SIMPLY CONNECTED TAME QUASI-TILTED ALGEBRAS

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Dedicated to Idun Reiten for her 60^{th} birthday

Introduction.

Let k be an algebraically closed field. By algebra is meant a finite dimensional associative k-algebra with an identity. We are interested in studying the representation theory of A, that is, the category mod A of finitely generated right A-modules. For this purpose, we may assume that A is basic and connected. An algebra A is called triangular if its ordinary quiver Q_A has no oriented cycles. It is well-known that, if kQ_A denotes the path algebra of Q_A, then there exists a surjective algebra morphism $\nu : kQ_A \to A$, whose kernel is denoted by I_ν (see, for instance, [17]). For each pair (Q_A, I_ν), one can define a fundamental group $\pi_1(Q_A, I_\nu)$ (see [33] or (3.1) below) and A is called simply connected if it is triangular and, for each pair (Q_A, I_ν), we have $\pi_1(Q_A, I_\nu) = 1$ (see [9]). Simply connected algebras have played an important rôle in representation theory. A triangular algebra is simply connected if and only if it has no proper Galois covering. For any representation-finite algebra B, the indecomposable B-modules can be lifted to indecomposable modules over a simply connected algebra A (contained inside a certain Galois covering of the standard form of B, see [17,18]). Thus, covering techniques reduce many problems of the study of representation-finite algebras to the study of simply connected algebras, hence the importance of the latter. Representation-finite simply connected algebras are considered by now to be well-understood (see, for instance, [16,17]). While little is known about covering techniques in the representation-infinite case, many classes of representation-infinite simply connected algebras have been described (see, for instance, [5,9,37]). In particular, it was shown in [26,37,7] that there is a close connection between the simple connectedness of an algebra A, and the vanishing of the first Hochschild cohomology group $\mathrm{H}^1(A)$ (of the algebra A with coefficients in the bimodule ${}_AA_A$).

The class of quasi-tilted algebras, introduced by Happel, Reiten and Smalø in [30] is the generalisation of two well-known classes, namely, the class of tilted algebras of Happel and Ringel [31], and the class of canonical algebras of Ringel [36]. Since their introduction, quasi-tilted algebras have been the study of many investigations (see, for instance, [21,22,28,29,30,38]). In particular, it is shown in [38] that a tame quasi-tilted algebra is either tilted, or a semiregular enlargement of a tame concealed algebra (see (1.3) below for the definition). We conjecture that a quasi-tilted algebra A is simply connected if and only if $\mathrm{H}^1(A) = 0$. This generalises the conjecture saying that a tilted algebra is simply connected if and only if its type is a tree (see [5]). This conjecture is known to hold true in case A is a tame tilted algebra [5], and the first purpose of the present paper is to show that it holds true in case A is a tame quasi-tilted algebra. This also answers positively (for quasi-tilted algebras) Skowroński's question in [37], Problem 1, whether it is true that a tame triangular algebra A is simply connected if and only if $\mathrm{H}^1(A) = 0$. Namely, we prove the following theorem.

THEOREM (A). *Let A be a tame quasi-tilted algebra. The following conditions are equivalent:*

(a) *A is simply connected.*
(b) $\mathrm{H}^1(A) = 0$.
(c) *If A is tilted, then its type is a tree. If A is a semiregular enlargement of a tame concealed algebra, then A is not iterated tilted of type $\tilde{\mathbb{A}}$.*

One class of simply connected algebras has attracted much interest lately, this is the class of strongly simply connected algebras of [37]. The representation theory of strongly simply connected algebras seems to be relatively accessible, and some progress has been made towards understanding it in the tame case. Characterisations and construction techniques have been obtained in [37,3], and classes of strongly simply connected algebras have been completely described (see, for instance, [1,2,4]). In particular, it was asked in [37], Problem 2, whether it is true that an algebra is strongly simply connected if and only if it is simply connected and strongly $\tilde{\mathbb{A}}$-free, that is, contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}$. The answer is known to be positive if the algebra is iterated tilted of euclidean type [2], derived tubular [1] or tame tilted [5]. Also, it was shown in these papers that there is a close connection between the strong simple connectedness of an algebra, the shapes of the orbit graphs of the directed components of its Auslander-Reiten quiver on one hand, and the separation condition of [16] on the other hand. The second objective of this paper is to answer positively the aforementioned question for tame quasi-tilted algebras, and to relate the strong simple connectedness of a (non-tilted) tame quasi-tilted algebra A to that of two particular full convex subcategories A^+ and A^- (see (1.3) for the definitions).

THEOREM (B). *Let A be a tame quasi-tilted algebra which is not tilted. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *A^+ and A^- are strongly simply connected.*
(c) *A is strongly $\tilde{\mathbb{A}}$-free.*
(d) *A^+ and A^- are strongly $\tilde{\mathbb{A}}$-free.*
(e) *The orbit graph of each directed component of the Auslander-Reiten quiver of each of A^+ and A^- is a tree.*
(f) *A^+, $(A^+)^{\mathrm{op}}$, A^- and $(A^-)^{\mathrm{op}}$ satisfy the separation condition.*

Note that the strong simple connectedness of a tame tilted algebra has been characterised in [4,5], this justifies the assumption of the theorem.

As an application, we study the simple and strong simple connectedness of a natural generalisation of tame quasi-tilted algebras, namely the semiregular iterated tubular algebras, which form a subclass of the iterated tubular algebras of [34] and, in particular, are tame. We obtain results corresponding to the above two theorems.

The paper is organised as follows. After a preliminary section 1, the section 2 is devoted to lemmata showing how to compute the first Hochschild group in our case. Section 3 is devoted to the proof of theorem (A) and section 4 to the proof of theorem (B). Finally, the application to semiregular iterated tubular algebras is in section 5.

1. Preliminaries.

1.1. Notation. Throughout this paper, k denotes an algebraically closed field. By algebra is meant an associative, finite dimensional k-algebra with an identity, which we assume to be basic and, unless otherwise specified, connected.

We recall that a *quiver* Q is defined by its set of *points* Q_0 and its set of *arrows* Q_1. A *relation* from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each i such that $1 \leq i \leq m$, λ_i is a non-zero scalar and w_i is a path of length at least two from x to y. Assume that Q has no oriented cycles, then a set of relations generates an ideal I, called *admissible*, in the path algebra kQ. The pair (Q, I) is called a *bound quiver*. An algebra A is called *triangular* if its ordinary quiver Q_A has no oriented cycles. In this paper, we deal exclusively with triangular algebras. It is well-known that, for an algebra A, there exists a surjective morphism $\nu : kQ_A \to A$ of k-algebras (induced by the choice of a set of representatives of basis vectors in the k-vector space $\operatorname{rad} A / \operatorname{rad}^2 A$) whose kernel I_ν is admissible. Thus $A \cong kQ_A/I_\nu$. The bound quiver (Q_A, I_ν) is called a *presentation* of A. An algebra $A = kQ/I$ can equivalently be considered as a locally bounded k-category, whose object class, denoted by A_0, is the set Q_0, and where the set of morphisms $A(x, y)$ from x to y is the k-vector space $kQ(x, y)$ of all linear combinations of paths in Q from x to y modulo the subspace $I(x, y) = I \cap kQ(x, y)$, see [17]. A full subcategory B of A is called *convex* if any path in A with source and target in B lies entirely in B.

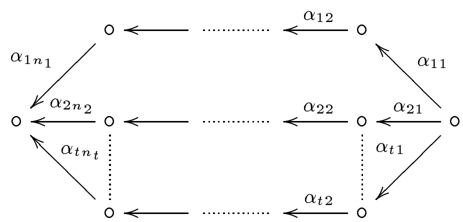
By A-module is meant a finitely generated right A-module. We denote by $\operatorname{mod} A$ their category. For $x \in A_0$, we denote by $S(x)$ the corresponding simple A-module, and by $P(x)$ (or $I(x)$) the projective cover (or injective envelope, respectively) of $S(x)$. We denote by $\mathrm{D} = \operatorname{Hom}_k(-, k)$ the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$, and by $\tau = \mathrm{D}\,\mathrm{Tr}$ and $\tau^{-1} = \mathrm{Tr}\,\mathrm{D}$ the Auslander-Reiten translations in $\operatorname{mod} A$. The Auslander-Reiten quiver of A is denoted by $\Gamma(\operatorname{mod} A)$ (for details, see [13,36]). A component Γ of $\Gamma(\operatorname{mod} A)$ is called *directed* if, for any indecomposable module M in Γ, there exists no sequence $M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = M$ of non-zero non-isomorphisms between indecomposable A-modules. Given a component Γ of $\Gamma(\operatorname{mod} A)$, its *orbit graph* $\mathcal{O}(\Gamma)$ has as points the τ-orbits M^τ of the modules M in Γ, there exists an edge M^τ — N^τ if there exist $m, n \in \mathbb{Z}$ and an irreducible morphism $\tau^m M \to \tau^n N$ or $\tau^n N \to \tau^m M$, and the number of such edges equals $\dim_k \operatorname{Irr}(\tau^m M, \tau^n N)$ or $\dim_k \operatorname{Irr}(\tau^n N, \tau^m M)$, respectively (here, $\operatorname{Irr}(X, Y)$ denotes the space of irreducible morphisms from X to Y).

