# STRONGLY SIMPLY CONNECTED ONE-POINT EXTENSIONS OF TAME HEREDITARY ALGEBRAS

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Abstract. We classify the strongly simply connected one-point extensions of tame hereditary algebras (over an algebraically closed field) by classifying the completely coseparating modules over an algebra whose quiver is a Dynkin or an Euclidean tree. We also obtain a complete classification of the completely coseparating modules over an algebra whose quiver is a star with three branches.

## Introduction

Covering techniques allow to reduce the representation theory of a (basic and connected) finite dimensional algebra to the representation theory of a simply connected algebra. While simply connected representation-finite algebras are now well-understood, little is known in the representation-infinite case. There exists however a subclass of the class of simply connected that seems more accessible, namely the strongly simply connected algebras introduced by Skowroński in [8]. The representation theory of strongly simply connected algebras was heavily investigated recently, and one of the most striking results states that a strongly simply connected algebra is of polynomial growth if and only if it is a multicoil algebra [9]. It was thus natural to look for explicit construction procedures for strongly simply connected algebras. Since the quiver of a strongly simply connected algebra has no oriented cycles, one can construct such an algebra by a sequence of one-point extensions (or coextensions). This point of view was taken in [1], where it is shown that, if $A$ is an algebra and $M$ is an $A$-module, then the one-point extension $A[M]$ is strongly simply connected if and only if $A$ is strongly simply connected and $M$ is completely coseparating in the sense of [1] (3.3).

The purpose of this paper is to classify the strongly simply connected one-point extensions of a tame hereditary algebra, that is, of a hereditary algebra whose quiver is a Dynkin or an Euclidean quiver. Applying the above characterisation, this amounts to computing all the completely coseparating modules over the tame hereditary algebras of type not equal to $\tilde{\mathbb{A}}_m$, that is, whose quiver is a tree.

The paper is organised as follows : in section 1, we recall the needed definitions and results of [1]; in section 2, we consider the Schurian one-point extensions of tame hereditary algebras; section 3 is devoted to preparatory lemmata; in section 4, we classify all the completely coseparating modules over the hereditary algebras whose quivers are stars with three branches. This class contains many wild hereditary algebras, but also all the algebras

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whose quiver is a Dynkin or an Euclidean tree distinct from $\tilde{\mathbb{D}}_n$. This last case is solved in section 5.

Clearly, the duals of the results obtained here, with one-point coextensions and completely separating modules, hold. They are omitted for the sake of brevity. For all notions and results not explicitly recalled here, we refer the reader to [2] and [7].

## 1. Preliminaries

1.1 **Notation.** Throughout this paper, $k$ denotes a fixed algebraically closed field. By algebra is meant an associative, finite dimensional $k$-algebra with an identity, assumed moreover to be basic.

We recall that a **quiver** $Q$ is defined by its set of points $Q_0$ and its set of arrows $Q_1$. For an algebra $A$, we denote by $Q_A$ the quiver of $A$. It is well-known that there exists an admissible ideal $I$ of the path algebra $kQ_A$ such that $A \cong kQ_A/I$ : the pair $(Q_A, I)$ is then called a **presentation** of $A$. An algebra $A = kQ/I$ can equivalently be considered as a $k$-category, of which the object class $A_0$ is $Q_0$, and the set $A(x, y)$ of morphisms from $x$ to $y$ is the $k$-vector space $kQ(x, y)$ of linear combinations of paths in $Q$ from $x$ to $y$ modulo the subspace $I(x, y) = I \cap kQ(x, y)$, see [4]. A full subcategory $B$ of $A$ is called **convex** if any path in $A$ with source and target in $B$ lies entirely in $B$. An algebra $A$ is called **triangular** if $Q_A$ has no oriented cycles.

By $A$-module is meant a finitely generated right $A$-module. We denote by $\operatorname{mod} A$ their category. If $A = kQ/I$, then $\operatorname{mod} A$ is equivalent to the category of all bound representations of $(Q, I)$, we may thus identify a module $M$ with the corresponding representation $(M_x, M_\alpha)_{x\in Q_0, \alpha\in Q_1}$, see [4]. For $x \in Q_0$, we denote by $S(x)$ the corresponding simple $A$-module, and by $P(x)$ the projective cover of $S(x)$.

