THE FUNDAMENTAL GROUPS OF A TRIANGULAR ALGEBRA

I. Assem and J. A. de la Peña

ABSTRACT. A finite dimensional algebra A (over an algebraically closed field) is called triangular if its ordinary quiver has no oriented cycles. To each presentation (Q, I) of A is attached a fundamental group $\pi_1(Q, I)$, and A is called simply connected if $\pi_1(Q, I)$ is trivial for every presentation of A. In this paper, we provide tools for computations with the fundamental groups, as well as criteria for simple connectedness. We find relations between the fundamental groups of A and the Hochschild cohomology $H^1(A)$.

Introduction.

Let k be an algebraically closed field. By algebra, we mean a finite dimensional associative k-algebra with an identity. We are interested in studying the representation theory of A, that is, in describing the category mod A of finitely generated left A-modules. For this purpose, we may assume that A is basic and connected. An algebra A is called triangular if its ordinary quiver Q_A contains no oriented cycles. It is well-known that, if kQ_A denotes the path algebra of Q_A, there exists a surjective algebra morphism $\nu: kQ_A \to A$ whose kernel will be denoted by I_ν. For each pair (Q_A, I_ν), called a presentation of A, one can define the fundamental group $\pi_1(Q_A, I_\nu)$ (see [18, 14], or (1.3) below). A triangular algebra A is called simply connected if, for every presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is trivial [1]. Simply connected algebras have played an important role in representation theory. A triangular algebra is simply connected if and only if it admits no proper Galois coverings, (see, for instance [18, 20, 12, 14]). For any representation-finite algebra B, it is well-known that the indecomposable B-modules can be lifted to indecomposable modules over a simply connected algebra A (contained inside a certain Galois covering of the so-called standard form of B, see [6]). Thus, covering techniques reduce many problems of the study of representation-finite algebras to the study of representation-finite simply connected algebras, hence the importance of the latter. Representation-finite simply connected algebras are by now well-understood (see, for instance [3, 5, 6]). While little is known about the use of covering techniques in the representation-infinite case (see however [2, 8, 9, 13]), many classes of simply connected representation-infinite algebras have been described (see [1, 2, 10, 22, 26, 27]).

The purpose of this paper is to provide tools for computations with the fundamental groups of triangular algebras. Since triangular algebras can be constructed inductively as one-point extensions, it is natural to study the relation between the fundamental groups of an algebra B and a one-point extension A of B by a B-module. Then, for any presentation (Q_A, I_ν) of A and the corresponding induced presentation $(Q_{B_i}, I_\nu^{(i)})$, of a connected component B_i of B, we have a canonical group morphism $c_i\colon \pi_1(Q_{B_i}, I_\nu^{(i)}) \to \pi_1(Q_A, I_\nu)$. In our theorem (2.4), we give an exact sequence allowing to compute the kernel and the cokernel of ~~$(\mathrm{Hom}(c_i, Z))_i$~~ $\mathrm{Hom}(c_i, Z)$, where Z is any abelian group.

Some applications in representation theory of the Hochschild cohomology groups $H^i(A)$ (of the algebra A with coefficients in the bimodule ${}_AA_A$) have been studied in [14, 26]. In certain cases, it was shown that A is simply connected whenever $H^1(A) = 0$, which suggested the existence of a relation between $H^1(A)$ and the fundamental groups of A. In our theorem (3.2), we show the existence of an injective morphism of abelian group $s\colon \mathrm{Hom}(\pi_1(Q_A, I_\nu), k^+) \to H^1(A)$, for any presentation (Q_A, I_ν) of A, where k^+ denotes the underlying additive group of the field k.

As consequences of these homological considerations, we recover most of the known results on simply connected algebras, obtained in [14, 26], as well as some new criteria. We also get a constructive procedure allowing to obtain all strongly simply connected algebras (in the sense of [26]). As a consequence, we show that if A is a schurian algebra all of whose indecomposable projective modules are directing (these algebras were studied in [28]), then A is simply connected if and only if it is strongly simply connected, if and only if it satisfies the separation property of [3], see (5.4).

The paper is organized as follows. In section 1, we fix the notation and briefly recall the definitions and results that will be needed in the sequel. Section 2 will be devoted to the study of the relation between the fundamental groups of a triangular algebra B and a one-point extension A of B. In section 3, we study the relation between $H^1(A)$ and the fundamental groups of A. In section 4, we consider the one-point extensions of an algebra by a faithful square-free module. Finally, section 5 is devoted to strongly simply connected algebras.

1. Preliminaries.

1.1. Notation. Throughout this paper, k will denote a fixed algebraically closed field. By algebra is always meant an associative, finite dimensional k-algebra with an identity, which we shall moreover assume to be basic.

We recall that a *quiver* Q is defined by its set of points Q_0 and its set of arrows Q_1. A *relation* from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each $1 \leq i \leq m$, λ_i is a non-zero scalar and w_i a path of length at least two from x to y. A set of relations on Q generates an ideal I, said to be *admissible*, in the path algebra kQ of Q. The pair (Q, I) is then called a *bound quiver*. For an algebra A, we denote by Q_A the ordinary quiver of A. It is well-known that, for every basic algebra A, there exists a surjective k-algebra morphism $\nu\colon kQ_A \to A$ (induced by the choice of a set of representatives of basis vectors in the k-vector space $\operatorname{rad} A/\operatorname{rad}^2 A$) whose kernel I_ν is admissible. We thus have $A \cong kQ_A/I_\nu$. The bound quiver (Q_A, I_ν) is called a *presentation* of A. An algebra $A = kQ/I$ can equivalently be considered as a k-linear category, of which the object class is the set Q_0, and the set of morphisms from x to y is the k-vector space $kQ(x,y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x,y) = I \cap kQ(x,y)$. A full subcategory B of A is called *convex* if any path in A with source and target in B lies entirely in B. An algebra A is called *triangular* in Q_A has no oriented cycles. The present work is devoted exclusively to triangular algebras.

By an A-module is meant a finitely generated left A-module. We denote by $\operatorname{mod} A$ their category. It is well-known that, if $A = kQ/I$, then $\operatorname{mod} A$ is equivalent to the category of all bound representations of (Q, I), we may thus identify a module M with the corresponding representations $(M(x), M(\alpha))_{x \in Q_0, \alpha \in Q_1}$. For each $x \in Q_0$, we denote by S_x the corresponding simple A-module, and by P_x the projective cover of S_x. The algebra A is called *schurian* if $\dim_k \operatorname{Hom}_A(P_x, P_y) \leq 1$ for all $x, y \in (Q_A)_0$.

1.2. One-point extensions. Let A be an algebra, and x be a source in Q_A. The full convex subcategory B of A consisting of all objects except x has as quiver Q_B the quiver obtained from Q_A by deleting x and all arrows starting with x. Any presentation (Q_A, I_ν) yields (by restriction) an induced presentation $(Q_B, I_{\nu'})$ of B and, clearly, all presentations of B are obtained in this way. The A-modules $M = \operatorname{rad} P_x$ has a canonical B-module structure, and A is isomorphic to the one-point extension algebra

$$B[M] = \begin{bmatrix} k & M \\ 0 & B \end{bmatrix}$$

where the operations are the usual addition of matrices, and the multiplication induced by the B-module structure of M. Dually, if x is a sink in Q_A, then A is isomorphic to the one-point coextension $[M]B$ of a full convex subcategory B by a B-module M.

1.3. Fundamental groups. Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^{m} \lambda_i \in I(x, y)$ is called *minimal* if $m \geq 2$ and, for every non-empty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse. A walk in Q for x to y is a formal composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \ldots \alpha_k^{\varepsilon_k}$ (where $\alpha_i \in Q_1, \varepsilon_i = \pm 1$ for $1 \leq i \leq k$) starting at x and ending at y. We denote by e_x the trivial path at x.

Let $\sim$ be the smallest equivalence relation on the set of all walks in Q such that:

(a) If $\alpha \colon x \to y$ is an arrow, then $\alpha^{-1}\alpha \sim e_x$ and $\alpha\alpha^{-1} \sim e_y$.

(b) If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all $1 \leq i, j \leq m$.

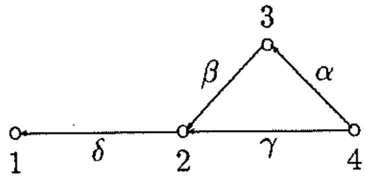
(c) If $u \sim v$, then $wuw' \sim wvw'$, whenever these compositions make sense.

We denote by $[u]$ the equivalence class of a walk u.

