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SEPARATING SPLITTING TILTING MODULES AND HEREDITARY ALGEBRAS

BY

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ABSTRACT. Let A be a finite-dimensional algebra over an algebraically closed field. By module is meant a finitely generated right module. A module T_A is called a tilting module if $\mathrm{Ext}_A^2(T, -) = 0 = \mathrm{Ext}_A^1(T, T)$ and there exists an exact sequence $0 \to A_A \to T' \to T'' \to 0$ with T', T'' direct sums of summands of T. Let $B = \mathrm{End}\, T_A \cdot T_A$ is called separating (respectively, splitting) if every indecomposable A-module M (respectively, B-module N) is such that either $\mathrm{Hom}_A(T, M) = 0$ or $\mathrm{Ext}_A^1(T, M) = 0$ (respectively, $N \otimes_B T = 0$ or $\mathrm{Tor}_1^B(N, T) = 0$). We prove that A is hereditary provided the quiver of A has no oriented cycles and every separating tilting module is splitting.

Introduction. Let k be an algebraically closed field, and A a finite-dimensional k-algebra (associative, and with an identity). By a module will always be meant a finitely generated right module. Following Happel and Ringel [7], we shall call a module T_A a tilting module if $\mathrm{Ext}_A^2(T, -) = 0$, $\mathrm{Ext}_A^1(T, T) = 0$ and there exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T' and T'' direct sums of summands of T. A tilting module T_A induces a torsion theory $(\boldsymbol{T}, \boldsymbol{F})$ in the category $\mathrm{mod}\, A$ of A-modules by:

$$\boldsymbol{T} = \boldsymbol{T}(T_A) = \{M_A \mid \mathrm{Ext}_A^1(T, M) = 0\}$$

$$\boldsymbol{F} = \boldsymbol{F}(T_A) = \{M_A \mid \mathrm{Hom}_A(T, M) = 0\}$$

and a torsion theory $(\boldsymbol{X}, \boldsymbol{Y})$ in $\mathrm{mod}\, B$, where $B = \mathrm{End}\, T_A$, by:

$$\boldsymbol{X} = \boldsymbol{X}(T_A) = \left\{N_B \mid N \underset{B}{\otimes} T = 0\right\}$$

$$\boldsymbol{Y} = \boldsymbol{Y}(T_A) = \{N_B \mid \mathrm{Tor}_1^B(N, T) = 0\}$$

The tilting module T_A is called separating [2] if $(\boldsymbol{T}, \boldsymbol{F})$ is a splitting torsion theory. Examples of separating tilting modules are provided by the APR tilts, introduced by Auslander, Platzeck and Reiten in [4]. The tilting module T_A is called splitting [1] if $(\boldsymbol{X}, \boldsymbol{Y})$ is a splitting torsion theory. It is well-known that, if A is a hereditary algebra,

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The author is an Alexander von Humboldt fellow at the University of Bielefeld (Federal Republic of Germany) then every tilting module is splitting [7]. The objective of this article is to show that, conversely, if A has no oriented cycles in its ordinary quiver and is such that every tilting module is splitting, then A is hereditary.

1. Preliminaries. In what follows, we shall assume that the algebra A is basic and connected, and will denote by Q_A its ordinary quiver. Recall that a relation on Q_A is a linear combination of paths in Q_A of length at least two having the same initial and terminal vertices. Thus A is isomorphic to the quotient of the quiver algebra kQ_A by an ideal generated by a set of relations on Q_A which we can assume to be minimal (that is, no proper subset generates the ideal) [6]. For each vertex i of Q_A, we shall denote by e_i the corresponding primitive idempotent of A, and by $S(i)$ the corresponding simple A-module. $P(i)$ (respectively, $I(i)$) will denote the projective cover (respectively, the injective envelope) of $S(i)$. We shall use freely properties of the Auslander–Reiten translations $\tau = D\,\mathrm{Tr}$ and $\tau^{-1} = \mathrm{Tr}\,D$, as in [6], as well as properties of tilting modules, for which we refer to [5] and [7].

We shall assume that Q_A has no oriented cycles. In particular, it contains at least one sink. A sink i will be called free if it is not the terminal point of a generating relation on Q_A, that is to say, if the canonical inclusion $P(i) \to \bigoplus_{j\to i} P(j)$ induces, for every vertex $h \neq i$, a vector space isomorphism

$$\mathrm{Hom}_A(P(i), P(h)) = e_hAe_i \xrightarrow{\sim} \bigoplus_{j\to i} e_hAe_j = \bigoplus_{j\to i} \mathrm{Hom}_A(P(j), P(h)).$$

Observe that this is equivalent to saying that i is free if and only if, for every vertex $h \neq i$,

$$\mathrm{Hom}_A(P(h), I(i)) \xrightarrow{\sim} \bigoplus_{j\to i} \mathrm{Hom}_A(P(h), I(j))$$

that is to say, if and only if $\bigoplus_{j\to i} I(j) \xrightarrow{\sim} I(i)/S(i)$.