1.2 **Strong simple connectedness.** Let $A$ be a triangular algebra. The **support** $\operatorname{Supp} M$ of an $A$-module $M$ is the full subcategory of $A$ with object class $\{x \in A_0 \mid M_x \neq 0\}$. We shall sometimes identify $\operatorname{Supp} M$ to its quiver. A module $M$ is called **separated** if the supports of the distinct indecomposable summands of $M$ lie in distinct connected components of $A$. For $x \in A_0$, let $A^x$ be the full subcategory of $A$ with objects all non-predecessors of $x$ in $Q_A$, then $x$ is called **separating** if the restriction to $A^x$ of $\operatorname{rad} P(x)_A$ is a separated $A^x$-module. We say that $A$ satisfies the **separation condition** if each $x \in A_0$ is separating, see [3].

A connected triangular algebra $A$ is called **strongly simply connected** if each connected full convex subcategory of $A$ satisfies the separation condition [8]. For other equivalent definitions, we refer the reader to [8] (4.1) and [1] (1.6).

Let $A$ be a triangular algebra and $M$ be an $A$-module. An ordering $\{x_1, \ldots, x_m\}$ of the objects of $\operatorname{Supp} M$ is an **admissible order** if $j > i$ implies that $x_j$ is not a successor of $x_i$. For such an admissible order, let $A^{(0)} = A$ and let, for each $1 \leq i < m$, $A^{(i)}$ be the full subcategory of $A$ with objects the non-successors of $x_1, \ldots, x_i$. Clearly, each $A^{(i)}$ is convex in $A$. An $A$-module $M$ is called **completely coseparated** if, for each admissible order $\{x_1, \ldots, x_m\}$ of the points of $\operatorname{Supp} M$ and each $0 \leq i < m$, the restriction $M^{(i)} = M \mid_{A^{(i)}}$ is a separated $A^{(i)}$-module. We may now state the criterion for the strong simple connectedness of the one-point extention $A[M]$ of $A$ by $M$.

**Theorem** [1] (3.4)**.** *Let $A$ be an algebra and $M$ be an $A$-module. Then $A[M]$ is strongly simply connected if and only if $A$ is strongly simply connected and $M$ is completely coseparated.* $\square$

It follows from this theorem and [8](4.2) that any completely coseparated module is a brick (that is, $\operatorname{End} M \cong k$), see [1] (3.4).

1.3 **Schurian strongly simply connected algebras.** An algebra $A$ is called **Schurian** if $\dim_k A(x,y) \leq 1$ for all $x, y \in A_0$. A Schurian strongly simply connected algebra has, by [1](2.4), a normed presentation (in fact, all algebras considered in this paper have normed presentations). We now describe all $A$-modules $M$ such that $A[M]$ is Schurian and strongly simply connected. Let $Q$ be a quiver without oriented cycles, and $Q'$ be a full subquiver of $Q$. An ordering $\{x_1, \ldots, x_m\}$ of the point of $Q'$ is an **admissible order** if $j > i$ implies that $x_j$ is not a successor of $x_i$. For such an admissible order, let $Q^{(0)} = Q$ and let, for each $1 \leq i < m$, $Q^{(i)}$ be the full subquiver of $Q$ with points the non-successors of $x_1, \ldots, x_i$. Clearly, each $Q^{(i)}$ is convex in $Q$ (that is, each path in $Q$ with source and target in $Q^{(i)}$ lies entirely in $Q^{(i)}$). We say that $Q'$ is **completely coseparated** if, for any admissible order $\{x_1, \ldots x_m\}$ of the points of $Q'$, and each $1 \leq i < m$, the intersection of $Q'$ with each of the connected components of $Q^{(i)}$ is empty or connected. If $A$ is strongly simply connected and $M$ is completely coseparated, then the quiver of $\operatorname{Supp} M$ is a completely coseparated subquiver of $Q_A$, see [1] (4.1).

