SOME CHARACTERISATIONS OF SUPPORTED ALGEBRAS

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Abstract. We give several equivalent characterisations of left (and hence, by duality, also of right) supported algebras. These characterisations are in terms of properties of the left and the right parts of the module category, or in terms of the classes $\mathcal{L}_0$ and $\mathcal{R}_0$ which consist respectively of the predecessors of the projective modules, and of the successors of the injective modules.

Introduction

Let A be an artin algebra. In order to study the representation theory of A, thus the category mod$A$ of finitely generated right $A-$modules, we consider a full subcategory ind$A$ of mod$A$ having as objects exactly one representative from each isomorphism class of indecomposable $A-$modules. Following Happel, Reiten and Smalø [15], we define the left part $\mathcal{L}_A$ of mod$A$ to be the full subcategory of ind$A$ having as objects the modules whose predecessors have projective dimension at most one. The right part $\mathcal{R}_A$ is defined dually. These classes were heavily investigated and applied (see, for instance, the survey [5]).

In particular, left (and right) supported algebras were defined in [4]: an artin algebra A is called left supported provided the additive full subcategory add$\mathcal{L}_A$ of mod$A$ having as objects the (finite) direct sums of modules in $\mathcal{L}_A$, is contravariantly finite in mod$A$ (in the sense of Auslander and Smalø [10]). Many classes of algebras are known to be

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left supported, such as those laura algebras which are not quasi-tilted (see [3], [19],[23]) as well as several classes of tilted algebras. Since, by definition, mod$A$ has a well-behaved left part when A is left supported, then this left part affords a reasonably good description, namely, it is contained in the left part of mod$B$, for some tilted algebra B, which is a full convex subcategory of A (see [4]).

The objective of this paper is to give several characterisations of left supported algebras. In our first main theorem, we prove that an artin algebra A is left supported if and only if $\mathcal{L}_A$ coincides with the full subcategory Pred$E$ of ind$A$ consisting of all predecessors of the direct sum E of all indecomposable Ext-injective modules in add $\mathcal{L}_A$ (these were characterised in [4],[7]). We also prove that A is left supported if and only if $\mathcal{L}_A$ equals the support $\mathrm{Supp}(-, E)$ of the contravariant Hom functor $\mathrm{Hom}_A(-, E)$ or, equivalently, equals $\mathrm{Supp}(-, L)$ for some suitably chosen module L. Other equivalent characterisations of left supported algebras involve the left support A_λ of A (in the sense of [4]). We now state our first main theorem (for the definition of almost directed and almost codirected modules, we refer the reader to (2.2)).

Theorem A. *The following conditions are equivalent for the artin algebra A:*

(a) *A is left supported.*
(b) $\mathcal{L}_A = \mathrm{Supp}(-, E)$.
(c) $\mathcal{L}_A = \mathrm{Pred}E$.
(d) *There exists an almost codirected $A-$module L such that* $\mathcal{L}_A = \mathrm{Supp}(-, L)$.
(e) *There exists an $A-$module L such that* $\mathrm{Hom}_A(\tau_A^{-1}L, L) = 0$ *and* $\mathcal{L}_A = \mathrm{Supp}(-, L)$.
(f) *E is a sincere $A_\lambda-$module.*
(g) $\mathcal{E} \cap \mathrm{mod}B \neq \emptyset$ *for each connected component B of A_λ.*
(h) *E is a cotilting $A_\lambda-$module.*
(i) *E is a tilting $A_\lambda-$module.*

All these characterisations are in terms of the left part of the module category. We also wish to have characterisations in terms of the remaining part of the module category. For this purpose, we define two new full subcategories of ind$A$: we let $\mathcal{L}_0$ (or $\mathcal{R}_0$) denote the full subcategory of ind$A$ consisting of the predecessors of projective modules (or the successors of injective modules, respectively). As we shall see, the class $\mathcal{R}_0$ is almost equal to the complement of $\mathcal{L}_A$ in ind$A$, in the sense that the intersection of $\mathcal{R}_0$ and $\mathcal{L}_A$ consists of only finitely many indecomposable modules. We describe the indecomposable Ext-projective modules in the class $\mathcal{R}_0$, and denote by U their direct sum. We are now able to state our second main result.

THEOREM B. *Let A be an artin algebra. The following conditions are equivalent:*

(a) *A is left supported.*
(b) $\mathrm{add}\mathcal{R}_0$ *is covariantly finite.*
(c) $\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}U$.
(d) *U is a tilting module.*
(e) $\mathcal{R}_0 = \mathrm{Supp}(U, -)$.
(f) *There exists an almost directed module R such that* $\mathcal{R}_0 = \mathrm{Supp}(R, -)$.
(g) *There exists a module R such that* $\mathrm{Hom}_A(R, \tau_A R) = 0$ *and* $\mathcal{R}_0 = \mathrm{Supp}(R, -)$.
(h) $\mathrm{add}\mathcal{R}_0 = \mathrm{Ker}\ \mathrm{Ext}^1_A(U, -)$.
(i) $\mathrm{Ker}\ \mathrm{Hom}_A(U, -) = \mathrm{add}(\mathcal{L}_A \setminus \mathcal{E}_1)$.

Clearly, the dual statements for right supported algebras hold as well. For the sake of brevity, we refrain from stating them, leaving the primal-dual translation to the reader. The paper is organised as follows. After a very brief preliminary section 1, devoted to fixing the notation and recalling some definitions, we study in section 2 those subcategories which are supports of Hom functors. In section 3, we recall known results on the Ext-injective modules in the left part. Section 4 is devoted to the proof of our first theorem (A). In section 5, we introduce the classes $\mathcal{L}_0$ and $\mathcal{R}_0$, study some of their properties, then prove our second theorem (B). Finally, in section 6, we characterise classes of algebras defined by finiteness or cofiniteness properties of the classes $\mathcal{L}_0$ and $\mathcal{R}_0$.

1. Preliminaries.

1.1. **Notation.** Throughout this paper, all our algebras are basic and connected artin algebras. For an algebra A, we denote by $\mathrm{mod}A$ its category of finitely generated right modules and by $\mathrm{ind}A$ a full subcategory of $\mathrm{mod}A$ consisting of one representative from each isomorphism class of indecomposable modules. Whenever we say that a given A-module is indecomposable, we always mean implicitly that it belongs to $\mathrm{ind}A$. Throughout this paper all modules considered belong to $\mathrm{mod}A$, that is, are finitely generated, unless otherwise specified. Also, all subcategories of $\mathrm{mod}A$ are full, and so are identified with their object classes. We sometimes consider an algebra A as a category, in which the object class A_0 is a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents of A, and the group of morphisms from e_i to e_j is $e_i A e_j$.

We say that a subcategory $\mathcal{C}$ of ind$A$ is *finite* if it has only finitely many objects, and that it is *cofinite* if $\mathcal{C}^c = \text{ind}A \setminus \mathcal{C}$ is finite. We sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. Further, we denote by add$\mathcal{C}$ the subcategory of mod$A$ having as objects the finite direct sums of objects in $\mathcal{C}$ and, if M is a module, we abbreviate add$\{M\}$ as add$M$. We denote the projective (or the injective) dimension of a module M as pd M (or id$M$, respectively). The global dimension of A is denoted by gl.dim. A. For a module M, the *support* $\text{Supp}(M,-)$ (or $\text{Supp}(-,M)$) of the functor $\text{Hom}_A(M,-)$ (or $\text{Hom}_A(-,M)$) is the subcategory of ind$A$ consisting of all modules X such that $\text{Hom}_A(M,X) \neq 0$ (or $\text{Hom}_A(X,M) \neq 0$, respectively). We denote by Gen$M$ (or Cogen$M$) the subcategory of mod$A$ having as objects all modules generated (or cogenerated, respectively) by M.

For an algebra A, we denote by $\Gamma(\text{mod}A)$ its Auslander-Reiten quiver, and by $\tau_A = \text{D Tr}$, $\tau_A^{-1} = \text{Tr D}$ its Auslander-Reiten translations. For further definitions and facts needed on mod$A$ or $\Gamma(\text{mod}A)$, we refer the reader to [9], [20], [22]. For tilting theory, we refer to [1], [20] and for quasi-tilted algebras to [15].

