STRONGLY SIMPLY CONNECTED ALGEBRAS

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Introduction

In the representation theory of finite dimensional algebras over an algebraically closed field, the simply connected representation-finite algebras introduced by Bongartz and Gabriel [6] have played an important rôle (see, for instance, [5,7]). The reason for their importance is that, for a representation-finite algebra A, the indecomposable A-modules can be lifted to indecomposable modules over a simply connected algebra $\tilde{A}$ (contained inside a certain Galois covering of the standard form of A, see [7]). Thus, covering techniques allow to reduce many problems of the study of representation-finite algebras to problems about simply connected representation-finite algebras. Little is known about covering techniques or simply connected algebras in the representation-infinite case. One class however of simply connected algebras has attracted much interest lately, this is the class of strongly simply connected algebras, introduced by Skowroński in [14]. The representation theory of strongly simply connected algebras seems to be relatively accessible, and some progress has been made in understanding it in the tame case (see, for instance, [13, 15]).

The purpose of this paper is to provide characterisations and construction techniques for strongly simply connected algebras. Since we are motivated by the study of coverings, we start by considering locally bounded k-categories [6] and give an alternative definition for the strong simple connectedness of a locally bounded category (1.3) which we believe is easier to handle than the one in [14]. We then show the equivalence of these two definitions and, while doing so, we obtain a handy criterion allowing to verify whether a locally bounded category is strongly simply connected or not (1.6). We next consider the case of Schurian locally bounded categories. We recall that the Schurian strongly simply connected algebras were already studied in [9], under the name of completely separating algebras. Here, we prove that a connected triangular locally bounded category is Schurian and strongly simply connected if and only if it has a presentation (called normed presentation, see [4]) such that all cycles are commutative (2.4). We deduce a new necessary and sufficient condition for a representation-finite algebra to be simply connected (2.5). We then turn our attention to the construction of strongly simply connected algebras. Since such an algebra is triangular, it can be constructed by repeated one-point extensions or coextensions. We define in (3.3) a notion of completely co-separated module, and the dual notion of completely separated module. Our main theorem (3.4) states that an algebra is strongly simply connected if and only if it is the one-point extension (or coextension) of a strongly simply connected algebra by a completely co-separated module (or a completely separated module, respectively). We end the paper with an inductive construction of the Schurian strongly simply connected algebras with a prescribed number of isomorphism classes of simple modules (4.2).

1. Strongly simply connected locally bounded categories

1.1. **Locally bounded categories.** Throughout this paper, k denotes a fixed algebraically closed field. We recall that a **k-category** A is a category where, for each pair of objets x, y of A, the set of morphisms $A(x, y)$ from x to y has a k-vector space structure such that the composition of morphisms is k-bilinear. Let A_0 denote the class of objects of A. A k-category A is called **locally bounded** [6] if : (a) for each $x \in A_0$, the endomorphism algebra $A(x, x)$ is local; (b) distinct objects are not isomorphic; and (c) for each $x \in A_0$, we have $\sum_{y \in A_0} \dim_k A(x, y) < \infty$ and $\sum_{y \in A_0} \dim_k A(y, x) < \infty$.

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Locally bounded categories are realised by locally finite quivers : if A is a locally bounded category, there exist a locally finite quiver Q_A and an admissible ideal I of the path category kQ_A of Q_A such that we have an isomorphism $A \cong kQ_A/I$, called a **presentation** of A. The pair (Q_A, I) is then called a **bound quiver**. We recall that a **quiver** Q is defined by its set of points Q_0, its set of arrows Q_1 and two mappings $Q_1 \to Q_0$ associating to each arrow its source and its target, respectively. If the quiver Q is finite and connected, a bound quiver category kQ/I can equivalently be viewed as a finite dimensional k-algebra which is moreover basic and connected. Conversely, any finite dimensional basic and connected k-algebra occurs in this way [10].

Let A be a locally bounded category, a full subcategory B of A is called **convex** if, for any path $x_0 \to x_1 \to \cdots \to x_t$ in (the quiver of) A with $x_0, x_t \in B_0$, we have $x_i \in B_0$ for all $1 \leq i < t$. The category A is called **triangular** if its quiver Q_A contains no oriented cycle.

By an A-module is meant a finitely generated right A-module. We denote here by mod A their category. It is well-known that, if $A = kQ/I$, then mod A is equivalent to the category of all bound (finite dimensional) representations of (Q, I), see [6,10]. For each $x \in Q_0$, we denote by $P(x)$ the corresponding indecomposable projective A-module.

1.2. **The fundamental group.** Let (Q, I) be a connected locally finite bound quiver. A **relation** from a point x to a point y is an element $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ such that, for each $1 \leq i \leq m$, λ_i is a non-zero scalar and w_i is a path of length at least two from x to y. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is called **minimal** if $m \geq 2$ and, for any proper non-empty subset $J \subseteq \{1, 2 \ldots m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse. A **walk** in Q from x to y is a formal composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \ldots \alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1, \varepsilon_i = \pm 1$ for all $1 \leq i \leq t$) starting at x and ending at y. We denote by e_x the trivial path at x. A walk in Q is called **reduced** if it contains no subwalk of one of the forms $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ with $\alpha \in Q_1$.

Let $\sim$ be the least equivalence relation in the set of all walks in Q such that :

(a) If $\alpha : x \to y$ is an arrow, then $\alpha\alpha^{-1} \sim e_x$ and $\alpha^{-1}\alpha \sim e_y$.
(b) If $\sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all $1 \leq i, j \leq m$.
(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions are defined.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes of all walks starting and ending at x has a group structure with operation induced from the composition of walks. Since, clearly, the group $\pi_1(Q, I, x)$ does not depend on the choice of x, we denote it by $\pi_1(Q, I)$ and call it the **fundamental group** of (Q, I), see [11,12].

1.3. **Strong simple connectedness.** Let Q be a locally finite quiver. A full subquiver Q' of Q is called **convex** if, for any path $x_0 \to x_1 \to \cdots \to x_t$ in Q with $x_0, x_t \in Q'_0$, we have $x_i \in Q'_0$ for all $1 \leq i < t$. A bound quiver (Q', I') is a full bound subquiver of a bound quiver (Q, I) if Q' is a full subquiver of Q, and $I' = I \cap kQ'$. We are now ready to define our object of study.

Definition. A connected triangular locally bounded k-category A is called **strongly simply connected** if there exists a presentation $A \cong kQ_A/I_A$ of A such that, for any connected full convex bound subquiver (Q, I) of (Q_A, I_A), we have $\pi_1(Q, I) = 1$.

Thus, if B is a full convex subcategory of A, and we denote by Q_B the full subquiver of Q_A generated by the set of points in Q_A which correspond to objects in B, and by I_B the ideal $I_B = kQ_B \cap I_A$, we have $\pi_1(Q_B, I_B) = 1$.

For example, a hereditary (or a monomial) locally bounded category is strongly simply connected if and only if its quiver is a tree.

We recall that Skowroński has given in [14] another definition of a strongly simply connected finite dimensional algebra : a triangular algebra A is called simply connected if, for any presentation $A \cong kQ_A/I_A$, we have $\pi_1(Q_A, I_A) = 1$ (see [3]); it is called strongly simply connected if every connected full convex subcategory of A is simply connected. Our first task is thus to show the equivalence of these two definitions.

1.4. For our first lemma, we need the following definition, due to Bautista, Larrión and Salmerón [5]. Let A be a triangular locally bounded k-category (not necessarily connected). An A-module M is called **separated** if, for each connected component C of A, the restriction $M\mid_C$ of M to C is either zero or indecomposable. This can be expressed in terms of supports : the **support** of an A-module M is the full subcategory Supp M of A generated by all $x \in A_0$ such that $M_x \neq 0$. Thus, an A-module M is separated if and only if the supports of the distinct indecomposable summands of M lie in distinct connected components of A. For each $x \in A_0$, let A^x denote the full subcategory of A generated by the non-predecessors of x in Q_A. The object x is called **separating** if the restriction to A^x of rad $P(x)_A$ is separated as an A^x-module. We say that A satisfies the **separation condition** if each $x \in A_0$ is a separating object. One defines dually co-separating objects and the co-separation condition.

We also need the following definition. Let Q be a locally finite quiver without oriented cycles. A **contour** (p,q) in Q from x to y is a pair of paths p,q of positive length having the same source x and the same target y.