To each sink i, we associate the tilting module:

$$T[i]_A = \tau^{-1}(e_iA) \oplus (1 - e_i)A$$

(where 1 denotes the identity of A) called the APR tilt corresponding to i [4]. Every APR tilt is a separating tilting module (in fact, the only torsion-free indecomposable module is $P(i) = e_iA$). It was proved by Hoshino [8] (see also [9]) that $T[i]$ is splitting if and only if the injective dimension of the simple projective e_iA is one. We shall now show:

LEMMA. *The APR tilt $T[i]$ is splitting if and only if i is a free sink. Moreover, in this case, the ordinary quiver Q_B of $B = \mathrm{End}\, T[i]_A$ has no oriented cycles and the vertex of Q_B corresponding to i is a source.*

PROOF. It follows from Hoshino's result that $T[i]$ is splitting if and only if $S(i)$ has a minimal injective resolution:

$$0 \to S(i) \to I(i) \to \bigoplus_{j\to i} I(j) \to 0$$

and it follows from the previous remarks that this is the case if and only if i is a free sink.

Let us now assume that i is a free sink. We shall denote by j' (for $j \neq i$) the vertex of Q_B corresponding to the indecomposable summand $P(j) = e_jA$ of $T[i]$ and by i' the vertex corresponding to $\tau^{-1}(e_iA)$. We claim that i' is a source. To an arrow $h \to i$ of Q_A through i correspond two irreducible maps $P(i)_A \to P(h)_A$ and $P(h)_A \to \tau^{-1}P(i)_A$. The latter induces, by application of the functor $\mathrm{Hom}_A(T[i], -)$, an irreducible map $P(h')_B \to P(i')_B$ in mod B and hence an arrow $i' \to h'$ in Q_B. On the other hand, to an arrow $j' \to i'$ in Q_B would correspond a non-zero homomorphism $f: \tau^{-1}P(i) \to P(j)$ in mod A. Since we have an Auslander–Reiten sequence:

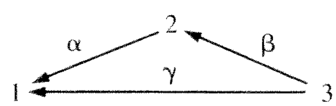
$$0 \to P(i) \xrightarrow{u} \bigoplus_{h \to i} P(h) \xrightarrow{v} \tau^{-1}P(i) \to 0$$

it follows that $fv \neq 0$. But $(fv)u = f(vu) = 0$ and this implies that there exists a (zero-) relation on Q_A of terminal point i, contrary to our hypothesis that the sink i is free. Thus i' is a source in Q_B.

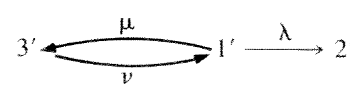
We shall now prove that Q_B has no oriented cycle. Indeed, if $i'_0 \leftarrow i'_1 \leftarrow i'_2 \leftarrow \ldots \leftarrow i'_t = i'_0$ is such a cycle, we must have $i_s \neq i$ for each $0 \leqslant s < t$ (because i' is a source). Therefore we have a chain of non-zero homomorphisms in mod B: $P(i'_0) \to P(i'_1) \to \ldots \to P(i'_t) = P(i'_0)$ where $P(i'_s) \xrightarrow{\sim} \mathrm{Hom}_A(T[i], P(i_s))$ for each $0 \leqslant s < t$. Applying the functor $- \otimes_B T[i]$, we obtain a chain of non-zero homomorphisms in mod A: $P(i_0) \to P(i_1) \to \ldots \to P(i_t) = P(i_0)$, and this is impossible, because Q_A has no oriented cycles.

REMARKS. 1. In fact, it is possible to prove that i' is a source if and only if the sink i is free.

2. If i is not free, Q_B may have oriented cycles. For instance, if A is given by the quiver:



bound by $\alpha\beta = 0$, then End $T[1]_A$ is given by the quiver:



bound by $\mu\nu = 0$.

2. The main result.

THEOREM. *Let A be a finite-dimensional k-algebra without oriented cycles in its ordinary quiver and such that every separating tilting module is splitting. Then A is hereditary.*