Let $A = kQ_A/I$. A full subquiver $Q$ of $Q_A$ is **zero-relation-free** if no path in $Q$ lies in $I$. Given a full subquiver $Q$ of $Q_A$, we denote by $U(Q)$ the representation of $Q_A$ defined by :

$$U(Q)_x = \begin{cases} k & \text{if } \ x \in Q_0 \\ 0 & \text{if } \ x \notin Q_0 \end{cases}$$

and

$$U(Q)_\alpha = \begin{cases} 1 & \text{if } \ \alpha \in Q_1 \\ 0 & \text{if } \ \alpha \notin Q_1 \end{cases}$$

If $A$ is Schurian strongly simply connected with normed presentation $A \cong kQ_A/I$, and $Q$ is a zero-relation-free connected full convex subquiver of $Q_A$, then $U(Q)$ is an indecomposable $A$-module [1] (4.2). In fact, we have the following.

**Theorem** [1] (4.3). *Let $A$ be Schurian strongly simply connected with normed presentation $A \cong kQ_A/I$ and $M$ be an $A$-module. Then $A[M]$ is Schurian strongly simply connected if and only if $M \cong U(Q)$ where $Q$ is a zero-relation-free completely coseparated connected full convex subquiver of $Q_A$.* $\square$

## 2. The Schurian case

In the case of tree algebras, one is able to make more precise the statement of (1.3) above. Observe that all tree algebras are Schurian and strongly simply connected.

**Lemma 2.1.** *Let $A \cong kQ_A/I$ be a tree algebra and $M$ be an $A$-module. Then $A[M]$ is Schurian and strongly simply connected if and only if $M \cong U(Q)$, where $Q (= \operatorname{Supp} M)$ is a zero-relation-free connected full subquiver of $Q_A$.*

*Proof.* Since necessity follows trivially from Theorem (1.3), let us show the sufficiency. Since $A$ is a tree algebra, it is Schurian, strongly simply connected and $A \cong kQ_A/I$ is a normed presentation. Let $M$ be as given, and $Q = \operatorname{Supp} M$. Since the intersection of two connected subquivers of a tree is empty or connected, then $Q$ is a completely coseparating subquiver of $Q_A$. Also, $Q$ is convex : let $x_0 \to x_1 \to \cdots \to x_t$ be a path in $Q_A$ with $x_0, x_t \in Q_0$, then this path coincides with the unique reduced walk joining $x_0$ to $x_t$ in the tree $Q_A$, and hence lies in $Q$, since $Q$ is connected. We then apply Theorem (1.3). $\square$

**Corollary 2.2.** *Let $A$ be a hereditary algebra of type $\mathbb{A}_m$, and $M$ be an $A$-module. The following conditions are equivalent :*

(a) *$M$ is indecomposable.*

(b) *$M$ is completely coseparating.*
(c) *$M \cong U(Q)$, where $Q$ is a connected full subquiver of $Q_A$.*

*Proof.* (c) implies (b) by (2.1), (b) implies (a) trivially, and (a) implies (c) by a well-known property of hereditary algebras of type $\mathbb{A}_m$. $\square$

*Remark* 2.3. Let $A$ be a tree algebra, and $M$ be an indecomposable $A$-module such that $\dim_k M_x \leq 1$ for each $x \in (Q_A)_0$. Then $M \cong U(\operatorname{Supp} M)$. This follows from the proof of [1] (4.3) or from [6] (2.9).

## 3. Preparatory results

**Lemma 3.1.** *Let $A$ be a triangular algebra and $M$ be an $A$-module. Then $M$ is completely coseparating if and only if $M$ is separated and, for every sink $x$ in $\operatorname{Supp} M$, the $A^{(x)}$-module $M^{(x)}$ is completely coseparated.*

*Proof.* This follows immediately from the definition. $\square$

**Lemma 3.2.** *Let $A$ be a hereditary tree algebra, $x$ be a sink in $Q_A$ having $n \geq 2$ neighbours $y_1, \ldots, y_n$ and $M_A$ be an indecomposable module such that $\dim_k M_{y_i} = 1$ for every $1 \leq i \leq n$. Then $\dim_k M_x \leq n-1$.*

*Proof.* Since $Q_A$ is a tree then, for each $i$, there exists a unique arrow $\alpha_i : y_i \to x$. We first note that $M_x = \sum\limits_{i=1}^{n} \operatorname{Im} M_{\alpha_i}$ : indeed, if this is not the case, then $S(x)$ is a proper direct summand of $M$, and this contradicts the indecomposability of $M$. Moreover,

$$\dim_k M_x = \dim_k \left( \sum_{i=1}^{n} \operatorname{Im} M_{\alpha_i} \right) \leq \sum_{i=1}^{n} \dim_k \left( \operatorname{Im} M_{\alpha_i} \right) \leq \sum_{i=1}^{n} \dim_k M_{y_i} = n.$$

