THE BOUND QUIVER OF A SPLIT EXTENSION

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Abstract. In this paper, we derive a necessary and sufficient condition on a set of arrows in the quiver of an algebra A so that A is a split extension of A/M, where M is the ideal of A generated by the classes of these arrows. We also compare the notion of split extension with that of semiconvex extension of algebras.

Let A and B be two finite dimensional algebras over an algebraically closed field such that there exists a split surjective algebra morphism $A \longrightarrow B$ whose kernel is a nilpotent ideal of A. We then say that A is a split extension of B. This situation has been studied, for instance, in [5, 8, 9, 17, 18, 19]. Examples of split extensions abound, the most important being that of the trivial extension algebras such as, for instance, the cluster-tilted algebras [2, 12].

Assume A is a split extension of B. It is reasonable to ask what is the relation between the bound quivers of A and B. It was shown in [9](1.3) that the quiver of B is obtained from that of A by deleting some arrows but, as pointed out there, these arrows cannot be taken arbitrarily. Our main result (2.4) gives an easily verified necessary and sufficient condition on a set S of arrows in the quiver of A so that A is a split extension of A/M, where M is the ideal of A generated by the classes of these arrows. This condition is expressed by saying that, if an arrow belonging to a minimal relation lies in this set S, then on each path of this minimal relation, there must be an arrow from S. This was already proven in the schurian triangular case in [3](3.2), but our proof here is completely different.

We then apply our result to compare the notion of split extension with that of semiconvex subcategories, introduced in [15]. We show in (3.3) that any semiconvex subcategory is a split extension, but the converse is not true. However, we characterise there the semiconvex subcategories in terms of a special kind of split extension, called elementary.

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This work consists of three sections. The first is devoted to preliminaries on split extensions, the second to the proof of our main theorem and the last to the connection with semiconvex subcategories.

1. Preliminaries on split extensions

1.1. **Notation.** Let k be an algebraically closed field. By algebra is meant a basic associative finite dimensional k-algebra with an identity. A *quiver* Q is a quadruple (Q_0, Q_1, s, t) where Q_0 is the set of *points* of Q, Q_1 is the set of *arrows* of Q, and s, t are functions from Q_1 to Q_0 which give, respectively, the source $s(\alpha)$ and the target $t(\alpha)$ of a given arrow α. By a well-known result of Gabriel, given an algebra A, there exists a (unique) finite quiver Q_A and (at least) a surjective algebra morphism $\eta\colon kQ_A \longrightarrow A$, where kQ_A denotes the path algebra of Q_A. Setting $I = \mathrm{Ker}(\eta)$, we then have $A \cong kQ_A/I$. The morphism η is called a *presentation* of A, and A is said to be given by the *bound quiver* (Q_A, I), see [7]. The ideal I is admissible, that is, there exists an $n \geq 2$ such that $kQ_A^{+n} \subset I \subset kQ_A^{+2}$, where kQ_A^{+l} is the ideal of kQ_A generated by the paths of length at least l in Q_A. Moreover, I is generated by a finite set of relations: a *relation* in Q_A from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} c_i w_i$, where the $c_i \in k$ are non-zero, and the w_i are paths of length at least two from x to y. A relation $\rho = \sum_{i=1}^{m} c_i w_i$ is *monomial* if $m = 1$ and *minimal* if $m \geq 2$ and, for every non-empty and proper subset J of $\{1, \cdots, m\}$, we have $\sum_{j \in J} c_j w_j \notin I$. Following [11], we sometimes consider equivalently the algebra $A = kQ_A/I$ as a k-category, of which the object class is the set $(Q_A)_0$ and where the set of morphisms from x to y is the quotient of the k-vector space $kQ_A(x, y)$ of all linear combinations of paths from x to y by the subspace $I(x, y) = I \cap kQ_A(x, y)$. The algebra A is *triangular* if Q_A is acyclic. For $x \in (Q_A)_0$, we let ϵ_x denote the corresponding stationary path, and $e_x = \epsilon_x + I$ the corresponding primitive idempotent of A. We also denote by S_x, P_x, respectively, the corresponding simple and indecomposable projective A-modules associated to x.

1.2. Let A, B be two algebras, we say that A is a *split extension of B by the nilpotent ideal M*, or briefly a *split extension of B* if there exists a split surjective algebra morphism $\pi\colon A \longrightarrow B$ whose kernel M is a nilpotent ideal. This means that there exists a short exact sequence of abelian groups

$$0 \longrightarrow M \xrightarrow{\iota} A \underset{\sigma}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

where ι denotes the inclusion and σ is an algebra morphism such that $\pi\sigma = 1_B$. In particular, σ identifies B with a subalgebra of A. Since M is nilpotent, we have $M \subset \text{rad}A$.

LEMMA. *Let A be a split extension of B by the nilpotent ideal M. The quiver Q_A of A is constructed as follows:*

(a) $(Q_A)_0 = (Q_B)_0$.

(b) *For $x, y \in (Q_A)_0$, the set of arrows in Q_A from x to y equals the set of arrows in Q_B from x to y plus*

$$dim_k\ e_x \frac{M}{M \cdot radB + radB \cdot M + M^2} e_y$$

additional arrows.