Let $x_0 \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x_0)$ of equivalence classes of all the closed walks starting and ending at x_0 has a group structure defined by the operation $[u][v] = [uv]$. Clearly the group $\pi_1(Q, I, x_0)$ does not depend on the choice of the base point x_0. We denote it simply by $\pi_1(Q, I)$ and calll it the *fundamental group* of (Q, I), see [18, 14].

1.4. Simple connectedness. Let A be a triangular algebra, and let (Q_A, I_ν) be a presentation of A. It follows from the above description that the fundamental group $\pi_1(Q_A, I_\nu)$ depends essentially on I_ν –thus it is not an invariant of A.

For example, let $A = kQ/I$, where Q is the quiver



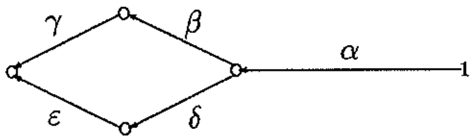
and I is generated by $\delta\beta\alpha - \delta\gamma$. Then $\pi_1(Q, I) = 0$. The choice of the presentation may be modified: indeed, $A \cong kQ/I'$, where I' is generated by $\delta\gamma$. Then $0 \neq [\gamma^{-1}\beta\alpha] \in \pi_1(Q, I')$ and $\pi_1(Q, I') \cong \mathbf{Z}$.

Definition [1]. A connected triangular algebra A is called *simply connected* if, for any presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is trivial.

Examples: (a) Any tree algebra (that is, an algebra whose quiver is a tree) is simply connected.

(b) Any algebra satisfying the separation condition of [3] is simply connected, see [26] (2.3) or (2.5) below. In particular, the good algebras of [21] and the completely separting algebras of [10] are simply connected.

(c) Let $A = kQ/I$, where Q is the quiver



and I is generated by $\gamma\beta\alpha - \varepsilon\delta\alpha$. Then $\pi_1(Q_A, I_\nu) = 0$ for every presentation of A, so that A is simply connected. On the other hand, the full convex subcategory B of A consisting of all objects except 1 is hereditary with fundamental group $\mathbf{Z}$, thus is not simply connected.

For more examples, we refer the reader to [1] (1.2) or [26].

1.5. Free connectedness. For our purposes, we shall need the following natural generalization of the notion of a simply connected algebra.

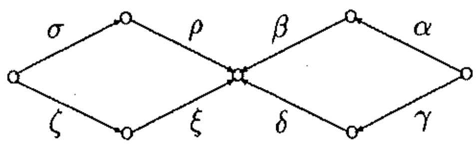
Definition. A connected triangular algebra A is called *freely connected* if, for any presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is free.

Examples: (a) Clearly, any simply connected algebra is freely connected.

(b) Any representation-finite algebra is freely connected [18] (4.3) and (4.4).

(c) Any monomial algebra (that is, an algebra A having a presentation (Q_A, I_ν), where I_ν is generated by finitely many paths in Q_A) is freely connected. This indeed follows at once from the definition of the fundamental group.

(d) Let $A = kQ/I$, where Q is the quiver



and I is generated by $\beta\alpha - \delta\gamma, \rho\sigma, \xi\zeta$. Clearly, A is freely (but not simply) connected. It is representation-infinite, and not a monomial algebra.

It is important to observe that there exist algebras which are not freely connected: in fact, for any finitely presented group G, there exists a triangular algebra A and a presentation (Q_A, I_ν) such that $\pi_1(Q_A, I_\nu) \cong G$, see [11] (7).

2. The fundamental groups of a one-point extension.

2.1. Throughout this section, we let A denote a triangular algebra, and x a source in Q_A. Let B denote the full convex subcategory of A consisting of all objects except x. Our first aim is to compute a direct sum decomposition of $M = \operatorname{rad} P_x$.

Let (Q_A, I_ν) be a presentation of A. We let $\approx$ denote the smallest equivalence relation on the set $x^\to$ of all arrows starting at x such that $\alpha \approx \beta$ (for $\alpha, \beta \in x^\to$) whenever there exist $y \in (Q_A)_0$ and a minimal relation $\sum_{i=1}^{m} \lambda_i w_i \in I_\nu(x,y)$ with $w_1 = v_1\alpha$ and $w_2 = v_2\beta$. We denote by $[\alpha]_\nu$ the equivalence class of $\alpha \in x^\to$.

Let $\alpha \in x^\to$, we define an A-module $M_{[\alpha]_\nu}$ as follows. Let $A_{[\alpha]_\nu}$ be the quotient algebra of A obtained by deleting all arrows $\beta \in x^\to$ such that $\beta \notin [\alpha]_\nu$ (thus, $A_{[\alpha]_\nu}$ is a specialization of A in the sense of [23]). Let $P_{x[\alpha]_\nu}$ be the indecomposable projective $A_{[\alpha]_\nu}$-module corresponding to the point x. We then define $M_{[\alpha]_\nu} = \operatorname{rad} P_{x[\alpha]_\nu}$. Since $A_{[\alpha]_\nu}$ is a quotient of A, the $A_{[\alpha]_\nu}$-module $M_{[\alpha]_\nu}$ can be canonically viewed as an A-module.

Proposition. *Let A be a triangular algebra, x be a source in Q_A and $M = \operatorname{rad} P_x$.*
a) *For any presentation (Q_A, I_ν) of A, we have $M = \bigoplus_{[\alpha]_\nu} M_{[\alpha]_\nu}$.*
b) *There exists a presentation (Q_A, I_ν) of A such that, for each $\alpha \in x^\to$, the module $M_{[\alpha]_\nu}$ is indecomposable. In particular, $M = \bigoplus_{[\alpha]_\nu} M_{[\alpha]_\nu}$ is an indecomposable decomposition.*

Proof: (a): For brevity, we write $[\alpha] = [\alpha]_\nu$. It follows from the well-known description of the indecomposable projectives over a bound quiver algebra that each $M_{[\alpha]}$ is a submodule of $M = \operatorname{rad} P_x$ and that $M = \sum_{[\alpha]} M_{[\alpha]}$. We must show that this sum is direct. Assume there exist $y \in (Q_A)_0 \setminus \{x\}$ and $0 \neq v \in M_{[\alpha]}(y) \cap \left(\sum_{[\beta]\neq[\alpha]} M_{[\beta]} \right)(y)$. Then there exists a linear combination $\rho_{[\alpha]} = \sum_{\varepsilon\in[\alpha]} \lambda_\varepsilon^{[\alpha]} w_\varepsilon \varepsilon$, where $w_\varepsilon\varepsilon$ is a path from

x to y and $\lambda_\varepsilon^{[\alpha]} \in k$, such that v, as an element of $M_{[\alpha]}(y)$, is induced by $\rho_{[\alpha]}$. For each $[\beta] \neq [\alpha]$, there also exists $\rho_{[\beta]} = \sum_{\sigma\in[\beta]} \lambda_\sigma^{[\beta]} w_\sigma \sigma$, where $w_\sigma\sigma$ is a path from x to y and $\lambda_\sigma^{[\beta]} \in k$, such that $-v$ as an element of $\sum_{[\beta]\neq[\alpha]} M_{[\beta]}(y)$, is induced by $\sum_{[\beta]\neq[\alpha]} a_{[\beta]}\rho_{[\beta]} = \sum_{[\beta]\neq[\alpha]} \sum_{\sigma\in[\beta]} a_{[\beta]}\lambda_\sigma^{[\beta]} w_\sigma\sigma$, where $a_{[\beta]} \in k$. Moreover, we get

$$\rho_{[\alpha]} + \sum_{[\beta]\neq[\alpha]} a_{[\beta]}\rho_{[\beta]} \in I_\nu(x,y).$$

As an $A_{[\alpha]}$-module, $M_{[\alpha]}$, satisfies

$$0 = P_{x[\alpha]}(\rho_{[\alpha]})(e_x) = \sum_{\varepsilon\in[\alpha]} \lambda_\varepsilon^{[\alpha]} M_{[\alpha]}(w_\varepsilon)(\varepsilon) = v$$

a contradiction. Therefore, $M = \bigoplus_{[\alpha]} M_{[\alpha]}$.

