STRONGLY SIMPLY CONNECTED TILTED ALGEBRAS

IBRAHIM ASSEM AND SHIPING LIU

Dedicated to the memory of Gilles Fournier

ABSTRACT. Let A be a tilted algebra. We prove that, if A is strongly simply connected, and C is a full convex subcategory of A, then the orbit graph of each of the directed components of the Auslander-Reiten quiver of C is a tree and, if A is tame, then the converse also holds. We further prove that, if A is an iterated tilted algebra of Euclidean type, then A is strongly simply connected if and only if the orbit graph of each of the directed components of its Auslander-Reiten quiver is a tree.

RÉSUMÉ. Soit A une algèbre inclinée. On montre que, si A est fortement simplement connexe, et C est une sous-catégorie pleine et convexe de A, alors le graphe orbital de chaque composante acyclique du carquois d'Auslander-Reiten de C est un arbre et, si A est docile, alors la réciproque est vraie. En outre, si A est pré-inclinée de type Euclidien, alors A est fortement simplement connexe si et seulement si le graphe orbital de chaque composante acyclique de son carquois d'Auslander-Reiten est un arbre.

INTRODUCTION

Among the well-known results of the representation theory of finite dimensional algebras over an algebraically closed field is the theorem, due to Bongartz and Gabriel [8], which states that a representation-finite algebra is simply connected if and only if the orbit graph of its Auslander-Reiten quiver is a tree. It is natural to ask whether a similar result holds for a representation-infinite algebra. In this case, the Auslander-Reiten quiver is no longer connected so one should consider the orbit graph of each of its connected components. On the other hand, we are particularly interested in one class of simply connected representation-infinite algebras, namely the class of strongly simply connected algebras introduced by Skowroński in [18]. This subclass seems to be the most accessible and has been the subject of many recent investigations : see, for instance [1, 2, 20]

In this paper, we seek a criterion for the strong simple connectedness of a tilted algebra. We prove the following theorem.

Theorem. *Let A be a tilted algebra. If A is strongly simply connected, and C is a full convex subcategory of A, then the orbit graph of each of the directed components of the Auslander-Reiten quiver of C is a tree. If A is tame, then the converse also holds.*

As a consequence of our proof, we also show that, if A is a representation-infinite iterated tilted algebra of Euclidean type, then A is strongly simply connnected if and only if the orbit graph of each of the directed components of its Auslander-Reiten quiver is a tree, or if and only if it does not contain $\tilde{\mathbb{A}}_m$ as a full convex subcategory.

Notice that if follows from [15] that the orbit graphs of the non-directed components of an arbitrary tilted algebra are of the form $\mathbb{A}_\infty$, $\tilde{\mathbb{A}}_m$ with $m \geq 2$, $\mathbb{A}_2$ or a loop.

This paper consists of three sections : in the first, we give the relevant definitions and prove some preliminary results, in the second, we consider the Euclidean case, and in the third, we prove our main results.

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1. Preliminary results

1.1. Notation and definitions. Throughout this paper, k denotes a fixed algebraically closed field. By algebra is meant a basic, connected, associative finite dimensional k-algebra with an identity, and by module a finitely generated right module. We sometimes consider an algebra A as a k-category, whose object set is denoted by A_0, see [8]. A full subcategory C of A is called *convex* if for any path $a_0 \to a_1 \to \cdots \to a_t$ in A, with $a_0, a_t \in C_0$, we have $a_i \in C_0$ for all i. For an algebra A, we denote by $\operatorname{mod} A$ its module category and by $P(a)$ (or $I(a)$, or $S(a)$) the indecomposable projective (or injective, or simple, respectively) module corresponding to $a \in A_0$. We use freely and without further reference properties of $\operatorname{mod} A$, the Auslander-Reiten translations $\tau_A = \mathrm{DTr}$ and $\tau_A^{-1} = \mathrm{TrD}$, and the Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$ of A, as can be found, for instance, in [7, 17]. For tilted algebras, we refer the reader to [16,14] and for iterated tilted algebras, to [3, 4, 5, 6].