1.2. **Paths.** Let A be an artin algebra. Given $M, N \in \text{ind}A$ we write $M \rightsquigarrow N$ in case there exists a *path*

$$(*) \qquad M = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = N$$

($t \geq 1$) from M to N in ind$A$, that is, the f_i are non-zero morphisms and the X_i lie in ind$A$. In this case, we say that M is a *predecessor* of N and N is a *successor* of M. A path from M to M involving at least one non-isomorphism is a *cycle*. An indecomposable module M lying on no cycle in ind$A$ is a *directed module*. When each f_i in $(*)$ is irreducible, we say that $(*)$ is a *path of irreducible morphisms*, or a path in $\Gamma(\text{mod}A)$. A path $(*)$ of irreducible morphisms is *sectional* if $\tau_A X_{i+1} \neq X_{i-1}$ for all i with $0 < i < t$. A *refinement* of $(*)$ is a path in ind$A$

$$M = X'_0 \xrightarrow{f'_1} X'_1 \xrightarrow{f'_2} \cdots \longrightarrow X'_{t-1} \xrightarrow{f'_t} X'_t = N$$

such that there exists an order-preserving injection $\sigma : \{1, \cdots, t-1\} \longrightarrow \{1, \cdots, s-1\}$ such that $X_i = X'_{\sigma(i)}$ for all i with $1 \leq i < t$. A subcategory $\mathcal{C}$ of mod$A$ is *convex* if, for any path $(*)$ in ind$A$ with M, $N \in \mathcal{C}$, all the X_i belong to $\mathcal{C}$.

Finally, $\mathcal{C}$ is said to be *closed under successors* if, whenever $M \rightsquigarrow N$ is a path in ind$A$ with M lying in $\mathcal{C}$, then N lies in $\mathcal{C}$ as well. Clearly, such a subcategory is then the torsion class of a split torsion pair. We define dually subcategories *closed under predecessors* which are then the torsion-free classes of split torsion pairs.

2. Supports of functors.

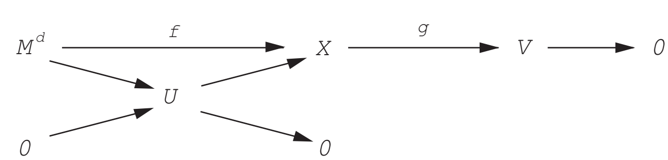
2.1. Let A be an artin algebra. We are interested in modules M having the property that $\mathrm{Hom}_A(M, \tau_A M) = 0$. These modules were studied in [11]. In particular, it is shown there that $\mathrm{Hom}_A(M, \tau_A M) = 0$ if and only if $\mathrm{Ext}^1_A(M, M') = 0$ for all quotient modules M' of M, or if and only if $\mathrm{Gen}M$ is closed under extensions (see [11] (5.5) (5.9)).

We recall that, if $\mathcal{C}$ is a subcategory of $\mathrm{mod}A$, closed under extensions, then a module $M \in \mathcal{C}$ is called *Ext-projective* (or *Ext-injective*) in $\mathcal{C}$ if $\mathrm{Ext}^1_A(M, -)|_{\mathcal{C}} = 0$ (or $\mathrm{Ext}^1_A(-, M)|_{\mathcal{C}} = 0$, respectively), see [11]. It is shown in [11] (3.3) (3.7) that if $\mathcal{C}$ is a torsion (or a torsion-free) class then an indecomposable module M is Ext-projective in $\mathcal{C}$ if and only if $\tau_A M$ is torsion-free (M is Ext-injective in $\mathcal{C}$ if and only if $\tau_A^{-1} M$ is torsion, respectively).

Proposition. *Let M be an $A-$module such that* $\mathrm{Hom}_A(M, \tau_A M) = 0$. *Then* $\mathrm{Supp}(M, -)$ *is closed under successors if and only if* add $\mathrm{Supp}(M, -) = \mathrm{Gen}M$. *Moreover, if this is the case, then* add $\mathrm{Supp}(M, -)$ *is a torsion class, and M is Ext-projective in* add $\mathrm{Supp}(M, -)$.

Proof. Assume first that $\mathrm{Supp}(M, -)$ is closed under successors. It is clear that $\mathrm{Gen}M \subseteq$ add $\mathrm{Supp}(M, -)$. In order to prove the reverse inclusion, let $X \in \mathrm{Supp}(M, -)$ and let $\{f_1, \cdots, f_d\}$ be a set of generators of the (non-zero) right $\mathrm{End}M-$module $\mathrm{Hom}_A(M, X)$. We claim that the morphism $f = [f_1, \cdots, f_d] : M^d \longrightarrow X$ is surjective.

Assume that this is not the case. Then $V = \mathrm{Coker}f \neq 0$. Also, clearly, $U = \mathrm{Im}f \neq 0$.



Since $\mathrm{Hom}_A(M, \tau_A M) = 0$ and U is a quotient of M, then $\mathrm{Ext}^1_A(M, U) = 0$, as we observed at the beginning of this section. Thus we have a short exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, U) \longrightarrow \mathrm{Hom}_A(M, X) \longrightarrow \mathrm{Hom}_A(M, V) \longrightarrow 0.$$

Since $\mathrm{Supp}(M, -)$ is closed under successors, then $V \in$ add $\mathrm{Supp}(M, -)$, and so there exists a non-zero morphism $h : M \longrightarrow V$. The exactness of the above sequence yields a morphism $h' : M \longrightarrow X$ such that $h = gh'$. By definition of f, there exist $u_1, \cdots, u_d \in \mathrm{End}M$ such that

$$h' = \sum_{i=1}^{d} f_i u_i = [f_1, \cdots, f_d] \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_d \end{bmatrix} = fu \quad , \qquad \text{where} \quad u = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ u_d \end{bmatrix} .$$

But this implies that $h = gh' = gfu = 0$, a contradiction which establishes our claim (and hence the necessity).

Conversely, assume that add $\mathrm{Supp}(M, -) = \mathrm{Gen}M$ and let $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = Y$ be a path in ind$A$, with $X \in \mathrm{Supp}(M, -)$. We prove by induction on j, with $0 \leq j \leq t$, that $X_j \in \mathrm{Supp}(M, -)$. So let $i < t$ and assume that $X_i \in \mathrm{Supp}(M, -)$. Since $X_i \in \mathrm{Gen}M$, there exist $d_i > 0$ and an epimorphism $p_i : M^{d_i} \longrightarrow X_i$. Therefore the composition $f_{i+1}p_i : M^{d_i} \longrightarrow X_{i+1}$ is non-zero and so $X_{i+1} \in \mathrm{Supp}(M, -)$. Thus $Y \in \mathrm{Supp}(M, -)$. This completes the proof of the sufficiency.

To show that add $\mathrm{Supp}(M, -) = \mathrm{Gen}M$ is a torsion class it suffices to observe that it is closed under quotients and extensions, since it is closed under successors.

There remains to prove that M is Ext-projective in add $\mathrm{Supp}(M, -)$. Assume that this is not the case. Then there is an indecomposable summand M_i of M such that $\tau_A M_i \in \mathrm{Supp}(M, -)$. Thus $\mathrm{Hom}_A(M, \tau_A M_i) \neq 0$, and this contradicts the hypothesis $\mathrm{Hom}_A(M, \tau_A M) = 0$. □

2.2. An $A-$module M (not necessarily indecomposable) is called *almost directed* if there exists no path $M_i \rightsquigarrow \tau_A M_j$ with M_i, M_j indecomposable summands of M. The reason for this terminology comes from the directing modules of [14]. Clearly, if M is directing in the sense of [14] then it is almost directed, but the converse is not true. Also, if M is directed, then $\mathrm{Hom}_A(M, \tau_A M) = 0$. The dual notion is that of an *almost codirected module.*

We recall from [11] (4.4) that, if $\mathcal{C}$ is a torsion class in mod$A$ of the form Gen$X$, then $\mathcal{C}$ has only finitely many isomorphism classes of indecomposable Ext-projective modules.

LEMMA. *Let $\mathcal{C}$ be an additive (full) subcategory of modA, closed under successors. Let M be the (not necessarily finite) sum of all the modules in* ind$A$ *which are Ext-projective in $\mathcal{C}$. Then the following conditions are equivalent:*

(a) *The module M is finitely generated and* $\mathcal{C} = \mathrm{Supp}(M, -)$.
(b) *There exists an almost directed (finitely generated) module R such that* $\mathcal{C} = \mathrm{Supp}(R, -)$.
(c) *There exists a (finitely generated) module R such that* $\mathrm{Hom}_A(R, \tau_A R) = 0$ *and* $\mathcal{C} = \mathrm{Supp}(R, -)$.

