Simply connected incidence algebras

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Dedicated to Julien Constantin

Abstract

It is known that the incidence algebra of a finite poset is not strongly simply connected if and only if its quiver contains a crown. We give a combinatorial condition on crowns which, if satisfied, forces the incidence algebra to be simply connected. The converse is not true, but we show that a simply connected incidence algebra which is not strongly simply connected always contains crowns satisfying this condition.

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1 Introduction.

The objective of this paper is to study whether the incidence algebra of a finite partially ordered set (poset) or, equivalently, a finite simplicial complex, is simply connected. This is well known to be an undecidable problem (because it can be reduced to a word problem) and therefore it is impossible to find a necessary and sufficient condition. We give here a sufficient condition, which also yields a necessary condition.

Our motivation comes from the representation theory of finite dimensional algebras over an algebraically closed field k. For such an algebra A, there exists a (uniquely determined) quiver Q_A and (at least) a surjective algebra morphism ν from the path algebra kQ_A of Q_A onto A, whose kernel is denoted by I_ν, see [BG]. The algebra A is called triangular if Q_A has no oriented cycles. For each pair (Q_A, I_ν), called a presentation of A, one can define the fundamental group $\pi_1(Q_A, I_\nu)$, see [G, MP]. A triangular algebra A is called simply connected if, for every presentation (Q_A, I_ν) of A, the group $\pi_1(Q_A, I_\nu)$ is trivial [AS]. If A is an incidence algebra, then all its presentations yield isomorphic fundamental groups [BM], and A is simply connected if and only if so is the associated simplicial complex (namely, the chain complex of the poset) [B, R]. Simply connected algebras have played an important rôle in representation theory: indeed, covering techniques allow to reduce many problems to problems about simply connected algebras.

In this paper, we are interested in finding conditions for simply connectedness. If the algebra A is representation–finite, that is, admits only finitely many isomorphism classes of indecomposable modules, then there exists a handy combinatorial criterion, known as the separation condition (see before (5.3) below) allowing to verify whether A is simply connected or not [BLS]. If, on the other hand, A is representation–infinite, then the separation condition is a sufficient condition for simple

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connectedness, but is not necessary [S]. On the other hand, it was shown by Dräxler [D] that an incidence algebra A is strongly simply connected (that is, the incidence algebra of every convex subposet of A is simply connected) if and only if the quiver of A contains no crowns, thus yielding another sufficient condition for A to be simply connected. Crowns are well–known in the combinatorics of posets, and are associated to their dismantlability (see, for instance, [CF, DR]). Here, we generalize this notion to that of a weak crown and describe a combinatorial operation, which we call suspension, and its dual, sustension, which we perform systematically on weak crowns (see (3.1) for the definitions). This allows us to define the notion of completeness of a weak crown (see (4.1)). We then prove the following theorem.

THEOREM. *Let A be the incidence algebra of a finite poset. If A is not strongly simply connected, then:*

(a) If every crown in A is complete, then A is simply connected.

(b) If A is simply connected, then there exists a complete crown in A.

(c) A is simply connected if and only if every crown in A is homotopic to a complete crown.

Note that there exist simply connected incidence algebras which satisfy condition (a) of the theorem but whose associated simplicial complex is not contractible (see (4.7)(c)).

Since our point of view and our intuition are mainly algebraic, we introduce in section 2 all the necessary terminology and results needed from the representation theory of algebras. Sections 3 and 4 are devoted to the proof of part (a) of the theorem and, after a section 5 devoted to reduction lemmata, we prove (b) and (c) in section 6.

2 Preliminaries

2.1 Algebras and quivers.

Throughout this paper, k will denote a fixed algebraically closed field. By algebra is meant an associative, finite dimensional k–algebra with an identity which we moreover assume to be basic (that is, $A/\operatorname{rad} A$ is a direct product of copies of k). Since we are interested in the representation theory of A, thus in the category $\operatorname{mod} A$ of finitely generated right A–modules, the latter hypothesis entails no loss of generality.

A (finite) **quiver** Q is a quadruple (Q_0, Q_1, s, t) consisting of two finite sets: Q_0 (the set of **points**) and Q_1 (the set of **arrows**) and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its **source** $s(\alpha) \in Q_0$ and its **target** $t(\alpha) \in Q_0$. Thus, one may think of a quiver as being a directed graph. A **relation** in a quiver Q from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each i with $1 \leq i \leq m$, λ_i is a non-zero scalar and w_i is a path of length at least two from x to y. A set of relations in Q generates an ideal I in the path algebra kQ of Q. We denote $kQ(x,y)$ the k–vector space generated by all paths in Q from x to y. For an algebra A, we denote by Q_A the ordinary quiver of A. For every basic algebra A, there exists a surjective k-algebra morphism $\nu : kQ_A \to A$ (induced by the choice of a set of representatives of basis vectors in the k–vector space $\operatorname{rad} A/\operatorname{rad}^2 A$) so that $A \simeq kQ_A/I_\nu$, where $I_\nu = \operatorname{Ker} \nu$ (see [BG]). The pair (Q_A, I_ν) is called a **presentation** of A. An algebra $A = kQ/I$ can equivalently be considered as a k–category of which the object class A_0 is the set Q_0, and the set of morphisms $A(x,y)$ from x to y is the quotient of $kQ(x,y)$ modulo the subspace $I(x,y) = I \cap kQ(x,y)$ (see [BG]). A full subcategory B of A is called **convex** if any path in A whith source and target in B lies entirely in B. An algebra A is called **triangular** if Q_A has no oriented cycles. The present work deals exclusively with triangular algebras.

2.2 Modules and representations.

Let $A = kQ/I$ be an algebra. A (finite dimensional) **representation** M of Q is defined by assigning to each $x \in Q_0$ a finite dimensional k–vector space $M(x)$, and to each arrow $\alpha : x \to y$ a k–linear map $M(\alpha) : M(x) \to M(y)$. The representation M of Q is said to be **bound by** I if, whenever $\rho = \sum_{i=1}^{m} \lambda_i \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_{t_i}}$ is a relation in I (with the λ_i non–zero scalars and the α_{i_j} arrows), then $\sum_{i=1}^{m} \lambda_i M(\alpha_{i_{t_i}}) \dots M(\alpha_{i_2}) M(\alpha_{i_1}) = 0$. A morphism $f : M \to N$ between bound representations is a family of k–linear maps $f_x : M(x) \to N(x)$ such that, if $\alpha : x \to y$, then $N(\alpha) f_x = f_y M(\alpha)$. Thus, bound representations of $A = kQ/I$ are just functors from the k–category A to $\operatorname{mod} k$. This yields a category of bound representations of A, which is equivalent to the category $\operatorname{mod} A$ (see [BG]). Accordingly, in the sequel, we identify these two categories, and view our modules as bound representations.

For an A–module M, we denote by $\operatorname{supp} M$ its **support**, that is, the full subcategory of A generated by the $x \in A_0$ such that $M(x) \neq 0$. For each $x \in Q_0$, we denote by S_x the corresponding simple A–module, and by P_x (or I_x) the projective cover (or the injective envelope, respectively) of S_x.

2.3 The fundamental group.

Let Q be a connected quiver without oriented cycles and I be an ideal of kQ generated by a set of relations. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i$ in $I(x, y)$ is called **minimal** if $m \geq 2$ and, for every non-empty proper subset $J \subset \{1, 2, \dots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$.

For an arrow $\alpha \in Q_1$, we denote by α^{-1} its formal inverse. A **walk** of length t in Q from x to y is a formal composition $\alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \dots \alpha_t^{\epsilon_t}$ (where $\alpha_i \in Q_1$ and $\epsilon_i \in \{1, -1\}$ for all i with $1 \leq i \leq t$) starting at x and ending at y. We also have walks of length zero, these are the trivial paths: we denote by e_x the **trivial path** at x.

We define the **homotopy relation** $\sim$ to be the smallest equivalence relation on the set of all walks in Q such that:

(a) If $\alpha : x \to y$ is an arrow, then $\alpha^{-1}\alpha \sim e_x$ and $\alpha\alpha^{-1} \sim e_y$.

(b) If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j such that $1 \leq i, j \leq m$.

c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions are defined.

We denote by $[u]$ the equivalence class of a walk u. Let $x_0 \in Q_0$ be arbitrary. The set $\pi_1(Q, I, x_0)$ of equivalence classes of all closed walks starting and ending at x_0 is a group under the operation induced from the composition of walks. Since, clearly, the group $\pi_1(Q, I, x_0)$ does not depend on the choice of x_0, we denote it by $\pi_1(Q, I)$ and call it the **fundamental group** of (Q, I), see [G, MP].

Let now A be a triangular algebra, and (Q_A, I_ν) be a presentation of A. The fundamental group $\pi_1(Q_A, I_\nu)$ depends essentially on I_ν, and thus is not an invariant of A, see, for instance, [AP, (1.4)] . A connected triangular algebra A is called **simply connected** if, for any presentation (Q_A, I_ν) of A, the fundamental group $\pi_1(Q_A, I_\nu)$ is trivial [AS].

