does not lie in ind$B$, then there exists an idempotent $e_c \in C$ such that there exists a non-zero morphism $P_c \longrightarrow X$. However, $P_c \notin \mathcal{L}_A$ and $X \in \mathcal{L}_A$ contradict the fact that $\mathcal{L}_A$ is closed under predecessors.
Let now $X \in \mathcal{L}_A$. We claim that the full subcategories $\text{Pred}_A X$ and $\text{Pred}_B X$ consisting of the predecessors of X in mod$A$ and mod$B$ respectively, coincide. It is clear that $\text{Pred}_B X \subseteq \text{Pred}_A X$. Let $Y \in \text{Pred}_A X$, then there exists a path $Y = Y_0 \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_t = X$ in ind$A$. However, $X \in \mathcal{L}_A$, hence each $Y_i \in \mathcal{L}_A$. Since $\mathcal{L}_A \subseteq \text{ind}B$, this means that each Y_i is a B-module. Therefore, $Y \in \text{Pred}_B X$. This establishes our claim.
In order to show that $X \in \mathcal{L}_B$, we assume that Y is a predecessor of X. Since Y precedes X in $\mathcal{L}_A$, there exists a minimal projective resolution in mod$A$

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow 0$$

Since $Y \in \mathcal{L}_A$, then $P_0, P_1 \in \mathrm{add}\mathcal{L}_A$. Therefore P_0, P_1 are projective B-modules. Hence $\mathrm{pd}Y_B \leq 1$ and $X \in \mathcal{L}_B$. Thus, $\mathcal{L}_A \subseteq \mathcal{L}_B$.

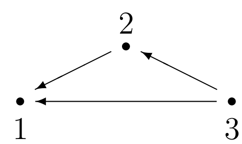
Necessity. Let $e_c \in C$ be a primitive idempotent such that $P_c \in \mathcal{L}_A$. Then $\mathcal{L}_A \subseteq \mathcal{L}_B$ yields $P_c \in \mathcal{L}_B$ and in particular is a B-module, an absurdity. □

2.2. Definition. (see [27] (3.1)) Let A be an artin algebra, and P denote the direct sum of a full set of representatives of the isomorphism classes of indecomposable projectives lying in $\mathcal{L}_A$. The algebra $A_\lambda = \mathrm{End}P_A$ is called the *left support of* A. We define dually the *right support* A_ρ *of* A.

Remarks. (a) A_λ is a full convex subcategory of A closed under successors. Indeed, if $P_x \in \mathcal{L}_A$ is projective and $P_{x_0} \longrightarrow P_{x_1} \longrightarrow \cdots \longrightarrow P_{x_t} = P_x$ is a path in $\mathrm{ind}A$ between projectives, then $P_{x_i} \in \mathcal{L}_A$ for each i.

(b) Any indecomposable in $\mathcal{L}_A$ has a canonical A_λ-module structure (that is, $\mathcal{L}_A \subseteq \mathrm{ind}A_\lambda$): indeed, if $X \in \mathcal{L}_A$ and P_x is an indecomposable projective module such that $\mathrm{Hom}_A(P_x, X) \neq 0$, then $P_x \in \mathcal{L}_A$.

(c) In general, the subcategories A_λ and A_ρ may intersect: if A is the radical square zero algebra given by the quiver



then A_λ is the full convex subcategory generated by the points $\{1, 2\}$ while A_ρ is generated by $\{2, 3\}$.

(d) The left support of a tilted (or, more generally, of a laura but not quasi-tilted) algebra does not coincide with the left end algebra (see [19] and [2], respectively): indeed, if A is tilted, then $A = A_\lambda = A_\rho$.

2.3. Corollary. ([27](3.1)) *Let A be an artin algebra. Then A_λ and A_ρ are direct products of connected quasi-tilted algebras.*

Proof. It follows from Remark (2.2)(a) that A can be written in triangular matrix form

$$A = \begin{bmatrix} A_\lambda & 0 \\ M & C \end{bmatrix}$$

By (2.1), $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$. Observe also that the projective A_λ-modules coincide with the projective A-modules. Thus, if P is an indecomposable

A_λ-module, we have $P \in \mathcal{L}_{A_\lambda}$. Applying [17](II.1.14) yields that each connected component of A_λ is quasi-tilted. □

2.4. THEOREM. *Let A be a triangular artin algebra. Then there exists a filtration $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_t = A$, with A_0 a direct product of connected quasi-tilted algebras, and modules M_i in modA_i such that $A_{i+1} = A_i[M_i]$ and $M_i \notin$ add$\mathcal{L}_{A_{i+1}}$, for each i such that $0 \leq i < t$.*

Proof. If A is quasitilted, then $A = A_0 = A_\lambda$. Thus, suppose it is not, and let $P_x \notin \mathcal{L}_A$ be indecomposable projective. Since A is triangular, we may assume that P_x has no projective successor. Hence we can write $A = A'[M]$. In particular, $M \notin$ add$\mathcal{L}_A$. □

REMARK. It follows from the proof of the theorem that $A_0 = A_\lambda$ and that $\mathcal{L}_{A_0} \subseteq \mathcal{L}_{A_1} \subseteq \cdots \subseteq \mathcal{L}_A$.