Proof. (a) implies (b). Assume (a). It suffices to show that M is almost directed. Let M_i, M_j be two indecomposable summands of M such that there exists a path $M_i \rightsquigarrow \tau_A M_j$. Since $\mathcal{C}$ is closed under successors and $M_i \in \mathcal{C}$, we have $\tau_A M_j \in \mathcal{C}$. On the other hand, M_j is Ext-projective in $\mathcal{C}$, and therefore $\tau_A M_j \notin \mathcal{C}$, a contradiction.

(b) implies (c). This is trivial.

(c) implies (a). Let R satisfy condition (c). Then, by Proposition (2.1), we know that $\mathrm{add}\,\mathcal{C} = \mathrm{GenR}$ and R is Ext-projective in $\mathrm{add}\mathcal{C}$. Hence, if we apply the above remarks with $X = R$, we obtain that M is finitely generated. Now, since $R \in \mathrm{add}M$, then $\mathrm{Supp}(R, -) \subseteq \mathrm{Supp}(M, -)$. Conversely, let $X \in \mathrm{Supp}(M, -)$. Since $M \in \mathcal{C}$, and $\mathcal{C}$ is closed under successors, we have $X \in \mathcal{C} = \mathrm{Supp}(R, -)$. $\square$

3. Ext-injectives in the left part.

3.1. Let A be an artin algebra. Following [15], we define the *left part of* $\mathrm{mod}A$ to be the (full) subcategory of $\mathrm{ind}A$ defined by

$$\mathcal{L}_A = \{M \in \mathrm{ind}A \mid \mathrm{pd}\, L \leq 1 \text{ for any predecessor } L \text{ of } M\}$$

Clearly, $\mathcal{L}_A$ is closed under predecessors. We refer to the survey [5] for characterisations of this class. The dual concept of $\mathcal{L}_A$ is the *right part* $\mathcal{R}_A$ *of* $\mathrm{mod}A$.

While the Ext-projectives in $\mathrm{add}\mathcal{L}_A$ are simply the projective modules lying in $\mathrm{add}\mathcal{L}_A$, the Ext-injectives are more interesting.

LEMMA [7] (3.2), [4] (3.1).
(a) The following conditions are equivalent for $M \in \mathcal{L}_A$ *:*

(i) *There exist an indecomposable injective module I and a path $I \rightsquigarrow M$.*
(ii) *There exist an indecomposable injective module I and a path of irreducible morphisms $I \rightsquigarrow M$.*
(iii) *There exist an indecomposable injective module I and a sectional path $I \rightsquigarrow M$.*
(iv) *There exists an indecomposable injective module I such that* $\mathrm{Hom}_A(I, M) \neq 0$.

(b) The following conditions are equivalent for $M \in \mathcal{L}_A$ which does not satisfy the conditions (a):

(i) *There exist an indecomposable projective module* $P \notin \mathcal{L}_A$ *and a path* $P \rightsquigarrow \tau_A^{-1}M$.
(ii) *There exist an indecomposable projective module* $P \notin \mathcal{L}_A$ *and a path of irreducible morphisms* $P \rightsquigarrow \tau_A^{-1}M$.
(iii) *There exist an indecomposable projective module* $P \notin \mathcal{L}_A$ *and a sectional path* $P \rightsquigarrow \tau_A^{-1}M$.
(iv) *There exists an indecomposable projective module* $P \notin \mathcal{L}_A$ *such that* $\mathrm{Hom}_A(P, \tau_A^{-1}M) \neq 0$.

Further, denoting by $\mathcal{E}_1$ *(or* $\mathcal{E}_2$*) the set of all* $M \in \mathcal{L}_A$ *satisfying conditions (a) (or (b), respectively), then* $X \in \mathcal{L}_A$ *is Ext-injective in* $\mathrm{add}\mathcal{L}_A$ *if and only if* $X \in \mathcal{E}_1 \cup \mathcal{E}_2$. □

Throughout this paper, we denote by E_1 (or E_2, or E) the direct sum of all $A-$modules lying in $\mathcal{E}_1$ (or $\mathcal{E}_2$, or $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, respectively).

3.2. The following lemma will also be useful.

Lemma [4] (3.4). *Assume that* $M \in \mathcal{E}$ *and that there exists a path* $M \rightsquigarrow N$ *with* $N \in \mathcal{L}_A$. *Then this path can be refined to a sectional path and* $N \in \mathcal{E}$. *In particular,* $\mathcal{E}$ *is convex in* $\mathrm{ind}A$. □

3.3. The endomorphism algebra A_λ of the direct sum of all projective $A-$modules lying in the left part $\mathcal{L}_A$ is called the *left support* of A (see [4], [23]). Since $\mathcal{L}_A$ is closed under predecessors, then A_λ is isomorphic to a full convex subcategory of A, closed under successors, and any module in $\mathcal{L}_A$ has a natural $A_\lambda-$module structure. It is shown in [4] (2.3), [23] (3.1) that A_λ is a product of connected quasi-tilted algebras and that $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda} \subseteq \mathrm{ind}A$. From this it follows easily that E is a convex partial tilting $A_\lambda-$module (see [4] (3.3)). Moreover, we can prove the following result.

Lemma. *The module* E *is a partial cotilting* $A_\lambda-$*module.*

Proof. It suffices to show that $\mathrm{id}_{A_\lambda}E \leq 1$. Let $E' \in \mathcal{E}$. Then $\tau_A^{-1}E' \notin \mathcal{L}_A$. Since $\tau_{A_\lambda}^{-1}E'$ is an epimorphic image of $\tau_A^{-1}E'$ (see[9], p.187), then $\tau_{A_\lambda}^{-1}E' \notin \mathcal{L}_A$. But $A_\lambda \in \mathrm{add}\mathcal{L}_A$. Hence $\mathrm{Hom}_A(\tau_{A_\lambda}^{-1}E', A_\lambda) = 0$ and $id_{A_\lambda}E \leq 1$. □

4. Left supported algebras.

4.1. Let $\mathcal{C} \subseteq \mathcal{D}$ be additive subcategories of $\mathrm{mod}A$. We recall from [10], [11] that $\mathcal{C}$ is called *contravariantly finite in* $\mathcal{D}$ if, for every $D \in \mathcal{D}$, there exists a morphism $f_D : C_D \longrightarrow D$ with $C_D \in \mathcal{C}$ such that, if $f : C' \longrightarrow D$ is a morphism with $C' \in \mathcal{C}$, then there exists $g : C' \longrightarrow C_D$

such that $f = f_D g$. Such a morphism f_D is called a *right approximation of D in $\mathcal{C}$*. The dual notion is that of a *covariantly finite subcategory.*

An algebra A is called *left supported* (see [4]) provided the subcategory $\mathrm{add}\mathcal{L}_A$ is contravariantly finite in $\mathrm{mod}A$. The following theorem characterises left supported algebras. Here, and in the sequel, we denote by F the sum of the projective $A-$modules in $\mathrm{ind}A \setminus \mathcal{L}_A$. It is shown in [4] (3.3) that $T = E \oplus F$ is a partial tilting module.

THEOREM [4] (4.2) (5.1). *Let A be an artin algebra. The following conditions are equivalent:*

(a) *A is left supported.*
(b) $\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}E$.
(c) *$T = E \oplus F$ is a tilting module.*
(d) *Each connected component B of the left support A_λ is tilted, and $\mathcal{E} \cap \mathrm{mod}B$ is a complete slice in* $\mathrm{mod}B$. □

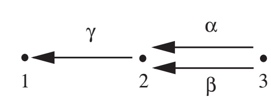
If A is left supported, then the module T is called the *canonical tilting module.*

4.2. We recall that, by (3.3), A_λ is a quasi-tilted algebra. We also have the following consequence of (4.1).

COROLLARY. *If A is left supported, then A_λ is a tilted algebra.* □

However, the converse is not true, as the following (counter)example shows. Left supported (quasi)tilted algebras were characterised in [25](3.8).

EXAMPLE. Let k be a field and A be the $k-$algebra given by the quiver



bound by the relation $\alpha\gamma = 0$.

$\left\{ 2 \;,\; \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \;,\; \begin{smallmatrix} & 3 & \\ 2 & & 2 \\ & 1 & \end{smallmatrix} \right\}$ is a complete slice in $\mathrm{mod}A$. Hence A is tilted and $A \simeq A_\lambda$. But $\mathcal{E} = \emptyset$, since $\mathcal{L}_A$ does not contain any injective modules. Therefore A is not left supported.