Lemma. *Let A be a strongly simply connected locally bounded k-category. Then any connected full convex subcategory of A satisfies the separation condition.*

Proof. Let $A \cong kQ_A/I_A$ be a presentation of A such that the fundamental group of any connected full convex bound subquiver of (Q_A, I_A) is trivial. In order to establish the lemma, it suffices to show that A itself satisfies the separation condition. Suppose on the contrary that there exists an object $x \in A_0$ which is not separating. Let $R(x) = \operatorname{rad} P(x)_A$. The k-vector space $R(x)$ has as basis the residue classes modulo I of the paths in Q_A of positive length and source x. Let B be a connected component of A^x such that $R(x)\mid_B$ is decomposable. Assume $R(x)\mid_B = R_1 \oplus R_2$, with R_1, R_2 non-zero. Let (Q, I) be the full bound subquiver of (Q_A, I_A) generated by B and x. Then (Q, I) is clearly connected and convex in (Q_A, I_A). Denote by K the Kronecker quiver

$$b\circ \underset{\delta}{\overset{\gamma}{\leftleftarrows}} \circ a\ .$$

We complete the proof by constructing a group epimorphism from $\pi_1(Q, I)$ onto $\pi_1(K)$, and this is a contradiction, because $\pi_1(Q, I) = 1$ by hypothesis, while clearly $\pi_1(K) \cong \mathbb{Z}$.

We define a surjective map φ from the set of walks in Q onto the set of walks in K as follows. We set $\varphi(e_x) = e_a$ and, for all $y \in Q_0$ such that $y \neq x$, we set $\varphi(e_y) = e_b$. For an arrow $\alpha : y \to z$ in Q, we let $\varphi(\alpha) = e_b$ if $y \neq x$ and, in the case where $y = x$, we define $\varphi(\alpha) = \gamma$ if z belongs to $\operatorname{Supp} R_1$, and $\varphi(\alpha) = \delta$ if z belongs to $\operatorname{Supp} R_2$. The map φ is well-defined since $R_1 \cap R_2 = 0$. Define $\varphi(\alpha^{-1}) = \varphi(\alpha)^{-1}$. For an arbitrary walk $w = \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\dots\alpha_t^{\varepsilon_t}$ in Q from y to z, say, with $\varepsilon_i = \pm 1, 1 \leq i \leq t$, it is easily shown that

$$\varphi(w) = \varphi(\alpha_1)^{\varepsilon_1}\varphi(\alpha_2)^{\varepsilon_2}\dots\varphi(\alpha_t)^{\varepsilon_t}$$

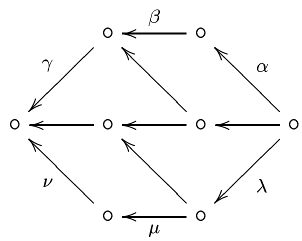
is a walk in K from the point corresponding to $\varphi(e_y)$ to the point corresponding to $\varphi(e_z)$.

Let now (p_1, p_2) be a contour in Q from y to z such that there exists a minimal relation $\sum_{i=1}^{m} \lambda_i p_i$. Write $p_1 = \alpha_1 q_1, p_2 = \alpha_2 q_2$ with $\alpha_1, \alpha_2 \in Q_1$. If $y \neq x$, then $\varphi(p_1) = \varphi(p_2) = e_b$ since x is a source in Q. Assume that $y = x$. Then $\varphi(p_1) = \varphi(\alpha_1)$ and $\varphi(p_2) = \varphi(\alpha_2)$. If the target y_1 of α_1 lies in $\operatorname{Supp} R_1$, then q_1 lies entirely in $\operatorname{Supp} R_1$. Hence q_2 also lies entirely in $\operatorname{Supp} R_1$, because $R_1 \cap R_2 = 0$. This implies that the target of α_2 lies in $\operatorname{Supp} R_1$. Therefore $\varphi(p_1) = \varphi(\alpha_1) = \varphi(\alpha_2) = \varphi(p_2) = \gamma$. Similarly, if y_1 lies in $\operatorname{Supp} R_2$, we have $\varphi(p_1) = \varphi(\alpha_1) = \varphi(\alpha_2) = \varphi(p_2) = \delta$. This shows that φ is compatible with the equivalence relation defined on (Q, I) and thus induces as required a group epimorphism $\pi_1(Q, I) \to \pi_1(K)$. □

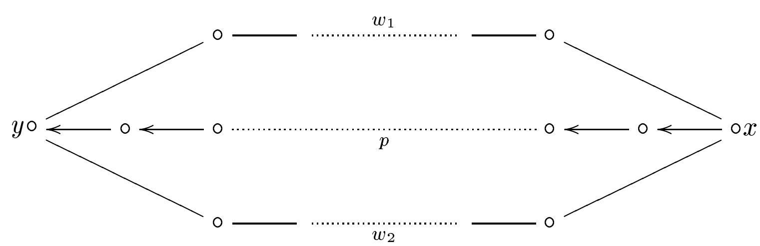
1.5. We need a few definitions and notations. Let Q be a locally finite quiver without oriented cycles. By cycle, we mean an unoriented simple cycle, that is, a subquiver C of Q is a **cycle** if each point in C is an end-point of exactly two arrows in C and there exists an enumeration $\{x_0, x_1, \dots x_{n-1}, x_n = x_0\}$ of the points of C such that there exists an edge between x_{i-1} and x_i on C, for all $1 \leq i \leq n$.

A contour (p, q) in Q from x to y is called **interlaced** if the paths p and q have a common point other than x and y. Thus, a contour is a cycle if and only if it is not interlaced. A contour (p, q) is called **reducible** if there exist paths $p = p_0, p_1, \dots p_m = q$ in Q from x to y such that, for each $1 \leq i \leq m$, the contour (p_{i-1}, p_i) is interlaced. In this case, we say that p is reducible to q. Otherwise, it is called **irreducible**.

In the following example, the contour $(\alpha\beta\gamma, \lambda\mu\nu)$ is reducible.

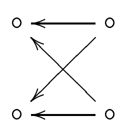


Let C be a cycle which is not a contour. Denote by $\sigma(C)$ the number of sources of C (which actually equals the number of sinks of C, by our definition of cycle). Thus $\sigma(C) > 1$. The cycle C is said to be **reducible** if there exist two points x, y in C, and a path $p : x \to \cdots \to y$ in Q as follows



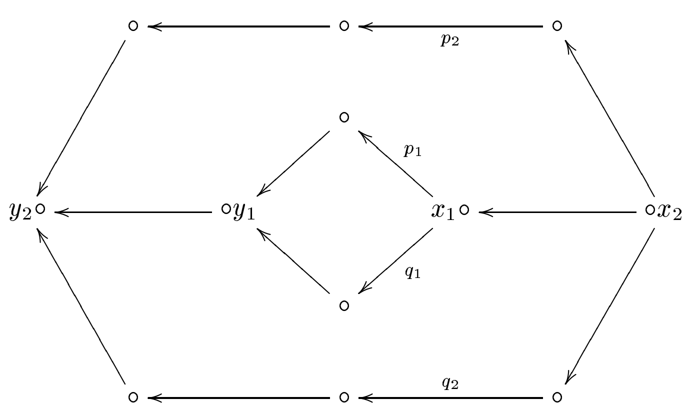
where the cycle C consists of the walks w_1 and w_2, such that both w_1p^{-1} and w_2p^{-1} are cycles and $\sigma(w_1p^{-1}) < \sigma(C), \sigma(w_2p^{-1}) < \sigma(C)$. We then say that a path such as p **reduces the cycle** C. A cycle C is said to be **irreducible** if it is either an irreducible contour, or it is not a contour, but it is not reducible in the above sense.

A typical example of an irreducible cycle which is not an irreducible contour is as follows.



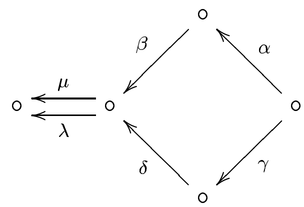
We also define a partial order on the contours in Q as follows. Let (p_1, q_1) and (p_2, q_2) be two contours from x_1 to y_1 and x_2 to y_2, respectively. Then $(p_1, q_1) \le (p_2, q_2)$ if either $(p_1, q_1) = (p_2, q_2)$ or $(x_1, y_1) \ne (x_2, y_2)$ and then x_1 is a successor of x_2, and y_1 is a predecessor of y_2.

In the following example, we have $(p_1, q_1) \le (p_2, q_2)$



The above definitions are purely quiver-theoretical. We also need a notion of contractibility of contours. Let Q be, as before, a locally finite quiver without oriented cycles, and I be an admissible ideal of kQ. Two paths p, q from x to y in Q are called **naturally homotopic** in (Q, I) if there exists a sequence of paths $p = p_0, p_1, \ldots p_m = q$ in Q such that, for each $0 \leq i < m$, p_i and p_{i+1} have subpaths q_i and q_{i+1}, respectively, which are involved in the same minimal relation in (Q, I). A contour (p, q) is called **naturally contractible** if the paths p, q are naturally homotopic in (Q, I).