3. EXT-INJECTIVES IN $\mathcal{L}_A$.

3.1. Let A be an artin algebra. We define two subclasses of $\mathcal{L}_A$:

- $\mathcal{E}_1 = \{M \in \mathcal{L}_A$: there exists an injective I in ind$A$ and a path $I \rightsquigarrow M\}$.
- $\mathcal{E}_2 = \{M \in \mathcal{L}_A \setminus \mathcal{E}_1$: there exists a projective P in ind$A \setminus \mathcal{L}_A$ and a sectional path $P \rightsquigarrow \tau_A^{-1}M\}$.

The definition of $\mathcal{E}_2$ makes sense: if $M \in \mathcal{L}_A \setminus \mathcal{E}_1$, then M is not injective, so $\tau_A^{-1}M$ exists. Finally, we set $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$.

We recall that, if $\mathcal{C}$ is a full subcategory of mod$A$, closed under extensions, then an indecomposable M in $\mathcal{C}$ is called an *Ext-projective* (or *Ext-injective*) in $\mathcal{C}$ if $\mathrm{Ext}_A^1(M,-)|_\mathcal{C} = 0$ (or $\mathrm{Ext}_A^1(-,M)|_\mathcal{C} = 0$, respectively). It is shown in [8](3.4) that, if $\mathcal{C}$ is a torsion-free class, then M is Ext-injective in $\mathcal{C}$ if and only if $\tau_A^{-1}M$ is not in $\mathcal{C}$. Finally, note that, since $\mathcal{L}_A$ is closed under predecessors, add$\mathcal{L}_A$ is a torsion-free class.

THEOREM. *Let A be an artin algebra, and M be an indecomposable A-module.*

(a) *M is Ext-projective in add$\mathcal{L}_A$ if and only if M is projective and lies in $\mathcal{L}_A$.*
(b) *M is Ext-injective in add$\mathcal{L}_A$ if and only if $M \in \mathcal{E}$.*

Proof. (a) Assume M to be Ext-projective in add$\mathcal{L}_A$ and let $f\colon P \longrightarrow M$ be a projective cover of M. Since $\mathcal{L}_A$ is closed under predecessors, then Ker$f \in$ add$\mathcal{L}_A$, so that $\mathrm{Ext}_A^1(M, \mathrm{Ker}f) = 0$, and the exact sequence $0 \longrightarrow \mathrm{Ker}f \longrightarrow P \xrightarrow{f} M \longrightarrow 0$ splits. Thus, M is projective.

Since the converse is clear, the statement is proved.

(b) *Necessity.* Clearly, $M \in \mathcal{E}$ implies that $M \in \mathcal{L}_A$. If $M \in \mathcal{E}_1$, then either M is injective (in which case it is automatically Ext-injective in $\mathcal{L}_A$) or, else, there exist an indecomposable injective I and a path $I \rightsquigarrow M \longrightarrow * \longrightarrow \tau_A^{-1}M$ in ind$A$. This path is not sectional, hence, by (1.4), $\tau_A^{-1}M \notin \mathcal{L}_A$. If $M \in \mathcal{E}_2$, there exists a path $P \rightsquigarrow \tau_A^{-1}M$ with $P \notin \mathcal{L}_A$ projective. Hence, $\tau_A^{-1}M \notin \mathcal{L}_A$.

Sufficiency. Assume $M \notin \mathcal{E}_1$ to be Ext-injective in add$\mathcal{L}_A$. We must show that $M \in \mathcal{E}_2$. Since $\tau_A^{-1}M \notin \mathcal{L}_A$, then $\tau_A^{-1}M$ has a predecessor L such that $\mathrm{pd}_A L \geq 2$.
Let $N_1 = \tau_A^{-1}M$. If $\mathrm{pd}_A N_1 \geq 2$, then

$$\mathrm{Hom}_A(\mathrm{D}(A), M) \cong \mathrm{Hom}_A(\mathrm{D}(A), \tau_A N_1) \neq 0$$

gives $M \in \mathcal{E}_1$, a contradiction. Thus, $\mathrm{pd}_A N_1 \leq 1$ and there exists a path $L \rightsquigarrow N_1$, which factors through one of the middle terms N_2 of the almost split sequence ending with N_1.
Since L precedes N_2, then $N_2 \notin \mathcal{L}_A$. We claim that $\mathrm{pd}_A N_2 \leq 1$. Indeed, if $\mathrm{pd}_A N_2 \geq 2$, then $\mathrm{Hom}_A(\mathrm{D}(A), \tau_A N_2) \neq 0$ gives an injective I and a path $I \rightsquigarrow \tau_A N_2 \longrightarrow M$ in ind$A$, so $M \in \mathcal{E}_1$, a contradiction which proves our claim.
Now, if N_2 is projective, then $M \in \mathcal{E}_2$ and we are done. We may thus assume that N_2 is not projective.
Inductively, if $N_1, \cdots, N_{i-1}$ are not projective, we construct in this way a path

$$(*) \qquad N_i \longrightarrow \cdots \longrightarrow N_2 \longrightarrow N_1 = \tau_A^{-1}M$$