2.4 Incidence algebras.

Let $(\Sigma, \leq)$ be a finite poset with n elements. The incidence algebra $A = A(\Sigma)$ of Σ is the subalgebra of the algebra $M_n(k)$ of $n \times n$ square matrices over k consisting of the matrices (x_{ij}) satisfying $x_{ij} = 0$ if $j \not\leq i$. In particular, $A(\Sigma)$ is a basic finite dimensional algebra.

An incidence algebra $A = A(\Sigma)$ can also be given by a quiver with relations. The quiver Q_A of $A(\Sigma)$ has as points the elements of Σ. For $x, y \in \Sigma$, there is (exactly) one arrow from x to y if and only if $y < x$ and there is no element $z \in \Sigma$ such that $y < z < x$ (we then say that x **covers** y).

In other words, Q_A is the Hasse diagram (also called covering diagram) of the poset Σ. Given two paths γ and γ' in Q_A, we say that γ and γ' are **parallel** if they have the same source and the same target. The ideal I is then the ideal generated by all differences $\gamma - \gamma'$, with γ, γ' parallel paths, and then $A \simeq kQ_A/I$ (see, for instance, [C, H, GR]).

Observe that, if $A = A(\Sigma)$ is an incidence algebra, then the full (or full convex) subcategories of A coincide with the incidence algebras of the full (or full convex, respectively) subposets of Σ.

It follows from [BM, (3.5)] that, if A is an incidence algebra then the fundamental group is independent of the presentation. Thus, for such an algebra, the notation $\pi_1(A)$ is not ambiguous.

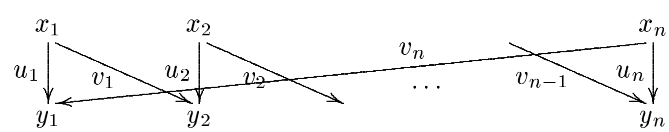
It is important to note that the fundamental group of an incidence algebra is also the fundamental group of a finite simplicial complex. Indeed, let $A = A(\Sigma)$ be an incidence algebra, and $|\Sigma|$ be the **chain complex** of Σ (that is, $|\Sigma|$ is the simplicial complex whose i–simplices are the chains $x_0 < x_1 < \ldots < x_i$ in Σ), then we have $\pi_1(A) \simeq \pi_1(|\Sigma|)$ (see [B, (2.2)], [R, (2.1)]). Conversely, if K is a finite simplicial complex, and Σ is the set of its non–degenerate simplices ordered by inclusion, then $\pi_1(K) \simeq \pi_1(A(\Sigma))$ (see [B, (3.5)]).

Finally, if A is an incidence algebra, then it is particularly easy to describe the simple modules and the indecomposable projective and injective modules. Indeed, let $x \in A_0$, then S_x is given by $S_x(x) = k$ and $S_x(y) = 0$ for $y \neq x$, and $S_x(\alpha) = 0$ for all arrows α. Its projective cover P_x is given by $P_x(y) = k$ if $x \geq y$ and $P_x(y) = 0$ if $x \not\geq y$; moreover, $P_x(\alpha) = id_k$ if $x \geq s(\alpha)$ and $P_x(\alpha) = 0$ if $x \not\geq s(\alpha)$. The injective envelope I_x of S_x is constructed dually.

3 Weak crowns

An algebra A is called **strongly simply connected** if every full convex subcategory of A is simply connected. It was shown in [D, (3.3)] that an incidence algebra is strongly simply connected if and only if it contains no crown. This leads to the following definitions.

DEFINITION 3.1 *Let Σ be a poset, and $A = A(\Sigma)$ be its incidence algebra. Let Γ be a full subcategory of A generated by $2n$ points $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ (with $n \geq 2$) and of the form*



$(*)$

1) *We say that Γ is a* **weak crown** *if:*

   (a) *For each i, the convex hull of $\{x_i, y_i\}$ in Σ intersects those of $\{x_{i-1}, y_i\}$ and $\{x_i, y_{i+1}\}$, and of no other $\{x_h, y_l\}$ (here, and in the sequel, we agree to set $y_{n+1} = y_1$ and $x_0 = x_n$).*

   (b) *The convex hulls of three distinct $\{x_h, y_l\}$ do not intersect.*

2) *A weak crown Γ is said to be a* **crown** *if, for each i, the intersection of the convex hulls of $\{x_i, y_i\}$ and of $\{x_i, y_{i+1}\}$ is x_i, and the intersection of the convex hulls of $\{x_{i-1}, y_i\}$ and of $\{x_i, y_i\}$ is y_i.*

3) *Given a weak crown Γ of the form $(*)$, we define its* **width** *$w(\Gamma)$ to be n. Thus, for any weak crown Γ, we have $w(\Gamma) \geq 2$.*

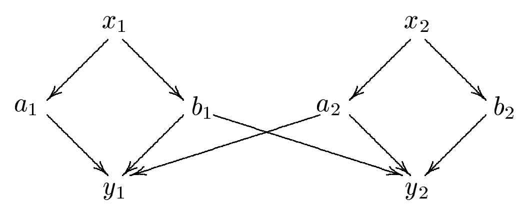
4) *Let Γ be a weak crown. A point x not in Γ is said to* **suspend** *Γ if x is a proper predecessor of at least two non–comparable points of Γ, and no proper successor of x is a predecessor of the same points of Γ. A suspending point x is said to be a* **top** *of Γ if x is a direct predecessor of all the maximal points of Γ. We define dually points which* **sustain** *Γ, or which lie at its* **bottom**.

5) *Let Γ be a weak crown, and let $x \in \Sigma$ suspend Γ. The* **suspension** Γ^x *of Γ is the full subcategory of A generated by x, all the minimal points of Γ and those of its maximal points which are not comparable to x. We define dually the* **sustension** Γ_x *of Γ at a point which sustains it.*

6) *A* **circumference** *of a weak crown Γ of the form $(*)$ above is a cyclic walk $w = w_1^{\epsilon_1} w_2^{\epsilon_2} \ldots w_{2n}^{\epsilon_{2n}}$ where, for each i with $1 \leq i \leq 2n$, we have that $\epsilon_i \in \{1, -1\}$ and w_i is a path parallel to one of the paths $u_1, \cdots, u_n, v_1, \cdots, v_n$ and such that, moreover, each u_i or v_i is parallel to exactly one of the w_j.*

Given a point x in a weak crown Γ, there exist many circumferences of Γ starting and ending at x. We refer to all of them as circumferences of Γ at x. For instance, $v_1 u_2^{-1} v_2 \cdots v_{n-1} u_n^{-1} v_n u_1^{-1}$ and $u_1 v_n^{-1} u_n v_{n-1}^{-1} \cdots v_2^{-1} u_2 v_1^{-1}$ are circumferences of Γ at x_1. We observe that any cirumference of Γ at x_1 is homotopic to one of these two circumferences. On the other hand, it is easily verified that, for any i, every circumference of Γ at x_i is homotopic to a conjugate of a circumference of Γ at x_{i-1}, and is also homotopic to a conjugate of a circumference of Γ at y_i. Thus, if a circumference of Γ is homotopic to a trivial walk, then so are all the circumferences of Γ.

In the poset Σ with Hasse diagram



the elements $\{x_1, x_2, y_1, y_2\}$ generate a weak crown of width two which is not a crown. We now show that the convex hull of a weak crown always contains a crown.

LEMMA 3.2 *The convex hull of a weak crown Γ contains a crown as a full subcategory, with a circumference homotopic to a circumference of Γ.*

Proof. Assume that Γ is not a crown. Then we may assume that either the intersection of the convex hulls of $\{x_1, y_1\}$ and $\{x_1, y_2\}$ contains a point $z \neq x_1, y_1, y_2$ or the intersection of the convex hulls of $\{x_1, y_1\}$ and $\{x_n, y_1\}$ contains a point $z \neq x_1, x_n, y_1$.

In the first case, the $2n$ points $z, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ generate a weak crown Γ'. This follows from the fact that the convex hull of $\{z, y_l\}$ is contained in the convex hull of $\{x_1, y_l\}$ for all l. In the second case, the points $x_1, x_2, \cdots, x_n, z, y_2, \cdots, y_n$ generate a weak crown Γ'. In either case, Γ' has a circumference homotopic to a circumference of Γ. If Γ' is not a crown, then we can iterate the procedure. Since the convex hull of Γ' is strictly contained in the convex hull of Γ, after a finite number of steps we obtain a crown , as desired. □

We now give a useful characterization of weak crowns.

LEMMA 3.3 *Let A be an incidence algebra, and Γ be a full subcategory of A, of the form $(*)$. Then Γ is a weak crown if and only if:*

(a) $n = 2$, and the convex hulls of $\{x_1, y_1\}$ and $\{x_2, y_2\}$ do not intersect, or

(b) $n > 2$ and the only pairs of distinct comparable points in Γ are of the form (x_i, y_i) and (x_i, y_{i+1}), or of the form (x_i, y_i) and (x_{i-1}, y_i), for each i.

Proof.

(a) Indeed, if the convex hulls of $\{x_1, y_1\}$ and $\{x_2, y_2\}$ intersect, then so do the convex hulls of $\{x_1, y_2\}$ and $\{x_2, y_1\}$.