Assume that $\dim_k M_x = n$. Then $M_{\alpha_1}, \ldots, M_{\alpha_n} : k \to k^n$ are linearly independent, hence the vectors $M_{\alpha_1}(1), \ldots, M_{\alpha_n}(1)$ form a basis of $M_x = k^n$. Consequently, $M_x = \bigoplus\limits_{i=1}^{n} \operatorname{Im} M_{\alpha_i}$ and $M$ is decomposable, a contradiction. $\square$

**Lemma 3.3.** *Let $A$ be an algebra and $M$ be an $A$-module such that the intersection of the support $B$ of $M$ with each connected component of $A$ is empty or connected. Then $M$ is separated as an $A$-module if and only if $M$ is separated as a $B$-module.*

*Proof.* Let $C$ be a connected component of $B = \operatorname{Supp} M$, and $C'$ be the unique connected component of $A$ that contains $C$. Then $C' \cap B = C$. Indeed, it is clear that $C \subseteq C' \cap B$. Conversely, let $x \in (C' \cap B)_0$ and $y \in C_0$. Since $C' \cap B$ is connected, there exists a walk from $x$ to $y$ in $C' \cap B$. Since $C$ is a connected component of $B$, this implies that $x \in C_0$. Hence $C' \cap B = C$.

Conversely, let $C'$ be a connected component of $A$ such that $C' \cap B \neq \emptyset$. By hypothesis $C' \cap B$ is connected so let $C$ be the unique connected component of $B$ that contains it. By the same reasoning as above, $C' \cap B = C$.

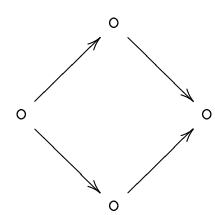
By definition, $M$ is separated as an $A$-module if and only if, for every connected component $C'$ of $A$, we have that $M\,|_{C'}$ is indecomposable or zero. The previous argument shows that this is the case if and only if, for every connected component $C$ of $B$, we have that $M\,|_{C}$ is indecomposable or zero, that is, if and only if $M$ is separated as a $B$-module. $\square$

**Proposition** 3.4. *Let $A$ be a tree algebra, and $M$ be an $A$-module such that the intersection of the support $B$ of $M$ with each connected component of $A$ is empty or connected. Then $M$ is completely coseparating as an $A$-module if and only if $M$ is completely coseparating as a $B$-module.*

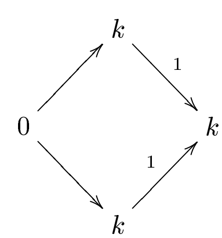
*Proof.* Let $\{x_1, \ldots, x_m\}$ be an admissible order of the objects of $B$(see (1.2)). Let $1 \leq i \leq m$ and $C$ be a connected component of $A^{(i)}$. We claim that $C \cap B^{(i)}$ is connected. Let $x, y$ be two objets in $C \cap B^{(i)}$. Then there exists a unique connected component $C'$ of $A$ that contains $C$. We have $C \cap B^{(i)} \subseteq C' \cap B$ and $C' \cap B$ is connected by hypothesis. Hence there exists a reduced walk $w$ joining $x$ to $y$ in $C' \cap B$. On the other hand, $C$ is connected, hence there exists a reduced walk $w'$ joining $x$ to $y$ in $C$. But $A$ is a tree algebra, hence $w = w'$ and lies in $C \cap B = C \cap A^{(i)} \cap B = C \cap B^{(i)}$. This establishes our claim. Since $\operatorname{Supp} M |_{A^{(i)}} = B^{(i)}$, it follows from (3.3) that $M |_{A^{(i)}}$ is a separated $A^{(i)}$-module if and only if $M |_{B^{(i)}}$ is a separated $B^{(i)}$-module. The conclusion follows. $\square$

*Remark* 3.5.