Following [18], we say that an algebra A is *strongly simply connected* if it satisfies the following equivalent conditions:

(a) Any full convex subcategory of A is simply connected.
(b) Any full convex subcategory of A satisfies the separation condition.
(c) For any full convex subcategory C of A, the first Hochschild cohomology space $H^1(C)$ of C with coefficients in ${}_CC_C$ vanishes.

For instance, a straightforward analysis of the lists in [13] shows that a tame concealed algebra is strongly simply connected if and only if it is simply connected, or if and only if it is not hereditary of type $\tilde{\mathbb{A}}_m$. There exist however simply connected (tilted) algebras which are not strongly simply connected (see (2.4) below or [18])

We need the following definitions and results from [2]. Let $A \cong kQ/I$ be a bound quiver presentation of an algebra A. A *contour* (p,q) in Q from x to y is a non-oriented cycle consisting of a pair of non-trivial paths p, q from x to y. A contour (p,q) is *interlaced* if p and q have a common point besides its end-points. It is *irreducible* if there does not exist a sequence of paths $p = p_0, p_1 \dots p_m = q$ from x to y such that each of the contours (p_i, p_{i+1}) is interlaced. A (simple, unoriented) cycle C in Q is *irreducible* if either C is an irreducible contour or C is not a contour, but satisfies the following condition and its dual : for each source x in C, no proper successor of x in Q is also a source in C, and exactly two proper successors of x in Q are sinks in C. Finally, a contour (p,q) from x to y is *naturally contractible* in (Q,I) if there exists a sequence of paths $p = p_0, p_1, \cdots p_m = q$ in Q such that, for each $0 \le i < m$, the paths p_i and p_{i+1} have subpaths q_i and q_{i+1}, respectively, which are involved in the same minimal relation in (Q,I). We have the following characterisation of strongly simply connected algebras (see [2] (1.6)).

Theorem. *An algebra A is strongly simply connected if and only if, for any presentation $A \cong kQ/I$, any irreducible cycle in Q is an irreducible contour, and any irreducible contour in Q is naturally contractible in (Q,I).* □

1.2. The orbit graph. Let A be an algebra and Γ be a connected component of $\Gamma(\operatorname{mod} A)$. The *orbit graph* $\operatorname{Gr}(\Gamma)$ of Γ is defined as follows : the points of $\operatorname{Gr}(\Gamma)$ are the τ-orbits ω_M of the A-modules $M \in \Gamma_0$, and there exists an edge ω_M —— ω_N if there exist $m, n \in \mathbb{Z}$ and an irreducible morphism of the form $\tau^m M \to \tau^n N$, or $\tau^n N \to \tau^m M$: in this case, the number of edges between ω_M and ω_N equals $\dim_k \operatorname{Irr}(\tau^m M, \tau^n N)$, or $\dim_k \operatorname{Irr}(\tau^n N, \tau^m M)$, respectively (here, $\operatorname{Irr}(X,Y)$ denotes the space of irreducible morphisms from the A-module X to the A-module Y).

By [8] (4.2), it is known that a representation-finite algebra is (strongly) simply connected if and only if the orbit graph of its Auslander-Reiten quiver is a tree. Here, we are mainly interested in representation-infinite algebras. We then consider the directed components of the Auslander-Reiten quiver. Let A be an algebra, a component Γ of $\Gamma(\operatorname{mod} A)$ is called *directed* if, for each M_0 in Γ, there exists no sequence of non-zero non-isomorphisms in $\operatorname{mod} A$ between indecomposable modules of the form $M_0 \to M_1 \to \cdots \to M_t = M_0$.

We conjecture that a tilted algebra is strongly simply connected if and only if for any full convex subcategory C of A, the orbit graph of each directed component of $\Gamma(\operatorname{mod} C)$ is a tree. We start by showing that the necessity part holds true.

1.3. Lemma. *Let A be a tilted algebra. If A is strongly simply connected, then the orbit graph of each directed component of $\Gamma(\operatorname{mod} A)$ is a tree.*

Proof. We may assume that A is representation-infinite. Let Γ be a directed component of $\Gamma(\operatorname{mod} A)$. We first claim that it suffices to prove the statement in case Γ is a connecting component of $\Gamma(\operatorname{mod} A)$. Indeed,the directed components of $\Gamma(\operatorname{mod} A)$ are the connecting component(s), the postprojective and preinjective components. Assume thus that Γ is a postprojective (or a preinjective) component but is not connecting. Then Γ is a standard component without injective (or projective, respectively) modules. Let $\operatorname{Ann}\Gamma$ be the intersection of the annihilators of all modules in Γ. By [16] (2.4) [19] (3.1), we have that $B = A/\operatorname{Ann}\Gamma$ is a tilted algebra, and Γ is a connecting component of $\Gamma(\operatorname{mod} B)$. On the other hand, it is easily seen that, since B is in fact the support algebra of Γ, then it is a full convex subcategory of A. Hence B is itself strongly simply connected. This establishes our claim.