(b) The proof is straightforward.

□

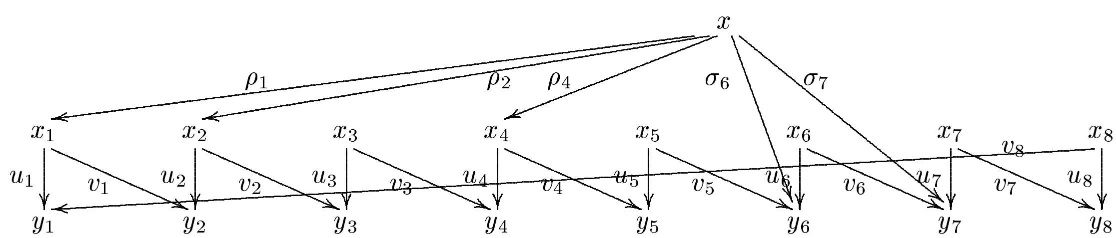
REMARK 3.4

(a) If Γ and Γ' are weak crowns and the points of Γ' are among those of Γ, then $\Gamma = \Gamma'$.

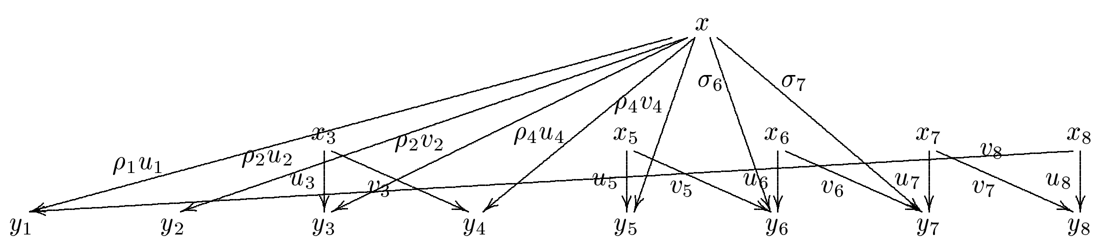
(b) The suspension Γ^x of a weak crown Γ of A at a point x is the full subcategory generated by the maximal and the minimal elements of the full subcategory of A generated by Γ and x.

The basic observation of our work is that the suspension of a weak crown decomposes uniquely as a union of weak crowns. We show the process on an example.

EXAMPLE 3.5 *Let*

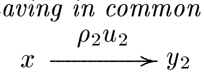


be a full subcategory of an incidence algebra A, and let Γ be the weak crown with points $x_1, \cdots, x_8, y_1, \cdots, y_8$. The suspension Γ^x is the full subcategory obtained by deleting the points x_1, x_2, x_4

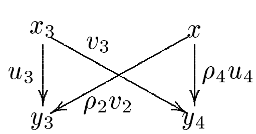


Thus, the suspension is the union of four weak crowns and a full subcategory Γ_2 generated by two points, all having in common the point x:

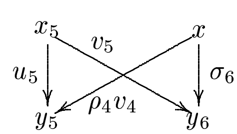
Γ_2



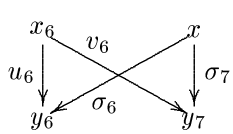
Γ_3



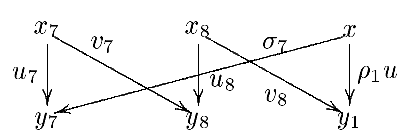
Γ_5



Γ_6



Γ_7

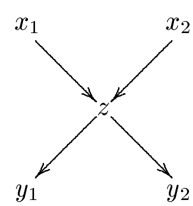


Moreover, if we consider the following circumference of Γ *starting and ending at* y_1 *containing the walk* u_1^{-1}*, then we have*

$$
\begin{aligned}
& u_1^{-1}v_1u_2^{-1}v_2u_3^{-1}v_3u_4^{-1}v_4u_5^{-1}v_5u_6^{-1}v_6u_7^{-1}v_7u_8^{-1}v_8 \\
\sim\ & u_1^{-1}\rho_1^{-1}\rho_2v_2u_3^{-1}v_3u_4^{-1}\rho_4^{-1}\rho_4v_4u_5^{-1}v_5\sigma_6^{-1}\sigma_6u_6^{-1}v_6\sigma_7^{-1}\sigma_7u_7^{-1}v_7u_8^{-1}v_8 \\
\sim\ & (\rho_1u_1)^{-1}(\rho_2v_2u_3^{-1}v_3u_4^{-1}\rho_4^{-1})(\rho_4v_4u_5^{-1}v_5\sigma_6^{-1})(\sigma_6u_6^{-1}v_6\sigma_7^{-1})(\sigma_7u_7^{-1}v_7u_8^{-1}v_8u_1^{-1}\rho_1^{-1})(\rho_1u_1)
\end{aligned}
$$

that is, this circumference is homotopic to a conjugate of the product of circumferences of each of the weak crowns in the decomposition, all starting and ending at x.

In the sequel, it is useful to refer to full subcategories of the form $x \to y$ as **sticks** and to full subcategories of the form



as **crosses**. The decomposition of the following proposition is referred to as the **canonical decomposition**.

PROPOSITION 3.6 *Let* Γ *be a weak crown in an incidence algebra* A*, and let* x *suspend* Γ*. Then:*

(a) The suspension Γ^x *of* Γ *at* x *can be written uniquely as a union of weak crowns, crosses and sticks all having in common the point* x.

(b) The width of each weak crown in the decomposition of (a) is smaller than the width of Γ*, unless* x *precedes no maximal point of* Γ*, and* x *precedes exactly two minimal points which are consecutive.*

(c) The product of circumferences, all starting and ending at x*, of the weak crowns in the above decomposition of* Γ^x *is homotopic to a conjugate of a circumference of* Γ.

Proof. Let, as in (3.1), $x_1, \ldots, x_n$ denote the maximal points of Γ, $y_1, \ldots, y_n$ denote its minimal points, and let $u_i : x_i \to y_i$, $v_i : x_i \to y_{i+1}$ be paths in A (where, as usual, $y_{n+1} = y_1$). Moreover, we define $\gamma_{i,j}$ to be the walk $u_i^{-1}v_iu_{i+1}^{-1}\ldots u_{j-1}^{-1}v_{j-1}$ from y_i to y_j.

(a) Assume that x precedes $y_{h_1}, y_{h_2}, \ldots, y_{h_r}$ and no other y_i, where $h_1 < h_2 < \ldots < h_r$ (and we agree that $h_{r+1} = h_1$). Let $C_0 = \{i : \ 1 \leq i \leq r \text{ and } x \not\geq x_{h_i}\}$ and let $C_0' = \{i : i \in C_0 \text{ and there exists a point } u \text{ such that } y_{h_i}, y_{h_{i+1}} \leq u \leq x, x_{h_i}\}$. Now, for each $i \in C_0'$, let Γ_i be the full subcategory of A generated by $x, x_{h_i}, y_{h_i}, y_{h_{i+1}}$ and, for each $i \in C_0 \setminus C_0'$, let Γ_i be the full subcategory of A generated by the points

$$x_{h_i}, x_{h_i+1}, \ldots, x_{h_{i+1}-1}, x, y_{h_i}, y_{h_i+1}, \ldots, y_{h_{i+1}}.$$

It follows from the definition of C_0' that Γ_i is a cross for each $i \in C_0'$. We prove next that, for each $i \in C_0 \setminus C_0'$, Γ_i is a weak crown contained in the suspension Γ^x. Since x precedes y_{h_i} and $y_{h_{i+1}}$, and no other y_i in Γ_i, we only have to consider the intersections of the convex hull of $\{x, y_l\}$, with that of $\{x_s, y_t\}$ in Γ_i, for $l = h_i$ and $l = h_{i+1}$. We study only the first case since the second one is analogous. So we assume that the convex hull of $\{x, y_{h_i}\}$ intersects the convex hull of a set $\{x_s, y_t\}$. Then x precedes y_t, so t is either h_i or h_{i+1}. From (3.3), s is h_i or $h_{i+1}-1$ respectively. Hence the convex hull of $\{x, y_{h_i}\}$ can only intersect with the convex hull of $\{x, y_{h_{i+1}}\}$, $\{x_{h_i}, y_{h_i}\}$ or $\{x_{h_{i+1}-1}, y_{h_{i+1}}\}$. If it intersects the convex hull of the third set, there exists a point u such that $y_{h_i}, y_{h_{i+1}} \leq u \leq x, x_{h_{i+1}-1}$. But in this case $h_{i+1} = h_i + 1$ and hence $i \in C'_0$, a contradiction. We only need to prove now that the convex hulls of three pairs of points of Γ_i do not intersect, and this reduces to prove that the intersection of the convex hulls of $\{x, y_{h_i}\}$, $\{x, y_{h_{i+1}}\}$ and $\{x_{h_i}, y_{h_i}\}$ is empty. Otherwise, we have $i \in C'_0$, again a contradiction. So Γ_i is a weak crown for each $i \in C_0 \setminus C'_0$.