(a) We shall apply this proposition to the case where $A$ is a tree algebra and $M$ is an indecomposable $A$-module : it allows us to assume that $M$ is sincere.
(b) Proposition (3.4) is not true if the quiver of $A$ is not a tree : let indeed $A$ be the tame hereditary algebra with quiver



then the indecomposable $A$-module defined by

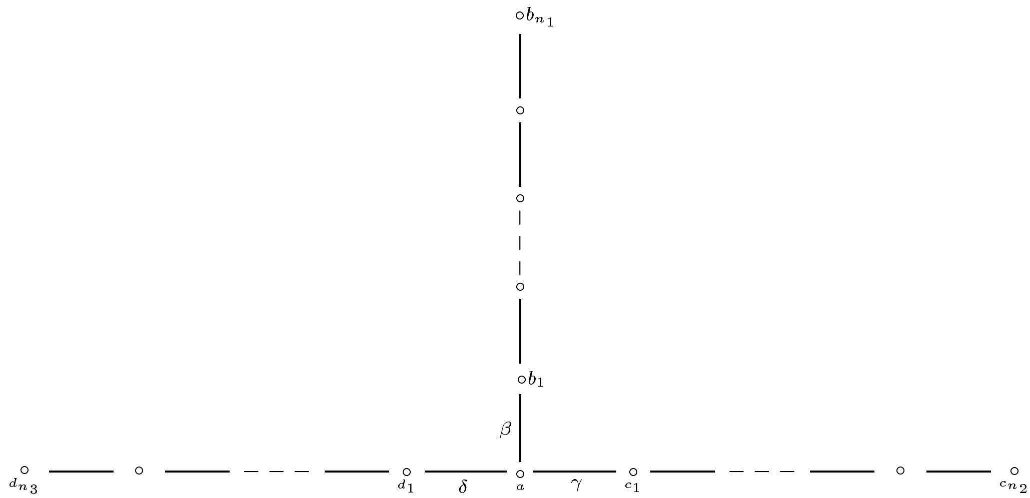



is completely coseparated over its support algebra, but clearly not over $A$.

## 4. Stars with three branches

The aim of this section is to classify the completely coseparating modules over the hereditary algebras $A$ whose quiver $Q_A$ has for underlying graph $\overline{Q}_A$ a star with three branches

$\mathbb{T}_{n_1,n_2,n_3}$ where $n_1, n_2, n_3 \geq 1$.




The unique point $a$ with three neighbours will be called the **node** of the quiver. In the sequel, we shall always use the notation of the above figure.

**Definition** 4.1. Let $A$ be a hereditary algebra of type $\mathbb{T}_{n_1,n_2,n_3}$ whose node is a sink. Let $Q$ be a connected full subquiver of $Q_A$ containing both the node and its neighbours. We define the $A$-module $V(Q)$ by :

$$V(Q)_x = \begin{cases} k^2 & x = a \\ k & x \in Q_0 \backslash \{a\} \\ 0 & x \notin Q_0 \end{cases}$$

$$V(Q)_\alpha = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \alpha = \beta \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \alpha = \gamma \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \alpha = \delta \\ 1 & \alpha \in Q_1 \backslash \{\beta, \gamma, \delta\} \\ 0 & \alpha \notin Q_1 \end{cases}$$

**Lemma** 4.2. *Let $A$ be a hereditary algebra of type $\mathbb{T}_{n_1,n_2,n_3}$ whose node is a sink. Let $Q$ be a connected full subquiver of $Q_A$ containing both the node and its neighbours. Then $V(Q)$ is completely coseparating.*

*Proof.* First, $V(Q)$ is clearly indecomposable. We show the statement by induction on $n = \mid Q_0 \mid$. If $n = 4$, the only sink in $Q$ is the node $a$. Also, $Q^{(a)} = Q_b \dot{\bigcup} Q_c \dot{\bigcup} Q_d$ where $Q_b$ (or $Q_c$, $Q_d$) is connected and contains as unique point $b_1$ (or $c_1$, $d_1$, respectively). Then $\overline{Q}_b = \overline{Q}_c = \overline{Q}_d = \mathbb{A}_1$ and $V(Q)^{(a)} = U(Q_b) \bigoplus U(Q_c) \bigoplus U(Q_d) = k \bigoplus k \bigoplus k$ (by (2.2)) which is completely coseparating. By (3.1), $V(Q)$ is completely coseparating over its support algebra hence, by (3.4), over $A$.