PROOF. Let A be such that Q_A has no oriented cycles. We shall assume that A is not hereditary, and construct a separating tilting module which is not splitting. We start by ordering the vertices of Q_A in an admissible sequence: $\{1, 2, \ldots n\}$ (that is to say, such that $e_tAe_s \neq 0$ implies $s \leq t$). If the sink 1 is free, the APR tilt:

$$T[1] = \tau^{-1}(e_1A) \oplus (1 - e_1)A$$

on $A = A_0$ is splitting, its endomorphism algebra $A_1 = \operatorname{End} T[1]_{A_0}$ has no oriented cycles in its ordinary quiver Q_{A_1} and 1 becomes a source in Q_{A_1}. Inductively, if the sink j in $Q_{A_{j-1}}$ is free, then the APR tilt:

$$T[j] = \tau^{-1}(e_jA_{j-1}) \oplus (1 - e_j)A_{j-1}$$

is splitting, the ordinary quiver Q_{A_j} of $A_j = \operatorname{End} T[j]_{A_{j-1}}$ has no oriented cycles, and j becomes a source in Q_{A_j}. Since A is not hereditary, its quiver is bound by at least one relation and, proceeding as above, we arrive at a first vertex i of Q_A which is not a free sink in $Q_{A_{i-1}}$. Putting $e = e_1 + e_2 + \ldots + e_i$, we define:

$$T_A = \tau^{-1}(eA) \oplus (1 - e)A.$$

Observe that eA is a hereditary projective: indeed, i is the first vertex of the sequence $1, 2 \ldots i$ which is not free, and hence each e_jA $(j \leq i)$ is a hereditary projective. This implies that $\operatorname{Hom}_A(DA, eA) = 0$ and so the projective dimension of $\tau^{-1}(eA)$ equals one. On the other hand, if $j \leq i$ and $\ell > i$, we have:

$$\operatorname{Ext}_A^1(\tau^{-1}P(j), P(\ell)) \xrightarrow{\sim} D\operatorname{Hom}_A(P(\ell), P(j)) = 0$$

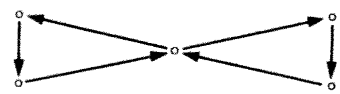
and also, if $h, j \leq i$, we have $\operatorname{Ext}_A^1(\tau^{-1}P(j), \tau^{-1}P(h)) = 0$, which gives $\operatorname{Ext}_A^1(T, T) = 0$. Since the number of non-isomorphic indecomposable summands of T equals n, T is indeed a tilting module. Let us prove that T is separating. $\boldsymbol{F}(T_A)$ is cogenerated by $\tau T \xrightarrow{\sim} eA$ [7] and thus has as only indecomposable modules $P(1), P(2), \ldots P(i)$. On the other hand, the isomorphisms:

$$\operatorname{Ext}_A^1(T, M) \xrightarrow{\sim} D\operatorname{Hom}_A(M, \tau T) \xrightarrow{\sim} D\operatorname{Hom}_A(M, eA)$$

show that, for an indecomposable module M_A, $M \notin \boldsymbol{T}(T_A)$ if and only if $\operatorname{Hom}_A(M, eA) \neq 0$, that is to say, if and only if $M \in \boldsymbol{F}(T_A)$.

There only remains to show that T is not splitting. Let $B = \operatorname{End} T_A$. We claim that $B \xrightarrow{\sim} A_i$. Indeed, the indecomposable summands of T_A are torsion in $(\boldsymbol{T}(T[1], \boldsymbol{F}(T[1]))$. Put $T_{A_1}^{(1)} = \operatorname{Hom}_A(T[1], T)$. Since $\boldsymbol{T}(T[1]) \xrightarrow{\sim} \boldsymbol{Y}(T[1])$, we have $\operatorname{End} T_{A_1}^{(1)} \xrightarrow{\sim} B$. Inductively, if $j < i - 1$, the indecomposable summands of $T_{A_j}^{(j)} = \operatorname{Hom}_{A_{j-1}}(T[j], T^{(j-1)})$ lie in $\boldsymbol{T}(T[j+1]) \xrightarrow{\sim} \boldsymbol{Y}(T[j+1])$ thus, putting $T_{A_{j+1}}^{(j+1)} = \operatorname{Hom}_{A_j}(T[j+1]), T^{(j)})$ we have $\operatorname{End} T_{A_{j+1}}^{(j+1)} \xrightarrow{\sim} B$. Since $T_{A_{i-1}}^{(i-1)} = T[i]$, we have $B \xrightarrow{\sim} \operatorname{End} T[i]_{A_{i-1}} = A_i$. In other words, the effect of the separating tilting module T_A on A is equivalent to the successive effect of the APR tilts $T[1], T[2], \ldots T[i]$. Now, it follows from the lemma that each of the tilting modules $T[1], \ldots T[i-1]$ is splitting, while $T[i]$ is not. Therefore, T_A is not splitting.

EXAMPLE. We now give an example of an algebra which is not self-injective and whose only tilting modules are the Morita progenerators (thus are both separating and splitting). Let A be the algebra with radical square zero given by the quiver:



Then every indecomposable non-projective A-module has infinite projective dimension and so the stated property is satisfied. Observe also that A is stably hereditary and representation-finite, admits oriented cycles in its Auslander–Reiten quiver but no short chains (and therefore its indecomposable modules are uniquely determined by their dimension-vectors [3]).

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