Assume now that Γ is a connecting component in $\Gamma(\operatorname{mod} A)$ and that Σ is a complete slice in Γ. The underlying graph $\overline{\Sigma}$ of Σ is equal to $\operatorname{Gr}(\Gamma)$. We thus need to show that $\overline{\Sigma}$ is a tree. Now, there exists a tilting $k\Sigma$-module T such that $A = \operatorname{End} T$. By [12] (4.2), we have $H^1(A) \cong H^1(k\Sigma)$. Since A is strongly simply connected, $H^1(A) = 0$. By [12] (1.6), $H^1(k\Sigma) = 0$ implies that $\overline{\Sigma}$ is a tree. □

1.4. **Corollary.** *Let A be a strongly simply connected tilted algebra, and C be a full convex subcategory of A. Then the orbit graph of each directed component of $\Gamma(\operatorname{mod} C)$ is a tree.*

Proof. This follows from the facts that C is itself strongly simply connected and, by [11] (III.6.5), is a tilted algebra. □

2. The Euclidean case

2.1. In this section, we show that the converse of (1.4) holds if A is a tilted algebra of Euclidean type. We first need some remarks about the Euler quadratic form of an iterated tilted algebra of type $\tilde{\mathbb{A}}_m$. Let C be an iterated tilted algebra of type $\tilde{\mathbb{A}}_m$ whose quiver is a (non-oriented) cycle. By [3], C is bound by zero-relations of length two, with as many relations in the clockwise sense as in the counterclockwise sense. Also, the Euler quadratic form χ_C of C is positive semidefinite of corank one. We wish to construct a radical vector v_C for χ_C. We need to recall a result on the Ext-spaces of C, see [10,9].

Proposition. *If $i, j \in C_0$ and $m \geq 2$, then $\operatorname{Ext}^m_C(S(i), S(j)) \neq 0$ if and only if there exists a path $i = i_0 \to i_1 \cdots \to i_m = j$ and, for each $0 < t < m$, there exists a relation from i_{t-1} to i_{t+1} with midpoint i_t. Moreover, in this case, $\dim_k \operatorname{Ext}^m_C(S(i), S(j))$ equals the number of such paths.* □

One case of interest is the following : if, in the situation of the lemma, there is no relation with midpoint i and target i_1, or with midpoint j and source i_{m-1}, then we say that the path from i to j is an *$(m-1)$-fold relation* from i to j.

We now proceed to define v_C. Let us fix a sink $x \in C_0$ and put $v_C(x) = 1$. Given a point $y \in C_0$, we consider the walk w from x to y in the counterclockwise sense, and let $\rho_+(y)$ (or $\rho_-(y)$) be the number of relations in the clockwise (or the counterclockwise, respectively) sense having their midpoint on w between x and y. We then set $v_C(y) = 0$ if y is the midpoint of a relation and $v_C(y) = (-1)^{\rho_+(y)-\rho_-(y)}$ otherwise. Since $\rho_+(x) = \rho_-(x)$, the vector v_C is well-defined.

Lemma. *With the above notation, the vector v_C is a radical vector for χ_C.*

Proof. Letting $C_0 = \{1, 2, \ldots n\}$ and $v_C = (v_1, v_2, \ldots v_n)$, we have

$$\chi_C(v_C) = \sum_{i=1}^{n} v_i^2 + \sum_{m \geq 0} \sum_{i,j=1}^{n} (-1)^m v_i v_j \dim_k \operatorname{Ext}^m_C(S(i), S(j)).$$

Since the conclusion is trivial if C is hereditary, we may assume that this is not the case. We may also, without loss of generality, assume that, for each arrow $i \to j$ of C, i or j is the midpoint of a relation. For, if this is not the case, then, since $v_i = v_j$ and $\operatorname{Ext}^m_C(S(i), S(j)) = 0 = \operatorname{Ext}^m_C(S(j), S(i))$ for all $m \geq 2$, the contribution of the arrow to $\chi_C(v_C)$ is $v_i^2 + v_j^2 - v_i v_j = 1$, that is, as much as the sole point i. This assumption implies that the total number of points of C which are not midpoints of relations equals the total number of m-fold relations on C, as m varies over all non-negative integers.