If $i \notin C_0$, then $h_{i+1} = h_i + 1$. We let $C_1 = \{i : 1 \leq i \leq r,\ i \notin C_0 \text{ and } i-1 \notin C_0\}$ and, for each $i \in C_1$, we let Γ_i be the stick $x \to y_{h_i}$.

We prove next that $\Gamma^x = \cup_{i \in C_0 \cup C_1} \Gamma_i$. We note that the points of Γ^x are x, all the x_j such that $x \not\geq x_j$ and all the minimal points $y_1, \ldots, y_n$ of Γ. So, let h be such that $1 \leq h \leq n$ and i be such that $h_i \leq h < h_{i+1}$. If $i \in C_0$ then, by definition, x_h and y_h are points in Γ_i. If $i \notin C_0$, then $x \geq x_h$ and hence x_h is not a point of Γ^x. We then have two cases. Firstly, if $i \notin C_1$, then $i-1 \in C_0$ and $h = h_i$ so that y_h belongs to Γ_{i-1}. Secondly, if $i \in C_1$, then Γ_i is the stick $x \to y_h$ which contains y_h. This establishes that $\Gamma^x = \cup_{i \in C_0 \cup C_1} \Gamma_i$.

Now, we show that this decomposition is unique. In view of (3.4), it suffices to prove that, if Γ' is a weak crown containing x and with maximal and minimal points among those of Γ^x, then Γ' is one of the weak crowns in the decomposition just described. So, let Γ' be such a weak crown. There exist two consecutive maximal points of Γ, say x_1 and x_2, such that x_1 does not belong to Γ', while x_2 does. Then the maximal points of Γ' are contained in the set $\{x_2, \ldots, x_n, x\}$. In this set, the only x_i preceding y_2 is x_2. Moreover, Γ' is a weak crown, so there are two maximal points preceding y_2. Therefore, x precedes y_2. Let i be the least index such that x_i belongs to Γ', while x_{i+1} does not belong to Γ' (recall that $x_{n+1} = x_1$). Then $x_i \geq y_{i+1}$ gives that y_{i+1} belongs to Γ'. Since y_{i+1} has exactly two predecessors among the maximal points, then $x \geq y_{i+1}$. Thus, $x_2, \ldots, x_i, x$ are maximal points of Γ', each of them having two minimal points of Γ as successors. Thus, necessarily, $y_2, \ldots, y_{i+1}$ are among the minimal points of the weak crown Γ'. Consequently, the weak crown Γ'' generated by the points $x_2, \ldots, x_i, x, y_2, \ldots, y_{i+1}$ has its points among those of Γ'. By (3.4)(a), we infer that $\Gamma'' = \Gamma'$. But Γ'' is one of the weak crowns in the described decomposition, and this proves that Γ' is one of them, as desired.

(b) For $i \in C_0$, we have $w(\Gamma_i) = h_{i+1} - h_i + 1 \leq n$. Assume that the equality holds for some i, and that $h_i = 1$, so $i = 1$. Then $h_{i+1} = n$, so that the only minimal points of Γ preceded by x are y_1 and y_n. If x precedes a maximal point x_h, then x precedes also y_h and y_{h+1}, and therefore $h = n$, since we are assuming that $1 \in C_0$. Since a proper successor of x, namely x_n, is a predecessor of the two non–comparable points y_1 and y_2, it follows that x does not suspend Γ, a contradiction. Therefore, x precedes no maximal points and it precedes exactly two minimal points which are consecutive, as desired.

(c) Let σ_i denote a fixed chosen path from x to y_{h_i}. When $i \notin C_0$, that is, when $x \geq x_{h_i}$, there is a path ρ_i from x to x_{h_i} and we have $\sigma_i \sim \rho_i u_{h_i}$, $\sigma_{i+1} \sim \rho_i v_{h_i}$. Thus $\sigma_i^{-1}\sigma_{i+1} \sim u_{h_i}^{-1} v_{h_i} = \gamma_{h_i, h_{i+1}}$. Moreover, if $i, i+1, \ldots, i+j-1 \notin C_0$, then

$$\sigma_i^{-1}\sigma_{i+j} \sim (\sigma_i^{-1}\sigma_{i+1})(\sigma_{i+1}^{-1}\sigma_{i+2}) \ldots (\sigma_{i+j-1}^{-1}\sigma_{i+j}) \sim \gamma_{h_i, h_{i+1}} \cdots \gamma_{h_{i+j-1}, h_{i+j}} = \gamma_{h_i, h_{i+j}}.$$

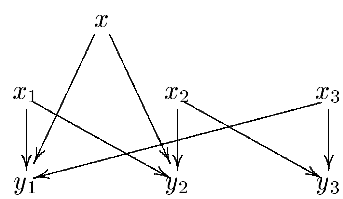
For any $i \in C_0$, we consider the walk $w_i = \sigma_i \gamma_{h_i, h_{i+1}} \sigma_{i+1}^{-1}$ of Γ_i. We now show that the product of the w_i, with $i \in C_0$, is homotopic to a conjugate of a circumference w of Γ. If $i, i+1 \in C_0$, then $w_i w_{i+1} \sim \sigma_i \gamma_{h_i, h_{i+2}} \sigma_{i+2}^{-1}$. If $i, i+j \in C_0$, and $i+1, \ldots, i+j-1 \notin C_0$, then, using the relation above, we get

$$w_i w_{i+j} = \sigma_i \gamma_{h_i, h_{i+1}} \sigma_{i+1}^{-1} \sigma_{i+j} \gamma_{h_{i+j}, h_{i+j+1}} \sigma_{i+j+1}^{-1} \sim \sigma_i \gamma_{h_i, h_{i+1}} \gamma_{h_{i+1}, h_{i+j}} \gamma_{h_{i+j}, h_{i+j+1}} \sigma_{i+j+1}^{-1}.$$

Hence $\prod_{i\in C_0} w_i \sim \sigma_1\gamma_{h_1,h_{r+1}}\sigma_{r+1}^{-1} = \sigma_1 w \sigma_1^{-1}$ with $w = \gamma_{h_1,h_{r+1}}$ (we recall that $h_{r+1} = h_1$). Finally we observe that if $i \in C'_0$, the walk $w_i \sim e_x$ because the underlying graph of the cross is a tree. So $\prod_{i\in C_0} w_i \sim \prod_{i\in C_0\setminus C'_0} w_i$.

□

We remark that the condition in (b) is in fact necessary and sufficient. Indeed, in the poset with Hasse diagram



the suspension Γ^x of the weak crown Γ generated by $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is a union of two weak crowns of respective widths two and three.

Let x suspend a weak crown Γ. We define the **width** $w(\Gamma^x)$ of Γ^x to be the maximal width of all the weak crowns in the canonical decomposition.

COROLLARY 3.7 *Assume that x suspends Γ, and precedes at least one maximal point, or two minimal non–consecutive points of Γ, then $w(\Gamma^x) < w(\Gamma)$.*

4 Completeness of a weak crown.

We are now able to define the notion of a complete weak crown, which is essential to our study, and we do it by induction on the width of a crown.

DEFINITION 4.1 *Let Γ be a weak crown.*

(a) If $w(\Gamma) = 2$, then Γ is said to be **complete** *if there exists a point which suspends Γ and precedes its two maximal points or, dually, there exists a point which sustains Γ and succedes its two minimal points.*

(b) If $w(\Gamma) > 2$, then Γ is said to be **complete** *provided one of the following conditions is satisfied:*

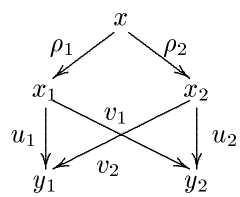
  (i) There exists a point x which suspends Γ and precedes at least two maximal points of Γ and, moreover, each weak crown in the canonical decomposition of Γ^x is complete or, dually,

  (ii) There exists a point x which sustains Γ and succedes at least two minimal points of Γ and, moreover, each weak crown in the canonical decomposition of Γ_x is complete.

EXAMPLE 4.2 *An inmediate example of a complete weak crown is that of a weak crown Γ such that there is a point x preceding all its maximal points. In this case, only sticks occur in the cannonical decomposition of Γ^x.*

LEMMA 4.3 *Let Γ be a complete weak crown. Then any circumference of Γ is homotopic to a trivial walk.*

Proof. Suppose $w(\Gamma) = 2$. Then, up to duality, we have the following picture



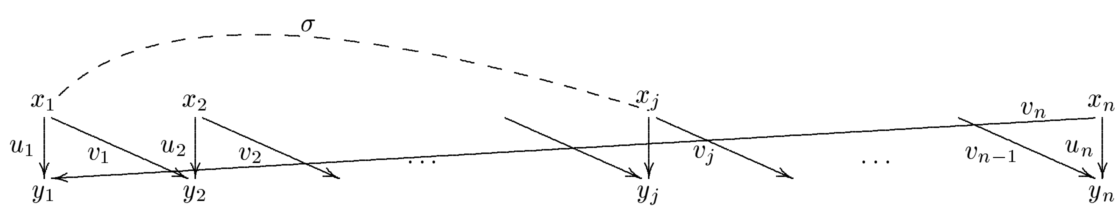
and $v_1 u_2^{-1} v_2 u_1^{-1} \sim \rho_1^{-1}\rho_2\rho_2^{-1}\rho_1 \sim e_{x_1}$.