Inductively, we assume that the statement holds for $t < n$. Let $x$ be a sink in $Q$. If $x = a$, then $Q^{(x)} = Q_b \dot{\bigcup} Q_c \dot{\bigcup} Q_d$, where $Q_b$ (or $Q_c$, $Q_d$) is connected and contains $b_1$ (or $c_1, d_1$, respectively) is completely coseparating. Then $\overline{Q}_b = \mathbb{A}_{n_1}$, $\overline{Q}_c = \mathbb{A}_{n_2}$, $\overline{Q}_d = \mathbb{A}_{n_3}$ and $V(Q)^{(x)} = U(Q_b) \bigoplus U(Q_c) \bigoplus U(Q_d)$ is completely coseparating. If $x \neq a$, then $x$ is not a neighbour of $a$, hence $Q^{(x)} = Q' \dot{\bigcup} Q''$ where $Q''$ is empty or $\overline{Q}'' = \mathbb{A}_\ell$ (for some $\ell$), while $Q'$ is a star with three branches and $\mid Q'_0 \mid < \mid Q_0 \mid$. By the induction hypothesis, $V(Q)^{(x)} = V(Q') \bigoplus U(Q'')$ is completely coseparating. The statement follows upon applying (3.1) and (3.4). $\square$

**Theorem** (4.3). *Let $A$ be a hereditary algebra of type $\mathbb{T}_{n_1,n_2,n_3}$ and $M$ be an $A$-module with $Q = \operatorname{Supp} M$ connected. Then $M$ is completely coseparating if and only if $M \cong U(Q)$, or the node of $Q_A$ is a sink contained in $Q$ together with its neighbours and $M \cong V(Q)$.*

*Proof.* Since the sufficiency follows from (2.1) and (4.2), let us show the necessity. Assume $M \not\cong U(Q)$. Then the node $a$ of $Q_A$ and its neighbours $b_1, c_1, d_1$ necessarily lie in $Q$: for, if this is not the case, then $\overline{Q} = \mathbb{A}_\ell$ (for some $\ell$) and, by (2.2), $M \cong U(Q)$, a contradiction. We now claim that $a$ is a sink in $Q$. Assume that this is not the case, we show by induction on $n = \mid Q_0 \mid$ that $M \cong U(Q)$.

If $n = 4$, let $x$ be a sink in $Q$, then $x$ is a neighbour of $a$. Moreover, $\overline{Q}^{(x)} = \mathbb{A}_m$ and, by (2.2), $M^{(x)} \cong U(Q^{(x)})$. Since $M$ is an indecomposable module over a hereditary algebra of type $\mathbb{D}_4$, we have $M \cong U(Q)$. Inductively, assume that $n > 4$ and that the statement holds for $t < n$. Let $x \in Q_0$ be a sink. Then $x \neq a$. Moreover, $Q^{(x)} = Q' \dot{\bigcup} Q''$ where $Q'$ is empty or $\overline{Q}' = \mathbb{A}_\ell$ and $Q''$ is a star with three branches and $\mid Q''_0 \mid < n$ or $\overline{Q}'' = \mathbb{A}_m$. Since $M^{(x)}$ is completely coseparating, the induction hypothesis and (2.2) yield $M^{(x)} \cong U(Q') \bigoplus U(Q'')$. Since $M$ is indecomposable, then $\dim_k M_x \leq 1$ by (3.2). By (2.3), we get $M \cong U(Q)$. This establishes our claim that $a$ is a sink in $Q$.

There remains to show that, if $a$ is a sink contained in $Q$ together with $b_1, c_1, d_1$ and $M \not\cong U(Q)$, then $M \cong V(Q)$. Since $M^{(a)}$ is completely coseparating and $Q^{(a)} = Q_b \dot{\bigcup} Q_c \dot{\bigcup} Q_d$ where $Q_b$ (or $Q_c$, $Q_d$) is connected and contains $b_1$ (or $c_1, d_1$, respectively) then $\overline{Q}_b = \mathbb{A}_{n_1}$, $\overline{Q}_c = \mathbb{A}_{n_2}$, $\overline{Q}_d = \mathbb{A}_{n_3}$ and (2.2) yield $M^{(a)} = U(Q_b) \bigoplus U(Q_c) \bigoplus U(Q_d)$. Since $M$ is indecomposable, then $\dim_k M_a \leq 2$ by (3.2). The hypothesis that $M \not\cong U(Q)$ and (2.3) yield $\dim_k M_a = 2$. Let $A'$ be the full subcategory of $A$ with objects $\{a, b_1, c_1, d_1\}$. Then $A'$ is hereditary of type $\mathbb{D}_4$ and its node is a sink. Since the restriction $M' = M \mid_{A'}$ is an indecomposable $A'$-module, it is isomorphic to $V(Q_{A'})$. Therefore, $M \cong V(Q)$. $\square$