(b): Let $M = M_1 \oplus \cdots \oplus M_s$ be an indecomposable decompositoin of M. We have $\bigoplus_{\alpha\in x^\to} k\alpha = \operatorname{top} M = \bigoplus_{i=1}^{s} \operatorname{top} M_i$. We can choose representatives $\{\mu(\alpha) \mid \alpha \in x^\to\}$ of the arrows in $x^\to$ (as elements in $\operatorname{rad} A/\operatorname{rad}^2 A$) in such a way that, for each $1 \le i \le s$, there is a subset $x_i^\to$ of $x^\to$ such that $\operatorname{top} M_i = \bigoplus_{\alpha\in x_i^\to} k\mu(\alpha)$. This selection may be completed to a surjetion $\mu: kQ_A \to A$ and therefore to a presentation (Q_A, I_μ). For this presentation, $M = \bigoplus_{[\alpha]_\mu} M_{[\alpha]_\mu}$. We shall show that, for each $\alpha \in x^\to$, the module $M_{[\alpha]_\mu}$ is indecomposable.

Assume that $M_{[\alpha]_\mu}$ decomposes. There exist non-empty disjoint sets $J_1, J_2 \subseteq \{1,2,\ldots,s\}$ such that $N_1 = \bigoplus_{i\in J_1} M_i$, $N_2 = \bigoplus_{j\in J_2} M_j$ and $\operatorname{top} M_{[\alpha]_\mu} = \bigoplus_{\beta\in[\alpha]_\mu} k\mu(\beta) = \operatorname{top} N_1 \oplus \operatorname{top} N_2$. Since $M_{[\alpha]_\mu} = \operatorname{rad} P_{x[\alpha]_\mu}$, then $M_{[\alpha]_\mu} = N_1 \oplus N_2$. On the other hand, for any $i \in J_1$ and $j \in J_2$, we have $\beta_i \approx \beta_j$. Therefore there exists a minimal relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \beta_i \in I_\mu(x,y)$ such that $1,\ldots,t \in J_1$ and $t+1,\ldots,m \in J_2$. Since ρ is a minimal relation, we have

$$v_1 = \sum_{i=1}^{t} \lambda_i N_1(w_i)(\beta_i) = \sum_{i=1}^{t} \lambda_i M_{[\alpha]_\mu}(w_i)(\beta_i) = \sum_{i=1}^{t} \lambda_i \mu(w_i\beta_i) \neq 0$$

and, similarly, $v_2 = \sum_{j=t+1}^{m} \lambda_j N_2(w_j)(\beta_j) \neq 0$. But then, in $(N_1 \oplus N_2)(y)$, we obtain the contradiction

$$0 \neq v_1 + v_2 = \sum_{i=1}^{m} \lambda_i M_{[\alpha]_\mu}(w_i)(\beta_i) = \mu(\rho) = 0.$$

Therefore, $M_{[\alpha]_\mu}$ is indecomposable. □

In the example of a non-simply connected algebra A in (1.4) above, we have two presentations (Q, I) and (Q, I') of A. For the first, we have $M_{[\alpha]} = M_{[\gamma]} = \operatorname{rad} P_4$. For the second, $M_{[\alpha]} = P_3$ while $M_{[\gamma]} = S_2$.

2.2. Let A be a connected algebra, and x be a source in Q_A. Let $c(x)$ denote the number of connected components of $Q_A \setminus \{x\}$. Given a presentation (Q_A, I_ν) of A, denote by $t(\nu)$ the number of equivalence classes $[\alpha]_\nu$ in $x^{\rightarrow}$. Finally, we denote by $t(x)$ the number of indecomposable direct summands in the decomposition of $\operatorname{rad} P_x$. With this notation, (2.1) can be restated as follows:

Corollary. *Let A be a triangular algebra, and x be a source in Q_A*
a) *For any presentation (Q_A, I_ν) of A, we have $c(x) \leq t(\nu) \leq t(x)$.*
b) *There exists a presentation (Q_A, I_μ) of A such that $t(\mu) = t(x)$.* □

2.3. Let (Q_A, I_ν) be a presentation of A, and recall the following construction from [20] (1.2).

Let Z be an arbitrary abelian group, denoted additively.

Let $C^0(A, \nu, Z)$ be the set of all Z-valued functions on $(Q_A)_0$.

Let $Z^1(A, \nu, Z)$ be the set of all Z-valued functions f on $(Q_A)_1$ such that $\sum_{i=1}^{s} f(\alpha_i) = \sum_{j=1}^{t} f(\beta_j)$ whenever there exists a minimal relation $\rho = \sum_{i=1}^{m} \lambda_i \mu$ such that $w_1 = \alpha_1\alpha_2 \ldots \alpha_s$ and $w_2 = \beta_1\beta_2 \ldots \beta_t$.

We have an exact sequence of abelian groups

$$0 \longrightarrow Z \xrightarrow{d^0} C^0(A, \nu, Z) \xrightarrow{d^1} Z^1(A, \nu, Z) \xrightarrow{p} \operatorname{Hom}(\pi_1(Q_A, I_\nu), Z) \longrightarrow 0$$

where d^0 associates to the integer m the constant function $f: (Q_A)_0 \to Z$ with value m; d^1 associates to a function $f: (Q_A)_0 \to Z$ the function $g: (Q_A)_1 \to Z$ which maps $\alpha: y \to z$ to $g(\alpha) = f(y) - f(z)$ and finally p maps a function g to the morphism of groups $h: \pi_1(Q_A, I_\nu) \to Z$ defined by $h([\alpha_1^{\varepsilon_1} \ldots \alpha_t^{\varepsilon_t}]) = \sum_{i=1}^{t} \varepsilon_i g(\alpha_i)$.

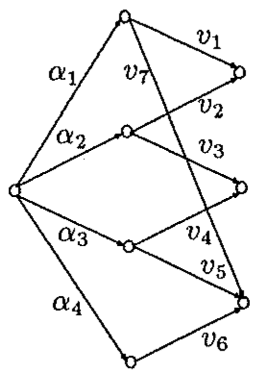
This construction originates from [19] (see also [6]) where it can be read as construction of a certain cohomology group $H^1(\pi_1(Q_A, I_\nu), Z)$ with coefficients in Z, in the case where A is triangular and schurian. We shall not pursue this approach.

2.4. Assume that A is connected and let B denote the full convex subcategory A consisting of all objects except the source x. Then $A = B[M]$, where $M = \operatorname{rad} P_x$. Let (Q_A, I_ν) be a presentation of A, and $(Q_B, I_{\nu'})$ be the induced presentation of B. Let $Q^{(1)}, \ldots, Q^{(m)}$ be the connected components of the quiver Q_B and $I_\nu^{(j)}$ be the restriction of $I_{\nu'}$ to $Q^{(j)}$. For each $1 \leq j \leq m$, the embedding of $Q^{(j)}$ inside Q_A induces a canonical group morphism $c_j: \pi_1(Q^{(j)}, I_\nu^{(j)}) \to \pi_1(Q_A, I_\nu)$. Our present aim is to compute the kernel and the cokernel of the induced morphism

$$c^* = (c_j^*)_j = (\operatorname{Hom}(c_j, Z))_j : \operatorname{Hom}(\pi_1(Q_A, I_\nu), Z) \to \prod_{j=1}^{m} \operatorname{Hom}(\pi_1(Q^{(j)}, I_\nu^{(j)}), Z),$$

for any abelian group Z.

For this purpose, we need some additional notation. Let x_0 be the base point of the fundamental group $\pi_1(Q_A, I_\nu)$ and for each $1 \leq j \leq m$, let x_j be the base point of the group $\pi_1(Q^{(j)}, I_\nu^{(j)})$. For each point $y \in (Q^{(j)})_0$, we fix a walk w_y in Q_B from the base point x_j to y, agreeing to take $w_{x_j} = e_{x_j}$, the trivial walk at x_j. Let $\beta_1, \ldots, \beta_{t(\nu)}$ be representatives of the classes $[\alpha]_\nu$ of the arrows $\alpha \in x^{\to}$. For each $1 \leq i \leq t(\nu)$, let $L(i)$ denote the set, and $\ell(i)$ denote the number, of those tuples of paths $(v_1, v_2, \ldots, v_{2s-1}, v_{2s})$ such that there are minimal relations $\lambda_1^{(1)} v_1 \alpha_1 + \lambda_2^{(1)} v_2 \alpha_2 + \sum_{j \geq 3} \lambda_j^{(1)} u_j^{(1)} \in I_\nu(x, y^{(1)}), \ldots, \lambda_1^{(s)} v_{2s-1} \alpha_s + \lambda_2^{(s)} v_{2s} \alpha_{s+1} + \sum_{j \geq 3} \lambda_j^{(s)} u_j^{(s)} \in I_\nu(x, y^{(s)})$ with $\alpha_1, \ldots, \alpha_s, \alpha_{s+1} = \alpha_1$ pairwise distinct arrows in $[\beta_i]$. In the figure below we illustrate the case $s = 3$.