1.2. Let A be an algebra. A module T_A is called a *tilting module* [31] if $\operatorname{pd} T_A \leq 1$, $\operatorname{Ext}^1_A(T, T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals the rank of the Grothendieck group $K_0(A)$.

Two algebras A and B are called *tilting-cotilting equivalent* if there exist a sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ and a sequence of tilting or cotilting modules $T^{(0)}_{A_0}$, $T^{(1)}_{A_1}, \ldots, T^{(m-1)}_{A_{m-1}}$ such that $A_{i+1} = \operatorname{End} T^{(i)}_{A_i}$, for each i. Let Q be a finite connected quiver without oriented cycles. An algebra A is called *iterated tilted of type* Q if A is tilting-cotilting equivalent to kQ, and it is called *tilted of type* Q if it is the endomorphism algebra of a tilting kQ-module. We need the following fact proved in [11](5.2): let A be an iterated tilted algebra of euclidean type (that is, of type Q such that the underlying graph of Q is an euclidean diagram), then any full convex subcategory of A is iterated tilted of Dynkin or of euclidean type. Also, it is well-known that an algebra A is iterated tilted of type Q if and only if there exists an equivalence of triangulated categories $\mathrm{D}^b(\operatorname{mod} A) \cong \mathrm{D}^b(\operatorname{mod} kQ)$ between the derived categories of bounded complexes over $\operatorname{mod} A$, and $\operatorname{mod} kQ$, respectively (see[27]).

An algebra A is called *quasi-tilted* if $\operatorname{gl.dim.} A \leq 2$ and, for each indecomposable module M_A, we have $\operatorname{pd} M \leq 1$ or $\operatorname{id} M \leq 1$ (see [30]). Tilted algebras furnish an example of quasi-tilted algebras, and a representation-finite algebra is tilted if and only if it is quasi-tilted [30](3.6). Another example is provided by the canonical algebras [36]. Let $t \geq 2$,

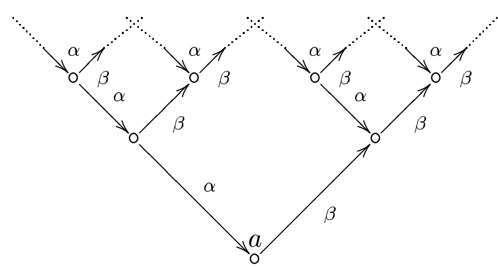
$n = (n_1, \dots, n_t)$ be a t-tuple of positive integers, and $\lambda = (\lambda_1, \dots, \lambda_t)$ be a t-tuple of pairwise distinct elements in $\mathbb{P}_1(k)$. The *canonical algebra* $C(t, \lambda, n)$ of type n is given by the quiver



bound by the $t-2$ relations $\alpha_{11}\alpha_{12}\cdots\alpha_{1n_1} + \alpha_{21}\alpha_{22}\cdots\alpha_{2n_2} + \lambda_i(\alpha_{i1}\alpha_{i2}\cdots\alpha_{in_i}) = 0$, with $3 \le i \le t$. If $n = (2,2,2,2)$, $(2,3,6)$, $(2,4,4)$ or $(3,3,3)$, then $C(t,\lambda,n)$ is called *tubular canonical*. An algebra A is called *derived canonical* (or *derived tubular*) if there exist a canonical algebra (or a tubular canonical algebra, respectively) C and an equivalence of triangulated categories $\mathrm{D}^b(\mathrm{mod}\, A) \cong \mathrm{D}^b(\mathrm{mod}\, C)$. If $t = 2$, then the canonical algebras are hereditary of type $\tilde{\mathbb{A}}$ and, thus, the derived canonical algebras coincide with the iterated tilted algebras of type $\tilde{\mathbb{A}}$.

1.3. An algebra C is called *tame concealed* if there exist a hereditary algebra A and a postprojective tilting A-module T such that $C = \mathrm{End}\, T_A$. Then, $\Gamma(\mathrm{mod}\, C)$ consists of a postprojective (also called preprojective) component $\mathcal{P}_C$, a preinjective component $\mathcal{Q}_C$, and a family $\mathcal{T}_C = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(k)}$ of stable tubes separating $\mathcal{P}_C$ from $\mathcal{Q}_C$ (see [36](4.3)).

We now define semiregular enlargements of a tame concealed algebra [38], see also [6]. A *branch* K with *root* a is a finite connected full bound subquiver, containing a, of the following infinite tree, bound by all possible relations of the form $\alpha\beta = 0$:



Let C be a tame concealed algebra, $(E_i)_{i=1}^m$ and $(F_j)_{j=1}^n$ be two families of simple regular C-modules, and $(K_i)_{i=1}^m$ and $(L_j)_{j=1}^n$ be two families of branches. For each i, we let a_i be the root of K_i. The *tubular extension* $A^+ = C[E_i, K_i]_{i=1}^m$ has as objects those of $C, K_1, \dots, K_m$ and as morphism spaces

$$A^+(x,y) = \begin{cases} C(x,y) & \text{if } x, y \in C_0 \\ K_i(x,y) & \text{if } x, y \in (K_i)_0 \\ K_i(x,a_i) \otimes_k E_i(y) & \text{if } x \in (K_i)_0,\, y \in C_0 \\ 0 & \text{otherwise.} \end{cases}$$

The *tubular coextension* $A^- = {}_{j=1}^{\,n}[L_j, F_j]C$ is defined dually. Finally, if the families $(E_i)_{i=1}^m$ and $(F_j)_{j=1}^n$ are *compatible*, that is, for any pair (i,j), the modules E_i and F_j do not lie in the same tube of $\Gamma(\mathrm{mod}\, C)$, then the *semiregular enlargement* $A = {}_{j=1}^{\,n}[L_j, F_j]C[E_i, K_i]_{i=1}^m$

is defined to have as objects those of A^+, A^- and as morphism spaces

$$A(x,y) = \begin{cases} A^+(x,y) & \text{if } x,y \in (A^+)_0 \\ A^-(x,y) & \text{if } x,y \in (A^-)_0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a tubular extension, or coextension, of C is trivially a semiregular enlargement of C.

For each $\lambda \in \mathbb{P}_1(k)$, let r_λ denote the rank of the stable tube $\mathcal{T}_\lambda$ in $\Gamma(\text{mod } C)$. The *tubular type* $n_A = (n_\lambda)_{\lambda\in\mathbb{P}_1(k)}$ is defined by:

$$n_\lambda = r_\lambda + \sum_{E_i\in\mathcal{T}_\lambda} |(K_i)_0| + \sum_{F_j\in\mathcal{T}_\lambda} |(L_j)_0|$$

(note that, for a given λ, at least one of the above two sums vanishes). Since all but at most finitely many n_λ equal 1, we write, instead of $(n_\lambda)_\lambda$, the finite sequence containing at least two n_λ, including all those larger than 1, in non-decreasing order. A tubular extension (or coextension) is tame if and only if its type is domestic, that is, one of (p,q), $(2,2,r)$, $(2,3,3)$, $(2,3,4)$ or $(2,3,5)$ (in which case, it is tilted of euclidean type $\tilde{\mathbb{A}}$, $\tilde{\mathbb{D}}$, $\tilde{\mathbb{E}}_6$, $\tilde{\mathbb{E}}_7$ or $\tilde{\mathbb{E}}_8$, respectively) or tubular, that is, one of $(2,2,2,2)$, $(2,3,6)$, $(2,4,4)$ or $(3,3,3)$ (in which case the algebra is called *tubular*, see [36]). Let A be a tubular extension of a tame concealed algebra C. If n_A is domestic, then C is the unique tame concealed full convex subcategory of A but, if n_A is tubular, then A contains (exactly) one other tame concealed full convex subcategory C' and is a tubular coextension of C'. Also, a tubular algebra is derived tubular.