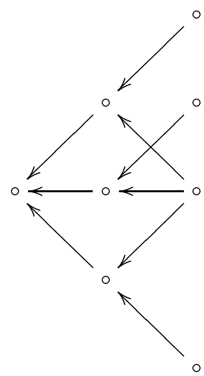
**Corollary** 4.4. *Let $A$ be a tame hereditary algebra such that $\overline{Q}_A \neq \tilde{\mathbb{A}}_m, \tilde{\mathbb{D}}_n$ and $M$ be an $A$-module with $Q = \operatorname{Supp} M$ connected. Then $M$ is completely coseparating if and only if either $M \cong U(Q)$ or else $Q$ contains a sink and three neighbours of this sink, and $M \cong V(Q)$.* $\square$

*Remark* 4.5.

(a) If $A$ is hereditary, and $M$ is a completely coseparating postprojective (or preinjective) $A$-module, then $A[M]$ is a tilted algebra having a complete slice in the postprojective (or the preinjective, respectively) component. This is the case, for instance, if $M$ is a completely coseparating module over a hereditary algebra of Dynkin type. We note that $A[M]$ may be representation-finite, tame, or even wild (assume indeed that $Q_A$ has at least five sources, and that $M$ is sincere).

(b) Assume that $A$ is hereditary of Euclidean type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$, that the node of $Q_A$ is a sink, and that $Q$ is a connected full subquiver of $Q_A$ containing the node and its three neighbours. An easy calculation, using the defect function of $A$, as in [5], shows that $V(Q)$ is regular if and only if $Q = Q_A$ and the node of $Q_A$ is its only sink, and this is the case if and only if $A[V(Q)]$ is tubular canonical (and of type (3,3,3), (2,4,4) or (2,3,6) according as $A$ is of type $\tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$, respectively). Otherwise, $V(Q)$ is postprojective.

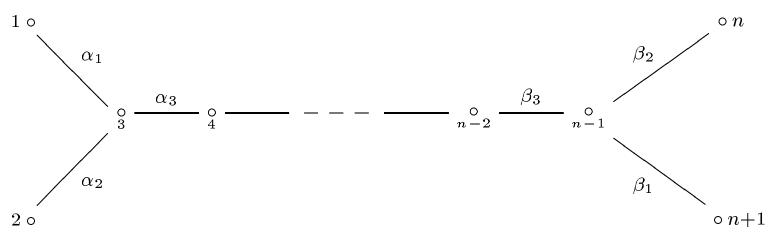
On the other hand, $U(Q)$ may be postprojective, preinjective, or regular. For example, the algebra given by the fully commutative quiver



(which is tubular of type (3,3,3)) is of the form $A[U(Q)]$, with $A$ hereditary of type $\tilde{\mathbb{E}}_6$.

## 5. The case $\tilde{\mathbb{D}}_n$

Assume first that $n \geq 5$. The ordinary quiver $Q$ of a hereditary algebra of type $\tilde{\mathbb{D}}_n$ has two points with three neighbours (also called **nodes**). We use the following notation for the underlying graph $\overline{Q} = \tilde{\mathbb{D}}_n$ of $Q$




This allows to define three types of sincere modules according to whether or not 3 and / or $n-1$ are sinks in $Q$.

**Definition** 5.1. Let $A$ be a hereditary algebra of type $\overline{Q} = \tilde{\mathbb{D}}_n$, with $n \geq 5$.

(i) If 3 is a sink in $Q$, we define the sincere $A$-module $V_3(Q)$ by

$$V_3(Q)_x = \begin{cases} k^2 & x = 3 \\ k & x \neq 3 \end{cases}$$

$$V_3(Q)_\alpha = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \alpha = \alpha_1 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \alpha = \alpha_2 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \alpha = \alpha_3 \\ 1 & \alpha \in Q_1 \backslash \{\alpha_1, \alpha_2, \alpha_3\} \end{cases}$$