The following example illustrates this definition. Let Q be the quiver



and I be the ideal generated by $\alpha\beta - \gamma\delta, \alpha\beta\lambda - \alpha\beta\mu$. The paths $\alpha\beta, \gamma\delta$ are naturally homotopic in (Q, I), and thus the contour $(\alpha\beta, \gamma\delta)$ is naturally contractible. On the other hand, the paths λ, μ are homotopic in (Q, I), but not naturally homotopic, hence the contour (λ, μ) is not naturally contractible.

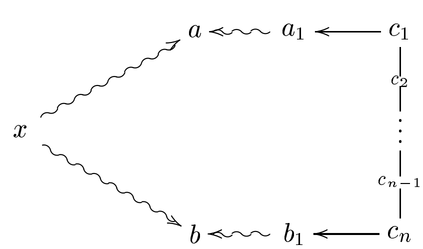
1.6. We are now able to state (and prove) the main result of this section, which asserts that our definition of strong simple connectedness is equivalent to that in [14], and also gives a handy criterion allowing to verify whether a locally bounded category is strongly simply connected or not.

Theorem. *Let A be a connected triangular locally bounded k-category. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *For any presentation $A \cong kQ_A/I_A$, the fundamental group of any connected full convex bound subquiver of (Q_A, I_A) is trivial.*
(c) *For any presentation $A \cong kQ_A/I_A$, any irreducible cycle in Q_A is an irreducible contour, and any irreducible contour is naturally contractible.*

Proof. Since (b) implies (a) trivially, it suffices to show that (a) implies (c) and that (c) implies (b).

Assume that A is strongly simply connected. By (1.4), A satisfies the separation condition. In order to show (c), let $A \cong kQ_A/I_A$ be an arbitrary presentation. Assume that there exists an irreducible cycle w in Q_A which is not an irreducible contour. Then w is not a contour, and hence is of the form $w = pp_1^{-1}vq_1q^{-1}$, where x is a source on w, and $p : x \to \cdots \to a$, $p_1 : c_1 \to a_1 \to \cdots \to a$ are paths, $v : c_1 — c_2 — \cdots — c_{n-1} — c_n$ is a reduced walk with c_1, c_n sources on w, and $q_1 : c_n \to b_1 \to \cdots \to b, q : x \to \cdots \to b$ are paths



Since w is irreducible, there exists no path in Q_A from x to c_i for each $1 \leq i \leq n$. If there exist non-trivial paths $p_2 : x \to \cdots \to y$, $p_3 : y \to \cdots \to a$ and $p_4 : y \to \cdots \to b$, then the cycle $p_3p^{-1}vq_1p_4^{-1}$ also satisfies the condition that there exists no path in Q_A from y to c_i for each $1 \leq i \leq n$. Since Q_A is locally finite, we may assume without loss of generality that any path in Q_A from x to a does not meet the paths from x to b. Let (Q, I) be the full bound subquiver of (Q_A, I_A) generated by the points lying on a path between points of the cycle w. Thus, Q is the convex hull of w in Q_A. Let $x \to \cdots \to z$ be a non-trivial path in Q, then z cannot be a predecessor of the c_i. Then z is a predecessor of exactly one of a, b, say of a. Let $\alpha : z \to z'$ be an arrow in Q, then again z' is a predecessor of a, and not a predecessor of b. It is now clear that x is not a separating object in the full convex subcategory kQ/I of A, a contradiction.

Supppose that there exists an irreducible contour (p, q) in Q_A from x to y which is not naturally contractible. We may assume that (p, q) is minimal with this property with respect to the partial order defined in (1.5). Let (Q, I) be the full bound subquiver of (Q_A, I_A) generated by the points lying on the paths in Q_A from x to y, and let $B = kQ/I$. Then B is a connected full convex subcategory of A. Let P_1 be the set of non-trivial paths in Q which start with x and are contained in a path naturally homotopic to p in (Q, I), and P_2 be the set of non-trivial paths in Q which start with x and are contained in a path which is not naturally homotopic to p in (Q, I). By the minimality of (p, q), we have $P_1 \cap P_2 = \emptyset$, and each path in Q which is reducible to p in Q is in P_1. Let $R(x) = \operatorname{rad} P(x)_B$. Then $R(x) = R_1 + R_2$ where, for each $i = 1, 2, R_i$ is the k-vector space with a basis consisting of the residue classes modulo I of the paths in P_i. Now, by definition, any two paths $p_1 \in P_1, p_2 \in P_2$ are not involved in any minimal relation simultaneously. Thus $R_1 \cap R_2 = 0$. Moreover, for any two paths $p_1 \in P_1$, $p_2 \in P_2$, we know that p_2 is not reducible to p_1, thus p_1, p_2 do not have a common point other than x, y. It follows that, if $p' : x \to \cdots \to z$ is a path in the k-basis of R_i, for some $i = 1, 2$, and $\alpha : z \to z'$ is an arrow in Q, then $p'\alpha = 0$ or is in the k-basis of R_i. Therefore, the R_i are submodules of $R(x)$. Thus x is not a separating object of B, a contradiction which completes the proof of (c).

We now show that (c) implies (b). Let $A \cong kQ_A/I_A$ be a presentation. It suffices to show that $\pi_1(Q_A, I_A) = 1$. It easily follows from the hypothesis that any contour (p, q) in Q_A is naturally contractible. Let w be a cycle in Q_A, and, as in (1.5), $\sigma(w)$ be the number of sources of w. If $\sigma(w) = 1$, then w is a contour, and hence is naturally contractible. Assume $\sigma(w) > 1$. Then w is not irreducible by hypothesis. Therefore $w \sim w_1 w_2$ where w_1, w_2 are cycles with $\sigma(w) > \sigma(w_1), \sigma(w) > \sigma(w_2)$. Thus w is naturally contractible by induction. It follows easily that any closed walk in Q is naturally contractible. $\square$

1.7. While proving the above theorem, we have shown that the equivalent conditions of the theorem are also equivalent to the statement that any connected full convex subcategory of our locally bounded category satisfies the separation condition. In fact, we have the following theorem of Skowroński [14] (4.1) whose proof, made for finite dimensional algebras, extends easily to the case of locally bounded categories.

Theorem. *Let A be a connected triangular locally bounded k-category. The following conditions are equivalent :*

(a) *A is strongly simply connected.*
(b) *For any connected full convex subcategory C of A, we have $H^1(C) = 0$.*
(c) *Any connected full convex subcategory of A satisfies the separation condition.*
(d) *Any connected full convex subcategory of A satisfies the co-separation condition.*

$\square$

Here, and in the sequel, $H^1(C)$ denotes the first Hochschild cohomology group of C with coefficients in the bimodule ${}_C C_C$, see [8].

1.8. **Corollary.** *Let A be a connected triangular locally bounded k-category. The following conditions are equivalent :*

(a) *A is strongly simply connected.*
(b) *There exists a presentation $A \cong kQ_A/I_A$ such that the fundamental group of any finite connected full convex bound subquiver of (Q_A, I_A) is trivial.*
(c) *For any presentation $A \cong kQ_A/I_A$, the fundamental group of any finite connected full convex bound subquiver of (Q_A, I_A) is trivial.*
(d) *Any connected full convex subcategory of A with finitely many objects satisfies the separation condition.*
(d) *Any connected full convex subcategory of A with finitely many objects satisfies the co-separation condition.*
(f) *For any connected full convex subcategory C of A with finitely many objects, we have $H^1(C) = 0$.*

Proof. It suffices to observe that the conditions stated are of a local nature (for instance, any indecomposable projective module is finite dimensional). $\square$

2. Schurian strongly simply connected locally bounded categories

2.1. A locally bounded k-category A is called **Schurian** if $\dim_k A(x, y) \leq 1$ for all $x, y \in A_0$. Schurian strongly simply connected finite dimensional algebras were studied in [9], where they are called completely separating algebras. Also, it is shown in [2] that, if A is a Schurian algebra all of whose indecomposable projective modules are directing, then the following conditions are equivalent :

(a) A is simply connected.
(b) A is strongly simply connected.
(c) A satisfies the separation condition.

Our aim is to find a criterion allowing to verify whether a Schurian locally bounded k-category is strongly simply connected or not. We start with the following lemma.