4.3. Now we show that all counterexamples to the converse of Corollary (4.2) must have $\mathcal{E} = \emptyset$, provided A_λ is connected. The next proposition generalises [15] (II,3.3) and its proof is inspired from the proof of the latter.

PROPOSITION. *Let B be a connected component of A_λ such that $\mathcal{E} \cap \mathrm{mod}B \neq \emptyset$. Then $\mathcal{E} \cap \mathrm{mod}B$ is a complete slice in* $\mathrm{mod}B$.

Proof. For simplicity, we assume that A_λ is connected and $\mathcal{E} \neq \emptyset$. We then show that E is a convex tilting $A_\lambda-$module. This is equivalent to proving that $\mathcal{E}$ is a complete slice in $\mathrm{mod}A_\lambda$, see [8]. It is easy to see that the argument carries on to the general case.

We know that E is a convex partial tilting $A_\lambda-$module. By counting the number of modules in $\mathcal{E}$, it suffices to prove that E is cotilting. By (3.3), E is a partial cotilting $A_\lambda-$module. Consequently, there exists a short exact sequence in mod A_λ

$(*)$ $\qquad 0 \longrightarrow E^d \longrightarrow X \longrightarrow \mathrm{D}A_\lambda \longrightarrow 0$

such that $E \oplus X$ is a cotilting $A_\lambda-$module (see [12] or [1] (1.7)). Let Y be an indecomposable summand of X. It follows from the exactness of $(*)$ that Y is $A_\lambda-$injective or $\mathrm{Hom}_{A_\lambda}(E, Y) \neq 0$ (as observed in [20], p. 167).

Assume first that $\mathrm{Hom}_{A_\lambda}(E, Y) \neq 0$. We claim that in this case, $Y \in \mathcal{E}$. To prove it, it suffices to show that $Y \in \mathcal{L}_A$, by (3.2). Now, suppose $Y \notin \mathcal{L}_A$ and let $f : E' \longrightarrow Y$ be a non-zero morphism, with $E' \in \mathcal{E}$. Then f factors through the $A_\lambda-$minimal left almost split morphism $g : E' \longrightarrow M$. Let M' be an indecomposable summand of M such that $\mathrm{Hom}_A(M', Y) \neq 0$. Since f is minimal, the morphism $\pi g : E' \longrightarrow M'$ is non-zero, where π is a projection of M onto M'. If $M' \in \mathcal{L}_A$ then $M' \in \mathcal{E}$, by (3.2). Hence, by factorising through minimal left almost split morphisms several times and using that $\mathrm{End}_{A_\lambda}E$ is a triangular algebra (by (3.2)) we can (and do) assume that $M' \notin \mathcal{L}_A$. In particular, M' is not A_λ-projective. Hence $\mathrm{Hom}_{A_\lambda}(\tau_{A_\lambda}M', E') \neq 0$ and thus $\tau_{A_\lambda}M' \in \mathcal{L}_A$. If $\tau_{A_\lambda}M' \notin \mathcal{E}$ then $\tau_A^{-1}\tau_{A_\lambda}M' \in \mathcal{L}_A \subseteq \mathrm{mod}A_\lambda$, whence $\tau_A^{-1}\tau_{A_\lambda}M' \simeq \tau_{A_\lambda}^{-1}\tau_{A_\lambda}M' \simeq M'$. This contradicts the hypothesis that $M' \notin \mathcal{L}_A$. Therefore $\tau_{A_\lambda}M' \in \mathcal{E}$ and f factors through $M' \in \tau_{A_\lambda}^{-1}\mathcal{E}$. Now, since $id_{A_\lambda}E \leq 1$, we have: $0 \neq \mathrm{Hom}_{A_\lambda}(\tau_{A_\lambda}^{-1}E, Y) \simeq \mathrm{D}\,\mathrm{Ext}_{A_\lambda}^1(Y, E)$, contradicting the fact that $E \oplus X$ is cotilting. Therefore $Y \in \mathcal{L}_A$ and our claim is established.

We have shown that the Bongartz sequence $(*)$ can be written in the form $0 \longrightarrow E_0 \longrightarrow E_1 \oplus J \longrightarrow \mathrm{D}A_\lambda \longrightarrow 0$, with $\mathrm{add}(E_0 \oplus E_1) = \mathrm{add}E$, and J an injective $A_\lambda-$module such that $\mathrm{Hom}_{A_\lambda}(E, J) = 0$. In order to complete the proof that E is cotilting, it suffices to show that $J = 0$. Assume that this is not the case. Since A_λ is a connected algebra and $E \oplus J$ is cotilting, then the algebra $\mathrm{End}_{A_\lambda}(E \oplus J)$ is also connected. Therefore there exists an indecomposable module J' which is a direct summand of J such that $\mathrm{Hom}_{A_\lambda}(J', E) \neq 0$ or $\mathrm{Hom}_{A_\lambda}(E, J') \neq 0$. Since $\mathrm{Hom}_{A_\lambda}(E, J) = 0$, we also have $\mathrm{Hom}_{A_\lambda}(E, J') = 0$. Therefore

$\mathrm{Hom}_{A_\lambda}(J', E) \neq 0$ and, in particular, $J' \in \mathcal{L}_A$. Since, by hypothesis, $J' \notin \mathcal{E}$, then $\tau_A^{-1} J' \in \mathcal{L}_A$. But then $\tau_A^{-1} J' \simeq \tau_{A_\lambda}^{-1} J' = 0$, a contradiction which completes the proof. □

4.4. Corollary. *If A_λ is connected and $\mathcal{E} \neq \emptyset$ then the cardinality $|\mathcal{E}|$ of $\mathcal{E}$ coincides with the rank of the Grothendieck group $K_0(A_\lambda)$ of A_λ.* □

4.5. For any module M, we let $\mathrm{Pred}M$ denote the subcategory of $\mathrm{ind}A$ having as objects all the predecessors of indecomposable summands of M.

Using Proposition (4.3) we can now give the following characterisations of left supported algebras.

Theorem. *The following conditions are equivalent for the artin algebra A:*

(a) *A is left supported.*
(b) $\mathcal{L}_A = \mathrm{Supp}(-, E)$.
(c) $\mathcal{L}_A = \mathrm{Pred}E$.
(d) *There exists an almost codirected $A-$module L such that* $\mathcal{L}_A = \mathrm{Supp}(-, L)$.
(e) *There exists an $A-$module L such that* $\mathrm{Hom}_A(\tau_A^{-1} L, L) = 0$ *and* $\mathcal{L}_A = \mathrm{Supp}(-, L)$.
(f) *E is a sincere $A_\lambda-$module.*
(g) *$\mathcal{E} \cap \mathrm{mod}B \neq \emptyset$ for each connected component B of A_λ.*
(h) *E is a cotilting $A_\lambda-$module.*
(i) *E is a tilting $A_\lambda-$module.*

Proof. (a) implies (b) implies (c) follows from (4.1) and the fact that $\mathrm{Cogen}E \subseteq \mathrm{add}\,\mathrm{Supp}(-, E) \subseteq \mathrm{add}\,\mathrm{Pred}E \subseteq \mathrm{add}\mathcal{L}_A$.

The equivalence of (b), (d), (e) is just the dual of (2.2).

(b) implies (f). This follows from the fact that every projective $A_\lambda-$module lies in $\mathrm{add}\mathcal{L}_A$.

Let now B be a connected component of A_λ and P be an indecomposable projective $B-$module. Since $P \in \mathcal{L}_A$, if (c) holds there exist $E' \in \mathcal{E}$ and a non-zero path $P \rightsquigarrow E'$. On the other hand, if (f) holds there exist $E' \in \mathcal{E}$ and a nonzero morphism $P \longrightarrow E'$. In either case we obtain that $E' \in \mathrm{mod}B$, and so (g) holds. Thus (c) implies (g), and also (f) implies (g).

(g) implies (h). This was established in Proposition (4.3).

(h) implies (i). This follows by counting the elements of the set $\mathcal{E}$, since E is a partial tilting $A_\lambda-$module.