Finally, let $L_\nu = \prod_{i=1}^{t(\nu)} Z^{\ell(i)}$.

Theorem. *Let A be connected and Z be any abelian group. There exists an exact sequence of abelian groups*

$$0 \longrightarrow Z^{t(\nu)-c(x)} \longrightarrow \mathrm{Hom}(\pi_1(Q_A, I_\nu), Z) \xrightarrow{c^*} \prod_{j=1}^{m} \mathrm{Hom}\,(\pi_1(Q^{(j)}, I_\nu^{(j)}), Z) \longrightarrow L_\nu.$$

Proof: It follows from (2.3) that there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & Z & \xrightarrow{d^0} & C^0(A,\nu,Z) & \xrightarrow{d^1} & Z^1(A,\nu,Z) & \xrightarrow{p} & \mathrm{Hom}\,(\pi_1(Q_A, I_\nu), Z) & \to 0 \\ & \Delta\downarrow & & h^0\downarrow & & h^1\downarrow & & c^*\downarrow & \\ 0 \to & Z^m & \xrightarrow{d'^0} & C^0(B,\nu',Z) & \xrightarrow{d'^0} & Z^1(B,\nu',Z) & \xrightarrow{p'} & \prod_{j=1}^{m} \mathrm{Hom}\,(\pi_1(Q^{(j)}, I_\nu^{(j)}), Z) & \to 0 \end{array}$$

where h^0 and h^1 denote the restriction morphisms, and Δ the diagonal inclusion. Applying the snake lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{d^0} & C^0(A,\nu,Z) & \longrightarrow & \mathrm{Coker}\, d^0 & \longrightarrow & 0 \\ & & \Delta\downarrow & & h^0\downarrow & & \bar{h}^0\downarrow & & \\ 0 & \longrightarrow & Z^m & \xrightarrow{d'^0} & C^0(B,\nu',Z) & \longrightarrow & \mathrm{Coker}\, d'^0 & \longrightarrow & 0 \end{array}$$

(where $\bar{h}^0$ denotes the induced morphism) yields that $\bar{h}^0$ is surjective and that there exists an exact sequence

$$0 \to Z \to \mathrm{Ker}\,\bar{h}^0 \to Z^{m-1} \to 0.$$

Since h^0, d^0, d'^0 split as morphisms of abelian groups, so does $\bar{h}^0$. Hence the above sequence splits so that $\mathrm{Ker}\,\bar{h}^0 \cong Z^m$. Applying now the snake lemma to the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Coker}\, d^0 & \longrightarrow & Z^1(A,\nu,Z) & \xrightarrow{p} & \mathrm{Hom}\,(\pi_1(Q_A, I_\nu), Z) & \longrightarrow & 0 \\ & & \bar{h}^0\downarrow & & h^1\downarrow & & c^*\downarrow & & \\ 0 & \longrightarrow & \mathrm{Coker}\, d'^0 & \longrightarrow & Z^1(B,\nu',Z) & \xrightarrow{p'} & \prod_{j=1}^{m} \mathrm{Hom}\,(\pi_1(Q^{(j)}, I_\nu^{(j)}), Z) & \longrightarrow & 0 \end{array}$$

yields an isomorphism $\mathrm{Coker}\, h^1 \cong \mathrm{Coker}\, c^*$ and an exact sequence

$$0 \longrightarrow Z^m \xrightarrow{\bar{d}} \mathrm{Ker}\, h^1 \longrightarrow \mathrm{Ker}\, c^* \longrightarrow 0$$

Now we describe $\mathrm{Ker}\, h^1$. We consider the fixed representatives β_i, of the classes $[\alpha]_\nu$, with $\alpha \in x^{\rightarrow}$. For any function $g \in Z^{t(\nu)}$, we define $\hat{g}\colon (Q_A)_1 \to Z$ by

$$\hat{g}(\beta) = \begin{cases} 0 & \text{if } \beta \notin x^{\rightarrow}, \\ g(i) & \text{if } \beta \in [\beta_i]. \end{cases}$$

Thus $\hat{g} \in Z^1(A,\nu,Z)$ and $h^1(\hat{g}) = 0$. Hence $\hat{g} \in \operatorname{Ker} h^1$. Moreover, any $f \in \operatorname{Ker} h^1$ is of the form $f = \hat{g}$ for some $g \in Z^{t(\nu)}$. The map $g \mapsto \hat{g}$ is injective by definition. Thus $\operatorname{Ker} h^1 = Z^{t(\nu)}$.

The morphism $\bar{d}: Z^m \to \operatorname{Ker} h^1$ is defined as follows. For each $\beta_i: x \to y_i$, the point y_i belongs to the component $Q^{(s(i))}$ of Q_B. Then, for $g \in Z^m$, we have that $\bar{d}(g)(i) = g(s(i))$. Clearly, $\bar{d}$ splits. Hence $\operatorname{Ker} c^* = Z^{t(\nu)-m}$. Moreover, by definition, $m = c(x)$.

To complete the proof, since $\operatorname{Coker} h^1 \cong \operatorname{Coker} c^*$, it suffices to construct an exact sequence

$$Z^1(A,\nu,Z) \xrightarrow{h^1} Z^1(B,\nu',Z) \xrightarrow{m} L_\nu.$$

Let $\beta_1,\ldots,\beta_{t(\nu)}$ be as before, and $1 \le i \le t(\nu)$. Let $g \in Z^1(B,\nu',Z)$ and $(v_1,v_2,\ldots,v_{2s-1},v_{2s}) \in L(i)$ be such that there exist minimal relations $\lambda_1^{(1)}v_1\alpha_1 + \lambda_2^{(1)}v_2\alpha_2 + \sum_{j\ge3}\lambda_j^{(1)}u_j^{(1)} \in I_\nu(x,y^{(1)}), \ldots, \lambda_1^{(s)}v_{2s-1}\alpha_s + \lambda_2^{(s)}v_{2s}\alpha_{s+1} + \sum_{j\ge3}\lambda_j^{(s)}u_j^{(s)} \in I_\nu(x,y^{(s)})$ with $\alpha_1,\ldots,\alpha_s \in [\beta_i]$. We set

$$m(g)(v_1,v_2,\ldots,v_{2s-1},v_{2s}) = \sum_{i=1}^{s}(-1)^i(g(v_{2i}) - g(v_{2i-1})) \in Z.$$

Clearly m is a well-defined morphism of abelian groups. Moreover if $f \in Z^1(A,\nu,Z)$ and $(v_1,v_2,\ldots,v_{2s-1},v_{2s}) \in L(i)$ as above, then

$$mh^1(f)(v_1,v_2,\ldots,v_{2s-1},v_{2s}) = \sum_{i=1}^{s}(-1)^i(f(v_{2i}\alpha_{i+1}) - f(v_{2i-1}\alpha_i)) = 0$$

by definition of $Z^1(A,\nu,Z)$. Thus $mh^1 = 0$.

Let $g \in Z^1(A,\nu,Z)$ be such that $m(g) = 0$. We extend g to a function $f: (Q_A)_1 \to Z$ by setting $f(\beta_i) = 0$ for every $1 \le i \le t(\nu)$; for an arrow $\alpha \in [\beta_i]$ such that there are minimal relations $\lambda_1^{(1)}v_1\alpha_1 + \lambda_2^{(1)}v_2\alpha_2 + \sum_{j\ge3}\lambda_j^{(1)}u_j^{(1)} \in I_\nu(x,y^{(1)}), \ldots, \lambda_1^{(s)}v_{2s-1}\alpha_s + \lambda_2^{(s)}v_{2s}\alpha_{s+1} + \sum_{j\ge3}\lambda_j^{(s)}u_j^{(s)} \in I_\nu(x,y^{(s)})$ such that $\alpha_1 = \beta_i$, $\alpha_{s+1} = \alpha$ and $f(\alpha_1),\ldots,f(\alpha_s)$ have already been defined, then we set $f(\alpha) = g(v_{2s-1}) - g(v_{2s}) + f(\alpha_s)$. Since $m(g) = 0$, this yields a well-defined function $f \in Z^1(A,\nu,Z)$ with $h^1(f) = g$. □

2.5. Following [3], we say that a source x in Q_A is a *separating point* if $t(x) = c(x)$. In general, a point y in Q_A (not necessarily a source) is called a separating point if y is separating as a source in the full convex subcategory of A with objects all points of Q_A except the points z such that there exists a non-trivial path from z to y in Q_A. The algebra A is called *separated* (or satisfaying the (S)-condition) if all points in Q_A are separating.