in ind$A$, with $N_1, \cdots, N_i \notin \mathcal{L}_A$ and $\mathrm{pd}_A N_j \leq 1$ for all j. We now show that this path is sectional: indeed, if this is not the case, then there exists a least index $j \leq i-1$ such that $N_j \longrightarrow \cdots \longrightarrow N_1$ is sectional, while $N_{j+1} = \tau_A N_{j-1}$. But, in this case, we get a path

$$N_{j+1} = \tau_A N_{j-1} \longrightarrow \cdots \longrightarrow \tau_A N_2 \longrightarrow M = \tau_A N_1$$

in ind$A$, and this yields a contradiction, since $M \in \mathcal{L}_A$ would then imply $N_{j+1} \in \mathcal{L}_A$.
It follows from [10, 11, 18] and from the sectionality of the path $(*)$ that the modules N_j are pairwise non-isomorphic.
Assume now that $i \geq 1+ \mathrm{rk}(K_0(A))$. By [26], there exist p, q such that $\mathrm{Hom}_A(N_p, \tau_A N_q) \neq 0$. We then get, as above, $N_p \in \mathcal{L}_A$, a contradiction. Therefore $i \leq \mathrm{rk}(K_0(A))$. This shows that the above construction stops after at most $\mathrm{rk}(K_0(A))$ steps. Thus, there is an index j such that N_j is a projective module. Since $N_j \notin \mathcal{L}_A$, this implies $M \in \mathcal{E}_2$. □

3.2. Corollary. *Any path of irreducible morphisms in $\mathcal{E}$ is sectional.*

Proof. Assume $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t$ is a path of irreducible morphisms, with $M_i \in \mathcal{E}$ for all i. If $M_{i+1} = \tau_A^{-1} M_{i-1}$, then since we have $\tau_A^{-1} M_{i-1} \notin \mathcal{L}_A$ because of the Ext-injectivity of M_{i-1}, the fact that $M_{i+1} \in \mathcal{L}_A$ yields a contradiction. □

3.3. We now denote by E the direct sum of all indecomposable A-modules lying in $\mathcal{E}$, by F the direct sum of a full set of representatives of the isomorphism classes of indecomposable projective A-modules not lying in $\mathcal{L}_A$, and we set $T = E \oplus F$.

Lemma. *With the above notations, $T = E \oplus F$ is a partial tilting A-module. It is a tilting module if and only if the number of indecomposable summands of E equals the number of isomorphism classes of indecomposable projectives lying in $\mathcal{L}_A$.*

Proof. Clearly, $\mathrm{pd}_A T \leq 1$. It follows from the Ext-injectivity of E that $\mathrm{Ext}_A^1(E, E) = 0$. In order to prove that T is a partial tilting module there only remains to show that $\mathrm{Ext}_A^1(E, F) = 0$. Since $F \notin \mathrm{add}\mathcal{L}_A$, we have, for every $M \in \mathcal{L}_A$, that $\tau_A M \in \mathcal{L}_A$. Hence

$$\mathrm{Ext}_A^1(M, F) \cong \mathrm{D}\ \mathrm{Hom}_A(F, \tau_A M) = 0$$

because $\mathrm{pd}_A M \leq 1$ (see [23]). This implies our statement.
Now, the partial tilting module T is a tilting module if and only if the number of isomorphism classes of its summands equals the number of isomorphism classes of indecomposable projective A-modules. The second statement follows at once. □

3.4. Proposition. *Assume that $M \in \mathcal{E}$ and that there exists a path $M \rightsquigarrow N$, with $N \in \mathcal{L}_A$. Then this path can be refined to a sectional path of irreducible morphisms and $N \in \mathcal{E}$. In particular, $\mathcal{E}$ is convex in modA.*

Proof. There are two cases to consider. Assume first that $M \in \mathcal{E}_1$. Hence, there exist an injective I and a path $I \rightsquigarrow M$ in ind$A$. Since $N \in \mathcal{L}_A$, it follows from (1.4) that the path $I \rightsquigarrow M \rightsquigarrow N$ is refinable to a sectional path of irreducible morphisms. Hence $N \in \mathcal{E}_1 \subseteq \mathcal{E}$.
Assume now $M \in \mathcal{E}_2$. If the given path

$$(*) \qquad M = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t = N$$

factors through an indecomposable injective I, then the subpath $I \rightsquigarrow N$ lies in $\mathcal{E}_1$. We may thus assume that none of the M_i is injective and that none of the morphisms f_i factors through an injective. Hence, for each i, $\mathrm{Hom}_A(M_i, M_{i+1}) = \overline{\mathrm{Hom}}_A(M_i, M_{i+1})$. The Auslander-Reiten formula then yields

$$\underline{\mathrm{Hom}}_A(\tau_A^{-1}M_i, \tau_A^{-1}M_{i+1}) \cong \mathrm{Hom}_A(M_i, M_{i+1}) \neq 0$$