(i) implies (a). If (i) holds, then $T = E \oplus F$ is a tilting $A-$module (see [4] (3.3)). (a) follows from this and (4.1). □

5. The classes $\mathcal{L}_0$ and $\mathcal{R}_0$.

5.1. Let M be an A-module. Now we consider the subcategory $\mathrm{Succ}M = \mathrm{D}(\mathrm{Pred}\ \mathrm{D}M)$ of ind$A$ consisting of the successors of M. We define two (full) subcategories of ind$A$ as follows:

$\mathcal{L}_0 = \{M \in \mathrm{ind}A \mid$ there exists a projective P in ind$A$ and a path $M \rightsquigarrow P\}$

$\mathcal{R}_0 = \{M \in \mathrm{ind}A \mid$ there exists a injective I in ind$A$ and a path $I \rightsquigarrow M\}$

Then $\mathcal{L}_0 = \mathrm{Pred}A$, and $\mathcal{R}_0 = \mathrm{Succ}\ \mathrm{D}A$.

Thus, the class $\mathcal{L}_0$ contains all the projective modules of ind$A$ and is closed under predecessors. In particular, add$\mathcal{L}_0$ is the torsion-free class of a split torsion pair. Clearly, $\mathcal{L}_0$ coincides with the class of all projective modules in ind$A$ if and only if A is hereditary.

Dually, the class $\mathcal{R}_0$ contains all the indecomposable injectives and is closed under successors. In particular, add$\mathcal{R}_0$ is the torsion class of a split torsion pair.

Our first lemma gives the relationship between these classes and the classes $\mathcal{L}_A$ and $\mathcal{R}_A$. We only state the results for $\mathcal{R}_0$, and leave to the reader the formulation of the corresponding ones for $\mathcal{L}_0$.

Lemma. $\mathcal{R}_0 = \mathcal{E}_1 \cup (\mathcal{L}_A)^c$.

Proof. In order to prove that $(\mathcal{L}_A)^c \subseteq \mathcal{R}_0$, let $M \in (\mathcal{L}_A)^c$. Then there exists a predecessor L of M such that $\mathrm{pd}\, L > 1$. By [20] p. 74, there exists an injective $I \in \mathrm{ind}A$ such that $\mathrm{Hom}_A(I, \tau_A L) \neq 0$. The path $I \longrightarrow \tau_A L \longrightarrow * \longrightarrow L \rightsquigarrow M$ yields $M \in \mathcal{R}_0$.

On the other hand, it follows from the very definition of $\mathcal{E}_1$ (see (3.1)) that $\mathcal{E}_1 = \mathcal{L}_A \cap \mathcal{R}_0$. Therefore

$\mathcal{R}_0 = \mathcal{R}_0 \cap (\mathcal{L}_A \cup (\mathcal{L}_A)^c) = (\mathcal{R}_0 \cap \mathcal{L}_A) \cup (\mathcal{R}_0 \cap (\mathcal{L}_A)^c) = \mathcal{E}_1 \cup (\mathcal{L}_A)^c$. □

5.2. Corollary. *Let A be a quasi-tilted algebra which is not tilted. Then $\mathcal{R}_0 = (\mathcal{L}_A)^c$.*

Proof. Since A is not tilted, then by [15] (II.3.3), $\mathcal{L}_A$ contains no injective. Therefore $\mathcal{E}_1 = \emptyset$. □

5.3. Recall from (3.1) and (4.1) that E_1 (or E_2) denotes the direct sum of all modules in $\mathcal{E}_1$ (or $\mathcal{E}_2$, respectively), and F denotes the direct sum of all projectives in $\mathrm{ind}A \setminus \mathcal{L}_A$.

From now on, we denote by U the direct sum $U = E_1 \oplus \tau_A^{-1} E_2 \oplus F$ (we recall that no summand of E_2 is injective).

LEMMA. *Let $M \in \text{ind}A$. Then:*

(a) *M is Ext-projective in* $\text{add}\mathcal{R}_0$ *if and only if* $M \in \text{add}U$.
(b) *M is Ext-injective in* $\text{add}\mathcal{R}_0$ *if and only if M is injective.*

Proof. (a) Necessity. Let M be Ext-projective in $\text{add}\mathcal{R}_0$. If $M \in \mathcal{L}_A$, then $M \in \mathcal{E}_1$ (so $M \in \text{add}U$). If $M \notin \mathcal{L}_A$ and is projective, then $M \in \text{add}F$ (so $M \in \text{add}U$). If $M \notin \mathcal{L}_A$ and is not projective, then $\tau_A M \neq 0$. Since M is Ext-projective in $\text{add}\mathcal{R}_0$ then $\tau_A M \notin \mathcal{R}_0$. Since $(\mathcal{L}_A)^c \subseteq \mathcal{R}_0$, we have $\tau_A M \in \mathcal{L}_A$. Then $\tau_A M$ is Ext-injective in $\text{add}\mathcal{L}_A$, that is, $\tau_A M \in \mathcal{E}$. If $\tau_A M \in \mathcal{E}_1$, then $\tau_A M \in \mathcal{R}_0$, a contradiction. Therefore $\tau_A M \in \mathcal{E}_2$, and so $M \in \tau_A^{-1}(\mathcal{E}_2) \in \text{add}U$.

Sufficiency. Assume $M \in \text{add}F$. Since M is projective and lies in $\mathcal{R}_0$, then it is Ext-projective in $\text{add}\mathcal{R}_0$.

Assume $M \in \mathcal{E}_1$. If $\tau_A M \in \mathcal{R}_0$, there exists an indecomposable injective I and a path $I \rightsquigarrow \tau_A M$, which we may assume to consist of irreducible morphisms, by (3.1). But then the composed path $I \rightsquigarrow \tau_A M \longrightarrow * \longrightarrow M$ consists of irreducible morphisms and is not sectional, contradicting [3] (1.6). Therefore $\tau_A M \notin \mathcal{R}_0$ and so M is Ext-projective in $\text{add}\mathcal{R}_0$.

Finally, assume $M \in \tau_A^{-1}(\mathcal{E}_2)$. Then $\tau_A M \in \mathcal{E}_2$. By (5.1), $\tau_A M \notin \mathcal{R}_0$ and so, again, M is Ext-projective in $\text{add}\mathcal{R}_0$.

(b) Assume that M is Ext-injective in $\text{add}\mathcal{R}_0$ and let $j : M \longrightarrow I$ be an injective envelope, so that we have a short exact sequence

$$0 \longrightarrow M \xrightarrow{j} I \longrightarrow \text{Coker} j \longrightarrow 0.$$

Since $\mathcal{R}_0$ is closed under successors, both I and $\text{Coker}j$ belong to $\text{add}\mathcal{R}_0$. Hence $\text{Ext}_A^1(\text{Coker}j, M) = 0$, the sequence splits, and so M is injective. The reverse implication is trivial. $\square$

5.4. LEMMA. *(a) U is a partial tilting module.*

(b) U is a tilting module if and only if $T = E_1 \oplus E_2 \oplus F$ is a tilting module, if and only if the number of (isomorphism classes of) indecomposable summands of $E_1 \oplus \tau_A^{-1} E_2$ equals the number of projectives lying in $\mathcal{L}_A$.

Proof. (a) Since U is Ext-projective in $\text{add}\mathcal{R}_0$, then $\text{Ext}_A^1(U, U) = 0$. We thus have to show that $\text{pd}\, U \leq 1$. Clearly, $\text{pd}(E \oplus F) \leq 1$. Let $M \in \tau_A^{-1}\mathcal{E}_2$. Then $\tau_A M \in \mathcal{E}_2$. Now, since $\tau_A M \in \mathcal{L}_A$, the existence of a morphism from an indecomposable injective I to $\tau_A M$ would imply

$I \in \mathcal{L}_A$, and then we would deduce that $\tau_A M \in \mathcal{E}_1$, a contradiction. Thus $\mathrm{Hom}_A(\mathrm{D}A, \tau_A M) = 0$, that is, $\mathrm{pd}\, M \leq 1$.

(b) We recall that, by [4] (3.3), T is a partial tilting module. Since no summand of E_2 is injective, we have $|\mathrm{ind}A \cap \mathrm{add}U| = |\mathrm{ind}A \cap \mathrm{add}T|$. This establishes the statement. □

5.5. We denote by $(\mathcal{T}(L), \mathcal{F}(L))$ the torsion pair determined by a tilting module L.

Lemma. *Assume that* $U = E_1 \oplus \tau_A^{-1} E_2 \oplus F$ *is a tilting module. Then* $\mathcal{T}(U) = \mathrm{add}\mathcal{R}_0$ *and* $\mathcal{F}(U) = \mathrm{add}(\mathrm{ind}A \setminus \mathcal{R}_0)$.