(ii) If $n-1$ is a sink in $Q$, we define the sincere $A$-module $V_{n-1}(Q)$ by

$$V_{n-1}(Q)_x = \begin{cases} k^2 & x = n-1 \\ k & x \neq n-1 \end{cases}$$

$$V_{n-1}(Q)_\alpha = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \alpha = \beta_1 \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \alpha = \beta_2 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \alpha = \beta_3 \\ 1 & \alpha \in Q_1 \backslash \{\beta_1, \beta_2, \beta_3\} \end{cases}$$

(iii) If both 3 and $n-1$ are sinks in $Q$, we define the sincere $A$-module $V_{3,n-1}(Q)$ by

$$V_{3,n-1}(Q)_x = \begin{cases} k^2 & x \in \{3, n-1\} \\ k & x \notin \{3, n-1\} \end{cases}$$

$$V_{3,n-1}(Q)_\alpha = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \alpha \in \{\alpha_1, \beta_1\} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \alpha \in \{\alpha_2, \beta_2\} \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \alpha \in \{\alpha_3, \beta_3\} \\ 1 & \alpha \in Q_1 \backslash \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} \end{cases}$$

**Lemma** 5.2. *Let $A = kQ$, with $\overline{Q} = \tilde{\mathbb{D}}_n$ and $n \geq 5$. Then the modules $V_3(Q), V_{n-1}(Q)$ and $V_{3,n-1}(Q)$ are completely coseparating (when they are defined).*

*Proof.* Clearly, the modules $V_3(Q), V_{n-1}(Q)$ and $V_{3,n-1}(Q)$ are indecomposable (when they are defined).

We assume that 3 is a sink and show that $V_3(Q)$ is completely coseparating. Let $x$ be a sink in $Q$. If $x = 3$, then $Q^{(x)} = Q(1) \dot{\bigcup} Q(2) \dot{\bigcup} Q(4)$ where $Q(i)$ is connected and contains $i$, with $i \in \{1, 2, 4\}$. Then $\overline{Q}(1) = \overline{Q}(2) = \mathbb{A}_1$ and $\overline{Q}(4) = \mathbb{D}_{n-2}$ (except if $n = 5$, in which case $\overline{Q}(4) = \mathbb{A}_3$), and $V_3(Q)^{(x)} = U(Q(1)) \bigoplus U(Q(2)) \bigoplus U(Q(4)) = k \bigoplus k \bigoplus U(Q(4))$ is completely coseparating. If $x = n-1$, then $Q^{(x)} = Q(n-2) \dot{\bigcup} Q(n) \dot{\bigcup} Q(n+1)$ where $Q(i)$ is connected and contains $i$, with $i \in \{n-2, n, n+1\}$. Then $\overline{Q}(n) = \overline{Q}(n+1) = \mathbb{A}_4$ and $\overline{Q}(n-2) = \mathbb{D}_{n-2}$ (indeed, $n > 5$ because both 3 and $n-1$ are sinks ) and $V_3(Q)^{(x)} = V(Q(n-2)) \bigoplus k \bigoplus k$ is completely coseparating. If $x$ has two neighbours, then $4 < x \leq n-2$ and $Q^{(x)} = Q' \dot{\bigcup} Q''$ where $3 \in Q'_0, \overline{Q}' = \mathbb{D}_\ell$ (for some $\ell$) and $n-1 \in Q''_0$, $\overline{Q}'' = \mathbb{D}_{n-\ell}$ (except if $x = n-2$, in which case $\overline{Q}'' = \mathbb{A}_3$). We have $V_3(Q)^{(x)} = V(Q') \bigoplus U(Q'')$ which is completely coseparating. Finally, if $x$ has just one neighbour, then $x \in \{n, n+1\}$ and $Q^{(x)}$ is connected with $\overline{Q}^{(x)} = \mathbb{D}_n$, so that $V_3(Q)^{(x)} = V(Q^{(x)})$ is completely coseparating. Since $V_3(Q)$ is indecomposable, applying (3.1) yields the statement. One shows similarly that $V_{n-1}(Q)$ is completely coseparating.

The remains to show that, if both 3 and $n-1$ are sinks, then $V_{3,n-1}(Q)$ is completely coseparating. Let $x$ be a sink in $Q$. Either $x \in \{3, n-1\}$ or $x$ has two neighbours. In