This yields a path

$$(**) \qquad \tau_A^{-1}M_0 \longrightarrow \tau_A^{-1}M_1 \longrightarrow \cdots \longrightarrow \tau_A^{-1}M_t = \tau_A^{-1}N$$

in ind$A$. Now, all modules on $(*)$ lie in $\mathcal{L}_A$ (because N does), while no module on $(**)$ belongs to $\mathcal{L}_A$ (because all are successors of a projective not in $\mathcal{L}_A$). Therefore, all M_i are Ext-injective in add$\mathcal{L}_A$, that is, all belong to $\mathcal{E}$. This shows, in particular, that $\mathcal{E}$ is convex in mod$A$.
In order to show that the path $(*)$ can be refined to a sectional path of irreducible morphisms, it suffices, in view of (3.2), to show that none of the f_i lies in the infinite radical $\mathrm{rad}^{\infty}(\mathrm{mod}A)$ of the category mod$A$. However, if $f_i \in \mathrm{rad}_A^{\infty}(M_{i-1}, M_i)$ then, for any $s \geq 1$, the given path can be refined to a path

$$M \rightsquigarrow M_{i-1} = L_0 \longrightarrow L_1 \longrightarrow \cdots \longrightarrow L_s = M_i \rightsquigarrow N$$

in ind$A$. The above reasoning then gives that all L_j belong to $\mathcal{E}$, and this contradicts the fact that, by (3.3), the number of indecomposables in $\mathcal{E}$ does not exceed $\mathrm{rk}(K_0(A))$. □

3.5. COROLLARY. *Assume that* Γ *is a component of* $\Gamma(\mathrm{mod}A)$ *which contains an injective, then the number of elements of* $\mathcal{E}$ *which lie in* Γ *is equal to the number of* τ_A*-orbits in* $\mathcal{L}_A \cap \Gamma$.

Proof. We recall from (1.5) that $\mathcal{L}_A \cap \Gamma$ is directed. We first claim that, if $M \in \mathcal{L}_A \cap \Gamma$, then there exists $m > 0$ such that $\tau_A^{-m}M \in \mathcal{E}$. Assume that, for all $m > 0$, we have $\tau_A^{-m}M \in \mathcal{L}_A$. The directedness of $\mathcal{L}_A \cap \Gamma$ implies that M is right stable. We know that Γ contains an injective hence a walk from this injective to the τ_A-orbit of M. Among all such injectives, choose one, denoted by I, such that there is a walk of least length between I and the orbit of M. This minimality implies that all modules on this walk except I are right stable. Hence there exists $m > 0$ such that there is a path from I to $\tau_A^{-m}M$. Since $\tau_A^{-m}M \in \mathcal{L}_A$, then $I \in \mathcal{L}_A$. Hence $I \in \mathcal{E}$. By (3.4), $\tau_A^{-m}M \in \mathcal{E}$, and this is a contradiction, since m can be taken arbitrarily large. This shows that there exists $m > 0$ such that $\tau_A^{-m}M \in \mathcal{L}_A$ but $\tau_A^{-m-1}M \notin \mathcal{L}_A$. But then $\tau_A^{-m}M \in \mathcal{E}$. This establishes our claim (see also [15](3.3)).
We have proven that each τ_A-orbit in $\mathcal{L}_A \cap \Gamma$ intersects $\mathcal{E}$. Furthermore it intersects it only once: if M and $\tau_A^{-t}M$ (with $t > 0$) belong to $\mathcal{E}$ then, by (3.4), all the modules on the path

$$M \longrightarrow * \longrightarrow \tau_A^{-1}M \longrightarrow \cdots \longrightarrow \tau_A^{-t}M$$

belong to $\mathcal{L}_A$, and $\tau_A^{-1}M \in \mathcal{L}_A$ contradicts the Ext-injectivity of M. Therefore, the number of τ_A-orbits in $\mathcal{L}_A \cap \Gamma$ does not exceed the cardinality of $\mathcal{E} \cap \Gamma$. Since, clearly, any element in $\mathcal{E} \cap \Gamma$ belongs to exactly one τ_A-orbit in $\mathcal{L}_A \cap \Gamma$, the statement follows. □

4. Left supported algebras

4.1. We recall that a tilting A-module T induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in mod$A$, where $\mathcal{T}(T) = \text{Gen}T$ is the class of all A-modules generated by T, and $\mathcal{F}(T) = \{M \in \text{mod}A\colon\ \text{Hom}_A(T, M) = 0\}$, see [1].

Lemma. *With the notations of (3.2), assume that* $T = E \oplus F$ *is a tilting module. Then* $\mathcal{F}(T) = add(\mathcal{L}_A \setminus \mathcal{E})$ *and* $\mathcal{T}(T) = add(indA \setminus \mathcal{F}(T))$.

Proof. Assume $M \in \mathcal{L}_A \setminus \mathcal{E}$, we claim that $M \in \mathcal{F}(T)$. If this is not the case, $\text{Hom}_A(T, M) \neq 0$. Since $F \notin \mathcal{L}_A$, we have $\text{Hom}_A(F, M) = 0$. Consequently, there exists an indecomposable summand E_0 of E such that $\text{Hom}_A(E_0, M) \neq 0$ and, since $M \in \mathcal{L}_A \setminus \mathcal{E}$, this yields a contradiction to (3.4). This shows that $\mathcal{L}_A \setminus \mathcal{E} \subset \mathcal{F}(T)$. Conversely, let N be an indecomposable A-module in $\mathcal{F}(T)$. Then N is cogenerated by $\tau_A T = \tau_A E$ (see [1](2.4)). Hence there exist an indecomposable non-projective summand E_0 of E and a path $N \longrightarrow \tau_A E_0 \longrightarrow * \longrightarrow E_0$. Since $E_0 \in \mathcal{L}_A$, then $N \in \mathcal{L}_A$. On the other hand, $N \notin \mathcal{E}$ since $\mathcal{E} \subset \text{add}T \subset \mathcal{T}(T)$. This shows the first equality, and the second follows by maximality (because $\mathcal{L}_A \setminus \mathcal{E}$ is closed under predecessors). □