Now, the first term in $\chi_C(v_C)$, namely $\sum_{i=1}^{n} v_i^2$, equals the number of points of C which are not midpoints of relations.

On the other hand, if $v_i v_j \neq 0$, then neither i nor j is the midpoint of a relation. Also, $\operatorname{Ext}^m_C(S(i), S(j)) \neq 0$ implies that there exists a path $i = i_0 \to i_1 \to \cdots \to i_m = j$ and, for each $0 < t < m$, a relation on this path with midpoint i_t. Thus $v_i v_j \dim_k \operatorname{Ext}^m_C(S(i), S(j)) \neq 0$ implies that there is an $(m-1)$-fold relation from i to j, consequently $\sum_{i,j=1}^{n} \dim_k \operatorname{Ext}^m_C(S(i), S(j))$ equals $(-1)^{m-1}$ times the number of $(m-1)$-fold relations on C. Finally $\sum_{m\geq 0}\sum_{i,j=1}^{n} v_i v_j \dim_k \operatorname{Ext}_C^m(S(i),S(j))$ equals (-1) times the number of $(m-1)$-fold relations, as m varies over all positive integers. This implies that $\chi_C(v_C)=0$. □

2.2. **Proposition.** *Let A be an iterated tilted algebra of Euclidean type. If the orbit graph of each directed component of $\Gamma(\operatorname{mod} A)$ is a tree, then A is strongly simply connected.*

Proof. We may assume that A is representation-infinite. Suppose that the orbit graph of each directed component of $\Gamma(\operatorname{mod} A)$ is a tree, but that A is not strongly simply connected. Then the bound quiver of A contains an irreducible cycle which is not a contour, or an irreducible contour which is not naturally contractible. Let C be the convex hull of the full subcategory of A generated by this cycle. By [6] (5.2), C is iterated tilted of Dynkin or Euclidean type. On the other hand, since the cycle defining C is not contractible (in C itself), then C is not simply connected. By [4], C is iterated tilted of type $\tilde{\mathbb{A}}_m$. By (2.1), the Euler quadratic form of C has a radical vector v_C such that, if $c \in C_0$, then $v_C(c) \neq 0$ if and only if c is not the midpoint of a zero-relation on C. We consider v_C as a vector in $\mathbb{Z}^{|A_0|}$, extending by zeros where necessary.

By [5] (2.5), A contains a unique tame concealed full convex subcategory B, and is an enlargement of B by branches, each of which is attached to B by a single point (the root). Also, the postprojective (or preinjective) component of A is a finite enlargement of the postprojective (or preinjective, respectively) component of B. In particular, B is tame concealed of type $\neq \tilde{\mathbb{A}}_m$ and hence is strongly simply connected. This implies that B cannot contain C as a full subcategory. Since C is clearly not contained entirely inside any individual branch and since each walk between two distinct branches passes through B, it follows that there exists at least one of the roots of the branches that belongs to C. Let a be such a root. We claim that a is either a source or a sink in C. If this is not the case, then there exist two points $a', a'' \in C_0$ and arrows $a' \to a \to a''$ on C. Since a root is either an extension or a coextension point for B, we have that either a' or a'' is contained in B, and the other in the branch K rooted at a. Since C is a cycle, there exists another branch L, rooted at b, say, such that $b \in C_0$, and this contradicts the fact that there is no walk between K and L not passing through B. This shows our claim which implies, in particular, that a is not a midpoint of a zero-relation on C, so that $v_C(a) \neq 0$. Let now v_B be a radical vector of the Euler form of B, considered as a vector in $\mathbb{Z}^{|A_0|}$, extending by zeros where necessary. Since $v_B(x) \neq 0$ if and only if $x \in B_0$, the vectors v_B and v_C are linearly independent. Since both are clearly radical vectors for the Euler form of A, which is positive semidefinite of corank one, we have reached a contradiction. □

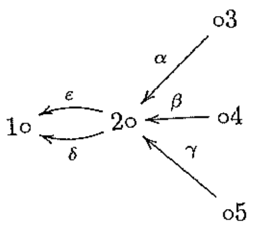
2.3. We recall that an algebra A is called $\tilde{\mathbb{A}}-free$ if there exists no full convex subcategory of A which is hereditary of type $\tilde{\mathbb{A}}_m$.