Lemma. *Let A be a triangular locally bounded k-category which is Schurian and strongly simply connected, and let $A \cong kQ_A/I_A$ be any presentation. Then, for any contour (p,q) in Q_A, we have that $p \in I_A$ if and only if $q \in I_A$.*

Proof. We note that, since A is Schurian, for any contour (u,v) with $u,v \notin I_A$, there exists a non-zero $\lambda \in k$ such that $u = \lambda v$. Assume that there exists a contour (p,q) in Q_A from x to y such that exactly one of p and q lies in I_A. We may assume that (p,q) is minimal with respect to the partial order defined in (1.5). Suppose that $p \notin I_A$ and that $q \in I_A$. If (p,q) is not irreductible, then there exist paths $p = p_0, p_1, \ldots p_{m-1}, p_m = q$ from x to y such that (p_{i-1}, p_i) is an interlaced contour for each $1 \le i \le m$. It follows from the minimality of (p,q) that $p_1 \notin I_A$ and, inductively, $q \notin I_A$. This contradiction shows that (p,q) is irreductible. By (1.6), the contour (p,q) must be naturally contractible, that is, there exist paths $p = p_0, p_1, \ldots p_{m-1}, p_m = q$ in Q_A from x to y such that for each $0 \le i < m, p_i$ and p_{i+1} contain subpaths q_i and q_{i+1}, respectively, which are involved in the same minimal relation in (Q_A, I_A). If $q_1 \ne p_1$, then (p_0, p_1) is an interlaced contour and hence $p_1 \notin I_A$ by the minimality of (p,q). If $q_1 = p_1$, then $p = p_0$ and p_1 are involved in the same minimal relation, and hence $p_1 \notin I_A$. Inductively, $q \notin I_A$. This contradiction completes the proof. □

2.2. **Lemma.** *Let A be a triangular locally bounded k-category which is Schurian and strongly simply connected, and let $A \cong kQ_A/I_A$ be any presentation. Then all irreducible cycles in Q_A are irreducible contours and, for each irreducible contour (p,q) in Q_A, we have $p,q \notin I_A$ and $p - \lambda q \in I_A$ for some non-zero $\lambda \in k$.*

Proof. By (1.6), all irreducible cycles in Q_A are irreducible contours and each irreducible contour is naturally contractible. Let (p,q) be an irreducible contour from x to y. Since A is Schurian, there exists $\lambda \in k$ such that $p - \lambda q \in I_A$. Assume that one of p,q lies in I_A and further assume that (p,q) is minimal with this property. Let $p \in I_A$. Since (p,q) is naturally contractible, there exist paths $p = p_0, p_1, \ldots p_{m-1}, p_m = q$ in Q_A from x to y such that, for each $1 \le i \le m$, p_{i-1} and p_i contain subpaths q_{i-1} and q_i, respectively, which are involved in the same minimal relation. Since (p,q) is irreducible, there exists $0 \le t < m$ such that p_t is reducible to p while p_{t+1} is not. By the minimality of (p,q), we may assume that $p_t \in I_A$. Since p_{t+1} is not reducible to p in Q_A, we see that p_t, p_{t+1} have no common point other than x, y. Thus p_t, p_{t+1} are involved in the same minimal relation in (Q_A, I_A), and this is impossible. Consequently, neither of p,q lies in I_A. □

2.3. **Lemma.** *Let Q be a connected locally finite quiver without oriented cycles. Then there exists an ascending chain $Q(n)$, with $n \ge 0$, of finite connected full convex subquivers of Q such that :*

(a) *$Q(0)$ consists of exactly one point.*
(b) *For each $n > 0$, if $Q(n-1) \underset{\ne}{\subset} Q(n)$, then all except one point x_n of $Q(n)$ belong to $Q(n-1)$, and x_n is either a source or a sink in $Q(n)$.*
(c) *$Q = \bigcup_{n \ge 0} Q(n)$.*

Proof. Choose any point x_0 in Q, and let $Q(0) = \{x_0\}$. Suppose that, for an even integer n, we have defined an ascending chain $Q(m)$ with $0 \le m \le n$ of finite connected full convex subquivers of Q satisfying (a) and (b). If there exists no arrow $a \to b$ in Q with b in $Q(n)$ and a not in $Q(n)$, then we define $Q(n+1) = Q(n)$. Otherwise, let $x_{n+1} \to b$ be an arrow in Q with b in $Q(n)$ and x_{n+1} not in $Q(n)$. Note that, since $Q(n)$ is finite, there exist only finitely many paths in Q starting at x_{n+1} and ending at a point in $Q(n)$. Therefore, we may assume that there exists no path in Q of length greater than one starting with x_{n+1} and ending at a point in $Q(n)$. Furthermore, we may assume that x_{n+1} is such that its distance to x_0 (that is, the least length of all the reduced walks from x_{n+1} to x_0) is minimal. Let $Q(n+1)$ be the full subquiver of Q

generated by $Q(n)$ and x_{n+1}. By our choice of x_{n+1} and the convexity of $Q(n)$, we conclude that $Q(n+1)$ is convex and has x_{n+1} as a source. We now construct $Q(n+2)$ from $Q(n+1)$ in a dual manner so that either $Q(n+2) = Q(n+1)$ or $Q(n+2)$ is generated by $Q(n+1)$ and an additional point x_{n+2} which is a sink of $Q(n+2)$. By induction, we have defined an ascending chain $Q(n)$, with $n \geq 0$, satisfying (a) and (b). Suppose that there exists a point x in Q but not in $\bigcup_{n\geq 0} Q(n)$. Clearly, we may assume that there exists an edge x — a with a in $Q(m)$ for some $m \geq 0$. Assume first that there is an arrow $x \to a$ in Q. It follows from our construction that there will be infinitely many arrows starting with x, which is impossible. A similar impossibility arises if there is an arrow $a \to x$ in Q. $\square$

2.4. The main result of this section characterises the Schurian strongly simply connected locally bounded categories in terms of their presentations. In particular, it asserts that a connected triangular locally bounded category is Schurian and strongly simply connected if and only if there exists a presentation such that all irreducible cycles are commutative contours. This result implies that such a category always has a multiplicative basis [4]. A presentation of a Schurian strongly simply locally bounded category A such as in (b) below will be called in the sequel a **normed presentation** of A.

Theorem. *Let A be a connected triangular locally bounded k-category. The following conditions are equivalent :*

(a) *A is Schurian and strongly simply connected.*
(b) *There exists a presentation $A \cong kQ_A/I_A$ such that all irreducible cycles in Q_A are irreducible contours and, for each irreducible contour (p,q), we have $p, q \notin I_A$ and $p - q \in I_A$.*
(c) *For any presentation $A \cong kQ_A/I_A$, all irreducible cycles in Q_A are irreducible contours and, for each irreducible contour (p,q), we have $p, q \notin I_A$ and $p - \lambda q \in I_A$ for some non-zero $\lambda \in k$.*

Proof. If follows from (2.2) that (a) implies (c). Since (b) implies (a), by (1.6) and the definition of Schurian, it suffices to prove that (c) implies (b), that is, to construct a normed presentation of A. Given two points x, y in Q_A, there is at most one arrow $\alpha : x \to y$ to which we must associate an element $\varphi(\alpha) \in \operatorname{rad} A(x,y) \backslash \operatorname{rad}^2 A(x,y)$. We say that a path p

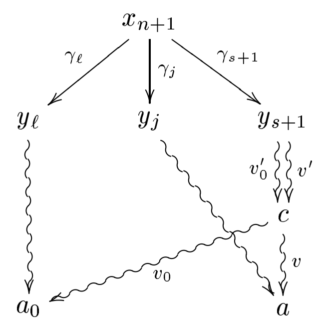
$$x_0 \overset{\alpha_1}{\to} x_1 \overset{\alpha_2}{\to} x_2 \to \ldots \overset{\alpha_m}{\to} x_m$$

is non-zero if, for each $1 \leq i \leq m$, there exists $\varphi(\alpha_i) \in \operatorname{rad} A(x_{i-1}x_i) \backslash \operatorname{rad}^2 A(x_{i-1}, x_i)$ such that the composite $\varphi(\alpha_1)\varphi(\alpha_2)\ldots\varphi(\alpha_m)$ (which we write $\varphi(p)$ for the sake of brevity) is non-zero. We call a contour (p,q) non-zero if both p and q are non-zero. Given a contour (p,q) from x to y, we say that (p,q) starts with a pair (α, β) of arrows if α, β are the unique arrows of source x such that $p = \alpha p', q = \beta q'$ with p', q' paths of target y.