4.2. We recall that a full additive subcategory $\mathcal{C}$ of mod$A$ is called *contravariantly finite* if, for any A-module M, there exists a morphism $f_{\mathcal{C}}\colon M_{\mathcal{C}} \longrightarrow M$ such that $M_{\mathcal{C}} \in \mathcal{C}$ and, if $f\colon N \longrightarrow M$ is any morphism with $N \in \mathcal{C}$, then there exists $g\colon N \longrightarrow M_{\mathcal{C}}$ such that $f = f_{\mathcal{C}}g$ (see [9]). The dual notion is that of *covariantly finite*. We observe that, since the class $\mathcal{L}_A$ is closed under predecessors, then add$\mathcal{L}_A$ is trivially covariantly finite.

Definition. An artin algebra A is called *left supported* provided the class add$\mathcal{L}_A$ is contravariantly finite in mod$A$. We define dually *right supported algebras*.

Obviously, any representation-finite algebra is both left and right supported. We defer to later further remarks and examples, we wish to prove first our main theorem of this section.

Theorem. *Let* A *be an artin algebra. The following conditions are equivalent:*

(a) *A is left supported.*
(b) *$\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}E$.*
(c) *$T = E \oplus F$ is a tilting module.*

Proof. (a) implies (b). By [28], there exists a module N in mod$A$ such that $\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}N$. We claim that $\mathrm{Cogen}N = \mathrm{Cogen}E$. By [8](5.3), N is Ext-injective in $\mathrm{add}\mathcal{L}_A$, so that $N \in \mathrm{add}E$. Hence $E \in \mathrm{add}\mathcal{L}_A$ implies

$$\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}N \subseteq \mathrm{Cogen}E \subseteq \mathrm{add}\mathcal{L}_A$$

and (b) follows.

(b) implies (c). By (3.1), the number of isomorphism classes of indecomposable summands of E equals the number of isomorphism classes of indecomposable Ext-injectives in $\mathrm{add}\mathcal{L}_A$. Since $\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}E$, the latter equals the number of isomorphism classes of indecomposable Ext-projectives in $\mathcal{L}_A$ (by [8](A.6)) that is, by (3.1) again, the number of isomorphism classes of indecomposable projective modules in $\mathcal{L}_A$. The result then follows from (3.3).

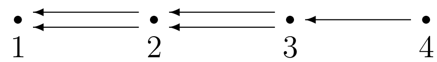
(c) implies (a). Since T is a tilting module then, by [28], $\mathcal{F}(T)$ is contravariantly finite. By (4.1), $\mathcal{L}_A = \mathrm{ind}\mathcal{F}(T) \cup \mathcal{E}$. Therefore, $\mathrm{add}\mathcal{L}_A$ is contravariantly finite. □

4.3. Remarks and Examples. (a) The tilting module $T = E \oplus F$ constructed above is a generalization of the one constructed in [12] in the context of shod algebras, and our main result is a generalization of [12](3.1,3.6). For this reason, T is called the *canonical tilting module* (see also (5.4) below).

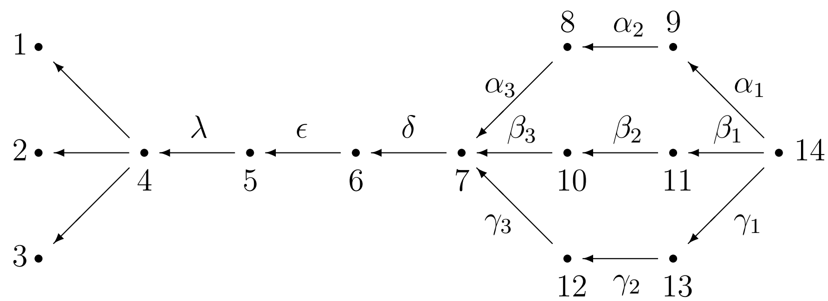
(b) Let A be a tubular algebra [23], then A is not left supported. Similarly, if A is the endomorphism algebra of a regular tilting module over a wild hereditary algebra [24], then A is not left supported. On the other hand, if A is tilted and has an injective in the connecting component, then it is left supported.

(c) The class of left supported algebras is not closed under tilting: let A be tubular, then there exists a sequence of tilts so that A tilts to a representation-finite algebra A'. Then A' is left supported, but A is not.

(d) As we shall see in (4.4) below, the classes of laura, weakly shod and shod algebras which are not quasi-tilted, are all left supported. The following is an example of a left supported algebra which belongs to neither of these classes: let A be the radical square zero algebra given by the quiver



(e) There exist left supported algebras which are not right supported, such as the algebra A given by the quiver



bound by the relations $\alpha_1\alpha_2\alpha_3+\beta_1\beta_2\beta_3+\gamma_1\gamma_2\gamma_3=0$, $\alpha_3\delta=0$, $\beta_3\delta=0$, $\gamma_3\delta=0$, $\delta\epsilon=0$, $\epsilon\lambda=0$. The same example shows that the class of left supported algebras is not closed under taking full (even convex!) subcategories (compare with [3]).