Suppose $w(\Gamma) > 2$. By (3.6)(c), a circumference of Γ is homotopic to a conjugate of the product of circumferences of the weak crowns in the canonical decomposition of Γ^x. By induction, this latter product is homotopic to a trivial walk. Hence the statement. □

LEMMA 4.4 *Let w be a cyclic walk in an incidence algebra A having least number of sinks (or, equivalently, of sources) among the cyclic walks which are not homotopic to a trivial walk. Then the full subcategory Γ of A generated by the sinks and the sources of w is a weak crown.*

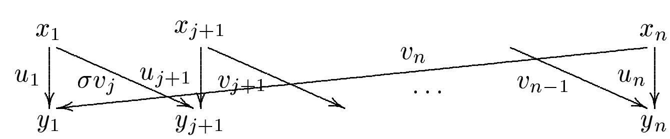
Proof. We may assume that $w = v_1 u_2^{-1} \cdots v_n u_1^{-1}$ with u_i a non–trivial path from x_i to y_i, and v_i a non–trivial path from x_i to y_{i+1}, for each i with $1 \le i \le n$ (where we set $y_{n+1} = y_1$). Then Γ is generated by the sinks and sources $x_1, \ldots, x_n, y_1, \ldots, y_n$. Furthermore, we may assume that these points are all different. Otherwise, we can replace w by a walk w' passing exactly once through its sinks and sources, and not homotopic to a trivial walk. Then the full subcategory of A generated by the sinks and the sources of w' coincides with Γ.

Clearly, $n > 1$ since otherwise w would be homotopic to a trivial walk. If $n = 2$ and the intersection of the convex hulls of $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is non–empty, then w is homotopic to a trivial walk, contradicting our assumption. Therefore, such an intersection is empty and w is a weak crown, by (3.3)(a). So, let $n > 2$. According to (3.3)(b), in order to prove that w is a weak crown, we have to prove that the only pairs of comparable elements among the x_i and the y_i are of the form (x_i, y_i) and (x_i, y_{i+1}). Assume that x_i and x_j are comparable. We may assume without loss of generality that $1 = i < j$ and that $x_1 \ge x_j$. Let σ be a path from x_1 to x_j .

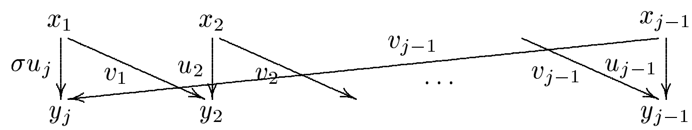


Then w equals the union of the two cycles w_1 and w_2 represented by:

w_1

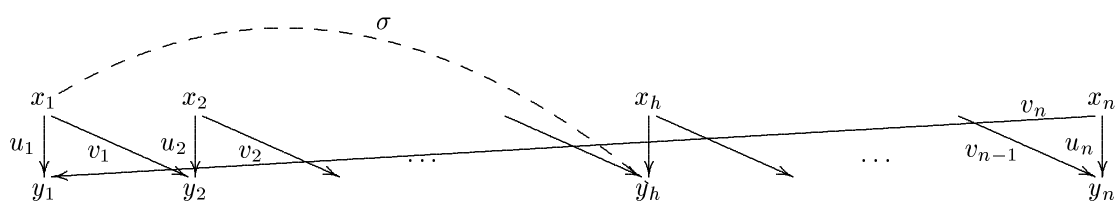


w_2



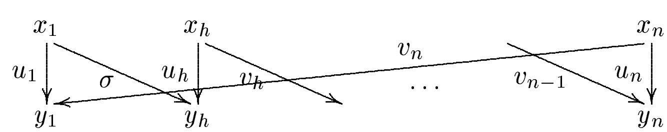
Let $w_1 = (\sigma v_j)u_{j+1}^{-1}v_{j+1}\cdots u_n^{-1}v_n u_1^{-1}$ and $w_2 = v_1 u_2^{-1}\cdots u_{j-1}^{-1}v_{j-1}(\sigma u_j)^{-1}$. Then w_1 and w_2 are cyclic walks each having less sinks than w and $w_2w_1 \sim w$. Since w is not homotopic to a trivial walk, then this is the case for at least one of w_1 and w_2, contradicting our minimality assumption.

We treat by duality the case where y_i and y_j are comparable, and we assume next that x_i and y_h are comparable, where $h \notin \{i, i+1\}$. We may assume without loss of generality that $1 = i < h-1$ and x_1 is not comparable with x_h. We can furthermore assume, by duality, that $x_1 \geq y_h$. Let σ be a path from x_1 to y_h.

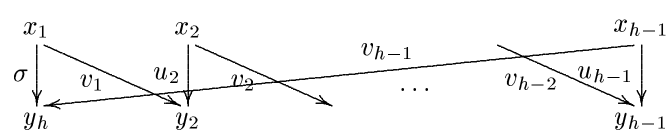


Then w equals the union of two cycles w_1 and w_2 represented by

w_1



w_2



Setting $w_1 = \sigma u_h^{-1}v_h\cdots u_n^{-1}v_n u_1^{-1}$ and $w_2 = v_1 u_2^{-1}\cdots u_{h-1}^{-1}v_{h-1}\sigma^{-1}$, we have $w \sim w_2w_1$ and the argument continues as before. $\square$

PROPOSITION 4.5 *Let A be an incidence algebra. If A is not simply connected, then A contains a crown Γ whose circumference is not homotopic to a trivial walk.*

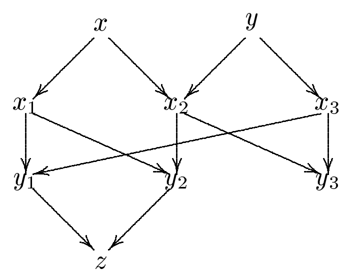
Proof. Since A is not simply connected, it contains cyclic walks which are not homotopic to trivial walks. Consider all such cyclic walks in A having least number of sinks (or, equivalently, of sources), then choose among them one of minimal length, and call it w. By (4.4), the sinks and the sources of w generate a weak crown Γ in A. Moreover the minimality of the length of w implies that, for each i, the convex hulls of $\{x_i, y_i\}$ and $\{x_i, y_{i+1}\}$ intersect only at x_i and, dually, the convex hulls of $\{x_{i-1}, y_i\}$ and $\{x_i, y_i\}$ intersect only at y_i Thus, Γ is a crown. $\square$

COROLLARY 4.6 *Let A be an incidence algebra, and assume that every crown in A is complete. Then A is simply connected.*

Proof. If this is not the case, then, by (4.5) there exists a crown Γ whose circumference is not homotopic to a trivial walk. By (4.3), this implies that Γ is not complete, a contradiction. $\square$

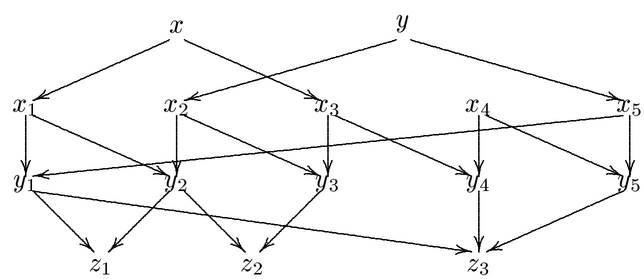
EXAMPLE 4.7 *The first two examples show that the converse of (4.6) is not true, and the third shows an incidence algebra satisfying the hypothesis of (4.6) but whose associated simplicial complex is not contractible.*

(a) Let A be given by the poset with Hasse diagram



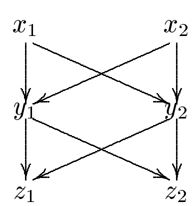
Then A is simply connected, but the crown generated by the points $\{x, x_3, y_1, y_3\}$ is not complete.

(b) Let A be given by the poset with Hasse diagram



Then A is simply connected. Let Γ be the crown generated by the points $x_1, \ldots, x_5, y_1, \ldots, y_5$. Then the suspension Γ^x is complete, while Γ^y is not.

(c) Let A be given by the poset with Hasse diagram



Clearly, every crown in A is complete and therefore A is simply connected. On the other hand, the geometric realisation of the associated simplicial complex is an octahedron in three dimensional space, and therefore is not contractible.

5 Reduction lemmata.

In order to prove our main theorem, we need a couple of lemmata which allow us to do induction on the number of points in A. The first reduction is well–known (see for instance, [CF, (3.7)] or [B, (3.4)]), we give however an independent proof for the convenience of the reader.

LEMMA 5.1 *Let A be an incidence algebra and x be a point in A such that there exists a unique arrow $\alpha : x \to y$ with source x, and let B be the full subcategory of A generated by all objects except x. Then $\pi_1(B) \cong \pi_1(A)$.*

Proof. Let y be the base point of both $\pi_1(A)$ and $\pi_1(B)$. We construct group morphisms $\psi : \pi_1(A) \to \pi_1(B)$ and $\phi : \pi_1(B) \to \pi_1(A)$, and show that they are inverse isomorphisms by showing that ψ is surjective and that $\phi\psi = 1$.