There are close relations between the separation property and simple connectedness. For instance, a representation-finite algebra is separated if and only if it is simply connected (this is essentially shown in [3], the definition of simple connectedness given there coincides with ours because of [5, 18]). On the other hand, any separated algebra is simply connected [26] (2.3). Actually we have:

Lemma. *Let $A = B[M]$, where $M = \operatorname{rad} P_x$. If B is simply connected and x is separating, then A is simply connected.*

Proof: This indeed follows from the proof of [26] (2.3). □

2.6. Corollary. *Let A be a triangular algebra. Assume there exists a non-trivial abelian group Z such that, for every presentation (Q_A, I_ν) of A, we have $\operatorname{Hom}(\pi_1(Q_A, I_\nu), Z) = 0$. Then all sources in Q_A are separating. In particular, if A is simply connected, all sources in Q_A are separating.*

Proof: Let x be a source in Q_A. By (2.2), there exists a presentation (Q_A, I_μ) of A such that $t(\mu) = t(x)$. By (2.4), there exists an injective group morphism $Z^{t(\mu)-c(x)} \to \operatorname{Hom}(\pi_1(Q_A, I_\mu), Z)$. Since the latter vanishes, we have $t(\mu) = c(x)$. Consequently, $t(x) = c(x)$ and x is separating. □

2.7. Let $P_1(Q_A, I_\nu)$ denote the abelianisation of the group $\pi_1(Q_A, I_\nu)$. For any abelian group Z, we have a functorial isomorphism

$$\operatorname{Hom}(\pi_1(Q_A, I_\nu), Z) \cong \operatorname{Hom}_{\mathbb{Z}}(P_1(Q_A, I_\nu), Z).$$

Moreover, let $A = B[M]$ (where $M = \operatorname{rad} P_x$) be a one-point extension. Consider $(Q_B, I_{\nu'})$ the induced presentation of B and $Q^{(1)}, \ldots, Q^{(m)}$ be the connected components of Q_B. Then there is a morphism of abelian groups

$$\bar{c}\colon \prod_{j=1}^{m} P_1(Q^{(j)}, I_\nu^{(j)}) \to P_1(Q_A, I_\nu)$$

induced from the morphisms c_j. One can easily compute the cokernel of $\bar{c}$. Indeed, it follows from (2.4) that we have an exact sequence of abelian groups

$$0 \to Z^{t(\nu)-c(x)} \to \mathrm{Hom}_{\mathbf{Z}}(P_1,(Q_A,I_\nu),Z) \xrightarrow{\mathrm{Hom}_{\mathbf{Z}}(\bar{c},Z)} \prod_{j=1}^{m} \mathrm{Hom}_{\mathbf{Z}}(P_1(Q^{(j)},I_\nu^{(j)}),Z)$$

for any abelian group Z. Consequently, there exists an exact sequence of abelian groups

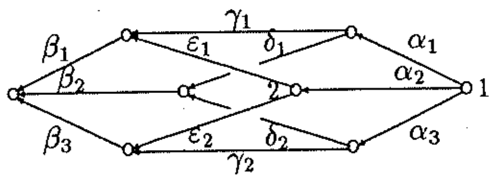
$$\prod_{j=1}^{m} P_1(Q^{(j)},I_\nu^{(j)}) \xrightarrow{\bar{c}} P_1(Q_A,I_\nu) \longrightarrow \mathbf{Z}^{t(\nu)-c(x)} \longrightarrow 0$$

that is, $\mathrm{Coker}\,\bar{c} \cong \mathbf{Z}^{t(\nu)-c(x)}$.

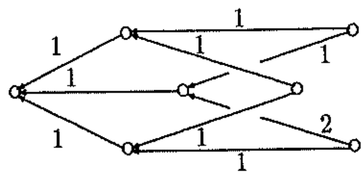
Corollary. *The following are equivalent for a freely connected algebra A.*
a) *A is simply connected.*
b) *For any presentation (Q_A,I_ν) of A, we have $P_1(Q_A,I_\nu)=0$.*
c) *There is a non-trivial abelian group Z such that, for any presentation (Q_A,I_ν) of A, we have* $\mathrm{Hom}\,(\pi_1(Q_A,I_\nu),Z)=0$.

Proof: Let (Q_A,I_ν) be a presentation of A. Assume that $\pi_1(Q_A,I_\nu)$ is the free group in n generators. Then $P_1(Q_A,I_\nu) \cong \mathbf{Z}^m$ and $\mathrm{Hom}\,(\pi_1(Q_A,I_\nu),Z) \cong \mathrm{Hom}_{\mathbf{Z}}(P_1(Q_A,I_\nu),Z) \cong Z^m$ for any abelian group Z. Consequently, $\pi_1(Q_A,I_\nu)=0$ if and only if $m=0$, if and only if $P_1(Q_A,I_\nu)=0$, if and only if $\mathrm{Hom}\,(\pi_1(Q_A,I_\nu),Z)=0$ for some non-trivial abelian group Z. □

2.8. *Example:* Consider the algebra $A=kQ/I_\nu$, where Q is the quiver



and I_ν is generated by $\gamma_1\alpha_1-\varepsilon_1\alpha_2$, $\gamma_2\alpha_3-\varepsilon_2\alpha_2$, $\beta_1\gamma_1-\beta_2\delta_1$, $\beta_3\gamma_2-\beta_2\delta_2$. Take $Z=\mathbf{Z}$. Consider the source 1 and the algebra B such that $A=B[M]$, with $M=\mathrm{rad}\,P_1$. The point 1 is separating and $t(\nu)=1$. Clearly, $\pi_1(Q_B,I_{\nu'})\cong\mathbf{Z}$. Moreover, it is easy to see that $L_\nu=0$ in this case. Hence $\pi_1(Q_A,I_\nu)\cong\mathbf{Z}$. Let now $A'=kQ/I'_\mu$ where I'_μ is generated by I_ν and the additional relation $\delta_1\alpha_1-\delta_2\alpha_2$. We still write $A'=B[N]$ with $N=\mathrm{rad}_{A'}P_1$. As before, the point 1 is separating a $t(\mu)=1$. Also $\pi_1(Q_B,I'_{\mu'})\cong\mathbf{Z}$. We have that $(\gamma_1, \varepsilon_1, \varepsilon_2, \gamma_2, \delta_2, \delta_1)$ is an element in L_μ. Thus $L_\mu = \mathbf{Z}$. Take $g \in Z^1(B, \mu', \mathbf{Z})$, as follows



Then $m(g) \neq 0$. Thus $m\colon \mathbf{Z} \to \mathbf{Z}$ is injective and $\pi_1(Q_{A'}, I'_\mu) = 0$, that is, A' is simply connected.

3. The fundamental groups and Hochschild cohomology.

3.1. Given an algebra A and the bimodule ${}_AA_A$, the *Hochschild complex* $C = (C^i, d^i)_{i \in \mathbf{Z}}$ is defined as follows: $C^i = 0$, $d^i = 0$ for $i < 0$, $C^0 = {}_AA_A$, $C^i = \mathrm{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product $A \otimes_k \cdots \otimes_k A$, $d^0\colon A \to \mathrm{Hom}_k(A, A)$ with $(d^0x)(a) = ax - xa$ (for $a, x \in A$) and $d^i\colon C^i \to C^{i+1}$ with

$$\begin{aligned}(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) \\ &+ \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ &+ (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}\end{aligned}$$

for $f \in C^i$ and $a_1, \ldots, a_{i+1} \in A$. Then $H^i(A) = H^i(C^0)$ is the i^{th} *Hochschild cohomology group* of A (with coefficients in the bimodule ${}_AA_A$), see [7].

The following theorem of Happel [15] (5.3) is useful for the calculation of the Hochschild cohomology groups for triangular algebras:

Theorem. *Let $A = B[M]$. There exists a long exact sequence*

$$0 \to H^0(A) \to H^0(B) \to \mathrm{Hom}_B(M, M)/k \to H^1(A) \xrightarrow{e} H^1(B) \to \mathrm{Ext}^1_B(M, M) \to \cdots$$

$$\cdots \to \mathrm{Ext}^i_B(M, M) \to H^{i+1}(A) \to H^{i+1}(B) \to \mathrm{Ext}^{i+1}_B(M, M) \to \cdots \qquad \square$$

3.2. It follows from [15, 26] that there is a close relation between the first Hochschild cohomology group $H^1(A)$ and the fundamental groups $\pi_1(Q_A, I_\nu)$ of A. The following result makes this relation explicit:

Theorem. *Let A be a triangular algebra, and (Q_A, I_ν) be a presentation of A. There exists an injective group morphism*

$$s\colon \mathrm{Hom}\,(\pi_1(Q_A, I_\nu), k^+) \to H^1(A)$$

(where k^+ denotes the underlying additive group of the field k).