Proof. Since $M \subset \text{rad}A$, the quivers of A and B have the same points. The arrows in Q_A correspond to a k-basis of the vector space $\text{rad}A/\text{rad}^2A$. Now, $\text{rad}A = \text{rad}B \oplus M$ as a vector space, and hence

$$\text{rad}^2A = \text{rad}^2B \oplus [M \cdot \text{rad}B + \text{rad}B \cdot M + M^2].$$

Since $\text{rad}^2B \subset \text{rad}B$, and $M\cdot$ $\text{rad}B+$ $\text{rad}B \cdot M + M^2 \subset M$, and since the arrows of Q_B correspond to a basis of $\text{rad}B/\text{rad}^2B$, the additional arrows of Q_A correspond to a basis of $M/[M \cdot \text{rad}B + \text{rad}B \cdot M + M^2]$. The arrows from x to y are obtained by multiplying by e_x on the left and by e_y on the right. □

1.3. By (1.2), Q_B is a (non-full) subquiver of Q_A. We now show that, if they are equal, then $M = 0$.

LEMMA. *Let A be a split extension of B by the nilpotent ideal M. If $Q_A = Q_B$, then $A \cong B$.*

Proof. Set $Q = Q_A = Q_B$. There exists presentations $A \cong kQ_A/J$ and $B \cong kQ_B/I$ such that $J \subseteq I$. Here, J and I are admissible ideals of kQ, which we may assume to be generated by monomial and minimal relations. Let now ρ be the preimage in kQ of a generator of I. If ρ is monomial, then ρ is a path

$$\rho \;=\; \alpha_1\alpha_2\cdots\alpha_m \quad \text{with} \quad \alpha_i \in Q_1 \quad \text{for all } i = 1, \cdots m.$$

Thus, in B,

$$0 \;=\; \rho + I \;=\; (\alpha_1 + I)(\alpha_2 + I)\cdots(\alpha_m + I).$$

On the other hand, B is a subalgebra of A. Hence

$$0 \;=\; \rho + J \;=\; (\alpha_1 + J)(\alpha_2 + J) \cdots (\alpha_m + J),$$

so that $\rho \in J$. If ρ is minimal, then ρ is a linear combination of paths w_i of length at least two

$$\rho \;=\; \sum_{i=1}^{m} \lambda_i w_i, \quad \text{with} \quad \lambda_i \in k^* \text{ for all } i.$$

Thus, in B,

$$0 \;=\; \rho + I \;=\; \sum_{i=1}^{m} \lambda_i (w_i + I).$$

Again, B being a subalgebra of A, we have

$$0 \;=\; \rho + J \;=\; \sum_{i=1}^{m} \lambda_i (w_i + J)$$

and so, again, $\rho \in J$. This shows that $I = J$ and hence $A \cong B$. □

1.4. As a consequence, we show that, if $M \neq 0$, then A admits a presentation such that M is generated by arrows.

Corollary. *Let A be a split extension of B by M such that $A \ncong B$. Then there exists a presentation of A such that M is generated by the classes of arrows of Q_A which are not in Q_B.*

Proof. By (1.3), we have $Q_A \neq Q_B$. Hence, using (1.2), we get $(Q_B)_1 \neq (Q_A)_1$. Let $A \cong kQ_A/I$ be any presentation of A, and let $\{\rho_1, \cdots, \rho_s\}$ be the preimage modulo I of any finite set of generators of M. Left and right-multiplying, if necessary, by stationary paths, we may assume that each ρ_i is a linear combination of paths having the same source and the same target in Q_A, and all the paths involved in these linear combinations have length at least one. As is clear from the proof of (1.3) above, we may assume that $\rho_i + I \in \mathrm{rad}A/\mathrm{rad}^2 A$ for each i with $1 \leq i \leq s$. Thus, we can write

$$\rho_i \;=\; \alpha^i + \sum_j \lambda'_j w'_j$$

where $\alpha^i \in (Q_A)_1$ and $\sum_j \lambda'_j w'_j$ a linear combination of paths of length at least one. Since all $\rho_i + I$ are linearly independent, we define a new presentation by replacing the arrow α^i by

$$\alpha'_i \;=\; \alpha^i + \sum_j \lambda'_j w'_j.$$

With this definition, we indeed get $M = < \alpha'_1, \cdots, \alpha'_s >$. □

1.5. As a consequence of the above remarks, any presentation of B can be extended to a "nice" presentation of A.

COROLLARY. *Let A be a split extension of B by M. Given a presentation $\eta_B\colon kQ_B \longrightarrow B$, there exists a presentation $\eta_A\colon kQ_A \longrightarrow A$ such that M is an ideal of A generated by classes of arrows of Q_A and there is a commutative diagram of abelian groups with exact rows and columns*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M}\cap I_A & \longrightarrow & I_A & \longrightarrow & I_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M} & \longrightarrow & kQ_A & \underset{\widetilde{\sigma}}{\overset{\widetilde{\pi}}{\rightleftarrows}} & kQ_B & \longrightarrow & 0 \\
 & & \downarrow & & \eta_A\downarrow & & \downarrow\eta_B & & \\
0 & \longrightarrow & M & \longrightarrow & A & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{\sigma}, \widetilde{\pi}$ are algebra morphisms such that $\widetilde{\pi}\widetilde{\sigma} = 1_{kQ_B}$ and $\eta_A\widetilde{\sigma} = \sigma\eta_B$. In particular, $\widetilde{\sigma}(I_B) \subset I_A$.