Corollary. *Let A be a representation-infinite tilted algebra of Euclidean type having a complete slice in its preinjective component. The following conditions are equivalent :*

(a) *A is strongly simply connected.*
(b) *The orbit graph of each directed component of $\Gamma(\operatorname{mod} A)$ is a tree.*
(c) *The orbit graph of the postprojective component is a tree.*
(d) *A is $\tilde{\mathbb{A}}$-free.*

Proof. The equivalence of (a) and (b) readily follows from (1.4) and (2.2), and it is trivial that (b) implies (c). If (c) holds, then the unique tame concealed full convex subcategory of A (which is the support algebra of the postprojective component) cannot be hereditary of type $\tilde{\mathbb{A}}_m$, thus showing (d). Finally, if (d) holds, then A itself has at least three non-homogeneous tubes and consequently neither A nor its unique full convex subcategory B is tilted of type $\tilde{\mathbb{A}}_m$. By [4], both A and B are simply connected so that the orbit graph of each of the postprojective component of $\Gamma(\operatorname{mod} A)$ (which coincides with that of $\Gamma(\operatorname{mod} B)$) and the preinjective component of $\Gamma(\operatorname{mod} A)$ is also a tree. We have thus shown (b). □

2.4. **Example.** Let A be given by the quiver



bound by $\alpha\delta = 0$, $\beta\delta = \beta\varepsilon$, $\gamma\varepsilon = 0$. Then A is simply connected but not strongly simply connected : indeed, the orbit graph of its postprojective component is $\tilde{\mathbb{A}}_1$. The same example shows that the class of strongly simply connected algebras is not closed under tilting : indeed, A is tilted of type $\tilde{\mathbb{D}}_4$ and a hereditary algebra of type $\tilde{\mathbb{D}}_4$ is always strongly simply connected.

3. The results

3.1. In order to prove our main theorem, we use essentially Kerner's description of the module category of a tilted algebra (see [14]). In particular, we start by showing that each of the left, and the right end algebra of tame a tilted algebra may be used to define a natural labelling of the points in the orbit graph of the connecting component. Let indeed A be a tame tilted algebra, B be the left end algebra of A, and Γ be a connecting component of $\Gamma(\operatorname{mod} A)$. Then the underlying graph of a complete slice Σ in Γ is equal to $\operatorname{Gr}(\Gamma)$. In particular, the number of points in Σ is equal to the number of objects in A_0. For a point $M \in \Sigma_0$, one of the following two cases holds:

(a) There exists $a \in A_0$ such that $P(a) \in \Gamma_0$ and $M \cong \tau_A^{-t}P(a)$ for some $t \geq 0$. In this case, we label the point in $\operatorname{Gr}(\Gamma)$ corresponding to M as ω_a.

(b) M is left stable, that is, $\tau_A^t M \neq 0$ for all $t \geq 0$, but in this case, there exists $t > 0$ large enough so that $\tau_A^t M$ is a B-module, and hence there exists a unique component B' of B, and a unique indecomposable injective B'-module $I'(b)$ such that $\tau_A^t M$, considered as a B'-module, lies in the $\tau_{B'}$-orbit of $I'(b)$. In this case, we label the point in $\operatorname{Gr}(\Gamma)$ corresponding to M as ω_b.

This natural bijection between the points in A and in $\operatorname{Gr}(\Gamma)$ is called the *left label* of $\operatorname{Gr}(\Gamma)$. We may construct similarly its *right label* using the right end algebra.