To construct ψ, we first let W_A and W_B denote respectively the set of walks in A and in B, then define $\overline{\psi} : W_A \to W_B$ as follows

$$\begin{array}{llll} \overline{\psi}(e_x) = \overline{\psi}(e_y) = e_y & \\ \overline{\psi}(e_z) = e_z & \text{if } z \neq x, y \\ \overline{\psi}(\alpha) = e_y & \\ \overline{\psi}(\beta) = \beta & \text{for any arrow } \beta \neq \alpha \text{ such that } t(\beta) \neq x \end{array}$$

Let δ be an arrow with $t(\delta) = x$. If there exists at least one path v in A which is parallel to $\delta\alpha$, we choose one such v and set $\overline{\psi}(\delta) = v$. Otherwise, we set $\overline{\psi}(\delta) = \delta'$, where δ' is the arrow in B from $s(\delta)$ to y.

For any arrow ξ, we set $\overline{\psi}(\xi^{-1}) = \overline{\psi}(\xi)^{-1}$ and, for any walk $w = \xi_1^{\epsilon_1}\xi_2^{\epsilon_2}\ldots\xi_m^{\epsilon_m}$ (where the ξ_i are arrows, and $\epsilon_i \in \{+1, -1\}$ for all i such that $1 \leq i \leq m$), we set

$$\overline{\psi}(\xi_1^{\epsilon_1}\xi_2^{\epsilon_2}\ldots\xi_m^{\epsilon_m}) = \overline{\psi}(\xi_1)^{\epsilon_1}\overline{\psi}(\xi_2)^{\epsilon_2}\ldots\overline{\psi}(\xi_m)^{\epsilon_m}.$$

Then $\overline{\psi}$ induces a group morphism $\psi : \pi_1(A) \to \pi_1(B)$: indeed, we must check that, if w and w' are parallel paths in A, then $\overline{\psi}(w)$ and $\overline{\psi}(w')$ are parallel in B, and this is clear from the above definition, because $\overline{\psi}(w)$ and $\overline{\psi}(w')$ have the same endpoints. Moreover, $\overline{\psi}$ is surjective, hence so is ψ.

We now construct a group morphism $\phi : \pi_1(B) \to \pi_1(A)$ as follows. Let $\overline{\phi} : W_B \to W_A$ be such that $\overline{\phi}(e_z) = e_z$ for all z, $\overline{\phi}(\beta) = \beta$ for all $\beta \neq \delta'$ (where δ' is as above) and, for any such δ', let $\overline{\phi}(\delta') = \delta\alpha$. We extend $\overline{\phi}$ to walks in the obvious way. Again, $\overline{\phi}$ induces a group morphism $\phi : \pi_1(B) \to \pi_1(A)$: if w and w' are parallel paths in B, then $\overline{\phi}(w)$ and $\overline{\phi}(w')$ have the same endpoints.

To finish, we must prove that $\phi\psi = 1$. Let w be a closed walk in W_A through y. If w does not pass through α and the arrows δ of terminus x, then, clearly, $\overline{\phi}\,\overline{\psi}(w) = w$. If $w = w_1\delta_1\alpha$ (where δ_1 is an arrow of target x and we can assume that the walk w_1 does not pass through α) then we have two cases to consider: $\overline{\psi}(\delta_1) = \delta_1'$ or $\overline{\psi}(\delta_1) = v$ (where v is parallel to $\delta_1\alpha$). In the first case, we have $(\overline{\phi}\,\overline{\psi})(w_1\delta_1\alpha) = \overline{\phi}(w_1\delta_1') = w_1\delta_1\alpha$. In the second case, we have $(\overline{\psi}\,\overline{\phi})(w_1\delta_1\alpha) = \overline{\psi}(w_1 v) = w_1 v$ which is homotopic in A to $w_1\delta_1\alpha$. The last case we have to consider is that of w not passing through α but passing through arrows with target x. Then $w = w_1\delta_1\delta_2^{-1}w_2$ (where we can assume that neither w_1 nor w_2 passes through an arrow of target x). We then have to consider three cases, up to duality:

1) $\overline{\phi}(\delta_1) = \delta_1'$, $\overline{\phi}(\delta_2) = \delta_2'$.

2) $\overline{\phi}(\delta_1) = v_1$, $\overline{\phi}(\delta_2) = \delta_2'$.

3) $\overline{\phi}(\delta_1) = v_1$, $\overline{\phi}(\delta_2) = v_2$.

We have, respectively,

1) $(\overline{\psi\phi})(w_1\delta_1\delta_2^{-1}w_2) = \overline{\psi}(w_1\delta_1'\delta_2'^{-1}w_2) = w_1\delta_1\alpha\alpha^{-1}\delta_2^{-1}w_2 \sim w_1\delta_1\delta_2^{-1}w_2$.

2) $(\overline{\psi\phi})(w_1\delta_1\delta_2^{-1}w_2) = \overline{\psi}(w_1v_1\delta_2'^{-1}w_2) = w_1v_1(\delta_2\alpha)^{-1}w_2 = w_1v_1\alpha^{-1}\delta_2^{-1}w_2 \sim w_1\delta_1\alpha\alpha^{-1}\delta_2^{-1}w_2 \sim w_1\delta_1\delta_2^{-1}w_2$.

3) $(\overline{\psi\phi})(w_1\delta_1\delta_2^{-1}w_2) = \overline{\psi}(w_1v_1v_2^{-1}w_2) = w_1v_1v_2^{-1}w_2 \sim w_1(\delta_1\alpha)(\delta_2\alpha)^{-1}w_2 \sim w_1\delta_1\delta_2^{-1}w_2$.

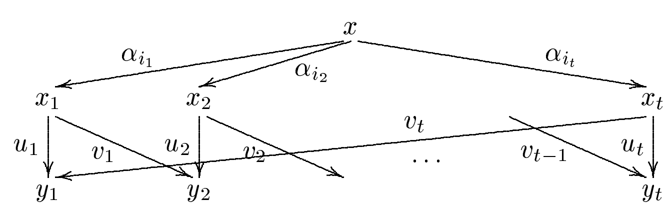
□

The reader will observe that, in the situation of the lemma, the simplicial complex corresponding to B is a deformation retract of the one corresponding to A and, in fact, the proof consists in constructing a retraction to the inclusion of B into A.

Before stating our second reduction procedure, we need a lemma, which generalizes an idea used in [AMP, (2.2)(2.3)]. Let x be a source in A and $x^{\rightarrow}$ be the set of all arrows starting at x. Following [AP, (2.1)], we let $\approx$ be the smallest equivalence relation on $x^{\rightarrow}$ such that $\alpha \approx \beta$ whenever there exist paths u, v in A such that αu and βv are parallel paths. We denote by $[\alpha]$ the equivalence class of an arrow $\alpha \in x^{\rightarrow}$ and we associate to $[\alpha]$ a graph $G[\alpha]$ as follows. Assume that $[\alpha] = \{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$. Then $G[\alpha]$ has r vertices. Let $M_j = \{z \in A_0 : z \leq t(\alpha_j)\}$. Then the number of edges connecting α_{i_2} and α_{i_1} is equal to the number of maximal points in $M_{i_2} \cap M_{i_1}$. The number of edges connecting α_{i_3} and α_{i_1} is equal to the number of maximal points in $M_{i_3} \cap M_{i_1}$ and the number of edges connecting α_{i_3} and α_{i_2} is equal to the number of maximal points in $M_{i_3} \cap M_{i_2}$, which are not comparable with the maximal elements in $M_{i_3} \cap M_{i_1}$. Assume that we have already constructed the edges connecting $\alpha_{i_{j-1}}$ and the α_{i_h} with $1 \leq h \leq j-2$. Then the number of edges connecting α_{i_j} with α_{i_h} for $1 \leq h \leq j-1$ is equal to the number of maximal points in $M_{i_j} \cap M_{i_h}$ which are not comparable with the maximal elements in $M_{i_j} \cap M_{i_l}$ with $1 \leq l \leq h-1$.

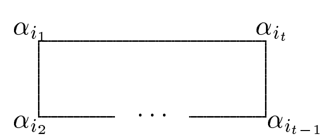
LEMMA 5.2 *With the above notation, $G[\alpha]$ is a tree if and only if $x = s(\alpha)$ tops no weak crown.*

Proof. Suppose that x tops a weak crown



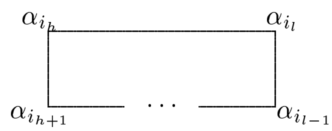
This gives $\alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_t} - \alpha_{i_1}$ in $G[\alpha]$, which therefore is not a tree.

Conversely, if $G[\alpha]$ is not a tree, then it contains a circuit



where we can assume that t is minimum. Let $x_j = t(\alpha_{i_j})$, and $M_h = \{z \in A_0 : z \leq t(\alpha_h)\}$. If $t = 2$, then we have two edges connecting α_{i_1} and α_{i_2}. These edges correspond to two maximal elements y_1, y_2 in $M_{i_1} \cap M_{i_2}$. Then the convex hulls of $\{x_1, y_1\}$ and $\{x_2, y_2\}$ do not intersect, and, by (3.3), $\{x_1, x_2, y_1, y_2\}$ generate a weak crown topped by x.