Proof. Let M be an indecomposable module in $\mathcal{T}(U)$. Then $\mathrm{Hom}_A(U, M) \neq 0$. Since $U \in \mathrm{add}\mathcal{R}_0$ which is closed under successors, then $M \in \mathcal{R}_0$. Assume conversely that $M \in \mathcal{R}_0$. If $M \notin \mathcal{T}(U)$, then $\mathrm{Hom}_A(M, \tau_A U) \simeq \mathrm{D}\,\mathrm{Ext}^1_A(U, M) \neq 0$. Since $\tau_A U \in \mathrm{add}\mathcal{L}_A$ which is closed under predecessors, then $M \in \mathcal{L}_A$. Therefore $M \in \mathcal{R}_0 \cap \mathcal{L}_A = \mathcal{E}_1$ and hence there exist an injective I in $\mathrm{ind}A$ and a path $I \rightsquigarrow M$. Since $\mathrm{Ext}^1_A(E_1, M) = 0$ (because E_1 is a partial tilting module), then the condition $\mathrm{Ext}^1_A(U, M) \neq 0$ implies the existence of $E_0 \in \mathcal{E}_2$ such that $\mathrm{Hom}_A(M, E_0) \simeq \mathrm{D}\,\mathrm{Ext}^1_A(\tau_A^{-1} E_0, M) \neq 0$. Hence our path can be extended to a path $I \rightsquigarrow M \longrightarrow E_0$. But this yields $E_0 \in \mathcal{E}_1$, a contradiction. This shows the first equality. The second follows by maximality (because $\mathcal{R}_0$ is closed under successors). □

5.6. We are now able to prove our second main theorem. Observe that, since $\mathcal{R}_0$ is closed under successors, then it is trivially contravariantly finite. Here and in the sequel, for a functor $F : \mathrm{mod}A \longrightarrow \mathrm{mod}A$, we denote by $\mathrm{Ker}F$ the full subcategory having as objects the A-modules M such that $F(M) = 0$.

theorem. *Let A be an artin algebra. The following conditions are equivalent:*

(a) *A is left supported.*
(b) $\mathrm{add}\mathcal{R}_0$ *is covariantly finite.*
(c) $\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}U$.
(d) *U is a tilting module.*
(e) $\mathcal{R}_0 = \mathrm{Supp}(U, -)$.
(f) *There exists an almost directed module R such that* $\mathcal{R}_0 = \mathrm{Supp}(R, -)$.
(g) *There exists a module R such that* $\mathrm{Hom}_A(R, \tau_A R) = 0$ *and* $\mathcal{R}_0 = \mathrm{Supp}(R, -)$.
(h) $\mathrm{add}\mathcal{R}_0 = \mathrm{Ker}\ \mathrm{Ext}^1_A(U, -)$.
(i) $\mathrm{Ker}\ \mathrm{Hom}_A(U, -) = \mathrm{add}(\mathcal{L}_A \setminus \mathcal{E}_1)$.

Proof. (a) is equivalent to (d). By (4.1), A is left supported if and only if $T = E \oplus F$ is a tilting module. By (5.4) T is tilting if and only if so is U.

(d) implies (c), (h), (i). This follows from (5.5) (Note that $\mathrm{ind}A \setminus \mathcal{R}_0 = \mathcal{L}_A \setminus \mathcal{E}_1$).

(h) implies (a). Since we always have $\mathrm{add}\mathcal{R}_0 \subseteq \mathrm{Ker}\ \mathrm{Ext}^1_A(U, -)$ (because U is Ext-projective in $\mathrm{add}\mathcal{R}_0$), (h) says that if $X \in \mathrm{ind}A$ is such that $\mathrm{Hom}_A(X, \tau_A U) \simeq \mathrm{D}\,\mathrm{Ext}^1_A(U, X) = 0$, then $X \in \mathcal{R}_0$, or, equivalently, if $X \notin \mathcal{R}_0$, then $\mathrm{Hom}_A(X, \tau_A U) \neq 0$. Now assume (h) holds and let $X \in \mathcal{L}_A$. If $X \notin \mathcal{E}_1$, then $X \in \mathcal{L}_A \setminus \mathcal{E}_1 = (\mathcal{R}_0)^c$. Hence $\mathrm{Hom}_A(X, \tau_A E_1 \oplus E_2) = \mathrm{Hom}_A(X, \tau_A U) \neq 0$, and so $X \in \mathrm{Pred}E$. If $X \in \mathcal{E}_1$ then we also have $X \in \mathrm{Pred}E$. Thus $\mathcal{L}_A \subseteq \mathrm{Pred}E$, and so $\mathcal{L}_A = \mathrm{Pred}E$. Now (a) follows from Theorem (4.5).

(i) implies (c). Assume (i). Since U is a partial tilting module, it induces the torsion class $\mathrm{Gen}U$. We claim that the torsion pair $(\mathrm{Gen}U, \mathrm{Ker}\,\mathrm{Hom}_A(U, -))$ is split. To prove this, it suffices to show that $\mathcal{L}_A \setminus \mathcal{E}_1$ is closed under predecessors. Let $X \rightsquigarrow Y$, with $Y \in \mathcal{L}_A \setminus \mathcal{E}_1$. Since $Y \in \mathcal{L}_A$, then $X \in \mathcal{L}_A$. Suppose $X \in \mathcal{E}_1$. Then there exist an indecomposable injective A-module I and a path $I \rightsquigarrow X$. But then the composed path $I \rightsquigarrow X \rightsquigarrow Y$ yields $Y \in \mathcal{E}_1$, a contradiction. Hence $X \in \mathcal{L}_A \setminus \mathcal{E}_1$, as required. The pair being split, we deduce that $\mathrm{Gen}U = \mathrm{add}(\mathrm{ind}A \setminus (\mathcal{L}_A \setminus \mathcal{E}_1)) = \mathrm{add}\mathcal{R}_0$ (by (5.1)).

(c) implies (d). Since $\mathrm{add}\mathcal{R}_0$ is a torsion class which contains the injectives, then $\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}U$ implies that $\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}V$ for some tilting module V (see [1] (3.2)). Since $\mathrm{add}V = \mathrm{add}\{M \mid M \text{ is Ext-projective in } \mathrm{add}\mathcal{R}_0\} = \mathrm{add}U$ and U is a partial tilting module, then we obtain that U is a tilting module by counting the indecomposable summands of $\mathrm{add}U$.

(b) implies (c). Since $\mathrm{add}\mathcal{R}_0$ is covariantly finite and is the torsion class of a torsion pair, then, by [24], there exists an Ext-projective V in $\mathrm{add}\mathcal{R}_0$ such that $\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}V$. Thus $V \in \mathrm{add}U$, and so

$$\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}V \subseteq \mathrm{Gen}U \subseteq \mathrm{add}\mathcal{R}_0$$

implying the result.

(c) implies (b). This follows directly from [10] (4.5).

(c) implies (e). Assume (c). Then (e) follows from

$$\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}U \subseteq \mathrm{add}\ \mathrm{Supp}(U, -) \subseteq \mathrm{add}\mathcal{R}_0.$$

(e) implies (c). If (e) holds, then $\mathrm{Supp}(U, -)$ is closed under successors. So, by (2.1), $\mathrm{add}\,\mathrm{Supp}(U, -) = \mathrm{Gen}U$. Therefore $\mathrm{add}\mathcal{R}_0 = \mathrm{Gen}U$.

The equivalence of (e), (f), (g) follows from (2.2). □

5.7. The following technical lemma is a consequence of [10] (3.13).

Lemma. *Let $\mathcal{B}, \mathcal{C}$ be (full) subcategories of ind$A$ such that the symmetric difference $\mathcal{B} \triangle \mathcal{C}$ is finite and* $\mathrm{add}(\mathcal{B} \cup \mathcal{C})$ *has left almost split morphisms. Then* $\mathrm{add}\mathcal{B}$ *is covariantly finite in* $\mathrm{mod}A$ *if and only if so is* $\mathrm{add}\mathcal{C}$.