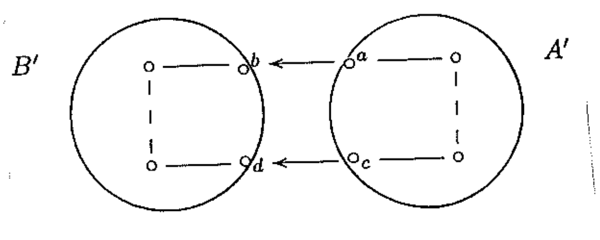
3.2. Our main theorem shows that the converse of (1.4) holds in the tame case.

Theorem. *Let A be a tame tilted algebra. Then A is strongly simply connected if and only if for each full convex subcategory C of A, the orbit graph of each directed component of $\Gamma(\operatorname{mod} C)$ is a tree.*

Proof. Since the necessity of the condition follows from (1.4), we merely have to prove the sufficiency. We proceed by induction on the number of isomorphism classes of simple A-modules. We may again assume that A is representation-infinite. Since the statement is clear if A is tame concealed, we may assume that this is not the case. By duality, we may suppose that the left end algebra B of A is not equal to A. Since A is tame, B is a direct product of tilted algebras of Euclidean type each having a complete slice in its preinjective component.

Assume that for each full convex subcategory C of A, the orbit graph of each directed component of $\Gamma(\operatorname{mod} C)$ is a tree, but that A is not strongly simply connected. Then the bound quiver of A contains an irreducible cycle which is not a contour, or an irreducible contour which is not naturally contractible. The hypothesis implies that, for each connected component B' of B, the orbit graph of each directed component of $\Gamma(\operatorname{mod} B')$ is a tree. Consequently, B' is strongly simply connected by (2.3). Thus the cycle in question cannot lie completely inside any of these connected components. We now show that it cannot actually intersect any of these components.

Let thus B' be a connected component of B such that the given cycle intersects B'. Since A is an iterated one-point extension of B, we have the following situation



where A' is the full subcategory of A generated by all objects except those lying in B' and we have, on the given cycle, arrows $a \to b$ and $c \to d$, with $a, c \in A'_0$ and $b, d \in B'_0$. Clearly, A' is convex. Since A' contains the part of the cycle which does not lie inside B', this part lies completely in one of the connected components of A'. By [11] (III.6.5), each of the connected components of A' (thus in particular the one containing that part of the cycle) is a tilted algebra. Let Σ be a complete slice in the connecting component Γ of $\Gamma(\operatorname{mod} A)$, and assume that $\operatorname{Gr}(\Gamma) = \overline{\Sigma}$ is given its left label. Further, let $\bar{\Sigma}' = \{\omega_x \in \bar{\Sigma}_0 \mid x \notin B_0'\}$. Since, by hypothesis, $\overline{\Sigma}$ is a tree, then $\bar{\Sigma}'$ is a disjoint union of trees, which (by [14] (4.1), Remark (1)) can be identified with the underlying graphs of complete slices in the connecting components of the Auslander-Reiten quiver of each of the connected components of A'. Since a and c belong to the same connected component of A', then ω_a and ω_c belong to the same tree in $\overline{\Sigma}'$. Consequently, there exists a unique (reduced) walk $\omega_a - \cdots - \omega_c$ in $\bar{\Sigma}'$. On the other hand, B' is tilted of Euclidean type and strongly simply connected so that there exists a complete slice Σ'' in the preinjective component of $\Gamma(\operatorname{mod} B')$ whose underlying graph is identified to the orbit graph of this component.

Since $\operatorname{rad} P(a)$ and $\operatorname{rad} P(c)$ have indecomposable summands which are preinjective B'-modules, there exist $x, y \in B_0'$ such that we have edges $\omega_x - \omega_a$ and $\omega_y - \omega_c$ in $\overline{\Sigma}$. Now $\overline{\Sigma}''$ is a tree, hence there exists in $\overline{\Sigma}''$ a unique (reduced) walk $\omega_x - \cdots - \omega_y$. Embedding Σ'' and Σ' inside Σ, we obtain a cycle in $\overline{\Sigma}$, namely $\omega_a - \omega_x - \cdots - \omega_y - \omega_c - \cdots - \omega_a$. This contradiction shows our claim that the cycle cannot intersect B. It therefore lies completely inside one of the connected components C of the full convex subcategory of A generated by all objects except those lying in B. The conclusion now follows at once from the hypothesis and induction. □