Assume now that $t > 2$. Since we have an edge connecting α_{i_j} and $\alpha_{i_{j+1}}$, there is at least a maximal point in $M_{i_j} \cap M_{i_{j+1}}$ which is not comparable with the maximal points in $M_{i_l} \cap M_{i_{j+1}}$ for all l such that $1 \leq l \leq j-1$. Let y_{j+1} be such a point and $x_j = t(\alpha_{i_j})$. We now prove that the full subcategory of A generated by $\{x_1, \cdots, x_t, y_1, \cdots, y_t\}$ with $y_1 = y_{t+1}$ is a weak crown. Invoking (3.3), we need to prove that the only pairs of comparable elements among the x_j and the y_j are of the form (x_j, y_j) and (x_j, y_{j+1}) (where, as usual, we set $x_0 = x_t$ and $y_1 = y_{t+1}$). Now, two points x_h, x_l are not comparable, because α_{i_h} and α_{i_l} are arrows. By definition, y_h, y_l are not comparable for $h \neq l$. Now, $y_h \geq x_l$ implies $x_h \geq x_l$ so $h = l$, a contradiction. On the other hand, if $x_l \geq y_h$ for $h \notin \{l, l+1\}$, then we get a circuit



contradicting the assumed minimality of t. □

We now recall the separation property [BLS]. Let x be a source in A, and B be the full subcategory of A generated by all objects except x. Then x is said to be **separating** if the number of indecomposable summands of $\operatorname{rad} P_x$ (that is, the kernel of the canonical surjection $P_x \to S_x$) equals the number of connected components of B. In general, a point y in A (not necessarily a source) is called a separating point if y is separating as a source in the full subcategory of A generated by all objects except the points z such that there exists a non–trivial path from z to y in A. The algebra A is said to satisfy the **separation property** or is called **separated** if all the points in A are separating. It is known that, if A is separated, then it is simply connected [S, (2.3)]. The dual notions are those of coseparating points and coseparated algebras.

LEMMA 5.3 *Assume that A is an incidence algebra and that x is a source in A such that $\operatorname{rad} P_x$ is indecomposable, and x tops no weak crown. Let B be the full subcategory of A generated by all objects except x. Then $\pi_1(B) \simeq \pi_1(A)$.*

Proof. Since $\operatorname{rad} P(x)$ is indecomposable, x is separating. By [AP, (2.2)] all arrows α in $x^{\rightarrow}$ are equivalent under the relation $\approx$. Let G be the graph associated to this equivalence class, as described before (5.2), whose vertices are in one–to–one correspondence with the arrows $\alpha_i : x \to x_i$ starting at x (where $1 \leq i \leq n$). Since x tops no weak crown, G is a tree (by (5.2)).

We may assume that α_1 corresponds to a simple vertex of G. Given any arrow $\alpha_r : x \to x_r$, there exists a unique sequence of vertices in G associated to the arrows $\alpha_1 = \alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_t} = \alpha_r$ with t minimun such that there is an edge in G connecting α_{i_j}, $\alpha_{i_{j+1}}$ for each j with $1 \leq j \leq r$. Thus, each pair $(\alpha_{i_j}, \alpha_{i_{j+1}})$ is involved in a commutativity relation $\alpha_{i_j} v_{i_j, i_{j+1}} = \alpha_{i_{j+1}} u_{i_j, i_{j+1}}$.

We take x_1 as base point for both $\pi_1(A)$ and $\pi_1(B)$ and define $\overline{\varphi} : W_A \to W_B$ as follows. We set

$$
\begin{array}{lcll}
\overline{\varphi}(e_x) & = & \overline{\varphi}(e_{x_1}) = e_{x_1} & \\
\overline{\varphi}(e_y) & = & e_y & \text{for all } y \neq x, x_1 \\
\overline{\varphi}(\alpha_1) & = & e_{x_1} & \\
\overline{\varphi}(\alpha_r) & = & v_{i_1,i_2} u^{-1}_{i_1,i_2} v_{i_2,i_3} u^{-1}_{i_2,i_3} \cdots v_{i_{t-1},i_t} u^{-1}_{i_{t-1},i_t} & \text{for all } r \text{ such that } 1 < r \leq n \\
\overline{\varphi}(\beta) & = & \beta & \text{for all } \beta \neq \alpha_r.
\end{array}
$$

We extend $\overline{\varphi}$ to walks as usual. Assume w and w' are two parallel paths in A, then clearly, $\overline{\varphi}(w)$ and $\overline{\varphi}(w')$ are parallel in B. Therefore, $\overline{\varphi}$ induces a group morphism $\varphi : \pi_1(A) \to \pi_1(B)$. Also, $\overline{\varphi}$ is surjective, hence so is φ.

On the other hand, the inclusion of B as a full subcategory of A induces a map $\overline{\psi} : W_B \to W_A$ and a group morphism $\psi : \pi_1(B) \to \pi_1(A)$. We claim that $\psi\varphi = 1$, and this will finish the proof. Let indeed w be a closed walk in A through x_1. If w does not factor through any of the arrows starting at x, then it is clear that $\overline{\psi}\overline{\varphi}(w) = w$. Otherwise, there exist r, s such that $1 \leq r, s \leq n$ and $w = w_1\alpha_r^{-1}\alpha_s w_2$ and we can assume that w_1, w_2 do not factor through any of the arrows starting at x. Then $\overline{\psi}\overline{\varphi}(w_1\alpha_r^{-1}\alpha_s w_2) = w_1\overline{\psi}\overline{\varphi}(\alpha_r^{-1}\alpha_s)w_2$. On the other hand, let $\alpha_1 = \alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_p} = \alpha_r$ and $\alpha_1 = \alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_q} = \alpha_s$ be the unique sequences of arrows in $x^{\rightarrow}$ corresponding respectively to α_r and α_s above defined, with the commutativity relations $\alpha_{i_h}v_{i_h,i_{h+1}} = \alpha_{i_{h+1}}u_{i_h,i_{h+1}}$ and $\alpha_{j_l}v_{j_l,j_{l+1}} = \alpha_{j_{l+1}}u_{j_l,j_{l+1}}$ for all h, l. Then we have

$$\overline{\psi}\overline{\varphi}(\alpha_r^{-1}) = (v_{i_1,i_2}u_{i_1,i_2}^{-1}v_{i_2,i_3}u_{i_2,i_3}^{-1}\cdots v_{i_{t-1},i_t}u_{i_{p-1},i_p}^{-1})^{-1} \sim (\alpha_1^{-1}\alpha_{i_2}\alpha_{i_2}^{-1}\alpha_{i_3}\cdots\alpha_{i_{p-1}}^{-1}\alpha_{i_p})^{-1} \sim \alpha_{i_p}^{-1}\alpha_1 = \alpha_r^{-1}\alpha_1,$$

$$\overline{\psi}\overline{\varphi}(\alpha_s) = v_{j_1,j_2}u_{j_1,j_2}^{-1}v_{j_2,j_3}u_{j_2,j_3}^{-1}\cdots v_{j_{l-1},j_l}u_{j_{q-1},j_q}^{-1} \sim \alpha_1^{-1}\alpha_{j_2}\alpha_{j_2}^{-1}\alpha_{j_3}\cdots\alpha_{j_{q-1}}^{-1}\alpha_{j_q} \sim \alpha_1^{-1}\alpha_{j_q} = \alpha_1^{-1}\alpha_s$$

so that $\overline{\psi}\overline{\varphi}(w_1\alpha_r^{-1}\alpha_s w_2) \sim w_1\alpha_r^{-1}\alpha_1\alpha_1^{-1}\alpha_s w_2 \sim w_1\alpha_r^{-1}\alpha_s w_2 = w$. □

LEMMA 5.4 *Let A be an incidence algebra, x be a source in A, B be the full subcategory of A generated by all objects of A except x, and A' be the full subcategory of A generated by x and one of the connected components of B. Then the morphism $\psi : \pi_1(A') \to \pi_1(A)$ induced by the inclusion admits a retraction. In particular, if A is simply connected, then so is A'.*

Proof. We take x as a base point for both A and A' and define $\overline{\varphi} : W_A \to W_{A'}$ as follows. We set

$$\begin{array}{rcll} \overline{\varphi}(e_y) & = & e_y & \text{if } y \in A'_0 \\ \overline{\varphi}(e_z) & = & e_x & \text{if } z \notin A'_0 \\ \overline{\varphi}(\alpha) & = & \alpha & \text{if } \alpha \text{ is an arrow in } A' \\ \overline{\varphi}(\beta) & = & e_x & \text{if } \beta \text{ is an arrow not in } A'. \end{array}$$

We extend $\overline{\varphi}$ to walks as usual. If w and w' are parallel paths in A, then $\overline{\varphi}(w)$ and $\overline{\varphi}(w')$ are parallel in A'. Therefore, $\overline{\varphi}$ induces a group morphism $\varphi : \pi_1(A) \to \pi_1(A')$. Also, $\overline{\varphi}$ is surjective, hence so is φ.