Proof. Let $\mathcal{B}, \mathcal{C}$ be as above. By symmetry, we assume without loss of generality that $\mathrm{add}\mathcal{B}$ is covariantly finite in $\mathrm{mod}A$, and show that then so is $\mathrm{add}\mathcal{C}$. From [10] (3.13), we deduce that $\mathrm{add}\mathcal{C}$ is covariantly finite in $\mathrm{add}(\mathcal{B} \cup \mathcal{C})$. Since $\mathrm{add}\mathcal{B}$ is covariantly finite in $\mathrm{mod}A$ by hypothesis, and $\mathrm{add}(\mathcal{C} \setminus \mathcal{B})$ is covariantly finite in $\mathrm{mod}A$ (because $\mathcal{C} \setminus \mathcal{B}$ is a finite set), then $\mathrm{add}(\mathcal{B} \cup \mathcal{C}) = \mathrm{add}(\mathcal{B} \cup (\mathcal{C} \setminus \mathcal{B}))$ is covariantly finite in $\mathrm{mod}A$. Then, by transitivity, $\mathrm{add}\mathcal{C}$ is covariantly finite in $\mathrm{mod}A$. □

The dual of the preceding lemma is also valid. We leave the primal-dual translation to the reader.

5.8. With the aid of the preceding lemma, we obtain the following corollary of (4.5) and (5.6).

Proposition. *The class* $\mathrm{add}\mathcal{L}_A$ *is contravariantly finite in modA if and only if* $\mathrm{add}((\mathcal{L}_A)^c)$ *is covariantly finite in modA.*

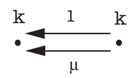
Proof. Indeed, $\mathrm{add}\mathcal{L}_A$ is contravariantly finite in $\mathrm{mod}A$ if and only if A is left supported, if and only if $\mathrm{add}\mathcal{R}_0$ is covariantly finite in $\mathrm{mod}A$. Since, by (5.1), $\mathcal{R}_0 = (\mathcal{L}_A)^c \cup \mathcal{E}_1$, then $(\mathcal{L}_A)^c \triangle \mathcal{R}_0 = \mathcal{R}_0 \setminus (\mathcal{L}_A)^c = \mathcal{E}_1$ is a finite set, and $\mathrm{add}((\mathcal{L}_A)^c \cup \mathcal{R}_0) = \mathrm{add}\mathcal{R}_0$ has left almost split morphisms, since it is closed under successors. Then the result follows from (5.7). □

5.9. Let $\mathcal{C}$ be a subcategory of ind$A$. It follows from [10] (4.1) (4.2) that if $\mathcal{C}$ is finite or cofinite, then $\mathrm{add}\mathcal{C}$ is contravariantly and covariantly finite in $\mathrm{mod}A$. From this, and our preceding proposition, it may be asked whether $\mathrm{add}\mathcal{C}$ is covariantly finite in $\mathrm{mod}A$ if and only if $\mathrm{add}(\mathcal{C}^c)$ is contravariantly finite in $\mathrm{mod}A$. This is not true though, as the following example shows.

Example. Let A be the Kronecker algebra over an algebraically closed field k. This algebra can be described as the path algebra of the quiver

$$\underset{1}{\bullet} \leftleftarrows \underset{2}{\bullet}$$

Let M_μ be the indecomposable representation



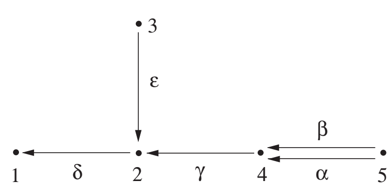
Consider the full subcatefory $\mathcal{C}$ of ind$A$ having as objects all M_μ in ind$A$, with $\mu \in k$. Then, since length$(M_\mu) = 2$, it follows from [10] (4.1) that add$(\text{ind}A \setminus \mathcal{C})$ is functorially finite in mod$A$. However $\mathcal{C}$ is neither covariantly nor contravariantly finite in mod$A$. For instance, the injective hull I_2 of M_μ (the same module for every μ) does not admit a left approximation $C \longrightarrow I_2$ in add$\,\mathcal{C}$, for $\text{Hom}_A(M_\mu, M_\nu) = 0$ if $\mu \neq \nu$.

5.10. We now show that, if A is left supported, then the tilting module U has a property also enjoyed by the canonical tilting module T (see [4] (5.3)). Recall from [6] (4.3) that the torsion classes having a given partial tilting module M as Ext-projective form a complete lattice under inclusion, having as largest element the class $\mathcal{T}_1(M) = \{N \in \text{mod}A \mid \text{Ext}^1_A(M, N) = 0\}$ and furthermore, $\mathcal{T}_1(M) = \text{Gen}(M \oplus X)$, where X is the Bongartz complement of M (see [1] (1.7)).

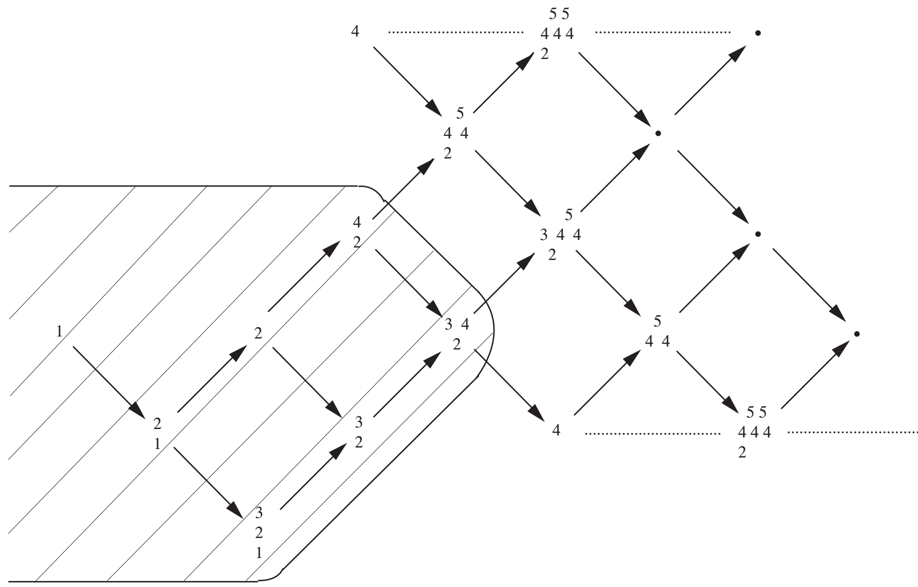
Corollary. *Let A be left supported. Then F is the Bongartz complement of $E_1 \oplus \tau_A^{-1} E_2$.*

Proof. Let X denote the Bongartz complement of $E_1 \oplus \tau_A^{-1} E_2$. Since $\text{Ext}^1_A(E_1 \oplus \tau_A^{-1} E_2 \oplus F, -) = \text{Ext}^1_A(E_1 \oplus \tau_A^{-1} E_2, -)$, we deduce that $\text{Gen}(E_1 \oplus \tau_A^{-1} E_2 \oplus F) = \mathcal{T}(U) = \text{Gen}(E_1 \oplus \tau_A^{-1} E_2 \oplus X)$. Since add$U$ = $\{M \mid M$ is Ext-projective in $\mathcal{T}(U)\} = \text{add}(E_1 \oplus \tau_A^{-1} E_2 \oplus X)$, looking at the number of isomorphism classes of indecomposable summands of U and of $E_1 \oplus \tau_A^{-1} E_2 \oplus X$, we conclude that $E_1 \oplus \tau_A^{-1} E_2 \oplus X = E_1 \oplus \tau_A^{-1} E_2 \oplus F$. $\square$

5.11. Example. Let k be a field and A be the finite dimensional $k-$algebra given by the quiver



bound by the relations $\alpha\gamma = 0$, $\gamma\delta = 0$. Then the beginning of the postprojective component of $\Gamma(\text{mod}A)$ has the following shape:



where modules are represented by their composition factors and we identify along the horizontal dotted lines. The shaded area represents $\mathcal{L}_A$. Clearly, here $\mathcal{E}_1 = \left\{ \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 3\,4 \\ 2 \end{smallmatrix} \right\}$ and $\mathcal{E}_2 = \left\{ \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} \right\}$. Indeed, $F = \begin{smallmatrix} 5 \\ 4\,4 \\ 2 \end{smallmatrix}$. The module $U = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3\,4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 3\,4\,4 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4\,4 \\ 2 \end{smallmatrix}$ is clearly a tilting module. Thus A is left supported.

6. Algebras determined by the classes $\mathcal{L}_0$ and $\mathcal{R}_0$.

6.1. Many classes of algebras have been characterised by finiteness or cofiniteness properties of the classes $\mathcal{L}_A$ and $\mathcal{R}_A$; see, for instance, the survey [5]. It is natural to seek similar characterisations using the classes $\mathcal{L}_0$ and $\mathcal{R}_0$. Our first proposition is a restatement of many known results. For the definitions and properties of left glued, right glued and laura algebras, we refer to [5]. We denote by μ the Gabriel-Roiter measure of a module [21].