Proof. As in (1.2), we identify B with a subalgebra of A via σ. This identifies Q_B to a subquiver of Q_A. Therefore, the inclusion $Q_B \hookrightarrow Q_A$ induces an algebra morphism $\widetilde{\sigma}\colon kQ_B \longrightarrow kQ_A$ by setting $\widetilde{\sigma}(\epsilon_x) = \epsilon_x$ for every $x \in (Q_B)_0$ and $\widetilde{\sigma}(\alpha) = \alpha$ for every $\alpha \in (Q_B)_1$. Letting $\eta_A\colon kQ_A \longrightarrow A$ be a presentation constructed as in (1.2) and (1.4), we then have $\eta_A\widetilde{\sigma} = \sigma\eta_B$. By (1.4), there exists a set of arrows S in Q_A such that M is the ideal generated by the classes $\alpha + I$, with $\alpha \in S$. Let $\widetilde{M}$ be the ideal of kQ_A generated by all arrows in S, and let $\widetilde{\pi}\colon kQ_A \longrightarrow kQ_B$ be the algebra morphism defined by setting $\widetilde{\pi}(\epsilon_x) = \epsilon_x$, for every $x \in (Q_A)_0$ and $\widetilde{\pi}(\beta) = \beta$, for every $\beta \in (Q_A)_1 \setminus S$, while $\widetilde{\pi}(\alpha) = 0$, for every $\alpha \in S$. We then have a short exact sequence of abelian groups

$$0 \longrightarrow \widetilde{M} \longrightarrow kQ_A \xrightarrow{\widetilde{\pi}} kQ_B \longrightarrow 0$$

and moreover, $\eta_B\widetilde{\pi} = \pi\eta_A$. This yields the required commutative diagram. Clearly, $\widetilde{\pi}\widetilde{\sigma} = 1_{kQ_B}$. The last statement follows by passing to the kernels. □

1.6. We finish this section by showing that taking split extensions is a transitive operation. We need this fact in section 3.

LEMMA. *If A is a split extension of B, and B is a split extension of C, then A is a split extension of C.*

Proof. There exist short exact sequences of abelian groups

$$0 \longrightarrow M \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$$

$$0 \longrightarrow M' \longrightarrow B \xrightarrow{\pi'} C \longrightarrow 0$$

with π, π' algebra morphisms such that there exist algebra morphisms $\sigma\colon B \longrightarrow A$ and $\sigma'\colon C \longrightarrow B$ satisfying $\pi\sigma = 1_B$ and $\pi'\sigma' = 1_C$. Moreover, there exist $m, n > 0$ such that $M'^m = 0$ and $M^n = 0$.

We thus get a short exact sequence of abelian groups

$$0 \longrightarrow \pi^{-1}(M') \longrightarrow A \xrightarrow{\pi'\pi} C \longrightarrow 0.$$

We claim that $\pi^{-1}(M')^{mn} = 0$. Let then $x_j^i \in \pi^{-1}(M')$, with $1 \leq i \leq n$, $1 \leq j \leq m$. Observe that

$$\prod_{i=1}^{n}(x_1^i x_2^i \cdots x_m^i) = 0$$

Indeed, we have $\pi(x_j^i) \in M'$ for all i, j, and hence

$$\pi(x_1^i x_2^i \cdots x_m^i) = \pi(x_1^i)\pi(x_2^i) \cdots \pi(x_m^i) \in M'^m = 0.$$

So $x_1^i x_2^i \cdots x_m^i \in \mathrm{Ker}\pi = M$, for each i. Therefore,

$$\prod_{i=1}^{n}(x_1^i x_2^i \cdots x_m^i) \in M^n = 0,$$

and this establishes our claim. Since $(\pi'\pi)(\sigma\sigma') = 1_C$, the statement follows. □

2. The main results

2.1. We start by showing that if M is an ideal of $A \cong kQ_A/I_A$ generated by a set of classes of arrows in Q_A, then we can deduce a "nice" presentation of $B = A/M$.

PROPOSITION. *Let $\eta_A\colon kQ_A \longrightarrow A$ be a presentation of A, let M be an ideal of A generated by the classes modulo $I_A = Ker(\eta_A)$ of a set S of arrows, and let $B = A/M$. Then there exists a presentation*

$\eta_B\colon kQ_B \longrightarrow B$ such that we have a commutative diagram of abelian groups with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M}\cap I_A & \longrightarrow & I_A & \longrightarrow & I_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M} & \longrightarrow & kQ_A & \underset{\widetilde{\sigma}}{\overset{\widetilde{\pi}}{\rightleftarrows}} & kQ_B & \longrightarrow & 0 \\
 & & \downarrow & & \eta_A\downarrow & & \downarrow\eta_B & & \\
0 & \longrightarrow & M & \longrightarrow & A & \overset{\pi}{\longrightarrow} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\widetilde{\pi}, \widetilde{\sigma}$ are algebra morphisms such that $\widetilde{\pi}\widetilde{\sigma} = 1_{kQ_B}$.

Proof. Let Q be the quiver defined by $Q_0 = (Q_A)_0$ and $Q_1 = (Q_A)_1 \setminus S$. We first construct a surjective algebra morphism $\eta_B\colon kQ \longrightarrow B$. We obviously have a surjective algebra morphism $\widetilde{\pi}\colon kQ_A \longrightarrow kQ$ given by $\widetilde{\pi}(\epsilon_x) = \epsilon_x$ for every $x \in (Q_A)_0$ and $\widetilde{\pi}(\beta) = \beta$ for $\beta \in (Q_A)_1 \setminus S$ while $\widetilde{\pi}(\alpha) = 0$ for all $\alpha \in S$. Now, let $b \in B$ and let $\pi\colon A \longrightarrow B$ be the canonical projection with kernel M. There exists $a \in A$ such that $b = \pi(a)$. Since $A \cong kQ_A/I_A$, there exists $w \in kQ_A$ such that $b = \pi\eta_A(w)$. On the other hand, let $\widetilde{M}$ be the ideal of kQ_A generated by the arrows in S (thus, clearly, $\widetilde{M} = \mathrm{Ker}\widetilde{\pi}$), then we have $\pi\eta_A(\widetilde{M}) = \pi(M) = 0$. Hence $\pi\eta_A\colon kQ_A \longrightarrow B$ factors uniquely through $\widetilde{\pi}$, that is, there exists a unique algebra morphism $\eta_B\colon kQ \longrightarrow B$ such that $\eta_B\widetilde{\pi} = \pi\eta_A$.