Proof: Let us fix a base point $x_0 \in (Q_A)_0$ for the fundamental group and walks w_y from x_0 to y for each $y \in (Q_A)_0$, where $w_{x_0} = e_{x_0}$. We define $s^1\colon \mathrm{Hom}\,(\pi_1(Q_A, I_\nu), k^+) \to \mathrm{Hom}_k(A, A)$ as follows: let $f \in \mathrm{Hom}\,(\pi_1(Q_A, I_\nu), k^+)$ for $a \in A$, write $a = \sum_{i=1}^{s} \lambda_i \bar{v}_i$ where v_i is a path in Q_A from x_i to y_i, with residual class $\bar{v}_i$ in $A \cong kQ_A/I_\nu$ and $\lambda_i \in k$ $(1 \le i \le s)$. We then define

$$s^1(f)(a) = \sum_{i=1}^{s} \lambda_i f([w_{y_i}^{-1} v_i w_{x_i}])\bar{v}_i \in A.$$

To check that this indeed defines a function, let $\rho = \sum_{i=1}^{m} \lambda_i v_i \in I_\nu(x, y)$ be minimal relation, then $[w_y^{-1} v_i w_x] = [w_y^{-1} v_j w_x]$ in $\pi_1(Q_A, I_\nu)$ for any $1 \le i \le m$. Hence $f([w_y^{-1} v_i w_x]) = f([w_y^{-1} v_j w_x])$ for any $1 \le i, j \le m$, and $s^1(f) = 0$. Clearly, $s^1(f) \in \mathrm{Hom}_k(A, A)$.

We now verify that $s^1(f) \in \mathrm{Ker}\, d^1$. Let $a, b \in A$ and write $a = \sum_{i=1}^{s} \lambda_i v_i$, $b = \sum_{j=1}^{t} \mu_j \bar{u}_j$ where v_i (or u_j) is a path in Q_A from x_i to y_i (or from x'_j to y'_j, respectively), and $\lambda_i, \mu_j \in k$. Then

$$d^1 s^1(f)(a \otimes b) = \sum_{i,j} \lambda_i \mu_j f([w_{y_i}^{-1} v_i w_{x_i}])\overline{v_i u_j} + \sum_{i,j} \lambda_i \mu_j f([w_{y'_j}^{-1} u_j w_{x'_j}])\overline{v_i u_j}$$
$$- \sum_{x_i = y'_j} \lambda_i \mu_j f([w_{y_i}^{-1} v_i u_j w_{x'_j}])\overline{v_i u_j}.$$

But, whenever $x_i = y'_j$ we have

$$f([w_{y_i}^{-1} v_i u_j w_{x'_j}]) = f([w_{y_i}^{-1} v_i w_{x_i}][w_{y'_j}^{-1} u_j w_{x'_j}]) = f([w_{y_i}^{-1} v_i w_{x_i}]) + f([w_{y'_j}^{-1} u_j w_{x'_j}]).$$

Therefore, $d^1 s^1(f)(a \otimes b) = 0$ and $s^1(f) \in \mathrm{Ker}\, d^1$. The required morphism s: $\mathrm{Hom}\,(\pi_1(Q_A, I_\nu), k^+) \to H^1(A)$ is induced from s^1 by passing to the quotient $H^1(A) = \mathrm{Ker}\, d^1/\mathrm{Im}\, d^0$.

We now show that s is injective. Let $f \in \mathrm{Hom}\,(\pi_1(Q_A, I_\nu), k^+)$ be such that $s(f) = 0$. Then $s^1(f) \in \mathrm{Im}\, d^0$ and there exists $a \in A$ such that $s^1(f)(b) = ab - ba$ for all

$b \in A$. Considering the trivial paths e_x, we get $0 = s^1(f)(\bar{e}_x) = e_x a - a e_x$. Since Q_A has no oriented cycles, this easily implies that $a = \sum_{x \in (Q_A)_0} \lambda_x e_x$, for some $\lambda_x \in k$. Hence, for any arrow $\alpha: x \to y$, we get $f([w_y^{-1} \alpha w_x]) = \lambda_y - \lambda_x$. This implies $f = 0$: indeed, let $w = \beta_1 \dots \beta_s$ be a closed walk with β_i and arrow or the formal inverse of an arrow from x_i to y_i, then $x_s = x_0 = y_1$ implies that $f([w]) = f([w_{y_1}^{-1} \beta_1 w_{x_1}] \dots [w_{y_1}^{-1} \beta_s w_{x_s}]) = \sum_{i=1}^{s} (\lambda_{x_i} - \lambda_{y_i}) = \lambda_{x_s} - \lambda_{y_1} = 0$. □

3.3. As a first corollary, we obtain the following generalization of [15] (5.5) (where the result is shown for algebras which have no oriented cycles of non-zero non-isomorphisms in their module categories, and, in particular, are representation-finite and hence freely connected, see (1.5)).

Corollary. *Let A be a freely connected algebra with $H^1(A) = 0$. Then A is simply connected.*

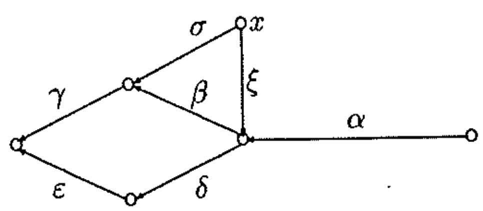
Proof: By (3.2), $\mathrm{Hom}(\pi_1(Q_A, I_\nu), k^+) = 0$ for any presentation (Q_A, I_ν) of A. By (2.7), A is simply connected. □

3.4. *Example:* The converse of this corollary is not true. We now give a procedure for constructing examples of simply connected algebras A such that $H^1(A) \neq 0$ (see also [26] (3.4)). Let B be a simply connected triangular algebra with $H^1(B) = 0$, and M be an indecomposable B-module such that $\mathrm{End}_B M \neq k$. Then $A = B[M]$ is simply connected and $H^1(A) \neq 0$. Indeed, the exact sequence

$$0 \to H^0(A) \to H^0(B) \to \mathrm{End}_B M/k \to H^1(A) \to H^1(B) = 0$$

already shows that $H^1(A) \neq 0$. On the other hand, since M is indecomposable, the extension point x such that $M = \mathrm{rad}\, P_x$ is separating. Then (2.5) implies that A is simply connected.

An explicit version of the example is the algebra $A = kQ/I$, where Q is the quiver



and I is generated by $\gamma\beta\alpha - \varepsilon\delta\alpha$, $\gamma\sigma - \varepsilon\delta\xi$, $\gamma\beta\xi - \gamma\sigma$. Taking $M = \operatorname{rad} P_x$, we get $A = B[M]$, with B simply connected and $H^1(B) = 0$. Moreover, $\dim_k \operatorname{End}_B M = 2$.

3.5. Corollary [26] (3.2). *Let A be a triangular algebra with $H^1(A) = 0$, then all sources of Q_A are separating.*

Proof: Let x be a source in Q_A. By (3.2), $H^1(A) = 0$ implies that we have $\operatorname{Hom}(\pi_1(Q_A, I_\nu), k^+) = 0$ for any presentation (Q_A, I_ν) of A. By (2.6), x is separating. □

4. Faithful square-free radicals.

4.1. Let A be a triangular algebra, and x be a source in Q_A. Let B denote the full convex subcategory of A consisting of all objects except x, and $M = \operatorname{rad} P_x$ so that $A = B[M]$. We thus have a surjective algebra morphism $\pi: A \to B$ with kernel the two-sided ideal $\langle e_x \rangle$ of A generated by e_x. We have an exact sequence

$$H^1(A) \xrightarrow{e} H^1(B) \xrightarrow{d} \operatorname{Ext}^1_B(M, M)$$

where the morphisms e and d are as follows. For every $f \in H^1(A)$, it is easily seen that $f(\langle e_x \rangle) \subset \langle e_x \rangle$, thus there exists a unique $g: B \to B$ such that $g\pi = \pi f$. It is readily checked that $g \in H^1(B)$. We set $e(f) = g$. Then define d, let $b_1 = 1, \ldots, b_s$ be a k-basis of B (such that $b_1 = a_1, \ldots, b_s = a_s, a_{s+1}, \ldots, a_t$ is a k-basis of A). We identify the m-dimensional B-module M with a k-linear representation $B \to \operatorname{End} k^m$, thus with the s-tuple $(M(b_1) \ldots M(b_s))$ of $m \times m$-matrices satisfying the structure equations of the algebra B. Now for $g \in H^1(B)$, we let $d(g)$ denote the extension of M by M

$$0 \to M \to E_g \to M \to 0$$

where $E_g(b_i) = \begin{bmatrix} M(b_i) & M(g(b_i)) \\ 0 & M(b_i) \end{bmatrix}$.