4.4. We recall that an artin algebra A is called a *laura algebra* [2, 27] provided that the class $\mathcal{L}_A\cup\mathcal{R}_A$ is cofinite in ind$A$. The following proposition is contained in the proof of the main theorem of [27]. Our proof is however different.

PROPOSITION. *Let A be a laura algebra which is not quasi-tilted. Then A is left and right supported.*

Proof. Let Γ denote the unique faithful, quasi-directed non-semiregular component of $\Gamma(\text{mod}A)$. Then Γ contains an injective. By (3.5), the number of τ_A-orbits of $\mathcal{L}_A\cap\Gamma$ equals the number of elements of $\mathcal{E}\cap\Gamma$. Since all injectives which lie in $\mathcal{L}_A$, and all indecomposable projectives which do not lie in $\mathcal{L}_A$, belong to Γ, then $\mathcal{E}\subseteq\Gamma$, so $\mathcal{E}\cap\Gamma=\mathcal{E}$. In view of (3.3), there remains to show that the number of τ_A-orbits of $\mathcal{L}_A\cap\Gamma$ equals the number of isomorphism classes of indecomposable projective modules lying in $\mathcal{L}_A$. Since, by (1.5), $\mathcal{L}_A\cap\Gamma$ is directed, and since A is a laura not quasi-tilted algebra, the number of τ_A-orbits of $\mathcal{L}_A\cap\Gamma$ equals the sum of the number of isomorphism classes of indecomposable projectives lying in $\mathcal{L}_A\cap\Gamma$ plus the number of τ_A-orbits of the left stable part Γ_l of Γ. By [2](4.3), the number of τ_A-orbits of Γ_l equals the number of isomorphism classes of the indecomposable projectives over the left end algebra ${}_\infty A$ of A. Thus, the number of τ_A-orbits of $\mathcal{L}_A \cap \Gamma$ equals the sum of the number of isomorphism classes of indecomposable projectives lying in $\mathcal{L}_A$ plus the number of isomorphism classes of indecomposable projectives over ${}_\infty A$, and this sum is indeed equal to the number of isomorphism classes of indecomposable projective modules in $\mathcal{L}_A$, as desired. □

4.5. We recall that an A-module T is called a *generalized cotilting module* provided $\mathrm{id}T_A < \infty$, $\mathrm{Ext}^i_A(T,T) = 0$ for all $i \geq 1$, and there exists an exact sequence

$$0 \longrightarrow T_m \longrightarrow \cdots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow \mathrm{D}(A) \longrightarrow 0$$

with $T_j \in \mathrm{add}T$ for all j.

Corollary. *Let A be a left supported algebra of finite global dimension, then $T = E \oplus F$ is a generalized cotilting module.*

Proof. Clearly, $\mathrm{gl.dim}A < \infty$ implies that $\mathrm{id}T_A < \infty$. Also, $\mathrm{Ext}^1_A(T,T) = 0$ and $\mathrm{pd}T_A \leq 1$ imply that $\mathrm{Ext}^i_A(T,T) = 0$ for all $i \geq 1$.
Since T is a tilting module, then $\mathrm{D}(A) \in \mathcal{T}(T)$. Applying [1](1.8) repeatedly yields an exact sequence

$$T_l \xrightarrow{f_l} \cdots \longrightarrow T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} \mathrm{D}(A) \longrightarrow 0$$

with $T_j \in \mathrm{add}T$ for all j. Set, for each j, $K_j = \mathrm{Ker}f_j$ and also $K_{-1} = \mathrm{D}(A)$. Thus $K_j \in \mathcal{T}(T)$ for all j. Let $M \in \mathcal{T}(T)$. Applying $\mathrm{Hom}_A(-,M)$ to the exact sequences $0 \longrightarrow K_j \longrightarrow T_j \longrightarrow K_{j-1} \longrightarrow 0$ yields, for each $i \geq 1$, an isomorphism $\mathrm{Ext}^{i+1}_A(K_{j-1},M) \cong \mathrm{Ext}^i_A(K_j,M)$. Let now $m = \mathrm{pd}\ \mathrm{D}(A)$. Then $\mathrm{Ext}^{m+1}_A(\mathrm{D}(A),K_m) = 0$. The above isomorphisms yield $\mathrm{Ext}^1_A(K_{m-1},K_m) = 0$. Thus, $K_{m-1} \in \mathrm{add}T$ and we have the desired sequence. □

5. The left support of a left supported algebra.

5.1. Before proving the main result of this section, we recall the following fact from [7]: let C be a quotient algebra of A, and $0 \longrightarrow \tau_A X \longrightarrow Y \longrightarrow X \longrightarrow 0$ be an almost split sequence in $\mathrm{mod}A$, with both X and $\tau_A X$ indecomposable C-modules. Then this sequence is also almost split in $\mathrm{mod}C$. In particular, $\tau_C X = \tau_A X$.