Proposition. *Let A be an artin algebra.*

(a) *A is left (or right) glued if and only if the class $\mathcal{L}_0$ (or $\mathcal{R}_0$, respectively) is finite.*
(b) *A is concealed if and only if the class $\mathcal{L}_0 \cup \mathcal{R}_0$ is finite.*
(c) *The following conditions are equivalent:*
   (i) *A is a laura algebra.*

(ii) *$\mathcal{L}_0 \cap \mathcal{R}_0$ is finite.*
(iii) *The set $\{\mu(M) \mid M \in \mathcal{L}_0 \cap \mathcal{R}_0\}$ is finite.*
(iv) *There exists an m such that any path in $\mathcal{L}_0 \cap \mathcal{R}_0$ contains at most m hooks.*

Proof. (a). By [3] (2.2), the algebra A is left glued if and only if $\mathcal{R}_A$ is cofinite, thus if and only if $(\mathcal{R}_A)^c$ is finite. By the dual of (5.1), this amounts to saying that $\mathcal{L}_0$ is finite. The proof is similar for right glued algebras.

(b) By [2] (3.4), A is concealed if and only if it is both left and right glued, thus if and only if both $\mathcal{L}_0$ and $\mathcal{R}_0$ are finite.

(c) The equivalence of (i) and (ii) follows from [3] (2.4) (or directly from the definition and (5.1)). The equivalence of (i) and (iii) follows from [16], and the equivalence of (i) and (iv) from [17]. □

6.2. The following proposition is a reformulation of part of a result of D. Smith [25], Theorem 2. For quasi-directed components, see [5, 25].

Proposition. *Let A be an artin algebra, and Γ be a non-semiregular connected component of $\Gamma(\mathrm{mod}A)$. The following conditions are equivalent:*

(a) *Γ is quasi-directed and convex.*
(b) *There exists an n_0 such that any path in $\Gamma \cap \mathcal{L}_0 \cap \mathcal{R}_0$ contains at most n_0 distinct modules.*
(c) *There exists an m_0 such that any path in $\Gamma \cap \mathcal{L}_0 \cap \mathcal{R}_0$ contains at most m_0 distinct hooks.*

Furthermore, if $\mathrm{Ann}\Gamma$ is the annihilator of Γ and $B = A/\mathrm{Ann}\Gamma$, then B is a laura algebra and Γ is the unique non-semiregular and faithful component of $\Gamma(\mathrm{mod}B)$. □

6.3. The following is a restatement of [25] (1.4).

Lemma. *Let A be an artin algebra, Γ be a non-semiregular component of $\Gamma(\mathrm{mod}A)$ having only finitely many τ_A-orbits, and $X \in \Gamma$ be a non-directed module. Then $X \in \mathcal{L}_0 \cap \mathcal{R}_0$.* □

6.4. We now look at what happens when the classes $\mathcal{L}_0$ and $\mathcal{R}_0$ are cofinite, that is, when $\mathcal{L}_A$ and $\mathcal{R}_A$ are finite.

Proposition. *Let A be an artin algebra. The class $\mathcal{R}_0$ is cofinite if and only if the left support A_λ is a product of connected tilted algebras, each of which has an injective in its corresponding postprojective component.*

Proof. Sufficiency. Assume that A_λ satisfies the stated condition. Then, for each connected component B of A_λ there is a complete slice in a postprojective component of $\Gamma(\text{mod}B)$, which is thus unique. Then, clearly, $\mathcal{L}_{A_\lambda}$ is finite. Hence $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$ is finite. But then $\mathcal{R}_0 = (\mathcal{L}_A)^c \cup \mathcal{E}_1$ is cofinite.

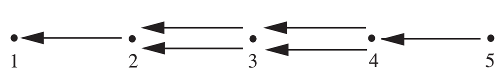
Necessity. If $\mathcal{R}_0$ is cofinite then, by [10] (4.1), add$\mathcal{R}_0$ is covariantly finite. By (5.6), A is left supported. By (4.1), A_λ is a product of connected tilted algebras. We may, without loss of generality, assume that A_λ is connected. By [4] (5.4), the Auslander-Reiten quiver $\Gamma(\text{mod}A)$ has a postprojective component containing at least one injective module I. We may, without loss of generality, assume that I is minimal with respect to the natural order in the component. Hence $I \in \mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$. Since I is injective as an $A-$module, it is also injective as an $A_\lambda-$module. This completes the proof. Observe that the postprojective component containing I is the unique connecting component of $\Gamma(\text{mod}A_\lambda)$. □

6.5. The dual notion of the left support algebra A_λ of an artin algebra A is called its *right support* and is denoted by A_ρ. The following corollary is a direct consequence of (6.4) and its dual.

Corollary. *Let A be an artin algebra. The following conditions are equivalent:*

(a) *$\mathcal{L}_0 \cap \mathcal{R}_0$ is cofinite.*
(b) *$\mathcal{L}_A \cup \mathcal{R}_A$ is finite.*
(c) *A_λ is a product of connected tilted algebras, each of which has an injective in its corresponding postprojective component, and A_ρ is a product of connected tilted algebras, each of which has a projective in its corresponding preinjective component.* □

Example. The following is an example of an artin algebra satisfying the conditions of the corollary. Let k be a field, and A be the radical-square zero algebra given by the quiver:



6.6. It is an interesting problem to determine which algebras have the property that the class $\mathcal{L}_0 \cup \mathcal{R}_0$ is cofinite. We solve here this problem in the case of laura algebras.

PROPOSITION. *Let A be a laura algebra. The following conditions are equivalent:*

(a) $\mathcal{L}_0 \cup \mathcal{R}_0$ *is cofinite.*
(b) $\Gamma(\text{mod}A)$ *has a non-semiregular component.*
(c) A *is left and right supported but not concealed.*

Proof. Assume first that A is a laura algebra which is not quasi-tilted. Then all three statements clearly hold true (see [3] (4.6), [4] (4.4)). We may thus assume that A is quasi-tilted. It was shown by Smith in [25] (3.8) that a quasi-tilted algebra A is left supported if and only if A is tilted having an injective module in a connecting component of $\Gamma(\text{mod}A)$. Thus (b) and (c) are equivalent, and we just have to prove that (a) holds if and only if A is tilted having both an injective and a projective in a connecting component of $\Gamma(\text{mod}A)$.

Clearly, if the latter condition is satisfied, then $\mathcal{L}_0 \cup \mathcal{R}_0$ is cofinite.

Conversely, assume that A is tilted and $\Gamma(\text{mod}A)$ has a connecting component Γ containing no injective. Let Σ be a complete slice in Γ. We have to prove that $(\mathcal{L}_0 \cup \mathcal{R}_0)^c$ is not finite. Clearly, it suffices to show that all proper successors in Σ of Γ lie in $(\mathcal{L}_0 \cup \mathcal{R}_0)^c$. Indeed, let $M \in \text{Succ}\Sigma \cap \Gamma$. Hence $M \in \tau_A^{-k}\Sigma$, for some $k \geq 0$. Since there are no injectives in $\Gamma, \tau_A^{-k}\Sigma$ is also a complete slice. If $M \in \mathcal{L}_0$, there exists a projective $P \in \text{ind}A$ and a path $M \rightsquigarrow P \longrightarrow S$, with $S \in \tau_A^{-k}\Sigma$ (by sincerity of $\tau_A^{-k}\Sigma$). Hence, using the convexity of $\tau_A^{-k}\Sigma$, we obtain that $M \in \tau_A^{-k}\Sigma$. If $M \in \mathcal{R}_0$, there exist an injective $I \in \text{ind}A$ and a path $S \longrightarrow I \rightsquigarrow M$, with $S \in \tau_A^{-k}\Sigma$, and so we reach the contradiction $I \in \tau_A^{-k}\Sigma \subseteq \Gamma$. The case when A is tilted and $\Gamma(\text{mod}A)$ has a connecting component containing no projective module is dual. Finally, assume that A is not tilted. By Happel's theorem [13], A is of canonical type. By [18] (3.4), $\Gamma(\text{mod}A)$ contains infinitely many stable tubes which lie neither in $\mathcal{L}_0$ nor in $\mathcal{R}_0$. This completes the proof. $\square$

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