A semiregular enlargement A is tame if and only if both A^+ and A^- are tame or, equivalently, are tilted of euclidean type or tubular. The following theorem, due to Skowroński [38], will play an essential rôle in the sequel.

THEOREM. *Let A be a tame algebra. Then A is quasi-tilted if and only if it is a tilted algebra, or a semiregular enlargement of a tame concealed algebra.* □

2. Hochschild cohomology and semiregular enlargements.

2.1. Given an algebra A, the *Hochschild complex* $C^\bullet = (C^i, d^i)_{i\in\mathbb{Z}}$ is defined as follows: $C^i = 0$, $d^i = 0$ for $i < 0$, $C^0 = {}_AA_A$, $C^i = \operatorname{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product $A \otimes_k \cdots \otimes_k A$, $d^0 : A \to \operatorname{Hom}_k(A,A)$ with $(d^0x)(a) = ax - xa$ (for $a, x \in A$) and $d^i : C^i \to C^{i+1}$ with

$$(d^i f)(a_1 \otimes \cdots \otimes a_{i+1}) = a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) \\ + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) + (-1)^{i+1} f(a_1 \otimes \cdots \otimes a_i) a_{i+1}$$

for $f \in C^i$ and $a_1, \ldots, a_{i+1} \in A$. Then $\mathrm{H}^i(A) = \mathrm{H}^i(C^\bullet)$ is the i^{th} *Hochschild cohomology group* of A with coefficients in the bimodule ${}_AA_A$, see [20].

Recall that an algebra A is the *one-point extension* of an algebra B by a module M_B if

$$A = B[M] = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

with the usual matrix addition and the multiplication induced from the B-module structure of M. The one-point coextension $[M]B$ is defined dually.

THEOREM [26]. *(a) Let Q be a finite and connected quiver without oriented cycles, then* $\mathrm{H}^0(kQ) = k$, $\mathrm{H}^1(kQ) = 0$ *if and only if Q is a tree, and* $\mathrm{H}^i(kQ) = 0$ *for all* $i \geq 2$.

(b) Let A be an algebra, T_A be a tilting module and $B = \mathrm{End}\, T_A$, then $\mathrm{H}^i(A) \cong \mathrm{H}^i(B)$ *for all* $i \geq 0$.

(c) Let $A = B[M]$ be a one-point extension algebra. There exists an exact sequence

$$0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to \mathrm{End}\, M/k \to \mathrm{H}^1(A) \to \mathrm{H}^1(B) \to \mathrm{Ext}^1_B(M,M) \to \mathrm{H}^2(A) \to \cdots$$
$$\cdots \to \mathrm{Ext}^{i-1}_B(M,M) \to \mathrm{H}^i(A) \to \mathrm{H}^i(B) \to \mathrm{Ext}^i_B(M,M) \to \cdots \qquad \square$$

If, in particular, A is iterated tilted of type Q, then, by (a) and (b), $\mathrm{H}^1(A) = 0$ if and only if Q is a tree.

2.2. Let A be a triangular algebra (not necessarily connected). An A-module M is *separated* if, for each connected component C of A, the restriction $M\mid_C$ of M to C is zero or indecomposable. A point $x \in A_0$ is *separating* if the restriction of $\mathrm{rad}\, P(x)_A$ to the full subcategory A^x of A generated by the non-predecessors of x in Q_A, is separated as an A^x-module. The algebra A satisfies the *separation condition* if each $x \in A_0$ is separating [16]. We define dually *coseparated points*, and the *coseparation condition*.

Let A be an algebra. A module M_A is a *brick* if $\mathrm{End}\ M = k$.

The following has been used implicitly in [26,12,37].

LEMMA. *Let $A = B[M]$ be a one-point extension algebra. Then the morphism $f : \mathrm{H}^1(A) \to \mathrm{H}^1(B)$ in the exact sequence of (2.1)(c) is injective if and only if the extension point is separating, and M is a direct sum of bricks.*

Proof. Let $B = B_1 \times \cdots \times B_t$, where each B_i is connected, and $M = M_1 \oplus \cdots \oplus M_t$ where each M_i is a B_i-module. Since M is the radical of the unique indecomposable projective A-module which is not a B-module, then $M_i \neq 0$ for all i. The morphism f is injective if and only if the sequence

$$0 \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(B) \longrightarrow \mathrm{End}\ M/k \longrightarrow 0$$

is (right) exact or, equivalently, if and only if $\dim_k \mathrm{H}^0(B) = \dim_k \mathrm{H}^0(A) + \dim_k \mathrm{End}\ M - 1$. We have $\dim_k \mathrm{H}^0(A) = 1$, because $\mathrm{H}^0(A)$ is the centre of the connected algebra A. Similarly, $\dim_k \mathrm{H}^0(B) = t$. Thus, the morphism $f : \mathrm{H}^1(A) \to \mathrm{H}^1(B)$ is injective if and only if $\dim_k \mathrm{End}\ M = t$. Now, $\dim_k \mathrm{End}\ M = \dim_k \mathrm{End}\left(\bigoplus_{i=1}^t M_i\right) \geq t$, and equality holds if and only if, for each i, we have $\mathrm{End}\ M_i = k$ and, for $i \neq j$, we have $\mathrm{Hom}_B(M_i, M_j) = 0$. This establishes the statement. $\square$

2.3. COROLLARY. *Let A be a semiregular enlargement of a tame concealed algebra C, and B be a full convex subcategory of A containing C. If $\mathrm{H}^1(B) = 0$, then $\mathrm{H}^1(A) = 0$.*

Proof. There exists a sequence of full convex subcategories $A = A_0 \supset A_1 \supset \cdots \supset A_t = B$ where, for each i, A_i is obtained from A_{i+1} by a one-point extension with separating extension point, and the extension module is a direct sum of bricks, or else A_i is obtained from A_{i+1} by a one-point coextension with coseparating coextension point, and the coextension module is a direct sum of bricks. Now, for each i, (2.2) or its dual yields a monomorphism $\mathrm{H}^1(A_i) \to \mathrm{H}^1(A_{i+1})$. Thus $\mathrm{H}^1(A_{i+1}) = 0$ implies $\mathrm{H}^1(A_i) = 0$. The statement follows from an obvious induction. $\square$

2.4. The following lemma should be compared with [25](2.2) and (2.4).

LEMMA. *Let A be derived equivalent to a canonical algebra $C(t, \lambda, n)$. The following conditions are equivalent:*

(a) $\mathrm{H}^1(A) = 0$,
(b) $t > 2$,
(c) *A is not iterated tilted of type $\tilde{\mathbb{A}}$.*

Proof. We have already seen in (1.2) that (b) and (c) are equivalent. We now show the equivalence of (a) and (c). If A is iterated tilted of type Q, where the underlying graph of Q is $\tilde{\mathbb{A}}$, then, by (2.1)(a) and (b), we have $\mathrm{H}^1(A) \cong \mathrm{H}^1(kQ) \neq 0$.

Conversely, assume first that A is a canonical algebra which is not hereditary of type $\tilde{\mathbb{A}}$, then $A = B[M]$, where B is a hereditary algebra whose quiver is a tree, and M is a brick. Moreover, the extension point is separating. Hence, by (2.2), there is a monomorphism $\mathrm{H}^1(A) \to \mathrm{H}^1(B)$. Since the quiver of B is a tree, we have $\mathrm{H}^1(B) = 0$, and hence $\mathrm{H}^1(A) = 0$.

Let now A be a derived canonical algebra which is not iterated tilted of type $\tilde{\mathbb{A}}$. Then there exists a canonical algebra C, which is not hereditary of type $\tilde{\mathbb{A}}$, and an equivalence of triangulated categories $\mathrm{D}^b(\mathrm{mod}\ A) \cong \mathrm{D}^b(\mathrm{mod}\ C)$. Then A and C are tilting-cotilting equivalent (see, for instance, [32]) hence $\mathrm{H}^1(A) \cong \mathrm{H}^1(C) = 0$. □

REMARK. The above proof can be shortened using the known fact that derived equivalence preserves the Hochschild groups.