We now claim that $I_B = \mathrm{Ker}(\eta_B)$ is an admissible ideal of kQ (see [7]). Let kQ^{+n} be the ideal of kQ generated by the paths of length at least n. We must prove that there exists an n such that $kQ^{+n} \subset I_B \subset kQ^{+2}$. We first show that $I_B \subset kQ^{+2}$. If not, let $\gamma \in I_B \setminus kQ^{+2}$. There exist arrows $\beta_1, \cdots, \beta_t$, non-zero scalars $c_1, \cdots, c_t$ (with $t \geq 2$) and $\rho \in kQ^{+2}$ such that $\gamma = \sum_{j=1}^{t} c_j\beta_j + \rho$. Considering γ

as an element of kQ_A, we have

$$\pi\eta_A(\gamma) = \eta_B\widetilde{\pi}(\gamma) = \eta_B(\gamma) = 0.$$

Hence, $\eta_A(\gamma) = \gamma + I_A \in \mathrm{Ker}\pi = M$. Therefore, there exist scalars $d_1, \cdots, d_m$ not all zero, and arrows $\alpha_1, \cdots, \alpha_m \in S$, such that $\gamma + I_A = \sum_{i=1}^{m} d_i\alpha_i + I_A$. Since I_A is an admissible ideal and $\rho \in kQ^{+2}$, the equation

$$\sum_{j=1}^{t} c_j\beta_j + \rho + I_A = \sum_{i=1}^{m} d_i\alpha_i + I_A$$

yields, by reasons of grading, $\sum_{j=1}^{t} c_j\beta_j = \sum_{i=1}^{m} d_i\alpha_i$ and this is impossible, because the arrows α_i do not belong to Q. Thus, $I_B \subset kQ^{+2}$. On the other hand, there exists n such that $kQ_A^{+n} \subset I_A$. Because Q is a subquiver of Q_A, we have $kQ^{+n} \subset kQ_A^{+n}$ whence $kQ^{+n} \subset I_A$. By definition of η_A, this implies that $kQ^{+n} \subset I_B$. This establishes our claim and hence that $\eta_B \colon kQ \longrightarrow B$ is a presentation of B.

Since the quiver of an algebra is uniquely determined, we have that $Q = Q_B$ and we deduce the exactness of the two right columns of the required diagram. Since $\eta_B\widetilde{\pi} = \pi\eta_A$, and $M, \widetilde{M}$ are the respective kernels of $\pi, \widetilde{\pi}$, the diagram is indeed commutative with exact rows and columns. Finally, the inclusion $Q_B \hookrightarrow Q_A$ induces an algebra morphism $\widetilde{\sigma} \colon kQ_B \longrightarrow kQ_A$ by setting $\widetilde{\sigma}(\epsilon_x) = \epsilon_x$ for any $x \in (Q_B)_0$ and $\widetilde{\sigma}(\alpha) = \alpha$ for any $\alpha \in (Q_B)_1$. Clearly, $\widetilde{\pi}\widetilde{\sigma} = 1_{kQ_B}$. □

2.2. The algebra morphisms $\widetilde{\pi} \colon kQ_A \longrightarrow kQ_B$ and $\widetilde{\sigma} \colon kQ_B \longrightarrow kQ_A$ defined above are called respectively the morphism *induced by the projection* and the morphism *induced by the inclusion*. We have the following corollary.

Corollary. *Under the hypothesis of (2.1), if A is triangular, then kQ_A is a split extension of kQ_B by $\widetilde{M}$.*

Proof. In this case, kQ_A and hence kQ_B are finite dimensional. □

2.3. Corollary. *Let A be a split extension of B by M. Then there exists a bijection between presentations $\eta_A \colon kQ_A \longrightarrow A$ of A such that M is generated by classes of arrows of Q_A and presentations $\eta_B \colon kQ_B \longrightarrow B$ of B such that we have a commutative diagram of abelian groups with exact rows and columns*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M} \cap I_A & \longrightarrow & I_A & \longrightarrow & I_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M} & \longrightarrow & kQ_A & \underset{\widetilde{\sigma}}{\overset{\widetilde{\pi}}{\rightleftarrows}} & kQ_B & \longrightarrow & 0 \\
 & & \downarrow & & \eta_A \downarrow & & \downarrow \eta_B & & \\
0 & \longrightarrow & M & \longrightarrow & A & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where $\widetilde{M}$ is generated by arrows, and $\widetilde{\pi}, \widetilde{\sigma}, \pi, \sigma$ are algebra morphisms such that $\widetilde{\pi}\widetilde{\sigma} = 1_{kQ_B}$ and $\pi\sigma = 1_B$.

Proof. This follows from (1.5) and (2.1). □

2.4. Before stating and proving our main theorem, we need a notation: let w be a path in Q_A and α be an arrow such that there exists subpaths w_1, w_2 of w satisfying $w = w_1 \alpha w_2$, we then write $\alpha | w$. Also, when we speak about a relation, we assume, as may be done without loss of generality, that it is either monomial or minimal.