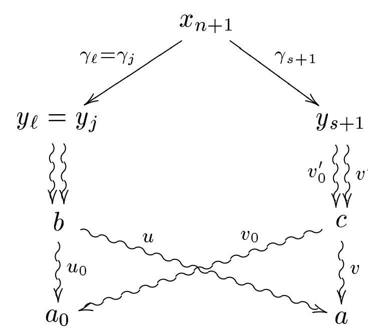
To construct the required normed presentation, we consider an ascending chain $Q(n)$, with $n \geq 0$, of finite connected full convex subquivers of Q_A satisfying the conditions of (2.3), and construct φ by induction on n. Assume thus that for each arrow $\alpha : x \to y$ in $Q(n)$, we have chosen $\varphi(\alpha) \in \operatorname{rad} A(x,y) \backslash \operatorname{rad}^2 A(x,y)$ such that, for any non-zero contour (p,q) in $Q(n)$, we have $\varphi(p) = \varphi(q)$. Assume that $Q(n+1) \neq Q(n)$. It suffices, by duality, to consider the case where x_{n+1} is a source of $Q(n+1)$. Let $\gamma_i : x_{n+1} \to y_i$, with $1 \leq i \leq t$, be all the arrows in $Q(n+1)$ having x_{n+1} as a source. We may clearly assume the γ_i to be ordered so that, for each $1 \leq i \leq t$, if there is no non-zero contour in $Q(n+1)$ starting with (γ_i, γ_{j_0}), then there is no non-zero contour starting with (γ_i, γ_j) for any $j_0 < j \leq t$. We define $\varphi(\gamma_i)$ by induction on i, where $1 \leq i \leq t$.

We choose arbitrarily $\varphi(\gamma_1) \in \operatorname{rad} A(x_{n+1}, y_1) \backslash \operatorname{rad}^2 A(x_{n+1}, y_1)$. Let (p,q) be a non-zero contour in $Q(n+1)$ starting with (γ_1, γ_1), that is $p = \gamma_1 p'$, $q = \gamma_1 q'$ with (p', q') a non-zero contour in $Q(n)$. Thus $\varphi(p') = \varphi(q')$ and hence $\varphi(p) = \varphi(q)$. Assume that $1 \leq s \leq t$ and that, for each $1 \leq i \leq s$, we have chosen $\varphi(\gamma_i) \in \operatorname{rad} A(x_{n+1}, y_i) \backslash \operatorname{rad}^2 A(x_{n+1}, y_i)$ such that, for any non-zero contour in $Q(n+1)$ starting with (γ_i, γ_j) with $1 \leq i, j \leq s$, we have $\varphi(p) = \varphi(q)$. We wish to define $\varphi(\gamma_{s+1})$. If, for any $1 \leq i \leq s$, there is no non-zero contour in $Q(n+1)$ starting with (γ_i, γ_{s+1}), then we choose arbitrarily $\varphi(\gamma_{s+1}) \in \operatorname{rad} A(x_{n+1}, y_{s+1}) \backslash \operatorname{rad}^2 A(x_{n+1}, y_{s+1})$. Otherwise, let (p_0, q_0) be a non-zero contour from x_{n+1} to a_0 starting with $(\gamma_\ell, \gamma_{s+1})$ for some $1 \leq \ell \leq s$, which we can assume to be minimal with this property. We choose $\varphi(\gamma_{s+1})$ so that $\varphi(p_0) = \varphi(q_0)$. We claim that there exists no non-zero contour (p,q) starting with (γ_j, γ_{s+1}), where $1 \leq j \leq s+1$, such that $\varphi(p) = \lambda\varphi(q)$ for some $\lambda \neq 1$.

Indeed, assume on the contrary that such a contour (p, q) exists with source x_{n+1} and target a (say). We may also assume that (p, q) is minimal with this property. By the induction hypothesis, we have that $j \leq s$, and that a is neither a predecessor nor a successor of a_0 in Q_A since otherwise $\varphi(p_0) = \varphi(q_0)$ implies $\varphi(p) = \varphi(q)$. Thus, there exists a point c on both q_0 and q distinct from x_{n+1}, a_0, a such that the subpath v_0 of q_0 from c to a_0, and the subpath v of q from c to a have no common point except c. Let v', v'_0 be the subpaths of q, q_0 from y_{s+1} to c such that $q = \gamma_{s+1}v'v$ and $q_0 = \gamma_{s+1}v'_0v_0$.

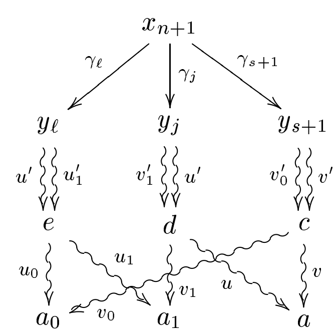


Suppose that $j = \ell$. Then there exists a point b on both p_0 and p other than x_{n+1}, a_0, a such that the subpath u_0 of p_0 from b to a_0 and the subpath u of p from b to a have no common points except b. By the minimality of (p_0, q_0), there is no path from b to c and no path from c to b in Q_A. Therefore $uv^{-1}v_0u_0^{-1}$ is an irreducible cycle in Q_A which is not a contour, a contradiction to our hypothesis.



Suppose now that $j \neq \ell$. By our hypothesis on the enumeration of the γ_i, there exists at least one non-zero contour starting with (γ_ℓ, γ_j). Thus there exist points d on p and e on p_0 such that there exists a non-zero contour (p_1, q_1) from x_{n+1} to a_1 (say) starting with (γ_ℓ, γ_j) and containing d, e, and any pair of points $(x, y) \neq (d, e)$ such that x is on the subpath of p from d to a, and y is on the subpath of p_0 from e to a_0, does not enjoy this property.

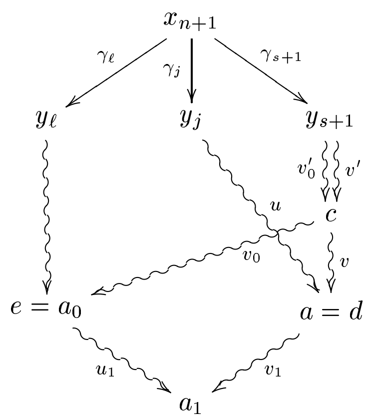
Write $p_1 = \gamma_\ell u'_1 u_1, q_1 = \gamma_j v'_1 v_1, p = \gamma_j u' u, q = \gamma_{s+1} v' v, p_0 = \gamma_\ell u'_0 u_0, q_0 = \gamma_{s+1} v'_0 v_0$ where v_1, u have source d; u_0, v_1 have source e and v, v_0 have source c.



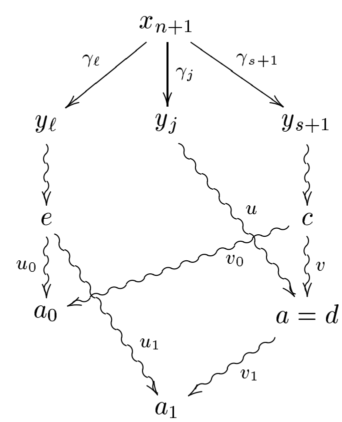
We then have four cases, and show that each leads to a contradiction.

(i) Assume $d = a$ and $e = a_0$. Note that $\varphi(v') = \varphi(v'_0)$ since both paths lie in $Q(n)$. Then $\varphi(pv_1) = \varphi(p_0u_1) = \varphi(q_0u_1) = \varphi(\gamma_{s+1}v'_0v_0u_1) = \varphi(\gamma_{s+1}v'vv_1) = \varphi(q)\varphi(v_1) = \lambda\varphi(p)\varphi(v_1) = \lambda\varphi(pv_1)$. By

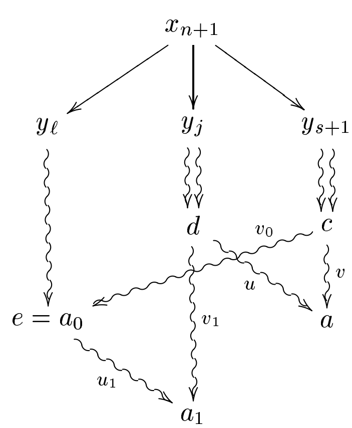
(2.1), we have $\varphi(pv_1) \neq 0$ because $\varphi(p_1) \neq 0$. Therefore $\lambda = 1$, a contradiction.



(ii) Assume $d = a$ and $e \neq a_0$. There is no path from a_0 to a_1, by the choice of e, and no path from a_1 to a_0, because a_0 is not a successor of a. Moreover, there are no paths from c to e or from e to c by the minimality of (p_0, q_0). Therefore $vv_1u_1^{-1}u_0v_0^{-1}$ is an irreducible cycle in Q_A which is not a contour, a contradiction to our hypothesis.



(iii) Assume $d \neq a$ and $e = a_0$. There is no path from a to a_1, by the choice of d, and no path from a_1 to a, because a is not a successor of a_0. Moreover, there are no paths from e to c or from c to d since otherwise $\varphi(p) = \varphi(q)$ by the minimality of (p, q) and the induction hypothesis. Therefore $vu^{-1}v_1u_1^{-1}v_0^{-1}$ is an irreducible cycle in Q_A which is not a contour, a contradiction to our hypothesis.