2.5. COROLLARY. *Let A be a derived canonical algebra which is not iterated tilted of type $\tilde{\mathbb{A}}$. Then:*

(a) *Every source in A is separating.*
(b) *There exists a connected algebra B and a brick M_B such that $A = B[M]$ or $A = [M]B$.*

Proof. By (2.4), we have $\mathrm{H}^1(A) = 0$. Hence (a) follows from the fact that, by [37](3.2), if A is a triangular algebra with $\mathrm{H}^1(A) = 0$, then every source in A is separating. In order to prove (b), we observe that there exists a connected algebra B and a module M_B such that $A = B[M]$ or $A = [M]B$. It remains to show that M is a brick. But now, $\mathrm{H}^1(A) = 0$ yields a short exact sequence

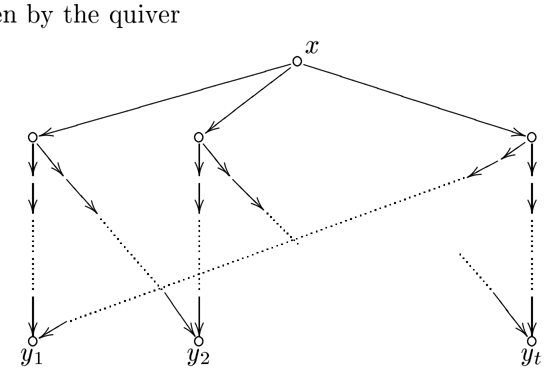
$$0 \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(B) \longrightarrow \mathrm{End}\ M/k \longrightarrow 0.$$

Hence $\mathrm{End}\ M = k$. □

REMARK. By [15], Theorem 1, the algebra B of (b) above is iterated tilted or derived canonical.

2.6. LEMMA. *Let A be a one-point extension of a hereditary algebra B of type $\tilde{\mathbb{A}}_{pq}$, with $p, q > 1$, by a simple homogeneous module. Then $\mathrm{H}^1(A) = 0$.*

Proof. Let x denote the extension point, and $y_1, y_2, \ldots, y_t$ be the sinks of B (hence of A). The algebra A is given by the quiver



bound by all possible commutativity relations. Let B' be the full convex subcategory of A generated by all points except $y_1, y_2, \ldots, y_t$ and, for each i such that $1 \leq i \leq t$, let $M_i = I(y_i)/S(y_i)$. Then $A = [M_t] \cdots [M_1]B'$. Since, for each i, M_i is a brick, and y_i is coseparating, the dual of (2.2) and induction yield a monomorphism $\mathrm{H}^1(A) \to \mathrm{H}^1(B')$. Since B' is a hereditary algebra whose quiver is a tree, $\mathrm{H}^1(B') = 0$. Hence $\mathrm{H}^1(A) = 0$. □

3. Simple connectedness of tame quasi-tilted algebras.

3.1. Let (Q, I) be a connected bound quiver. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is *minimal* if $m \geq 2$ and, for any non-empty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. We denote by α^{-1} the formal inverse of an arrow $\alpha \in Q_1$. A *walk* in Q from x to y is a formal composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{1, -1\}$ for all i) with source x and target y. We denote by e_x the trivial path at x. Let $\sim$ be the least equivalence relation on the set of all walks in Q such that:

(a) If $\alpha : x \to y$ is an arrow, then $\alpha^{-1}\alpha \sim e_y$ and $\alpha\alpha^{-1} \sim e_x$.

(b) If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j.

(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions make sense.

Let $x \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x)$ of equivalence classes $\tilde{u}$ of closed walks u starting and ending at x has a group structure defined by the operation $\tilde{u} \cdot \tilde{v} = \widetilde{uv}$. Since Q is connected, $\pi_1(Q, I, x)$ does not depend on the choice of x. We denote it by $\pi_1(Q, I)$ and call it the *fundamental group* of (Q, I), see [24,33].

Let $\pi_1(Q_A, I_\nu)$ be a presentation of a triangular algebra A. The fundamental group $\pi_1(Q_A, I_\nu)$ depends essentially on I_ν — thus is not an invariant of A (see, however, [14]). A triangular algebra A is *simply connected* if, for any presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is trivial [9].

The following result, of which the first part is [5] and the second part is [9], yields large classes of examples of simply connected algebras.

THEOREM. *Let A be an algebra.*

(a) *If A is tame tilted of type Q, then A is simply connected if and only if Q is a tree or, equivalently, if and only if $\mathrm{H}^1(A) = 0$.*

(b) *If A is iterated tilted of euclidean type or derived tubular, then A is simply connected if and only if it is not iterated tilted of type $\tilde{\mathbb{A}}$ or, equivalently, if and only if $\mathrm{H}^1(A) = 0$.* □

3.2. The following lemma follows from the proof of [37](2.3).

LEMMA. *Let $A = B[M]$ be a one-point extension algebra. If B is a product of simply connected algebras, and the extension point is separating, then A is simply connected.* □

3.3. One of the consequences of (3.2) is that, if A satisfies the separation condition, then A is simply connected [37](2.3). Another one is the following.

COROLLARY. *Let A be a semiregular enlargement of a tame concealed algebra C, and B be a full convex subcategory of A containing C. If B is simply connected, then so is A.*

Proof. There exists a sequence of full convex subcategories $A = A_0 \supset A_1 \supset \cdots \supset A_t = B$ where, for each i, A_i is obtained from A_{i+1} by a one-point extension with separating extension point lying in the branches or by a one-point coextension with coseparating point lying in the branches. Assume inductively that A_{i+1} is simply connected. Applying (3.2) or its dual yields that A_i is simply connected. The result follows by induction. □

3.4. We wish to apply (3.3) to a particular case. Let A be a semiregular enlargement of a tame concealed algebra C, and A^0 be the full convex subcategory of A generated by all points of C, as well as all the extension and all the coextension points of C inside A. We recall that a walk in a quiver is *reduced* if it contains no subwalk of one of the forms $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$, with α an arrow.

COROLLARY. *Let A be a semiregular enlargement of a tame concealed algebra.*

(a) *The non-contractible reduced cycles in A and A^0 coincide.*
(b) *A is simply connected if and only if so is A^0.*

Proof. (a) Let w be a non-contractible reduced cycle in A not lying entirely inside the tame concealed subcategory C. Then w contains a point x lying in a branch. But x must be connected to other points in w and w, being a cycle, cannot lie entirely in the branch containing x. Since the only walks between branches pass through C, then x must be the root of a branch. Thus, the only points of w not lying in C are in A^0, that is, w lies in A^0. On the other hand, by definition of a semiregular enlargement, if x is in a branch and is the starting (or ending) point of a relation ending (or starting) in C, then x is an extension point (or a coextension point, respectively) of C. That is, x lies in A^0. Thus, w is contractible in A if and only if it is contractible in A^0.

(b) Assume that A^0 is simply connected. Then, by (3.3), so is A. Conversely, if A^0 is not simply connected, then it contains a non-contractible cycle w. By (a), w is not contractible in A either. Thus, A is not simply connected. □

3.5. *Proof of Theorem (A).* If A is a tame tilted algebra of type Q, then, by (3.1)(a), A is simply connected if and only if Q is a tree or, equivalently, if and only if $\mathrm{H}^1(A) = 0$. Thus, the three conditions are equivalent in this case. Also, notice that a representation-finite quasi-tilted algebra is tilted (see (1.2)). In view of (1.3), we may assume from the start that A is a tame semiregular enlargement of a tame concealed algebra C, and that A is not tilted.

If A is iterated tilted of type $\tilde{\mathbb{A}}$ then, by (3.1)(b), A is not simply connected, and also, by (2.1), $\mathrm{H}^1(A) \neq 0$. Thus, either (a) or (b) implies (c). We have to show that (c) implies (a) and (b).

Suppose first that C is tame concealed of type different from $\tilde{\mathbb{A}}$. Then C is simply connected and $\mathrm{H}^1(C) = 0$. Applying (2.3) and (3.3), we infer that (a) and (b) hold. Also, in this case, it follows from [8] that A is not iterated tilted of type $\tilde{\mathbb{A}}$. Hence, the three conditions hold and they are equivalent.

Suppose now that C is tame concealed of type $\tilde{\mathbb{A}}$ (thus hereditary of type $\tilde{\mathbb{A}}$). Assume first that A^+ is not a tilted algebra of type $\tilde{\mathbb{A}}$. It follows from [10](2.5) that A is not iterated tilted of type $\tilde{\mathbb{A}}$. Furthermore, since A^+ is tilted of type $\neq \tilde{\mathbb{A}}$, or tubular, we have $\mathrm{H}^1(A^+) = 0$ and A^+ is simply connected. Hence, by (2.3) and (3.3), A is simply connected and $\mathrm{H}^1(A) = 0$. Thus the three conditions hold, and they are equivalent. This is in particular the case if $A = A^+$, because the assumption that A is not tilted implies that A^+ is tubular. Similarly, if A^- is not tilted of type $\tilde{\mathbb{A}}$ (in particular, if $A = A^-$), then the three conditions hold and they are equivalent. We may thus assume that $A \neq A^+, A^-$ and that each of A^+ and A^- is tilted of type $\tilde{\mathbb{A}}$. We now consider three cases:

Case 1. Assume C is of type $\tilde{\mathbb{A}}_{pq}$, with $p, q > 1$. Thus $\Gamma(\mathrm{mod}\, C)$ has two non-homogeneous tubes $\mathcal{T}_0$ and $\mathcal{T}_\infty$, of respective ranks p, q.