THEOREM. *Let $\eta_A \colon kQ_A \longrightarrow A$ be a presentation of A, let M be an ideal of A generated by the classes modulo $I_A = Ker(\eta_A)$ of a set S of arrows, and let $\pi \colon A \longrightarrow B = A/M$ be the projection. The following conditions are equivalent:*

(a) *The exact sequence $0 \longrightarrow M \longrightarrow A \xrightarrow{\pi} B \longrightarrow 0$ realises A as a split extension of B by M.*
(b) *Let $\widetilde{\sigma} \colon kQ_B \longrightarrow kQ_A$ be the morphism induced by the inclusion, and I_B be the kernel of the induced presentation $\eta_B \colon kQ_B \longrightarrow B$, then $\widetilde{\sigma}(I_B) \subset I_A$.*
(c) *Let $\widetilde{\pi} \colon kQ_A \longrightarrow kQ_B$ be the morphism induced by the projection. Then, for every relation $\rho \in I_A$, we have either $\widetilde{\pi}(\rho) = \rho$ or $\widetilde{\pi}(\rho) = 0$.*

(d) *If* $\rho = \sum_{i=1}^{m} c_i w_i$ *is a minimal relation in* I_A *such that there exist* i *and* $\alpha_i | w_i$ *satisfying* $\alpha_i \in S$ *then, for any* $j \neq i$*, there exists* $\alpha_j | w_j$ *satisfying* $\alpha_j \in S$.

Proof. We first observe that, by (2.1), there exists a commutative diagram of abelian groups with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{M} \cap I_A & \longrightarrow & I_A & \longrightarrow & I_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \lambda_B} & & \\
0 & \longrightarrow & \widetilde{M} & \longrightarrow & kQ_A & \overset{\widetilde{\pi}}{\underset{\widetilde{\sigma}}{\rightleftarrows}} & kQ_B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} & & \\
0 & \longrightarrow & M & \longrightarrow & A & \overset{\pi}{\underset{\sigma}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

(a) implies (b) is the last statement of (1.5).

(b) implies (a). By hypothesis, $\widetilde{\sigma}$ restricts to a morphism $\sigma' \colon I_B \longrightarrow I_A$ such that $\widetilde{\sigma}\lambda_B = \lambda_A \sigma'$. Hence there exists a morphism $\sigma \colon B \longrightarrow A$ of abelian groups such that $\sigma\eta_B = \eta_A\widetilde{\sigma}$. Since $M \subset \mathrm{rad}A$, it suffices to prove that σ is an algebra morphism and $\pi\sigma = 1_B$. Let w, w' be paths in Q_B, then

$$\sigma((w + I_B)(w' + I_B)) = \sigma(\eta_B(w)\eta_B(w')) = \sigma\eta_B(ww') = \eta_A\tilde{\sigma}(ww') =$$

$$= \eta_A\widetilde{\sigma}(w)\eta_A\widetilde{\sigma}(w') = \sigma\eta_B(w)\sigma\eta_B(w') = \sigma(w + I_B)\sigma(w' + I_B)$$

On the other hand, because $\widetilde{\pi}\widetilde{\sigma} = 1_{kQ_B}$, we have

$$\pi\sigma\eta_B = \pi\eta_A\widetilde{\sigma} = \eta_B\widetilde{\pi}\widetilde{\sigma} = \eta_B.$$

The surjectivity of η_B yields $\pi\sigma = 1_B$.

(b) implies (d). Let $\rho = \sum_{i=1}^{m} c_i w_i$ be a minimal relation in I_A such that there exist $i \in \{1, \cdots, m\}$ and $\alpha_i | w_i$ satisfying $\alpha_i \in S$. Assume that the proper subset $J \subset \{1, 2, \cdots, m\}$ of those j such that there is no $\alpha_j|w_j$satisfying $\alpha_j \in S$, is non-empty. Then, considering

$$\rho = \sum_{i \notin J} c_i w_i + \sum_{j \in J} c_j w_j$$

as an element of kQ_A and using the commutative diagram of (2.1) yield

$$\widetilde{\pi}(\rho) = \sum_{j \in J} c_j w_j$$

in kQ_B. Since $\rho \in I_A$, we have $\eta_B \widetilde{\pi}(\rho) = \pi\eta_A(\rho) = 0$. Hence $\widetilde{\pi}(\rho) = \sum_{j \in J} c_j w_j$ belongs to I_B. Applying $\widetilde{\sigma}(I_B) \subset I_A$, we get $\sum_{j \in J} c_j w_j \in I_A$, a contradiction to the minimality of the relation ρ.

(d) implies (c). Let $\rho = \sum_{i=1}^{m} c_i w_i$ be a minimal relation in I_A. Our hypothesis can be expressed by saying that if there exists i such that $\widetilde{\pi}(w_i) = 0$ then, for all $j \neq i$, we have $\widetilde{\pi}(w_j) = 0$ and so $\widetilde{\pi}(\rho) = 0$. This proves that either $\widetilde{\pi}(\rho) = 0$ or $\widetilde{\pi}(\rho) = \rho$. Since this is clearly true for monomial relations, our statement is proven.

(c) implies (b). Let $\gamma \in I_B$ be a non-zero element. We may suppose, without loss of generality, that γ is a relation. Since the restriction of π' to I_A gives a surjective map $\pi' \colon I_A \longrightarrow I_B$, there exists $\rho \in I_A$ such that $\pi'(\rho) = \gamma$. The element $\rho \in I_A$ can be written in the form $\rho = \varphi + \psi$ where $\varphi = \sum_i \varphi_i$ is a sum of monomial relations and $\psi = \sum_j \psi_j$ is a sum of minimal relations. By hypothesis, we have, for each i, either $\pi'(\varphi_i) = \varphi_i$ or $\pi'(\varphi_i) = 0$ and, for each j, either $\pi'(\psi_j) = \psi_j$ or $\pi'(\psi_j) = 0$. We consider two cases.

Assume first that γ is a monomial relation. Since each $\pi'(\varphi_i)$ and each $\pi'(\psi_j)$ is a summand of $\gamma = \pi'(\rho)$, then $\pi'(\psi_j) = 0$ for all j and there exists a unique i such that $\gamma = \pi'(\varphi_i) = \varphi_i$. We then have $\pi'(\gamma) = \gamma$ and so $\sigma'(\gamma) = \gamma \in I_A$.