Letting $\overline{\psi} : W_{A'} \to W_A$ be the map induced by the inclusion, it is clear that $\overline{\varphi}\overline{\psi} = 1$, and therefore $\varphi\psi = 1$. The last statement is obvious. □

6 Proof of the main theorem.

We recall a few facts about Hochschild cohomology. Given an algebra A, the Hochschild complex $C^\bullet = (C^i, d^i)_{i\in\mathbf{Z}}$ is defined as follows: $C^i = 0$, $d^i = 0$ for $i < 0$, $C^0 = {}_AA_A$, $C^i = \mathrm{Hom}_k(A^{\otimes i}, A)$ for $i > 0$, where $A^{\otimes i}$ denotes the i-fold tensor product $A \otimes_k \ldots \otimes_k A$, $d^0 : A \to \mathrm{Hom}_k(A, A)$ with $(d^0x)(a) = ax - xa$ (for $a, x \in A$) and $d^i : C^i \to C^{i+1}$ with

$$\begin{array}{rcl} (d^if)(a_1 \otimes \ldots \otimes a_{i+1}) & = & a_1 f(a_2 \otimes \ldots \otimes a_{i+1}) \\ & + & \displaystyle\sum_{j=1}^{i} (-1)^j f(a_1 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{i+1}) \\ & + & (-1)^{i+1} f(a_1 \otimes \ldots \otimes a_i)a_{i+1} \end{array}$$

for $f \in C^i$ and $a_1, \ldots, a_{i+1} \in A$. Then $H^i(A) = H^i(C^\bullet)$ is the **i-th Hochschild cohomology group** of A with coefficients in the bimodule ${}_AA_A$, see [CE].

There is a close relation between the first Hochschild cohomology group $H^1(A)$ and the fundamental group $\pi_1(Q, I)$ of a bound quiver presentation (Q, I) of a triangular algebra $A = kQ/I$. Indeed, denoting by k^+ the additive group of the field k, there exists a group monomorphism

$$\mathrm{Hom}(\pi_1(Q, I), k^+) \to H^1(A)$$

[AP, (3.2)]. Further, it follows from [PS, (3)] that, if A is an incidence algebra then this is an isomorphism. Consequently, if an incidence algebra A is simply connected, then $H^1(A) = 0$. For further results, see [H, GS].

Let now Σ be a poset, and $\hat{\Sigma}$ be obtained from Σ by adding two points a, b such that $a \geq x \geq b$ for all $x \in \Sigma$. We denote by A and $\hat{A}$ the respective incidence algebras of Σ and $\hat{\Sigma}$. It is shown in [IZ, (1.2)], [C, (2.1)] that $H^1(A) \simeq \mathrm{Ext}^3_{\hat{A}}(S_a, S_b)$.

LEMMA 6.1 *With the above notation, we have*

$$H^1(A) \simeq \mathrm{Ext}^1_{\hat{A}}(\mathrm{rad}\, P_a, I_b/S_b).$$

Proof. This follows from the aforementioned result and the short exact sequences

$$0 \to \mathrm{rad}\, P_a \to P_a \to S_a \to 0$$

and

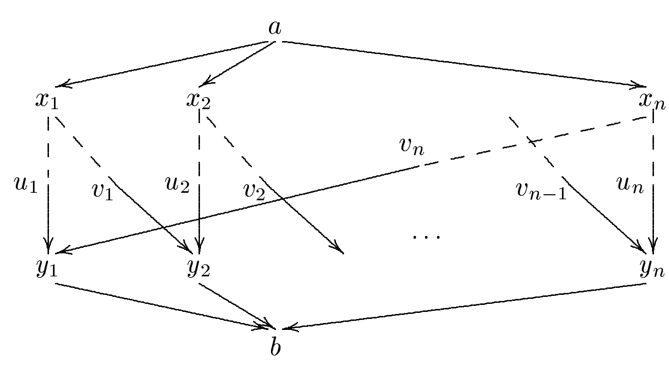
$$0 \to S_b \to I_b \to I_b/S_b \to 0.$$

□

PROPOSITION 6.2 *Let A be an incidence algebra which is the convex hull of a crown Γ. Then $H^1(A) \neq 0$. In particular, A is not simply connected.*

Proof. In view of (6.1), we only need to construct a non–split exact sequence $0 \to N \to E \to M \to 0$, where $N = I_b/S_b$ and $M = \mathrm{rad}\, P_a$.

It follows from (2.4) that $M(x) = k$ for each point $x \neq a$, $M(a) = 0$ and $M(\alpha) = 1$ for each arrow α such that $s(\alpha) \neq a$. Dually, $N(x) = k$ for each point $x \neq b$, $N(b) = 0$ and $N(\alpha) = 1$ for each arrow α such that $t(\alpha) \neq b$.



Consider in A all the paths $u_{n_1}, \cdots, u_{n_r}$ from x_n to y_n and, for each i such that $1 \leq i \leq r$, let γ_{n_i} be the unique arrow of u_{n_i} with origin x_n. Since $x_1, \cdots, x_n, y_1, \cdots, y_n$ generate a crown, no γ_{n_i} occurs in a path different from $u_{n_1}, \cdots, u_{n_r}$. Let $\lambda \in k$, and define an $\hat{A}$–module E_λ by:

$$
\begin{array}{lcll}
E_\lambda(a) & = & k & \\
E_\lambda(b) & = & k & \\
E_\lambda(x) & = & k^2 & \text{for any point } x \notin \{a,b\} \\
E_\lambda(\alpha) & = & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for any arrow } \alpha \text{ of source } a \\
E_\lambda(\beta) & = & (0 \quad 1) & \text{for any arrow } \beta \text{ of target } b \\
E_\lambda(\gamma_{n_i}) & = & \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} & \text{for each } i \text{ with } 1 \leq i \leq r \\
E_\lambda(\delta) & = & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for any other arrow } \delta
\end{array}
$$

A straightforward verification proves that E_λ is indeed a bound representation (thus an $\hat{A}$–module) and that, if $\lambda \neq \mu$ in k, then $E_\lambda \not\cong E_\mu$.

We now define a map $f_\lambda : N \to E_\lambda$ by $f_{\lambda,a} = 1$, $f_{\lambda,b} = 0$ and $f_{\lambda,x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for all $x \notin \{a,b\}$. It is easily shown that f_λ is a monomorphism of cokernel M. This shows that, for each $\lambda \in k$, we have an extension

$$0 \to N \xrightarrow{f_\lambda} E_\lambda \to M \to 0.$$

Since the E_λ are pairwise non–isomorphic, it follows that these extensions, except possibly one of them, do not split. This shows that $\mathrm{Ext}^1_{\hat{A}}(M,N) \neq 0$, thus ending the proof of the proposition. □

PROPOSITION 6.3 *Let A be an incidence algebra which is simply connected but is not strongly simply connected. Then A contains a complete crown.*

Proof. We prove the result by induction on the number $|A_0|$ of points of A. If $|A_0| \leq 4$, then the hypothesis is never satisfied, so the result trivially holds. Assume then that $|A_0| > 4$ and that A is not strongly simply connected, but that any incidence algebra B such that $|B_0| < |A_0|$ verifies the statement of the proposition.

Since A is not strongly simply connected, then A contains a crown Γ. If all the maximal and all the minimal points of A are in Γ, then A is the convex hull of Γ, thus A is not simply connected by (6.2). We may therefore assume, by duality, that there is a maximal point x of A which is not in Γ. Moreover, we assume that A is simply connected and that no crown in A is complete . Let B be the full convex subcategory of A generated by all objects except x. Since Γ is contained in B and is connected, there exists a connected component B_1 of B containing Γ. Then B_1 is not strongly simply connected. On the other hand, we know from [AP, (2.6)] that since A is simply connected, all sources in A_0 are separating. Hence x is separating. Denoting by A_1 the full subcategory of A generated by x and B_1, this means that the restriction of $\mathrm{rad}\, P_x$ to A_1 is indecomposable. Moreover, by (5.4), A_1 is simply connected.

Now, since we are assuming that no crown in A is complete, it follows that no crown in B_1 is complete. This implies that x tops no weak crown in B_1. Indeed, if x tops a weak crown Γ_1, by (3.2), there is a crown Γ_2 in the convex hull of Γ_1. Then x precedes all the maximal points of Γ_2, so Γ_2 is complete, contradicting our assumption. By the inductive hypothesis, it follows that the non–strongly simply connected algebra B_1 is not simply connected. On the other hand, since x tops no weak crown in B_1, we get from (5.3) that $\pi_1(B_1) \simeq \pi_1(A_1)$. Since A_1 is simply connected so is B_1. We have thus reached a contradiction which completes the proof. □

THEOREM 6.4 *Let A be an incidence algebra which is not strongly simply connected.*

(a) If every weak crown in A is complete, then A is simply connected.

(b) If A is simply connected, then there exists a complete crown in A.

(c) A is simply connected if and only if every crown of A is homotopic to a complete crown.

Proof. Part (a) follows from (4.6) and part (b) from (6.3) above. As for part (c), assume that A is simply connected then, clearly, every crown in A is homotopic to a complete crown (which exists by (b)), and the converse follows from (4.4). $\square$

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