(iv) Assume $d \neq a$ and $e \neq a_0$. There are no paths from d to a_0 or from e to a, by the choice of d, e. If there exists a path w from c to a_1, then $uv^{-1}wv_1^{-1}$ is an irreducible cycle which is not a contour (because the minimality of (p,q) implies that there is no path from d to c or from c to d, and the choice of d,e implies that there is no path from a_1 to a, or from a to a_1) and this is a contradiction. Thus, there is no path from c to a_1, and $uv^{-1}v_0u_0^{-1}u_1v_1^{-1}$ is an irreducible cycle which is not a contour, again a contradiction.

The theorem is now established by induction. □

2.5. Let A be a finite dimensional k-algebra which is representation-finite. It is well-known (and easy to see) that, if A is triangular, then A is Schurian. Hence, for any presentation $A \cong kQ_A/I_A$ and any contour (p,q) in Q_A with $p,q \notin I_A$, there exists a non-zero $\lambda \in k$ such that $p - \lambda q \in I_A$. On the other hand, A is simply connected if and only if it is strongly simply connected (see [7]). Hence (1.6) and (2.4) yield immediately the following new characterisation, in therms of their bound quivers, of simply connected representation-finite algebras.

Corollary. *Let A be a connected finite dimensional k-algebra which is representation-finite. The following conditions are equivalent :*

(a) *A is simply connected.*
(b) *A is triangular and there exists a presentation $A \cong kQ_A/I_A$ such that all irreducible cycles in Q_A are irreducible contours and, for each irreducible contour (p,q), we have $p,q \notin I_A$ and $p - \lambda q \in I_A$ for some non-zero $\lambda \in k$.*
(c) *A is triangular and, for any presentation $A \cong kQ_A/I_A$, all irreducible cycles in Q_A are irreducible contours and, for each irreducible contour (p,q), we have $p,q \notin I_A$ and $p-\lambda q \in I_A$ for some non-zero $\lambda \in k$.* □

3. Construction of strongly simply connected algebras

3.1. We recall that the one-point extension of a finite dimensional algebra B by a B-module M is the matrix algebra

$$A \quad = \quad B[M] \quad = \quad \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

where the operations are induced from the matrix operations and the module structure of M. The quiver Q_A of A then contains the quiver Q_B of B as a full convex subquiver and there is an additional (extension) point which is a source. Dually, one defines the one-point coextension $[M]B$ of B by M.

Let A be a strongly simply connected algebra. Since A is triangular, it can be constructed by repeated one-point extensions or coextensions. If $A = B[M]$ is strongly simply connected, then B is a full convex subcategory of A, hence is itself strongly simply connected. We are interested in finding necessary and sufficient conditions on a module M over a strongly simply connected algebra B so that the one-point extension $B[M]$, or the one-point coextension $[M]B$, is also strongly simply connected. This would give an inductive construction of strongly simply connected algebras. We start however by answering a more general question.

Theorem. *Let B be an algebra (not necessarily connected), and M be a B-module. Then :*

(a) *$A = B[M]$ satisfies the separation condition if and only if B satisfies the separation condition and M is a separated B-module.*
(b) *$A = [M]B$ satisfies the co-separation condition if and only if B satisfies the co-separation condition and M is a separated B-module.*

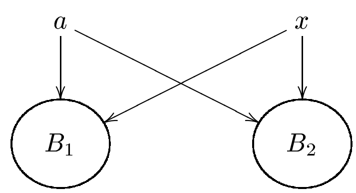
Proof. We only prove (a), since the proof of (b) is similar. Assume first that A satisfies the separation condition. Since the extension point a is separating as an object in A, the B-module M is clearly separated. In order to prove that B satisfies the separation condition, we must show that any $x \in B_0$ is separating. As usual, we denote by A^x (or B^x) the full subcategory of A (or B) generated by the non-predecessors of x in A (or B, respectively). Since there is no path from x to a, the indecomposable projective B-module $P(x)_B$, when considered as an A-module, is equal to $P(x)_A$. If a is a predecessor of x, then $B^x = A^x$ and x is separating in B, because it is so in A. If a is not a predecessor of x, then A^x is generated by B^x and a. Assume $\operatorname{rad} P(x)_B$ is not separated. Then there exist two distinct indecomposable summands R_1, R_2 of

$\operatorname{rad} P(x)$ whose supports lie in the same connected component of B^x. But R_1, R_2 lie in distinct connected components of A^x, an impossibility.

Conversely, assume that B satisfies the separation condition and that M is a separated B-module. Since the extension point a is clearly a separating object in A, we must prove that every $x \in B_0$ such that $x \neq a$ is also separating in A. If a is a predecessor of x, then clearly x is separating in A, since $(A^x)_0 \cup \{x\} = (B^x)_0 \cup \{x\}$ in this case. Thus, we need only consider the case where a is not a predecessor of x. In this case, again, A^x is generated by B^x and a. Assume on the contrary that $\operatorname{rad} P(x)_A$ is not a separated A^x-module.

Then there exist two distinct indecomposable summands R_1, R_2 (say) of $\operatorname{rad} P(x)_A$ whose supports are connected in A^x. Since they are not connected in B^x (because B satisfies the separation condition), there exist two distinct connected components of B^x, say B_1 and B_2, containing respectively the supports of R_1 and R_2, and connected in A^x through the extension point a. In fact, each of B_1 and B_2 is connected to a by a single arrow : let $a \to a_1 — a_2 — \cdots — a_t$, with a_t in B_i, be a walk of least length from a to B_i (where $i = 1, 2$). This hypothesis implies $a_j \neq a$ for all $1 \leq j \leq t$, thus a_j belongs to B^x for all j, and hence a_1 lie in B_i. Therefore the restriction of M to each B_i is non-zero.

We thus have the following situation

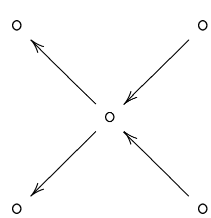


At this point, it is important to observe that B_1 and B_2 belong to the same connected component of B, since they are connected through x. Moreover, the restriction of M to this component is indecomposable since M is separated as a B-module. In particular, there exists a walk $b_1 — c_1 — \cdots — c_r — b_2$ in $\operatorname{Supp} M$, with b_1 in B_1, b_2 in B_2, and c_j not in B^x for all j $(1 \leq j \leq r)$ because B_1, B_2 are disconnected in B^x. Thus, each c_j is a predecessor of x. On the other hand, since c_j lies in the support of M, there exists a path from a to c_j. Hence a is a predecessor of x, which is the wanted contradiction. $\square$

Remarks.

(a) The above theorem generalises [2] (2.5).

(b) Let B be an algebra satisfying the separation condition, and M be a separated B-module. Then $[M]B$ usually does not satisfy the separation condition (even if B also satisfies the co-separation condition). Indeed, let B be the tame hereditary algebra given by the quiver



and H be a simple homogeneous B-module, then $[H]B$ does not satisfy the separation condition.

3.2. We have the following easy corollary.

Corollary. *A triangular algebra A satisfies the separation (or the co-separation) condition if and only if there exist a sequence of algebras $A_0, A_1, \ldots A_n = A$ with A_0 semisimple, and, for each $0 \leq i < n$, a separated A_i-module M_i such that $A_{i+1} = A_i[M_i]$ (or $A_{i+1} = [M_i]A_i$, respectively).* $\square$

3.3. Let B be a triangular algebra, and M be a B-module. An enumeration $\{x_1, \ldots x_m\}$ of the points of the support $\operatorname{Supp} M$ of M is called an **admissible ordering of sinks** (or **of sources**) if $j > i$ implies that x_j is not a successor (or predecessor, respectively) of x_i. The triangularity of B implies that, for each B-module M, there exists at least one admissible ordering of sinks and one admissible ordering of sources of the points of $\operatorname{Supp} M$. To each admissible ordering of sinks (or of sources) is associated a filtration of B

by a sequence of full convex subcategories : indeed, let $\{x_1, \dots x_m\}$ be such an admissible ordering of sinks (or of sources) of the points of $\operatorname{Supp} M$, then we define $B^{(0)} = B$ and, for each $0 < i < m$, we let $B^{(i)}$ be the full subcategory of B generated by the non-successors (or non-predecessors, respectively) of the points $x_1, \dots x_i$. Clearly, each $B^{(i)}$ is convex and we have $B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(m-1)}$.

Definition. Let B be a triangular algebra. A B-module M is called **completely co-separated** (or **completely separated**) if, for any admissible ordering of sinks (or of sources, respectively) of the points of $\operatorname{Supp} M$ and, for each $0 \leq i < m$, the restriction $M^{(i)} = M\mid_{B^{(i)}}$ of M to $B^{(i)}$ is separated as a $B^{(i)}$-module.