Theorem. *Let A be an artin algebra. Then A is left supported if and only if each connected component of its left support A_λ is a tilted algebra, and the restriction to this component of E is a slice module.*

Proof. We may, without loss of generality, assume that A_λ is connected. We set $B = A_\lambda$ for brevity. We first show the necessity, using the Liu-Skowroński criterion [21, 25]. This is done in the following steps:

(1) E is a faithful B-module: indeed, every indecomposable projective B-module is a projective A-module lying in $\mathcal{L}_A$, hence is cogenerated by E (see (4.2)).

(2) Let E_0, E_1 be two indecomposable summands of E. Thus $\mathrm{pd}_B E_1 \leq 1$ (because $\mathrm{pd}_A E_1 \leq 1$ and B is convex in A). Hence $D\mathrm{Hom}_B(E_0, \tau_B E_1) \cong \mathrm{Ext}^1_B(E_1, E_0) \cong \mathrm{Ext}^1_A(E_1, E_0) = 0$, because E_0 is Ext-injective in $\mathrm{add}\mathcal{L}_A$.

(3) $\mathcal{E}$ is convex in $\mathrm{mod}B$: let $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t$ be a path in $\mathrm{ind}B$, with $M_0, M_t \in \mathcal{E}$. Embedding this path in $\mathrm{ind}A$, $M_t \in \mathcal{E}$ implies that $M_i \in \mathcal{L}_A$, for all i. Moreover, (3.4) implies $M_i \in \mathcal{E}$ for all i (since $M_0 \in \mathcal{E}$).

(4) $\mathcal{E}$ is a subsection in $\mathrm{mod}B$: suppose that $E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_t$ is a path of irreducible morphisms in $\mathcal{E}$ and that there exists an i such that $E_{i-1} = \tau_B E_{i+1}$. By the discussion above, $\tau_B E_{i+1} = \tau_A E_{i+1}$. We then get a contradiction to (3.2).

(5) $\mathcal{E}$ intersects at least once each τ-orbit ot the Auslander-Reiten component of $\mathrm{mod}B$ where it lies:

(a) Suppose $M \in \mathcal{E}$, and $L \longrightarrow M$ is an irreducible morphism in $\mathrm{mod}B$. Then $L \in \mathcal{L}_A$. If L is injective, we are done. Assume it is not. If $\tau_A^{-1} L \in \mathcal{L}_A$, then, by (3.4), $M \longrightarrow \tau_A^{-1} L$ yields $\tau_A^{-1} L \in \mathcal{E}$. This implies that both L and $\tau_A^{-1} L$ are B-modules. Hence $\tau_B(\tau_A^{-1} L) = L$, so $\tau_B^{-1} L = \tau_A^{-1} L \in \mathcal{E}$. If, on the other hand, $\tau_A^{-1} L \notin \mathcal{L}_A$, then $L \in \mathcal{E}$.

(b) Suppose $M \in \mathcal{E}$, and $M \longrightarrow N$ is an irreducible morphism in $\mathrm{mod}B$. By (3.4), we may assume $N \notin \mathcal{L}_A$. This implies that N is not projective in $\mathrm{mod}B$ (if it were, it would be a projective A-module lying in $\mathcal{L}_A$). Let $X = \tau_B N$. Since $M \in \mathcal{E}$, then $X \in \mathcal{L}_A$. If X is injective in $\mathrm{mod}A$, then $X \in \mathcal{E}$. If not, assume that $\tau_B^{-1} X \in \mathcal{L}_A$, then, as in (a), we get $\tau_A^{-1} X = \tau_B^{-1} X = N$ and this contradicts the hypothesis that $N \notin \mathcal{L}_A$. Therefore, $\tau_A^{-1} X \notin \mathcal{L}_A$. But then $X \in \mathcal{E}$ and we are done.

(6) $\mathcal{E}$ intersects each τ_B-orbit at most once: suppose $M, \tau_B^{-t} M$ both belong to $\mathcal{E}$. Then we have a path $M \longrightarrow * \longrightarrow \tau_B^{-1} M \longrightarrow \cdots \longrightarrow \tau_B^{-t} M$ in $\mathrm{ind}B$, with $t \geq 1$, which induces a path

$$M \longrightarrow * \longrightarrow \tau_A^{-1} M \longrightarrow \tau_B^{-1} M \longrightarrow \cdots \longrightarrow \tau_B^{-t} M$$

in $\mathrm{ind}A$. Then the convexity of $\mathcal{E}$ in $\mathrm{mod}A$ yields a contradiction to the Ext-injectivity of M.

This concludes the proof of necessity.