Assume next that γ is a minimal relation. For each i, we must have $\pi'(\varphi_i) = 0$: indeed, if $\pi'(\varphi_i) = \varphi_i$, then it would be a summand of the minimal relation γ, a contradiction. Similarly, if $j_1 \neq j_2$ are such that $\pi'(\psi_{j_1}) = \psi_{j_1}$ and $\pi'(\psi_{j_2}) = \psi_{j_2}$, then $\psi_{j_1} + \psi_{j_2}$ would be a summand of γ, which yields another contradiction. Hence, there exists a unique j such that $\gamma = \pi'(\psi_j) = \psi_j$. We again have $\pi'(\gamma) = \gamma$ and $\sigma'(\gamma) = \gamma \in I_A$. □

2.5. Corollary. *Assume that the equivalent conditions of the theorem (2.4) are satisfied, then* $I_B = I_A \cap kQ_B$.

2.6. Condition (d) of the theorem is very easy to apply. If an arrow belongs to (only) a monomial relation, then there is no restriction at all. If, on the other hand, it belongs to a path w_i in a minimal relation $\sum c_i w_i$ then, when cutting it, one has to cut as well at least an arrow from each of the other paths w_j $(j \neq i)$. We now recall that an algebra A is called *monomial* if there is a bound quiver presentation $A \cong kQ_A/I$ with I generated by monomial relations. A very useful and heavily investigated class of monomial algebras is that of *string* algebras, see [14], an important subclass of which is the class of *gentle* algebras, see [7](Chapter IX) or [20]. Finally, the class of *special biserial* algebras is very close to the string algebras: they are no longer monomial, but the relations which are not monomial are commutativity relations [21]. The next corollary follows immediately from the above observations.

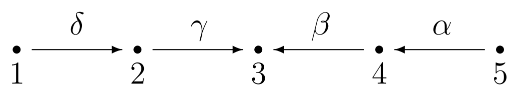
Corollary. *Let A be a split extension of B by M. If A is a special biserial (or monomial, or string, or gentle) algebra, then so is B.*

2.7. We now apply our results to cluster-tilted algebras. We recall that *cluster-tilted* algebras were defined in [12] as a by-product of the Fomin-Zelevinsky theory of cluster algebras [16]. In particular, it is shown in [2] that any cluster-tilted algebra is a trivial extension, hence a split extension, of a tilted algebra. For tilted and iterated tilted algebras, we refer the reader to [7, 4]. The following is a special case of a result obtained independently in [10].

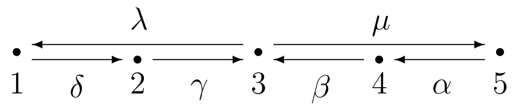
Corollary. *Let A be a split extension of B by M. If A is cluster-tilted of type $\mathbb{A}$ and B is triangular, then B is iterated tilted of type $\mathbb{A}$.*

Proof. By the classification of the cluster-tilted algebras of type $\mathbb{A}$ (see [1, 13]), A is gentle and all relations are contained in 3-cycles. Since B is triangular, then it is a gentle tree. Hence, using the classification in [4], it is iterated tilted of type $\mathbb{A}$. □

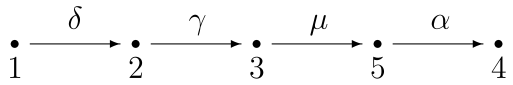
2.8. Examples. (a) Let C be the algebra given by the quiver



bound by $\delta\gamma = 0$, $\alpha\beta = 0$. By [7](Theorem IX.6.11), C is tilted of type $\mathbb{A}_5$. Its relation-extension (see [2]) is the cluster-tilted algebra A of type $\mathbb{A}_5$ given by the quiver

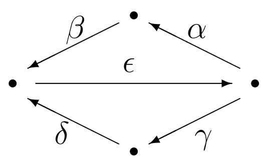


bound by $\lambda\delta = 0$, $\delta\gamma = 0$, $\gamma\lambda = 0$, $\mu\alpha = 0$, $\alpha\beta = 0$, $\beta\mu = 0$. Applying (2.4)(d), we see that A is a split extension of the algebra B given by the quiver



bound by $\delta\gamma = 0$, $\mu\alpha = 0$. Thus, applying [4] and [7](Theorem IX.6.11), we get that B is iterated tilted, but not tilted, of type $\mathbb{A}_5$.

(b) Let A be the (cluster-tilted) algebra given by the quiver



and I_A be the ideal generated by $\alpha\beta - \gamma\delta$, $\epsilon\alpha$, $\epsilon\gamma$, $\beta\epsilon$, $\delta\epsilon$. A set of arrows S satisfying (d) must either contain none of the four arrows $\alpha, \beta, \gamma, \delta$ or, if it contains at least one of α, β (or γ, δ) then it must contain at least one of γ, δ (or, α, β, respectively). Let, for instance, M be the ideal generated by $\alpha + I_A, \gamma + I_A, \epsilon + I_A$, then A is a split extension of A/M by M. On the other hand, if M' is generated by $\alpha + I_A, \epsilon + I_A$, then A is not a split extension of A/M' by M'.