Thus, any uniserial module (in particular any simple module) is completely co-separated and completely separated. In general, however, these two classes do not coincide. Examples are given later.

Clearly, any completely co-separated (or completely separated) module is separated. In particular, if B is connected, any completely co-separated (or completely separated) B-module is indecomposable.

3.4. We may now state and prove our main result.

Theorem. *Let B be a strongly simply connected algebra, and M be a B-module. Then :*

(a) $A = B[M]$ *is strongly simply connected if and only if M is a completely co-separated B-module.*
(b) $A = [M]B$ *is strongly simply connected if and only if M is a completely separated B-module.*

Proof. We only prove (a), since the proof of (b) is similar. For the necessity, assume that there exists an admissible ordering of sinks $\{x_1, \dots x_m\}$ of the points of $\operatorname{Supp} M$ with associated filtration $B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(m-1)}$.

For any $0 \leq j < m$, let $M_j = M\mid_{B^{(j)}}$. Then $B^{(j)}[M_j]$ is a full convex subcategory of A, and consequently satisfies the separation condition. By (3.1), M_j is separated as a $B^{(j)}$-module. This completes the proof of the necessity.

Conversely, we assume that M is completely co-separated and show that each connected full convex subcategory C of A satisfies the separation condition. If the extension point a is not in C, then C is a full convex subcategory of B, hence satisfies the separation condition. We thus assume that a lies in C.

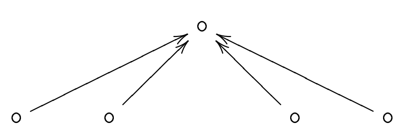
Let D be the full subcategory of B generated by all objects of C except a. Then C is the one-point extension of D by the restriction $M\mid_D$ of M to D. Moreover, D is a full convex subcategory of B and hence satisfies the separation condition. Let $\{x_1, \dots x_t\}$, with $t \geq 0$, be the points in $\operatorname{Supp} M$ which do not lie in C, and let $\{x_{t+1}, \dots x_m\}$ be those lying in C. Thus, all the x_i with $1 \leq i \leq m$, are successors of the extension point a. It then follows from the convexity of C that no point in C is a successor of the x_i, with $1 \leq i \leq t$. In particular, x_j is not a successor of x_i if $1 \leq i \leq t < j \leq m$. We may clearly assume further that x_j is not a successor of x_i whenever $1 \leq i < j \leq t$ or $t+1 \leq i < j \leq m$. Therefore $\{x_1, \dots x_t, x_{t+1} \dots x_m\}$ is an admissible ordering of sinks of the points of $\operatorname{Supp} M$. Let $B = B^{(0)} \supseteq B^{(1)} \supseteq \cdots \supseteq B^{(t)} \supseteq \cdots \supseteq B^{(m-1)}$ be the associated filtration of B. Then $M\mid_{B^{(t)}}$ is separated as a $B^{(t)}$-module. On the other hand, D is a full subcategory of $B^{(t)}$ and $M\mid_D = M\mid_{B^{(t)}}$. Thus, $M\mid_D$ is separated as a D-module. It follows from (3.1) that C satisfies the separation condition. The proof of the theorem is complete. □

3.5. **Corollary.** *Let B be a strongly simply connected algebra and M be a completely co-separated (or a completely separated) B-module. Then M is a brick (that is,* $\operatorname{End} M \cong k$*).*

Proof. This follows from our theorem and [14] (4.2). □

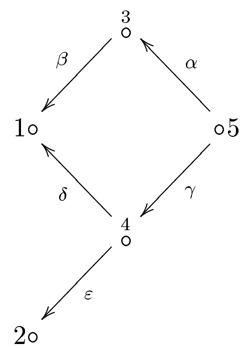
3.6. **Example.**

(a) Let B be the tame hereditary algebra given by the quiver

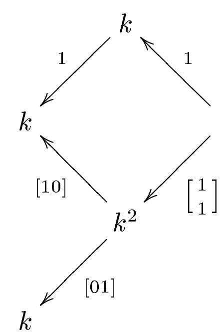


Each of the simple homogeneous B-modules H_λ (with $\lambda \in k \backslash \{0, 1\}$) is completely co-separated, but not completely separated. The algebras $B[H_\lambda]$ (which are just the canonical tubular algebras of type $(2, 2, 2, 2)$) are strongly simply connected. This furnishes an infinite family of strongly simply connected algebras with the same dimension and the same number of isomorphism classes of simple modules.

(b) Let B be given by the quiver



bound by $\alpha\beta = \gamma\delta$. The B-module M given by



is indecomposable (and, even, is a brick) but is not completely co-separated. Indeed, if one considers the shown admissible ordering of sinks for the points of $\operatorname{Supp} M = B$, then $B^{(1)}$ is generated by all points except 1, it is connected and $M^{(1)} = M \mid_{B^{(1)}}$ is decomposable, thus not separated. Note that B is strongly simply connected, but $B[M]$ is not.

3.7. The following corollary strengthens [2] (5.2) and [13] (2.2).

Corollary. *A connected algebra A is strongly simply connected if and only if there exist a sequence of algebras $A_0, A_1, \ldots A_n = A$ with $A_0 = k$, and for each $0 \le i < n$, an A_i-module M_i, such that either M_i is completely co-separated and $A_{i+1} = A_i[M_i]$, or M_i is completely separated and $A_{i+1} = [M_i]A_i$.*

Proof. Assume indeed that A is strongly simply connected. By (2.3), there exists a connected full convex subquiver Q of Q_A such that all the points of Q_A belong to Q except one, which is a source or a sink. Assume the former, and let B be the connected full convex subcategory of A generated by the points of Q, then A is a one-point extension of B by a B-module M, say. Since B is strongly simply connected (because A is), it follows from our theorem that M is completely co-separated. The proof is completed by induction. □

4. Construction of Schurian strongly simply connected algebras

4.1. Since there exists, so far, no general rule to decide whether a given module is indecomposable or not, we do not have any practical method to decide whether a given module is completely co-separated, or completely separated, or not. However, if A is a Schurian and strongly simply connected algebra, we can find all A-modules M such that $A[M]$, or $[M]A$, is Schurian and strongly simply connected. The result provides an inductive process to construct all Schurian and strongly simply connected algebras with a prescribed number of isomorphism classes of simple modules. In particular, one can obtain in this way all representation-finite simply connected algebras with a prescribed number of isomorphism classes of simple modules.

Let Q be a finite quiver without oriented cycles, and Q' be a full subquiver of Q. An enumeration $\{x_1, \ldots x_m\}$ of the points of Q' is called an **admissible ordering of sinks** (or **of sources**) if $j > i$ implies that x_j is not a successor (or predecessor, respectively). To each such admissible ordering, we associate a filtration of Q' by a sequence of full convex subquivers : indeed, let $\{x_1, \dots x_m\}$ be an admissible ordering of sinks (or of sources) of the points of Q', then we let $Q^{(0)} = Q$ and, for each $0 < i < m$, we let $Q^{(i)}$ be the full subquiver of Q generated by the non-successors (or non-predecessors, respectively) of $x_1, \dots x_i$ in Q. Clearly, each $Q^{(i)}$ is convex and we have $Q^{(0)} \supseteq Q^{(1)} \supseteq \cdots \supseteq Q^{(m-1)}$.

Definition. Let Q be a finite quiver without oriented cycle. A full subquiver Q' of Q is said to be **completely co-separated** (or **completely separated**) if, for each admissible ordering of sinks (or of sources, respectively) of the points of Q, and each $1 \leq i \leq m$, the intersection of Q' with each of the connected components of $Q^{(i)}$ is empty or connected.

Lemma. *Let A be a strongly simply connected algebra with ordinary quiver Q_A, and M be an A-module whose support has quiver Q. Then :*

(a) *If M is completely co-separated, then Q is a completely co-separated subquiver of Q_A.*
(b) *If M is completely separated, then Q is a completely separated subquiver of Q_A.*

Proof. We only prove (a), since the proof of (b) is similar. Let $\{x_1, \dots x_m\}$ be an admissible ordering of sinks of the points of Q, and let $A^{(i)}$ denote the full subcategory of A generated by the non-successors of $\{x_1, \dots x_i\}$. Then, by definition, $Q_A^{(i)}$ is the quiver of $A^{(i)}$. Since M is completely co-separated, $M^{(i)} = M\mid_{A^{(i)}}$ is separated as an $A^{(i)}$-module, that is, its restriction to each connected component of $A^{(i)}$ is indecomposable or zero. This implies the statement. □

4.2. Let A be an algebra with a presentation $A \cong kQ_A/I_A$ and let Q be a full subquiver of Q_A. We say that Q is **zero-relation-free** if no path in Q belongs to I_A. Given a full subquiver Q of Q_A, we denote by $U(Q)$ (see [9] (2.8)) the representation of Q_A defined by :

$$U(Q)_x = \begin{cases} k & \text{if} \quad x \in Q_0 \\ 0 & \text{if} \quad x \notin Q_0 \end{cases}$$
$$U(Q)_\alpha = \begin{cases} 1 & \text{if} \quad \alpha \in Q_1 \\ 0 & \text{if} \quad \alpha \notin Q_1. \end{cases}$$

We recall that, by (2.4), a Schurian strongly simply connected algebra always has a normed presentation.