We now prove the sufficiency. Assume the left support B of A to be tilted, having E as a slice module. We wish to show that add$\mathcal{L}_A =$ Cogen$E$. Since B is tilted, the class $\mathcal{C}$ of predecessors of $\mathcal{E}$ in mod$B$ equals the class of B-modules cogenerated by E. Let now $M \in \mathcal{L}_A$. Then M is a B-module. If M is a successor of $\mathcal{E}$, there exists a path $E_0 \rightsquigarrow M$, where $E_0 \in \mathcal{E}$, in ind$B$. Embedding this path in ind$A$ and using (3.4) yields $M \in \mathcal{E}$. This shows that add$\mathcal{L}_A \subseteq \mathcal{C}$. Since every A-module cogenerated by E is a B-module, we have $\mathcal{C} =$ Cogen$E$. Hence, add$\mathcal{L}_A \subset$ Cogen$E$. Since the reverse inclusion follows trivially from $E \in$ add$\mathcal{L}_A$, the theorem is proven. □

5.2. In the following, we generalize to left supported algebras parts (b) (c) of [20](2.4).

Corollary. *Let A be a left supported algebra, and $M \in add\mathcal{L}_A$ be such that $Ext^1_A(M, M) = 0$. Then $C = End\ M_A$ is a tilted algebra.*

Proof. Since $M \in$ add$\mathcal{L}_A$, then M is a B-module (we again set $B = A_\lambda$). Furthermore, $\mathrm{Ext}^1_B(M, M) = 0$ and $C = \mathrm{End}M_B$. By (5.1), there exist a hereditary algebra H and a tilting module U_H such that $B = \mathrm{End}U_H$. Then there exist a module $V \in \mathcal{T}(U)$ such that $M = \mathrm{Hom}_H(U, V)$. Furthermore, $\mathrm{Ext}^1_H(V, V) = 0$, so that V is a partial tilting module. By [16](III.6.5), $\mathrm{End}V_H$ is a tilted algebra. But, now, $C = \mathrm{End}V_H$. □

5.3. We recall from [4](4.3) that, if E is a partial tilting A-module, the torsion classes in mod$A$ having E as Ext-projective form a complete lattice under inclusion having as least element the class $\mathcal{T}_0(E) =$ Gen$E$ of the A-modules generated by E, and as largest element the class $\mathcal{T}_1(E) = \{M \in \mathrm{mod}A\colon\ \mathrm{Ext}^1_A(E, M) = 0\}$ and, furthermore, $\mathcal{T}_1(E)$ is the class generated by $E \oplus X$, where X is the Bongartz complement of E (see [1](1.7)). The first part of the following corollary shows how to recuperate the B-modules not in $\mathcal{F}(T)$ inside mod$A$, and the second gives another reason for the use of the name of canonical tilting module for T. We again write $B = A_\lambda$.

Corollary. *Let A be left supported, and $T = E \oplus F$. Then:*

(a) *The B-modules not in $\mathcal{F}(T)$ are precisely the modules in Gen$E$.*
(b) *F is the Bongartz complement of E.*

Proof. (a) This follows easily from the facts that B is tilted with E as slice module, and the modules generated by E in mod$B$ and mod$A$ are the same.

(b) Let X denote the Bongartz complement of E. Then, since $\mathrm{Ext}^1_A(E, M) = \mathrm{Ext}^1_A(E \oplus F, M)$, we have $\mathrm{Gen}(E \oplus F) = \mathrm{Gen}(E \oplus X)$. Since $F \in \mathrm{Gen}(E \oplus X)$ and is projective, then $F \in \mathrm{add}(E \oplus X)$. Hence $F \in \mathrm{add}X$. Looking at the number of isomorphism classes of indecomposable summands of E of X finishes the proof. □

5.4. Observe that Theorem (5.1) gives also information on the structure of the Auslander-Reiten components of a left supported algebra A.

Corollary. *Let A be a representation-infinite left supported connected algebra. Then the following statements are equivalent:*

(a) *$\mathcal{L}_A$ is infinite.*
(b) *There exists a component Γ of Γ(mod$A$) lying entirely in $\mathcal{L}_A$.*
(c) *Γ(mod$A$) has a postprojective component without injectives.*

Proof. (c) implies (b). Let Γ be a postprojective component of $\Gamma(\mathrm{mod}A)$ without injectives. Clearly, then, all modules in Γ have projective dimension at most one and, since such component is closed under predecessors, we infer that $\Gamma \subset \mathcal{L}_A$, which proves (b).

(b) implies (a). Since A is representation infinite and connected, then each component of $\Gamma(\mathrm{mod}A)$ is infinite. The result now follows easily.

(a) implies (c). By (5.1), A_λ is a product of connected tilted algebras. Since $\mathcal{L}_A$ is infinite, there exists connected summand B of A_λ such that $\mathcal{L}_B$ is infinite. Let Γ be a postprojective component of $\Gamma(\mathrm{mod}B)$ and assume that Γ has an injective module. Then Γ is a connecting component, it is the unique postprojective component and $\Gamma \cap \mathcal{L}_B$ is finite. Since $\mathcal{L}_B$ is infinite, there exists an indecomposable B-module $X \in \mathcal{L}_B$ not lying in Γ. But then, it is not difficult to get a non-zero morphism from a module in $\Gamma \setminus \mathcal{L}_B$ to X, a contradiction. So Γ is a postprojective component without injectives, as required. □

5.5. Corollary. *Let A be a representation-infinite left supported connected algebra which is not hereditary. If Γ is a component of Γ(mod$A$) lying entirely in $\mathcal{L}_A$, then Γ is one of the following: a postprojective component, a semiregular tube without injectives, a component of the form $\mathbf{Z}\ \mathbf{A}_\infty$ or a ray extension of $\mathbf{Z}\ \mathbf{A}_\infty$.*

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