In order to show that (c) implies (a), assume that A is not simply connected. Let M be a simple regular C-module such that $C[M]$ is a full convex subcategory of A^+. Assume $M \notin \mathcal{T}_0 \vee \mathcal{T}_\infty$. Then $C[M]$ has three non-homogeneous tubes and hence is simply connected (because it is a full convex subcategory of the tilted algebra A^+, hence it is tilted of euclidean type, see (1.2)). Therefore, by (3.3), A itself is simply connected, a contradiction. This shows that $M \in \mathcal{T}_0 \vee \mathcal{T}_\infty$. Similarly, if N is a simple regular C-module such that $[N]C$ is a full convex subcategory of A^-, then $N \in \mathcal{T}_0 \vee \mathcal{T}_\infty$. In particular, A has only two non-homogeneous tubes. Therefore, A is iterated tilted of type $\tilde{\mathbb{A}}$.

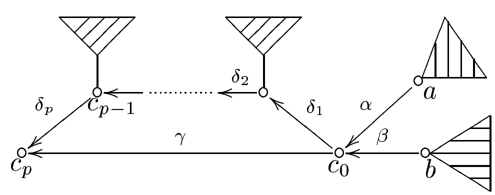
In order to show that (c) implies (b), assume that A is not iterated tilted of type $\tilde{\mathbb{A}}$. By [10](2.5), there exists a simple homogeneous C-module M (that is, $M \notin \mathcal{T}_0 \vee \mathcal{T}_\infty$) such that $C[M]$, or $[M]C$, is a full convex subcategory of A. By (2.6) or its dual, we have $\mathrm{H}^1(C[M]) = 0$, or $\mathrm{H}^1([M]C) = 0$, respectively. Applying (2.3) yields $\mathrm{H}^1(A) = 0$, as required.

Case 2. Assume C is of type $\tilde{\mathbb{A}}_{1p}$, with $p > 1$. Then $\Gamma(\mathrm{mod}\ C)$ has exactly one non-homogeneous tube $\mathcal{T}_0$.

In order to show that (c) implies (a), assume that A is not simply connected. Since A^+ and A^- are tilted of type $\tilde{\mathbb{A}}$, each is of tubular type (s, t), with $1 \leq s \leq t$. Assume first that the tube $\mathcal{T}_0$ is used for extensions, that is, there exists $M_0 \in \mathcal{T}_0$ such that $C[M_0]$ is a full convex subcategory of A. Assume next that $\mathcal{T}_1 \neq \mathcal{T}_0$ is also used for extensions and $\mathcal{T}_\infty \neq \mathcal{T}_1, \mathcal{T}_0$ for coextensions, that is, there exist $M_1 \in \mathcal{T}_1$ and $M_\infty \in \mathcal{T}_\infty$ such that $C[M_1]$ and $[M_\infty]C$ are full convex subcategories of A. Then the full convex subcategory $[M_\infty]C[M_0][M_1]$ of A is iterated tilted of type $(2, 2, p+1)$, hence is simply connected. Applying (3.3) yields that A is simply connected, a contradiction. Therefore we can only use one other tube, say $\mathcal{T}_\infty$, and, since $A \neq A^+$, this is necessarily for coextensions. Since A has only two non-homogeneous tubes, it is iterated tilted of type $\tilde{\mathbb{A}}$. The proof is dual if $\mathcal{T}_0$ is used for coextensions. We may therefore suppose that $\mathcal{T}_0$ is used neither for extensions nor for coextensions. Since $A \neq A^+, A^-$, there exist two simple regular C-modules $M_1 \in \mathcal{T}_1$, with $\mathcal{T}_1 \neq \mathcal{T}_0$, and $M_\infty \in \mathcal{T}_\infty$, with $\mathcal{T}_\infty \neq \mathcal{T}_0, \mathcal{T}_1$, such that $[M_\infty]C[M_1]$ is a full convex subcategory of A. Since $[M_\infty]C[M_1]$ is iterated tilted of type $(2, 2, p)$, it is simply connected. Hence, by (3.3), so is A, a contradiction. This completes the proof of this implication.

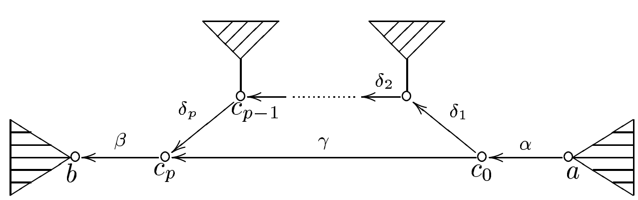
In order to show that (c) implies (b), we notice that, in this case, it is easily seen that the bound quiver of A is, up to duality, of one of the forms:

(i)



where the shaded triangles represent branches, bound by $\beta\gamma = 0$ and $\alpha\delta_1\delta_2 \cdots \delta_p = \alpha\gamma$ and possibly other relations in the branches, or

(ii)

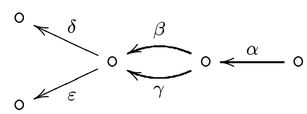


where the shaded triangles represent branches, bound by $\gamma\beta = 0$ and $\alpha\delta_1\delta_2 \cdots \delta_p = \alpha\gamma$, and possibly other relations in the branches.

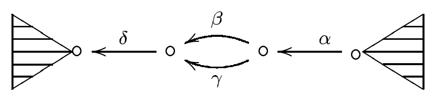
In each case, let B be the full convex subcategory of A generated by the points a, b, c_0, c_1, ..., c_p. Then B is iterated tilted of type $(2,2,p)$, that is, of type $\tilde{\mathbb{D}}$. Therefore, $\mathrm{H}^1(B)=0$ (see (2.1)). Applying (2.3) yields $\mathrm{H}^1(A)=0$, as required.

Case 3. Assume C is of type $\tilde{\mathbb{A}}_{11}$. Then all tubes in $\Gamma(\mathrm{mod}\ C)$ are homogeneous.

In order to show that (c) implies (a), assume that A is not simply connected. Since $A \neq A^+, A^-$ and each of A^+ and A^- is tilted of type $\tilde{\mathbb{A}}$, at most two tubes can be used for extensions, and at most two for coextensions. If at least three tubes are used, then A contains, up to duality, a full convex subcategory B given by the quiver

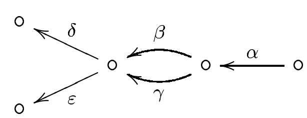


bound by $\alpha\beta=\alpha\gamma$, $\beta\delta=0$ and $\gamma\varepsilon=0$. Since B is simply connected, so is A, by (3.3), a contradiction. This shows that at most two tubes are used, so that the bound quiver of A is of the form



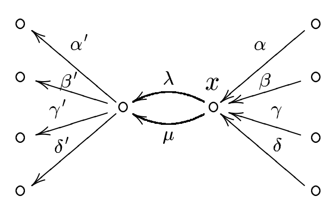
where the shaded triangles represent branches, bound by $\alpha\beta=0$, $\gamma\delta=0$ and possibly other relations in the branches. Therefore A is iterated tilted of type $\tilde{\mathbb{A}}$.

In order to show that (c) implies (b), assume that A is not iterated tilted of type $\tilde{\mathbb{A}}$. Then A contains, up to duality, a full convex subcategory B given by the quiver



bound by $\alpha\beta=\alpha\gamma$, $\beta\delta=0$ and $\gamma\varepsilon=0$. Then B is iterated tilted of type $\tilde{\mathbb{D}}_4$. Hence $\mathrm{H}^1(B)=0$. Applying (2.3) yields $\mathrm{H}^1(A)=0$, as required. □

REMARK AND EXAMPLE. In general, a simply connected tame quasi-tilted algebra does not satisfy the separation condition. Indeed, while the simple connectedness of an algebra implies that each source is separating [7](2.6), this is not true for the points which are not sources, as is shown by the following example. Let A be given by the quiver



bound by the relations $\alpha\lambda=c_1\cdot\alpha\mu$, $\beta\lambda=c_2\cdot\beta\mu$, $\gamma\lambda=c_3\cdot\gamma\mu$, $\delta\lambda=c_4\cdot\delta\mu$, $\lambda\alpha'=c_5\cdot\mu\alpha'$, $\lambda\beta'=c_6\cdot\mu\beta'$, $\lambda\gamma'=c_7\cdot\mu\gamma'$, $\lambda\delta'=c_8\cdot\mu\delta'$, where the c_i are pairwise distinct scalars. Clearly, the point x is not separating. On the other hand, each of A^+ and A^- is tubular of type $(2,2,2,2)$ so that A is a simply connected tame quasi-tilted algebra of type $(2,2,2,2,2,2,2,2)$. In particular, it is derived equivalent to a (wild) canonical algebra of that type (see [38], Corollary D).