Let now (Q_A, I_ν) be a presentation of A. Using (3.2), we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\operatorname{Hom}(\pi_1(Q_A, I_\nu), k^+) & \xrightarrow{c^*} & \prod_{j=1}^{m} \operatorname{Hom}(\pi_1(Q^{(j)}, I_\nu^{(j)}), k^+) & \longrightarrow & \operatorname{Coker} c^* & \to & 0 \\
{\scriptstyle s_A}\downarrow & & {\scriptstyle s_B}\downarrow & & {\scriptstyle s_x}\downarrow & & \\
H^1(A) & \xrightarrow{e} & H^1(B) & \xrightarrow{d} & \operatorname{Ext}^1_B(M, M) & &
\end{array}$$

Indeed, we just have to verify that $es_A = s_B c^*$: let $f \in \mathrm{Hom}(\pi_1(Q_A, I_\nu), k^+)$ then $es_A(f): B \to B$ satisfies $es_A(f)\pi = \pi s_A^1(f)$ which clearly implies $s_B c^*(f)\pi = \pi s_A^1(f)$, hence our claim.

4.2. Our present aim is to derive a sufficient condition for the map s_x to be a monomorphism. We need the following definition. Let A be an algebra, an A-module M is called *square-free* if $\dim_k M(y) \le 1$ for all $y \in (Q_A)_0$. It is called *locally faithful* if, for any arrow $\alpha: y \to z$ in Q such that $M(y) \neq 0$ and $M(z) \neq 0$, then $M(\alpha)$ is a non-zero map.

If, for instance, A is schurian, then, for any $y \in (Q_A)_0$, the radical P_y is square-free. On the other hand, any faithful module (over an arbitrary algebra) is clearly locally faithful.

Proposition. *Let $A = B[M]$ be a triangular algebra such that $M = \mathrm{rad}\, P_x$ for some source x in Q_A. Assume that M is a square-free locally faithful B-module. Then s_x is a monomorphism.*

Proof: We need to consider the explicit description of s_x. Let $g \in Z^1(B, \nu', k^+)$ where $(Q_B, I_{\nu'})$ is the presentation of B induced from a presentation (Q_A, I_ν) of A. For a $g \in H^1(B)$, then $s_x(g)$ is an extension

$$0 \to M \to E_g \to M \to 0$$

such that, for any arrow $\alpha: y \to z$ in Q_B, we have

$$E_g(\alpha) = \begin{bmatrix} M(\alpha) & g(\alpha)M(\alpha) \\ 0 & M(\alpha) \end{bmatrix}$$

this definition being independent of the class of g in $\mathrm{Coker}\, c^*$.

Assume $s_x(g) = 0$. There exists an isomorphism of A-modules $f: M \oplus M \to E_g$ such that for each $y \in (Q_A)$

$$f_y = \begin{bmatrix} 1_{M(y)} & m_y \\ 0 & 1_{M(y)} \end{bmatrix} : M(y) \oplus M(y) \to E_g(y).$$

Since M is square-free, $m_y \in k$ for each $y \in (Q_A)_0$. Since f is a morphism of A-modules, for each arrow $\alpha: y \to z$ in Q_A, we have

$$g(\alpha)M(\alpha) = (m_z - m_y)M(\alpha).$$

Hence, if $y, z \in \mathrm{Supp}\, M$, then $g(\alpha) = m_z - m_y \in k$.

We now extend g to $f \in Z^1(A, \nu, k)$ as follows: let $\beta_1, \ldots, \beta_t$ be representatives of the equivalence classes $[\alpha]_\nu$, for $\alpha \in x^\rightarrow$ and, for each $1 < i < t$, let y_i denote the target of β_i. Then, for any $\beta: x \to y$ with $\beta \in [\beta_i]$, we set $f(\beta) = m_y - m_{y_i}$. First, we show that $f \in Z^1(A, \nu, k)$. Indeed, we may assume that there exists a minimal relation $\lambda_1 v_1 \beta_i + \lambda_2 v_2 \beta + \sum\limits_{j \geq 3} \lambda_j u_j \in I_\nu(x, z)$ with $v_1 = \alpha_{11} \ldots \alpha_{1p}$ and $v_2 = \alpha_{21} \ldots \alpha_{2q}$. Then $M(v_1) \neq 0$, $M(v_2) \neq 0$ and

$$f(\beta) + \sum_{i=1}^{p} f(\alpha_{2i}) = m_y - m_{y_i} + \sum_{i=1}^{p} g(\alpha_{2i}) = (m_y - m_{y_i}) + (m_z - m_y)$$
$$= m_z - m_{y_i} = \sum_{j=1}^{q} g(\alpha_{1j}) = f(\beta_i) + \sum_{j=1}^{q} f(\alpha_{1j}).$$

Moreover, $p(f) \in \operatorname{Hom}(\pi_1(Q_A, I_\nu), k^+)$ satisfies $c^* p(f) = p' h^1(f) = p'(g)$ (notation as in (2.4)). Hence the image in $\operatorname{Coker} c^*$ of $p'(g) \in \operatorname{Hom}(\pi_1(Q_B, I_{\nu'}), k^+)$ is zero, and s_x is injective. □

4.3. Recall that a *cycle* in $\operatorname{mod} A$ is a sequence of non-zero non-isomorphic between indecomposable A-modules $M_0 \to M_1 \to \cdots \to M_s = M_0$. An indecomposable A-module M is called *directing* if there is no cycle passing through M.

Lemma. *Let M be an indecomposable directing module. Then M is locally faithful.*

Proof: Let B be the full subcategory of A with vertices $\{y \in (Q_A)_0 \mid M(y) \neq 0\}$ (=the support of M). It follows the well-known convexity argument of Bongartz [4] (3.2) that B is a full convex subcategory of A. Then M is a sincere indecomposable directing B-module and therefore, B is a tilted algebra [24]. By the same argument as in the proof of [16] M is faithful as a B-module, hence is locally faithful as an A-module. □

Corollary. *Assume that M is an indecomposable square-free directing module and that $A = B[M]$ is a triangular algebra. For any presentation (Q_A, I_ν) we have a group isomorphism $\operatorname{Hom}(\pi_1(Q_A, I_\nu), k^+) \cong \operatorname{Hom}(\pi_1(Q_B, I_{\nu'}), k^+)$. In particular, if B is freely connected, then A is simply connected if and only if B is simply connected.*

Proof: Since M is square-free and locally faithful, s_x is a monomorphism by (4.2). Moreover, M directing implies that $\operatorname{Ext}^1_A(M, M) = 0$. Hence $\operatorname{Coker} c^* = 0$, so that c^*

is surjective. On the other hand, since M is indecomposable, the extension point x is separating. By (2.4), c^* is injective and hence it is an isomorphism.

If B is simply connected, (2.5) implies that A is simply connected. Assume that B is freely connected and A is simply connected. Then $\operatorname{Hom}(\pi_1(Q_B, I_{\nu'}), k^+) = 0$ for any presentation $(Q_B, I_{\nu'})$ of B. By (2.7), B is simply connected. □

5. Strongly simply connected algebras.

5.1. Following [26], we say that a connected triangular algebra is *strongly simply connected* if every full convex subcategory of A is simply connected. We recall the following:

Theorem [26] (4.1). *The following are equivalent for a connected triangular algebra A:*

a) *A is strongly simply connected.*

b) *Every full convex subcategory of A is separated.*

c) *For every full convex subcategory C of A, we have $H^1(C) = 0$.* □

5.2. We now provide a simple method allowing to construct all strongly simply connected algebras.

Proposition. *A triangular algebra A is strongly simply connected if and only if there exist a sequence of connected algebras $A = A_0, A_1, \ldots, A_s = k$ and of indecomposable modules ${}_{A_1}M_1, \ldots, {}_{A_s}M_s$ such that either $A_{i-1} = A_i[M_i]$ or $A_{i-1} = [M_i]A_i$ for all $\leq i \leq s$.*

Proof: The necessity follows immediately form [22] (2.2). For the sufficiency, assume that we have sequences as in the statement, and let $B = A_1$. We may assume that B is strongly simply connected. Let $M_1 = M = \operatorname{rad} P_x$. Since M is indecomposable, x is separating. By (2.5), A is simply connected.