Lemma. *Let A be a Schurian and strongly simply connected algebra, with normed presentation $A \cong kQ_A/I_A$. Let Q be a connected full convex subquiver of Q_A which is zero-relation-free. Then $U(Q)$ has a natural A-module structure and is indecomposable.*

Proof. In order to show that $U(Q)$ is an A-module, is suffices to show that it is annibilated by the ideal I_A. Now, $A \cong kQ_A/I_A$ is a normed presentation, hence all relations are zero-relations or commutativity relations. Since Q is convex and zero-relation-free, the statement follows from the definition of $U(Q)$. Assume that $U(Q) = M \oplus N$, with $M, N \neq 0$. Since $\dim_k U(Q)_x \leq 1$ for all points x in Q_A, every point in Q either belongs to $\operatorname{Supp} M$ or $\operatorname{Supp} N$, and neither support is empty. Assume $x \in (\operatorname{Supp} M)_0$ and $y \in (\operatorname{Supp} N)_0$. Since Q is connected, there is a walk from x to y. We may clearly assume, without loss of generality, that there is an edge $x — y$, and, even, an arrow $\alpha : x \to y$. But the $U(Q)_\alpha$ must be equal to zero, a contradiction. □

4.3. We may now state, and prove, the main result of this section.

Theorem. *Let A be a Schurian and strongly simply connected algebra, with normed presentation $A \cong kQ_A/I_A$, and M be an A-module. Then :*

(a) *The algebra $A[M]$ is Schurian and strongly simply connected if and only if $M \cong U(Q)$, where Q is a completely co-separated convex subquiver of Q_A which is zero-relation-free.*
(b) *The algebra $[M]A$ is Schurian and strongly simply connected if and only if $M \cong U(Q)$, where Q is a completely separated convex subquiver of Q_A which is zero-relation-free.*

Proof. We only prove (a), since the proof of (b) is similar. We first prove the sufficiency. Assume that A satisfies the stated conditions, let $\{x_1, \dots x_m\}$ be an admissible ordering of sinks of the points of Q, and $A^{(i)}$

be the full subcategory of A generated by the non-successors of $\{x_1, \dots x_i\}$. Then $Q_A^{(i)}$ is the quiver of $A^{(i)}$. Let C be a connected component of $A^{(i)}$, then its quiver Q_C is a connected component of $Q_A^{(i)}$. Since Q is a completely co-separated subquiver of Q_A, the intersection $Q \cap Q_C$ is empty or is a connected subquiver of Q_C, and it is also zero-relation-free. By (4.2), $U(Q) \mid_C = U(Q \cap Q_C)$ is an indecomposable C-module. This shows that M is a completely co-separated A-module and hence, by (3.4), $A[M]$ is strongly simply connected. Since, clearly, $A[M]$ is Schurian, we are done.

We now prove the necessity. Assume that $B = A[M]$ is Schurian, strongly simply connected and given a normed presentation $B \cong kQ_B/I_B$ so that (Q_A, I_A) is a full bound subquiver of (Q_B, I_B). We denote by b the extension point of B. By (3.4), the A-module M is completely co-separated. Let Q be the quiver of $\operatorname{Supp} M$. Since A is Schurian, for any $x \in Q_0$, we have $\dim_k M_x \le 1$. By (4.1), the quiver Q is completely co-separated. We now prove that Q is convex and zero-relation-free. Let $p : x_0 \to x_1 \to \cdots \to x_t$, with $t \ge 1$, be a path in Q_A, with x_0 and x_t in Q. Then there exist paths $p_1 : b \to \cdots \to x_0$ and $p_2 : b \to \cdots \to x_t$ in Q_B such that $p_1, p_2 \notin I_B$. By (2.1), we have $p_1 p \notin I_B$. Consequently, $p \notin I_A$. Therefore Q is zero-relation-free. To prove it is convex, we observe that $p_1 p \notin I_B$ implies that, for each $1 \le i < t$, the composite of p_1 with the subpath $x_0 \to \cdots \to x_i$ is not in I_B. Hence $x_i \in Q_0$.

Finally, we want to prove that $U(Q)$ and M are isomorphic (for another proof, see [9](2.9)). We let $\tilde{Q}$ be the full subquiver of Q_B generated by b and the points of Q. Then $\tilde{Q}$ is clearly a connected full convex subquiver of Q_B. Moreover, let $p : b \to y_1 \to \cdots \to y_n$ be a path in $\tilde{Q}$. Then $y_n \in Q_0$. Therefore there exists a path q from b to y_n which is not in I_B. By (2.1), p is not in I_B either. This shows that $\tilde{Q}$ is zero-relation-free. By (4.2), $U(\tilde{Q})$ is a B-module. Notice that, since $\tilde{Q}$ is the quiver of $\operatorname{Supp} P(b)$, and B is Schurian, then $\dim_k P(b)_x = 1$ for each $x \in \tilde{Q}_0$. We construct an isomorphism of B-modules $\tilde{f} : U(\tilde{Q}) \to P(b)$ in the following way. We define $\tilde{f}_b : U(\tilde{Q})_b \to P(b)_b$ to be the identity on $k = U(\tilde{Q})_b = P(b)_b$. We now let $x \in Q_0$ be arbitrary. There exists a path in Q_B

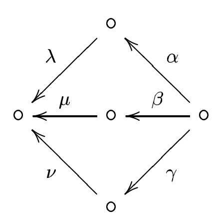
$$b = x_0 \stackrel{\alpha_1}{\to} x_1 \stackrel{\alpha_2}{\to} \cdots \stackrel{\alpha_t}{\to} x_t = x.$$

For each $1 \le i \le t$, there exists a non-zero scalar $\lambda_{\alpha_i} \in k$ such that $P(b)_{\alpha_i}$ equals the multiplication by λ_{α_i}. We then define $\tilde{f}_x : U(\tilde{Q})_x \to P(b)_x$ to be the multiplication by $\lambda_{\alpha_1}^{-1} \dots \lambda_{\alpha_t}^{-1}$. We must show that $\tilde{f}_x$ is well-defined. Assume that

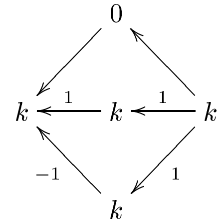
$$b = y_0 \stackrel{\beta_1}{\to} y_1 \stackrel{\beta_2}{\to} \cdots \stackrel{\beta_s}{\to} y_s = x$$

is another path in Q_B from b to x. Then $\lambda_{\alpha_1} \dots \lambda_{\alpha_t} = \lambda_{\beta_1} \dots \lambda_{\beta_s}$ because $P(b)$ is a B-module, and B is given a normed presentation. Hence $\lambda_{\alpha_1}^{-1} \dots \lambda_{\alpha_t}^{-1} = \lambda_{\beta_1}^{-1} \dots \lambda_{\beta_s}^{-1}$. Thus $\tilde{f}_x$ is well-defined. Clearly $\tilde{f}$ is an isomorphism of B-modules which restricts to an isomorphism of A-modules $f : U(Q) \to M = \operatorname{rad} P(b)$.
□

Example. In the non-Schurian case, the support of a completely co-separated module is not necessarily convex. Indeed, let A be given by the quiver



bound by $\alpha\lambda + \beta\mu + \gamma\nu = 0$, and M be the completely co-separated module given by



4.4. **Corollary.** *Let A be a triangular algebra. Then A is Schurian and strongly simply connected if and only if there exists a sequence of algebras $A_0, A_1, \dots A_n = A$ with $A_0 = k$ and, for each $0 \leq i < n$, a full convex zero-relation-free subquiver Q_i of Q_{A_i} such that either Q_i is completely co-separated and $A_{i+1} = A[U(Q_i)]$ or Q_i is completely separated and $A_{i+1} = [U(Q_i)]A_i$.*

Proof. This follows from (2.3) and (4.3). □

4.5. **Corollary.** *For each $n \geq 1$, there exist only finitely many non-isomorphic Schurian strongly simply connected algebras having n isomorphism classes of simple modules.*

Proof. This follows from (4.4) and induction. □

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