3.6. COROLLARY. *Let A be a tame quasi-tilted algebra which is not tilted. If A is strongly $\tilde{\mathbb{A}}$-free, then A is simply connected.*

Proof. If A is strongly $\tilde{\mathbb{A}}$-free, then the tame concealed full convex subcategory C of A (in the above notation) is not hereditary of type $\tilde{\mathbb{A}}$. It then follows from the proof above that A is simply connected. □

3.7. For representation-finite algebras, which are not necessarily (quasi-)tilted, we have the following result (for the general case, see [19]). The proof below is due to E. N. Marcos (private communication).

PROPOSITION. *Let A be a triangular representation-finite algebra. Then A is simply connected if and only if* $\mathrm{H}^1(A) = 0$.

Proof. Assume A to be simply connected. Since A is representation-finite, any full convex subcategory of A is simply connected [18](2.8). By [37](4.1), this implies $\mathrm{H}^1(A) = 0$. Conversely, assume that $\mathrm{H}^1(A) = 0$ and let (Q_A, I) be an arbitrary presentation of A. Since A is triangular, it is standard, hence it follows from [33](3.9) and (4.3) that $\pi_1(Q_A, I)$ is a free group. Then the monomorphism of abelian groups

$$0 \longrightarrow \mathrm{Hom}\left(\pi_1\left(Q_A, I\right), k^+\right) \longrightarrow \mathrm{H}^1(A)$$

(where k^+ denotes the additive group of the field k) of [35]§3, [23], [7](3.2) implies that $\pi_1(Q_A, I) = 1$. □

4. Strong simple connectedness of tame quasi-tilted algebras.

4.1. An algebra A is *strongly simply connected* [37] if it satifies the following equivalent conditions:

(a) Any full convex subcategory of A is simply connected.
(b) Any full convex subcategory of A satisfies the separation condition.
(c) Any full convex subcategory of A satisfies the coseparation condition.
(d) For any full convex subcategory C of A, we have $\mathrm{H}^1(C) = 0$

We need the following definitions and results from [3]. Let A be an algebra, and (Q_A, I) be a presentation of A. A *contour* (p, q) in Q_A from x to y consists of a pair of non-trivial paths p, q from x to y. It is *interlaced* if p, q have a common point besides x and y. It is *irreducible* if there exists no sequence of paths $p = p_0, p_1, \dots, p_m = q$ from x to y such that each of the contours (p_i, p_{i+1}) is interlaced. Let C be a simple cycle which is not a contour, and let $\sigma(C)$ denote the number of sources in C. Then C is *reducible* if there exist x, y on C and a path $p : x \to \cdots \to y$ in Q_A such that, if w_1 and w_2 denote the subwalks of C from x to y (so that $C = w_1 w_2^{-1}$), then $w_1 p^{-1}$ and $w_2 p^{-1}$ are cycles and $\sigma\left(w_1 p^{-1}\right) < \sigma(C)$, $\sigma\left(w_2 p^{-1}\right) < \sigma(C)$. A cycle C is *irreducible* if it is either an irreducible contour, or it is not a contour, but it is not reducible in the above sense. Finally, a contour (p, q) from x to y is *naturally contractible* in (Q_A, I) if there exists a sequence of paths $p = p_0, p_1, \dots, p_m = q$ in Q_A such that, for each i, the paths p_i and p_{i+1} have subpaths q_i and q_{i+1}, respectively, which are involved in the same minimal relation in (Q_A, I).

THEOREM [3](1.6). *An algebra A is strongly simply connected if and only if, for any presentation (Q_A, I) of A, any irreducible cycle in Q_A is an irreducible contour, and any irreducible contour in Q_A is naturally contractible in (Q_A, I).* □

4.2. THEOREM [4,5]. *Let A be a tame tilted algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *The orbit graph of each of the directed components of $\Gamma(\text{mod } A)$ is a tree.*
(c) *A is simply connected and strongly $\tilde{\mathbb{A}}$-free.*
(d) *A satisfies the separation condition and is strongly $\tilde{\mathbb{A}}$-free.* □

4.3. THEOREM [1]. *Let A be a tubular algebra, and $C^{(0)}$, $C^{(\infty)}$ denote its two tame concealed full convex subcategories. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *The orbit graph of each of the directed components of $\Gamma(\text{mod } A)$ is a tree.*
(c) *A is strongly $\tilde{\mathbb{A}}$-free.*
(d) *$C^{(0)}$ and $C^{(\infty)}$ are not hereditary of type $\tilde{\mathbb{A}}$.*
(e) *A and A^{op} satisfy the separation condition.* □

4.4. *Proof of Theorem (B).* Clearly, (a) implies each of (b) (c) and (f), and (c) implies (d). By (4.2) and (4.3), (b) implies (e). By (4.3) and [2](2.3), (b) is equivalent to (d). Since (e) implies (b) by (4.2), (4.3), we just have to show that (f) implies (b), and that (b) implies (a).

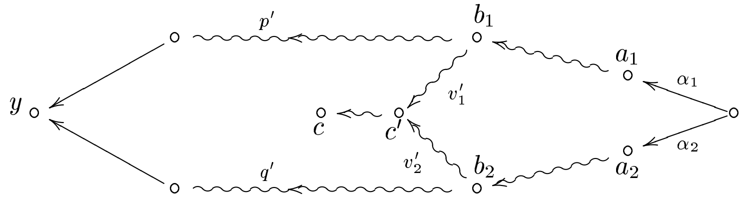
We first show that (f) implies (b). If A^+ (or A^-) is tubular, then it follows from (4.3) that the separation condition for A^+ and $(A^+)^{\text{op}}$ (or A^- and $(A^-)^{\text{op}}$) implies that A^+ (or A^-, respectively) is strongly simply connected. Let B be a representation-infinite tilted algebra of euclidean type having a complete slice in its preinjective component. By [1](1.5), if B satisfies the separation condition, then its unique tame concealed full convex subcategory is not hereditary of type $\tilde{\mathbb{A}}$. Hence B is strongly $\tilde{\mathbb{A}}$-free and consequently strongly simply connected by (4.2). Now, if B is representation-infinite tilted of euclidean type, either B or B^{op} has a complete slice in its preinjective component. Hence the separation condition for both B and B^{op} implies the strong simple connectedness of B. This completes the proof of this implication.

We now show that (b) implies (a). If A is not strongly simply connected, then, by (4.1), its bound quiver contains an irreducible cycle w which is not a contour, or an irreducible contour which is not naturally contractible. It follows from (b) that w does not lie entirely inside A^+, or inside A^-. We consider two cases. As usual, we denote by C the tame concealed full convex subcategory common to A^+ and A^-.

Case 1. Assume first that w is an irreducible contour (p,q) from x to y which is not naturally contractible. Since w lies neither inside A^+ nor inside A^-, we have $x \in (A^+)_0 \setminus (A^-)_0$ and $y \in (A^-)_0 \setminus (A^+)_0$. Applying (3.4)(a), we get that x is the root of an extension branch, thus is an extension point of C. Also, y is the root of a coextension branch, and is a coextension point of C.