3.3. **Proposition.** *Let A be a representation-infinite iterated tilted algebra of Euclidean type. The following conditions are equivalent :*

(a) *A is strongly simply connected.*

(b) *The orbit graph of each directed component of $\Gamma(\operatorname{mod} A)$ is a tree.*

(c) *A is $\tilde{\mathbb{A}}$-free.*

Proof. By (2.2), (b) implies (a). We now show that, conversely, (a) implies (b). For an iterated tilted algebra of Euclidean type, the directed components are just the postprojective and preinjective components. Further, it follows from [5] (2.5) that the support algebra B of, say, the preinjective component is a finite enlargement of a representation-infinite tilted algebra C having a complete slice in its preinjective component by branches rooted in the preinjective component of C. By duality, it suffices to show that the orbit graph of the preinjective component of $\Gamma(\operatorname{mod} B)$ is a tree. We may further assume that $B \neq C$.

We first claim that, for each extension point a of C inside B, the C-module $M = \operatorname{rad} P(a)$ is indecomposable. Indeed, since A is strongly simply connected, its two full convex subcategories C and $C[M]$ satisfy the separation condition. The indecomposability of M_C then follows directly from [2] (3.1).

Let now K be a branch of B, rooted at a to the tame concealed algebra C. We claim that the orbit graph of the preinjective component Γ of $\Gamma(\operatorname{mod} C[M, K])$ is a tree. Let ω_M denote the orbit of M in $\operatorname{Gr}(\Gamma)$. Then there is an edge $\omega_M - \omega_{P(a)}$ in $\operatorname{Gr}(\Gamma)$. Since K is a branch and, for each point $b \in K_0$, we have that $\operatorname{rad} P(b)$ is indecomposable or zero, then $\operatorname{Gr}(\Gamma)$ is indeed a tree. Inductively, and since there is no walk between two distinct branches, this completes the proof that (a) implies (b).

Clearly, if A is not $\tilde{\mathbb{A}}$-free then it cannot be strongly simply connected so that (a) implies (c). Conversely, if A is not strongly simply connected, then it contains a full convex subcategory C which is not simply connected. Since, by [6] (5.2), C is itself iterated tilted of Dynkin or Euclidean type, it must, by [4], be of type $\tilde{\mathbb{A}}_m$. By [3], either C contains a hereditary algebra of type $\tilde{\mathbb{A}}_m$ as its unique tame concealed full convex subcategory, or else C contains as a full convex subcategory a representation-finite iterated tilted algebra of type $\tilde{\mathbb{A}}_m$ whose quiver is a cycle. But then, as in the proof of (2.2), we obtain a contradiction to the fact that the quadratic form of A is positive semidefinite of corank one. □

Résumé substantiel

Soit A une algèbre de dimension finie sur un corps algébriquement clos. Un théorème dû à Bongartz et Gabriel [8] dit que, si A est de représentation finie, elle est simplement connexe si et seulement si le graphe orbital de son carquois d'Auslander-Reiten est un arbre. Dans cet article, nous considérons une généralisation possible de cet énoncé en cherchant un critère permettant de vérifier si une algèbre inclinée de représentation infinie est fortement simplement connexe au sens de [18]. Dans ce cas, le carquois d'Auslander-Reiten n'étant pas connexe, il faut considérer le graphe orbital de chacune de ses composantes connexes. Les composantes contenant des cycles orientés étant caractérisées dans [15], il reste à étudier les composantes acycliques. Nous prouvons le théorème suivant.

Theorem. *Soit A une algèbre inclinée. Si A est fortement simplement connexe et C est une sous-catégorie pleine et convexe de A, alors le graphe orbital de chacune des composantes acycliques du carquois d'Auslander-Reiten de C est un arbre. Si A est docile, la réciproque est aussi vraie.*

Il suit de notre preuve que, si A est pré-inclinée de type Euclidien et de représentation infinie, alors A est fortement simplement connexe si et seulement si le graphe orbital de chacune des composantes acycliques de son carquois d'Auslander-Reiten est un arbre, ou si et seulement si elle ne contient pas de sous-catégorie pleine et convexe qui est héréditaire de type $\tilde{\mathbb{A}}_m$

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