Let C be any full convex subcategory of A. We must prove that C is simply connected. We may assume that C is a proper subcategory of A and that C contains x (otherwise, $C = A$ or C is contained in B: in either case, we are done). Let D be the full convex subcategory of C consisting of all objects except x. Since D is contained in B, it is simply connected. Since x is separating in C, another application of (2.5) completes the proof. □

5.3. We now consider a class of schurian algebras for which simple connectedness and strong simple connectedness are equivalent: this is the class of schurian algebras all of whose indecomposable projective modules are directing, (such algebras are studied in [28]).

Proposition. *Let $A = B[M]$ be a simply connected schurian algebra all of whose indecomposable projective modules are directing. For any presentation (Q_A, I_ν) of A, we have $\operatorname{Hom}(\pi_1(Q_B, I_{\nu'}), k^+) = 0$. Consequently, all sources in Q_B are separating.*

Proof: Let x be an extension point of $A = B[M]$, and $M = \operatorname{rad} P_x = M_1 \oplus \cdots \oplus M_s$ be an indecomposable decomposition. Since P_x is directing, it follows from [17], theorem 1, that $\operatorname{rad} P_x$ is directing, that is, there are no chains of non-zero non-isomorphisms between indecomposable modules

$$M_i \to \cdots \to \tau N \to \bullet \to N \to \cdots \to M_j$$

for all $1 \leq i, j \leq n$. In particular, $\operatorname{Ext}^1_A(M, M) = 0$ and $\operatorname{End} M_i = k$ for all i. Moreover, the existence of a non-zero morphism $M_i \to M_j$ for $i \neq j$ would imply that $\operatorname{top} M_i \cap \operatorname{top} M_j \neq 0$, contradicting the assumption that A is schurian. Therefore $\operatorname{End}_A M = k^s$. On the other hand, by (2.5), x is separating. Since $H^0(B) = k^s$, the sequence

$$0 \to H^0(A) \to H^0(B) \to \operatorname{End} M/k \to 0$$

is exact. Therefore the morphism e in (3.1) is injective, and hence so is c^*. Since P_x is locally faithful, so is M. Since A is schurian, M is square-free as well. By (4.2), s_x is a monomorphism and hence c^* is surjective. Thus c^* is an isomorphism and $\operatorname{Hom}(\pi_1(Q_B, I_{\nu'}), k^+) \cong \operatorname{Hom}(\pi_1(Q_A, I_\nu), k^+) = 0$. The last statement follows from (2.6). □

5.4. The following corollary generalizes results obtained in [3, 6, 15] for representation-finite algebras:

Corollary. *Let A be a schurian algebra all of whose indecomposable projective modules are directing. The following are equivalent:*

a) *A is simply connected.*

b) *A is strongly simply connected.*

c) *A is separated.*

Proof: Clearly (b) implies (c). By (2.5), (c) implies (a). Assume that A is simply connected and let C be any full convex subcategory of A. By (5.3) and induction $\mathrm{Hom}(\pi_1(Q_C, I_{\nu'}), k^+) \cong \mathrm{Hom}(\pi_1(Q_A, I_\nu), k^+) = 0$ and hence, by (2.6), all sources in Q_C are separating. Now we may clearly assume by induction that any proper full convex subcategory of C is simply connected. Hence, by (2.5), C itself is simply connected. Thus (a) implies (b). Note that, in applying induction, we used the obvious observation that any full convex subcategory of A is itself schurian with directing indecomposable projective modules. □

Acknowledgements.

This paper was written while the second author was visiting the first. The first author gratefully acknowledges partial support from the Natural Sciences and Engineering Research Council of Canada and the Université de Sherbrooke, and the second author gratefully acknowledges partial support form the Consejo Nacional de Ciencia y Tecnología de México and the the Université de Sherbrooke.

References.

[1] Assem, I. And Skowroński, A.: *On some classes of simply connected algebras.* Proc. London Math. Soc. (3) 56 (1988) 417–450.

[2] Assem, I. and Skowroński, A.: *On tame repetitive algebras.* Fund. Math. 142 (1993) 59–84.

[3] Bautista, R., Larrión, F. and Salmerón, L.: *On simply connected algebras.* J. London Math. Soc. (2) 27 (1983), No. 2, 212–220.

[4] Bongartz, K.: *Algebras and quadratic forms.* J. London Math. Soc. (2) 28 (1983), 461–469.

[5] Bongartz, K. and Gabriel, P.: *Covering spaces in representation theory.* Invent. Math. 65 (1981/82) No. 3, 331–378.

[6] Bretscher, O. and Gabriel, P.: *The standard form of a representation-finite algebra.* Bull. Soc. Math. France 111 (1983) 21–40.

[7] Cartan, H. and Eilenberg, S.: *Homological algebra.* Princeton University Press (1956).

[8] Dowbor, P. and Skowroński, A.: *On Galois coverings of tame algebras.* Arch. Math. 44 (1985) 522–529.

[9] Dowbor, P. and Skowroński, A.: *Galois coverings of representation-infinite algebras.* Comment. Math. Helv. 62 (1987) 311–337.

[10] Dräxler, P.: *Completely separating algebras.* Preprint No. 91–100, Universitat of Bielefeld (1991).

[11] Fischbacher, U. and de la Peña, J. A.: *Algorithms in representation theory of algebras.* Proc. Ottawa 1984, Lecture Notes in Mathematics 1177, Springer-Verlag (1986) 115–134.

[12] Gabriel, P.: *The universal cover of a representation-finite algebra.* Proc. Puebla (1980), Lecture Notes in Mathematics 903, Springer-Verlag (1980) 68–105.

[13] Geiss, C. and de la Peña, J. A.: *An interesting family of algebras.* Arch. Math. 60 (1993) 25–35.

[14] Green, E. L.: *Graphs with relations, coverings and group-graded algebras.* Trans. Amer. Math. Soc. 297 (1983) 297–310.

[15] Happel, D.: *Hochschild cohomology of finite dimensional algebras.* Séminaire M.-P. Malliavin (Paris, 1987-88), Lecture Notes in Mathematics 1404, Springer-Verlag (1989) 108–126.

[16] Happel, D. and Ringel, C. M.: *Tilted algebras.* Trans. Amer. Math. Soc. 274 (1983) 399–443.

[17] Happel, D. and Ringel, C. M.: *Directing projective modules.* Arch. Math. 60 (1993) 237–246.

[18] Martínez-Villa, R. and de la Peña, J. A.: *The universal cover of a quiver with relations.* J. Pure Appl. Algebra 30 (1983) 277–292.

[19] Martínez-Villa, R. and de la Peña, J. A.: *Multiplicative basis for algebra whose universal cover has no oriented cycles.* J. Algebra 87 (1984) 389–395.

[20] de la Peña, J. A.: *On the abelian Galois coverings of an algebra.* J. Algebra 102, No. 1 (1986) 129–134.

[21] de la Peña, J. A.: *Algebras with hypercritical Tits form.* Topics in Algebra, Banach Center Publications, Vol. 26, Part 1, PWN - Polish Scientific Publishes, Warsaw (1990) 353–369.

[22] de la Peña, J. A. and Skowroński, A.: *Forbidden subalgebra of non-polynomial growth tame simply connected algebras.* Preprint, Universidad Nacional Autónoma de México (1994).

[23] Ringel, C. M.: *Tame algebras.* Proc. ICRA II (Ottawa, 1979), Lecture Notes in Mathematics 831, Springer-Verlag (1980) 137–287.

[24] Ringel, C. M.: *Tame algebras and integral quadratic forms.* Lecture Notes in Mathematics 1099, Springer-Verlag (1984).

[25] Skowroński, A.: *Algebras of polynomial growth.* Topics in Algebra, Banach Center Publications, Vol. 26, Part 1, PWN - Polish Scientific Publishers, Warsaw (1990) 535–5??.

[26] Skowroński, A.: *Simply connected algebras and Hochschild Cohomology.* Proc. ICRA IV (Ottawa, 1992), Can. Math. Soc. Proc. Vol. 14 (1993) 431–447.

[27] Skowroński, A.: *Loop-finite algebras with the separation property.* In preparation.

[28] Skowroński, A. and Wenderlich, M.: *Artin algebras with directing indecomposable projectives.* To appear J. Algebra (1994).

Ibrahim Assem
Departement de Mathématiques
et Informatique
Université de Sherbrooke
Sherbrooke, Québec
Canada, J1K 2R1
ibrahim.assem@dmi.usherb.ca

José Antonio de la Peña
Instituto de Matemáticas
Universidad Nacional
Autónoma de México
México 04510, D. F.
México
joseant@redvax1.dgsca.unam.mx