Now, x, being the root of a branch, is separating. Hence, by [7](2.2), if $\alpha_1 : x \to a_1$ and $\alpha_2 : x \to a_2$ are the arrows of source x on p and q, respectively, there exists a minimal relation $\lambda_1\alpha_1 v_1 + \lambda_2\alpha_2 v_2 + \sum_{j\geq 3} \lambda_j u_j$ from x to $c \in C_0$, say, when all the λ_i are non-zero scalars. Since $c \in (A^+)_0$ while $y \notin (A^+)_0$, we have $c \neq y$. Also, since A^+ is closed under predecessors, there is no path from y to c. Moreover, there is no path from c to y, because w is irreducible. Let b_1 (or b_2) be the last common point between v_1 and p (or v_2 and q, respectively) and c' be the first common point between v_1 and v_2. Call v_1' (or v_2') the subpath of v_1 (or v_2) from b_1 (or b_2, respectively) to c', and p' (or q') the subpath of p (or

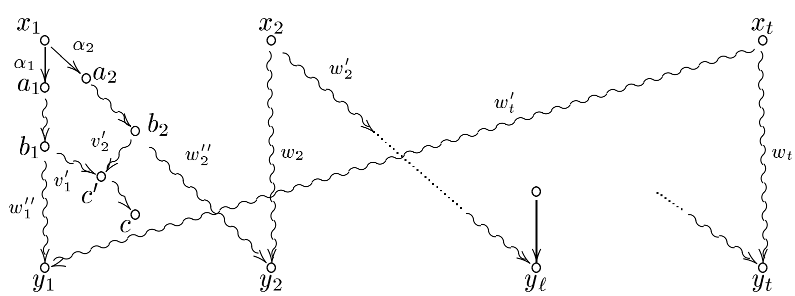
q from b_1 (or b_2, respectively) to y.



Consider now the walk $w' = {v'_2}^{-1} q' {p'}^{-1} v'_1$. This walk is a cycle: there is no intersection between p' and q', neither is there one between v'_1 and v'_2, and the existence of an intersection between $v'_2 {v'_1}^{-1}$ and $p' {q'}^{-1}$ would contradict the irreducibility of w. Further, w' is irreducible because w is. Finally, w' is not a contour, because it has two sinks y and c' and, since $c' \in C_0$, we have $c' \neq y$. Since w' lies entirely inside A^-, this contradicts (b).

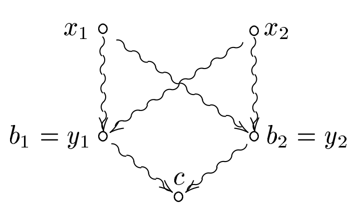
Case 2. Assume now that w is an irreducible cycle which is not a contour. We denote by $x_1, \ldots, x_t$ the sources of w, by $y_1, \ldots, y_t$ its sinks and, for each i with $1 \leq i \leq t$, by w_i the path from x_i to y_i, and by w'_i the path from x_i to y_{i+1} (where $y_{t+1} = y_1$). As before, one of the x_i (say x_1) lies in A^+ but not in A^-, and one of the y_i (say y_ℓ) lies in A^- but not in A^+. We may assume furthermore, without loss of generality, that w has the least possible number of sources lying in A^+ but not in A^-.

As before, x_1 is the root of a branch and, if $\alpha_1 : x_1 \to a_1$ and $\alpha_2 : x_1 \to a_2$ are the arrows of source x_1 on w_1 and w'_1, respectively, there exists a minimal relation $\lambda_1 \alpha_1 v_1 + \lambda_2 \alpha_2 + \sum_{j \geq 3} \lambda_j u_j$ from x_1 to $c \in C_0$, say, where the λ_i are non-zero scalars. Let b_1 (or b_2) be the last common point of v_1 (or v_2) and w_1 (or w'_1, respectively), and c' be the first common point of v_1 and v_2. Also, denote by v'_1 (or v'_2) the subpath of v_1 (or v_2) from b_1 (or b_2, respectively) to c' and by w''_1 (or w''_2) the subpath of w_1 (or w'_1) from b_1 (or b_2) to y_1 (or y_2, respectively)



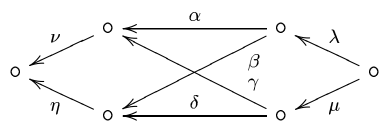
Again, $c \in (A^+)_0$ and $y_\ell \notin (A^+)_0$ imply that $y_\ell \neq c$, and there is no path from y_ℓ to c. Moreover, by irreducibility, there is no path from c to y_ℓ.

We claim that either $t \geq 3$ or, if $t = 2$, then $b_1 \neq y_1$ or $b_2 \neq y_2$. Indeed, if $t = 2$, $b_1 = y_1$ and $b_2 = y_2$ then, since c is a successor to both b_1 and b_2, we get a contradiction to the fact that y_1 or y_2 lies in A^-, but not in A^+, while $c \in (A^+)_0$, and A^+ is closed under predecessors. This establishes our claim.



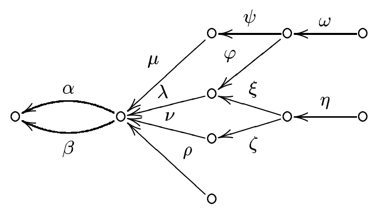
We now consider the walk $w' = w''_1 {w'_t}^{-1} w_t \cdots w_2 {w''_2}^{-1} v'_2 {v'_1}^{-1}$. This walk is a cycle: indeed, $w''_1 {w'_t}^{-1} w_t \cdots w_2 {w''_2}^{-1}$ has no self-intersection because it is a subwalk of w, the walk $v'_2 {v'_1}^{-1}$ has no self-intersection by definition, and these two do not intersect because w is irreducible. Further, w' is irreducible because w is. Finally, it is not a contour, because it has at least two sinks, namely c' (which lies in C, hence in A^+) and y_ℓ (which does not lie in A^+). Since the points of $v'_2 {v'_1}^{-1}$ belong to A^-, either w' lies entirely in A^-, a contradiction to (b), or else w' has one source less than w lying in A^+ but not in A^-, a contradiction to our minimality assumption on w. This completes the proof of this implication, and hence of the theorem. ☐

REMARKS AND EXAMPLES. (a) One cannot improve condition (e) of the theorem. The following example (borrowed from [1]) shows a tame quasi-tilted (derived tubular) algebra A such that the orbit graph of each directed component of $\Gamma(\text{mod } A)$ is a tree, but A is not strongly simply connected. Let A be given by the quiver



bound by $\lambda\alpha = \mu\gamma$, $\lambda\beta = \mu\delta$, $\alpha\nu = c \cdot \beta\eta$, $\gamma\nu = c \cdot \delta\eta$ (for some $c \in k \setminus \{0, 1\}$) and $\lambda\alpha\nu = 0$.

(b) One cannot improve condition (f) of the theorem. The following is an example of a tame quasi-tilted algebra such that each of A^+, $(A^+)^{\text{op}}$ and $(A^-)^{\text{op}}$ satisfies the separation condition, but A^- does not. Let A be given by the quiver



bound by $\lambda\alpha = 0$, $\rho\beta = 0$, $\mu\alpha = c \cdot \mu\beta$, $\nu\alpha = c' \cdot \nu\beta$ (where $c \neq c'$, $c, c' \in k \setminus \{0, 1\}$) $\phi\lambda = \psi\mu$, $\phi\lambda\beta = 0$, $\xi\lambda = \zeta\nu$ and $\xi\lambda\beta = 0$. Clearly, A is not strongly simply connected (but is simply connected).

4.5. COROLLARY. *Let A be a tame quasi-tilted algebra. Then A is strongly simply connected if and only if A is simply connected and strongly $\tilde{\mathbb{A}}$-free.*

Proof. If A is tame tilted, this follows from (4.2). Otherwise, this follows from (4.4). ☐

5. Semiregular iterated tubular algebras.

5.1. We start by defining these algebras. Let A_0 be a tame tubular coextension of a tame concealed algebra C_0, thus $A_0 = {}_{i_0=1}^{t_0}[K^0_{i_0}, E^0_{i_0}]\, C_0$ is either tubular or tilted of euclidean type having a complete slice in its postprojective component. We say that A_0 is a semiregular 0-iterated tubular algebra. Let $\{E^1_1, \ldots, E^1_{t_1}\}$ be a family of simple regular C_0-modules compatible with $\{E^0_1, \ldots, E^0_{t_0}\}$ that is, for any pair (i, j), E^0_i and E^1_j do not lie in the same tube of $\Gamma(\text{mod } C_0)$, and let $\{K^1_1, \ldots, K^1_{t_1}\}$ be a family of branches, and assume that $B_1 = C_0 \left[E^1_{i_1}, K^1_{i_1}\right]_{i_1=1}^{t_1}$ is a tame tubular extension of C_0. Then we say that the algebra

$$A_1 = {}_{i_0=1}^{t_0}\left[K^0_{i_0}, E^0_{i_0}\right] C_0 \left[E^1_{i_1}, K^1_{i_1}\right]_{i_1=1}^{t_1}$$