3. Split extensions and semiconvex extensions

3.1. Let C be an algebra and L be a C-module. The *one-point extension* of C by L is the matrix algebra

$$C[L] = \begin{pmatrix} C & 0 \\ L & k \end{pmatrix}$$

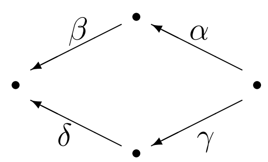
with the usual addition and the multiplication induced from the module structure of L_C. The quiver of $C[L]$ contains then Q_C as a full subquiver and there is an additional (extension) point x which is a source. Observe also that $\mathrm{rad} P_x = L$. The dual notion is that of a *one-point coextension*. We recall the following from [6](2.1). Let x denote the extension point of the one-point extension algebra $A = C[L]$ and suppose $A \cong kQ_A/I_A$. We denote by $\approx$ the least equivalence relation on the set $x^{\rightarrow}$ of all arrows starting at x such that $\alpha \approx \beta$ (for $\alpha, \beta \in x^{\rightarrow}$) whenever there exist $y \in (Q_A)_0$ and a minimal relation $\sum_{i=1}^{m} c_i w_i$ from x to y such that $w_1 = \alpha v_1$, $w_2 = \beta v_2$ for some subpaths v_1 and v_2.

Lemma. *Assume that $A = C[L' \oplus L'']$ and let x denote the extension point. If $\alpha\colon x \longrightarrow x'$, $\beta\colon x \longrightarrow x''$ are arrows in $x^{\rightarrow}$ such that $S_{x'}$ is a summand of the top of L', while $S_{x''}$ is a summand of the top of L'', then $\alpha \not\approx \beta$.*

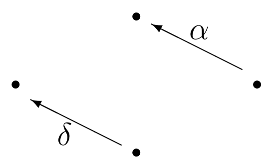
Proof. This follows easily from the definition above. □

3.2. Let A be an algebra and $\{e_1, e_2, \cdots, e_n\}$ be a set of primitive orthogonal idempotents of A (which are in one-to-one correspondence with the points of the quiver Q_A). An algebra C is a full subcategory of A if there exists an idempotent $e \in A$, sum of (some of) the distinguished idempotents e_i, such that $C = eAe$. A full subcategory C of A is *convex* if whenever there exists a sequence $e_{i_0}, e_{i_1}, \ldots, e_{i_t}$ of primitive orthogonal idempotents such that $e_{i_{l+1}} A e_{i_l} \neq 0$ for $0 \leq l < t$ and $ee_{i_0} = e_{i_0}$, $ee_{i_t} = e_{i_t}$, then $ee_{i_l} = e_{i_l}$ for each l. Also, if B is another algebra, then we say that A is a *semiconvex extension of* B if there exists a full convex subcategory C of both A and B and a C-module $L = L' \oplus L''$ with $L'' \neq 0$ such that $A = C[L] = C[L' \oplus L'']$ while $B = C[L']$, see [15]. If x is the extension point, we sometimes denote this situation by $A = B\{L'', x\}$. Dually, one can define the *semiconvex co-extension* of an algebra using the notion of one-point co-extension. We want here to relate the notion of split extensions to the one of semiconvex (co)-extensions. Although semiconvex (co)-extensions are split extensions (see result below), the converse is not true as shown by the following example.

Example. Let A be the algebra given by the commutative quiver



By (2.4), A is a split extension of the algebra given by the two arrows



but the latter is neither a semiconvex (co)-extension nor a semiconvex subategory, in the sense of (3.4) below, of the former.

3.3. Definition. A split extension A of B is called *elementary* provided $(Q_A)_1 \setminus (Q_B)_1$ consists of arrows either leaving a unique source or else targeting a unique sink of A.

Theorem. *An algebra A is an elementary split extension of B if and only if A is either a semiconvex extension or a semiconvex co-extension of B.*

Proof. Necessity. Let $\{\alpha_1, \cdots, \alpha_t\}$ be all the arrows in $(Q_A)_1 \setminus (Q_B)_1$. Assume first that there exists a source x such that $s(\alpha_i) = x$, for each $i = 1, \cdots, t$. Now, if P_x denotes the indecomposable projective A-module at x, then we have $\mathrm{rad} P_x = M \oplus N$, where the arrows α_i have targets y such that S_y is a direct summand of $\mathrm{top} M$ and those arrows in $(Q_B)_1$ with source x have targets z such that S_z is a direct summand of $\mathrm{top} N$, by (3.1). We then have $A = B\{M, x\}$. The case where there exists a unique sink y such that $t(\alpha_i) = y$, for each $i = 1, \cdots, t$, is dual and its proof is left to the reader.
Sufficiency. Assume first that A is a semiconvex extension of B. Then there exists a full convex subcategory C of both A and B and a C-module $L = L' \oplus L''$ with $L'' \neq 0$ such that $A = C[L]$ and $B = C[L']$. Let $A \cong kQ_A/I_A$ and M be the two-sided ideal of A generated by the classes (modulo I_A) of the arrows $\alpha\colon x \longrightarrow x''$ where x denotes the extension point and x'' is such that $S_{x''}$ is a summand of the top of L''. By (3.1), the condition (d) of (2.4) is satisfied, and therefore A is a split extension of $B = A/M$ by M. Clearly, the point x is a source and so the result is proven in this case. If now A is a semiconvex co-extension of B, a dual argument yields the result. □

3.4. Following [15], we say that an algebra B is a *semiconvex subcategory* of A provided there exists a sequence of subcategories

$$B = B_s \subset B_{s-1} \subset \cdots \subset B_0 = A$$

such that, for each i, B_i is either a semiconvex extension or a semiconvex co-extension of B_{i+1}.