is a semiregular 1-iterated tubular algebra. Assume that B_1 is a tubular algebra. Then there exist a tame concealed algebra C_1, a set of simple regular C_1-modules $\left\{F_1^1, \ldots, F_{s_1}^1\right\}$ and a set of branches $\left\{L_1^1, \ldots, L_{s_1}^1\right\}$ such that $B_1 = {}_{j_1=1}^{s_1}\left[L_{j_1}^1, F_{j_1}^1\right]C_1$. Let $\left\{E_1^2, \ldots, E_{t_2}^2\right\}$ be a set of simple regular C_1-modules compatible with $\left\{F_1^1, \ldots, F_{s_1}^1\right\}$, and $\left\{K_1^2, \ldots, K_{t_2}^2\right\}$ be a set of branches such that $B_2 = C_1\left[E_{i_2}^2, K_{i_2}^2\right]_{i_2=1}^{t_2}$ is a tame tubular extension of C_1. Then we say that the algebra

$$A_2 = A_1\left[E_{i_2}^2, K_{i_2}^2\right]_{i_2=1}^{t_2} = {}_{i_0=1}^{t_0}\left[K_{i_0}^0, E_{i_0}^0\right]{}_{j_1=1}^{s_1}\left[L_{j_1}^1, F_{j_1}^1\right]C_1\left[E_{i_2}^2, K_{i_2}^2\right]_{i_2=1}^{t_2}$$

is a semiregular 2-iterated tubular algebra.

Inductively, assume that

$$A_n = {}_{i_0=1}^{t_0}\left[K_{i_0}^0, E_{i_0}^0\right]\cdots{}_{j_{n-1}=1}^{s_{n-1}}\left[L_{j_{n-1}}^{n-1}, F_{j_{n-1}}^{n-1}\right]C_{n-1}\left[E_{i_n}^n, K_{i_n}^n\right]_{i_n=1}^{t_n}$$

is a semiregular n-iterated tubular algebra, and that $B_n = C_{n-1}\left[E_{i_n}^n, K_{i_n}^n\right]_{i_n=1}^{t_n}$ is tubular. There exist a tame concealed algebra C_n, a set of simple regular C_n-modules $\{F_1^n, \ldots, F_{s_n}^n\}$ and a set of branches $\{L_1^n, \ldots, L_{s_n}^n\}$ such that $B_n = {}_{j_n=1}^{s_n}[L_{j_n}^n, F_{j_n}^n]C_n$. Let $\left\{E_1^{n+1}, \ldots, E_{t_{n+1}}^{n+1}\right\}$ be a set of simple regular C_n-modules compatible with $\{F_1^n, \ldots, F_{s_n}^n\}$ and $\left\{K_1^{n+1}, \ldots, K_{t_{n+1}}^{n+1}\right\}$ be a set of branches such that $B_{n+1} = C_n\left[E_{i_{n+1}}^{n+1}, K_{i_{n+1}}^{n+1}\right]_{i_{n+1}=1}^{t_{n+1}}$ is a tame tubular extension of C_n. Then we say that the algebra

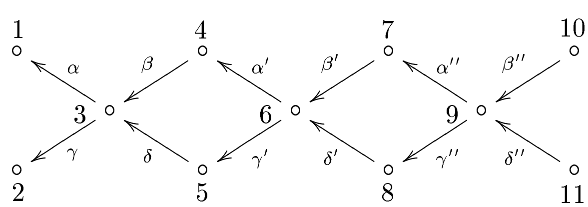
$$A_{n+1} = {}_{i_0=1}^{t_0}\left[K_{i_0}^0, E_{i_0}^0\right]\cdots{}_{j_n=1}^{s_n}\left[L_{j_n}^n, F_{j_n}^n\right]C_n\left[E_{i_{n+1}}^{n+1}, K_{i_{n+1}}^{n+1}\right]_{i_{n+1}=1}^{t_{n+1}}$$

is a *semiregular* $n+1$*-iterated tubular algebra.*

REMARKS. (a) Let A be a semiregular n-iterated tubular algebra, then A is n-iterated tubular in the sense of [34]. In particular, it follows from [34](2.4) that A is tame.

(b) The construction of the semiregular iterated tubular algebras generalises the one of (1.3). In fact, a semiregular n-iterated tubular algebra is quasi-tilted if and only if $n \leq 1$. If $n = 0$, then such an algebra is tilted of euclidean type or tubular. If $n = 1$, then it is a semiregular branch enlargement of a tame concealed algebra.

EXAMPLE. We borrow this example from [34]. Let A be given by the quiver



bound by $\beta''\alpha''\beta' = c_1\cdot\beta''\gamma''\delta'$, $\delta''\alpha''\beta' = c_2\cdot\delta''\gamma''\delta'$, $\alpha''\beta'\alpha' = c_3\cdot\gamma''\delta'\alpha'$, $\alpha''\beta'\gamma' = c_4\cdot\gamma''\delta'\gamma'$, $\beta'\alpha'\beta = c_5\cdot\beta'\gamma'\delta$, $\delta'\alpha'\beta = c_6\cdot\delta'\gamma'\delta$, $\alpha'\beta\alpha = c_7\cdot\gamma'\delta\alpha$, $\alpha'\beta\gamma = c_8\cdot\gamma'\delta\gamma$ (where the c_i are pairwise distinct scalars) and $\operatorname{rad}^4 A = 0$. Letting A_0 be the full convex subcategory generated by the points $1,2,3,4,5,6$, we see that A_0 is tubular. Then A_1, generated by the points $1,2,3,4,5,6,7,8$ is semiregular 1-iterated tubular (thus, tame quasi-tilted). The full convex subcategory A_2, generated by $1,2,3,4,5,6,7,8,9$ is semiregular 2-iterated tubular. Similarly, $A = A_3$ is semiregular 3-iterated tubular.

5.2. PROPOSITION. *Let A be a semiregular n-iterated tubular algebra. The following conditions are equivalent:*

(a) *A is simply connected.*
(b) $\mathrm{H}^1(A) = 0$.
(c) *$n \geq 2$ or, if $n \leq 1$, then A is not iterated tilted of type $\tilde{\mathbb{A}}$.*

Proof. Assume $n \leq 1$, then A is a tame quasi-tilted algebra, and the equivalence of (a)(b)(c) follows from (3.5). If $n \geq 2$, then A contains a tubular algebra B_1 as full convex subcategory, and is obtained from it by an iteration of one-point extensions (or coextensions) by separating (or coseparating) points, the extension (or coextension, respectively) modules being direct sums of bricks. Applying (2.3) and (3.3), the algebra A is simply connected and satisfies $\mathrm{H}^1(A) = 0$. □

5.3. Let A be a semiregular n-iterated tubular algebra, then we may assume that A_0 is a domestic tubular extension of C_0 (otherwise, A is also a semiregular $(n+1)$-iterated tubular algebra), and, similarly, B_n is a domestic tubular extension of C_{n-1}. When these assumptions are made, we say that A is a *semiregular strict n-iterated tubular algebra*.

PROPOSITION. *Let A be a semiregular strict n-iterated tubular algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected.*
(b) *A is strongly $\tilde{\mathbb{A}}$-free.*
(c) *Each B_i is strongly $\tilde{\mathbb{A}}$-free.*
(d) *No C_i is hereditary of type $\tilde{\mathbb{A}}$.*
(e) *For each i, the orbit graph of each of the postprojective and the preinjective components of $\Gamma(\mathrm{mod}\ B_i)$ is a tree.*
(f) *For each i, B_i and $(B_i)^{\mathrm{op}}$ satisfy the separation condition.*

Proof. Clearly, (a) implies (b) which implies (c). If $i < n$, then B_i is a tubular algebra, and the equivalence of (c)(d)(e)(f) follows from (4.3). On the other hand, B_n is tilted of euclidean type having a complete slice in its preinjective component, thus (c)(d)(e) are equivalent by [2](2.3). Finally, (c) clearly implies (f) and [1](1.6) yields that (f) implies (d). There only remains to show that (c) implies (a). We do it by induction on n. If $n \leq 1$, the statement holds by (4.4). Assume that $n \geq 2$ and that the statement holds for any $j \leq n-1$. If A is not strongly simply connected, then, by (4.1), it contains an irreducible cycle w which is not a contour, or an irreducible contour w which is not naturally contractible. By induction, w must contain a point lying in A_0 but not in C_0, and a point lying in B_n but not in C_{n-1}. Exactly as in the proof of (4.4), we replace w by an irreducible cycle w' which is not a contour, but lies in A_{n-1}, a contradiction to the induction hypothesis. We leave the easy details to the reader. □

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