Corollary. *If B is a semiconvex subcategory of A, then A is a split extension of B.*

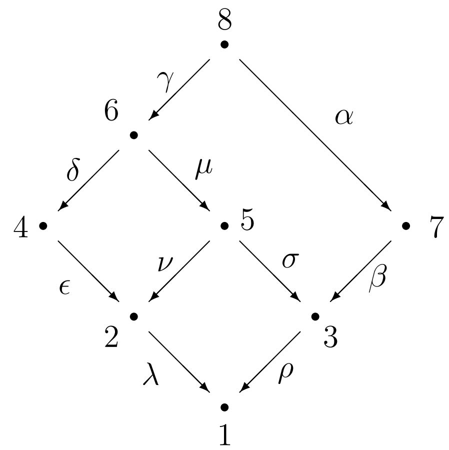
Proof. This follows from (3.3) and (1.6). □

3.5. The next result is also an easy consequence of (3.3).

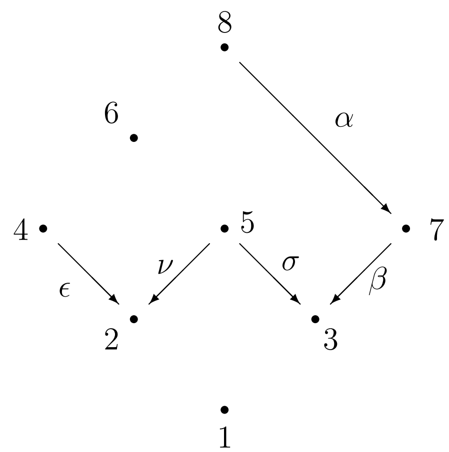
Corollary. *An algebra B is a semiconvex subcategory of A if and only if there exists a sequence of algebras $B = B_0 \subset B_1 \subset \cdots \subset B_t =$*

A such that, for each i with $0 \leq i < t$, B_{i+1} *is an elementary split extension of* B_i.

3.6. Example. Let A be given by the quiver

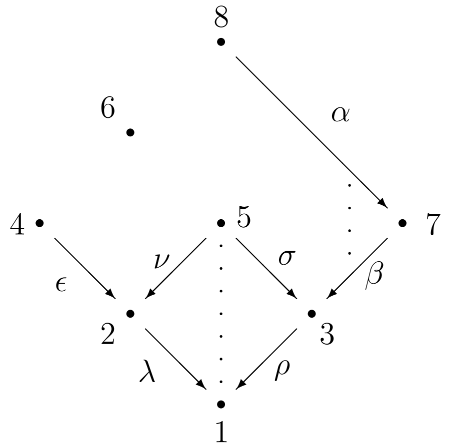


bound by $\alpha\beta = 0$, $\delta\epsilon = \mu\nu$, $\nu\lambda = \sigma\rho$. Using (2.4), we see easily that A is a split extension of the algebra B given by the quiver

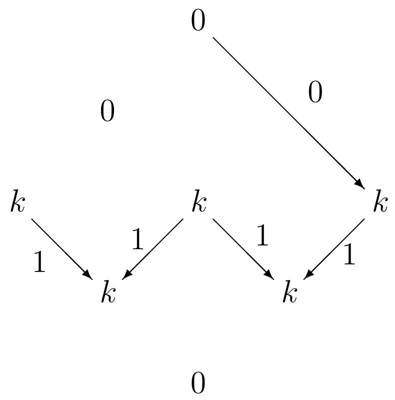


with $\alpha\beta = 0$. A possible sequence of algebras as in Corollary (3.5) can be constructed as follows. We denote relations by dotted lines and modules by the corresponding representations. Set $B_0 = B$.

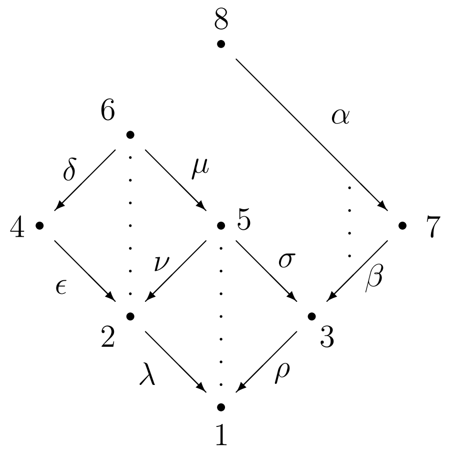
$B_1 = \{L_1, 1\}B_0$



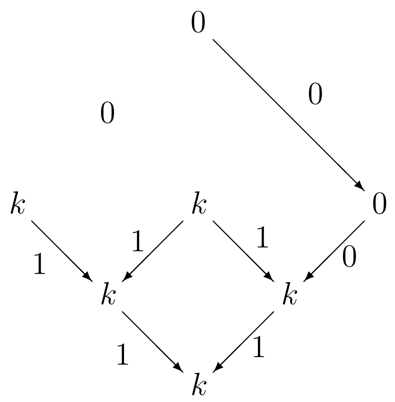
L_1 given by



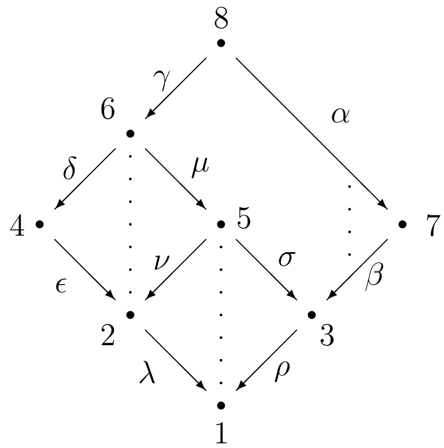
$B_2 = B_1\{L_2, 6\}$

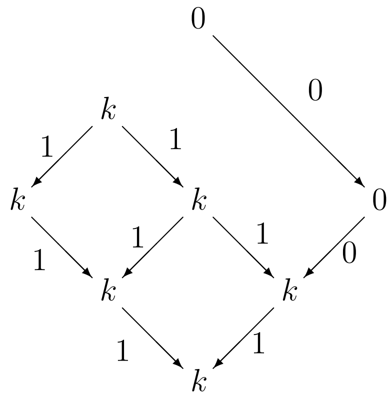


L_2 given by



$B_3 = B_2\{L_3, 8\} = A$ $\qquad\qquad$ L_3 given by





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