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Right Triangulated Categories with Right Semi-equivalences

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ABSTRACT. We show that a right triangulated category is best behaved when its shift satisfies conditions making it what we call a right semi-equivalence. We consider right triangulated categories constructed using the standard method of [8], and give a necessary and sufficient condition for the shift of such a right triangulated category to be a right semi-equivalence. We study examples where this condition is satisfied, then we apply our results to show that the unfolding of an APR-iterated tilted algebra is the postprojective component of the full subcategory of the homotopy category of bounded complexes of finitely generated projective modules having zero cohomology in the positive indices.

The notion of a triangulated category was introduced by Grothendieck and Verdier in the sixties [**22**]. At about the same time, the idea of a semi-triangulated category was implicit in Heller's construction of the suspension and the loop-space functors [**17**]. The explicit formulation of the axioms of a semi-triangulated category is due to Keller and Vossieck [**19**], it was used in [**18**] to provide a natural setting for an inductive construction of functors. On the other hand, triangulated categories have found many applications in the representation theory of finite dimensional algebras over a field (see, for instance, [**15**]). Our objective in this paper is to study those right triangulated categories which are as close as possible to triangulated ones.

We start by recalling the definition of a right triangulated category, and derive some elementary properties. It will be apparent from these properties that a right triangulated category is best behaved when the shift defining it is a right semi-equivalence (see (1.7)). This notion is further justified by our study of the Auslander-Reiten theory of a right triangulated category having a right semi-equivalence : this assumption allows us to prove in (2.4) the analog of Auslander-Reiten's characterisation of almost split sequences [**4**], thus slightly improving [**15**] (I.4). Since most known examples of right triangulated categories occur as subcategories of triangulated ones, we prove in (2.8) the analog of Auslander-Smalø's characterisation of almost split sequences in subcategories [**6**] (2.4).

We then turn to the consideration of examples. A standard construction of right triangulated categories, comprising most known examples, was given in [**8**]. Our main theorem (3.3) gives a necessary and sufficient condition for the shift defining

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this standard construction to be a right semi-equivalence. For instance, if A is an artin algebra, then the stable category $\overline{\text{mod}}A$ of finitely generated right modules modulo injectives is right triangulated, but its shift is a right semi-equivalence if and only if A is self-injective (in which case $\overline{\text{mod}}A$ is triangulated). There exist however non-self-injective locally bounded categories [**10**] over an algebraically closed field A such that the shift in $\overline{\text{mod}}A$ is a right semi-equivalence (3.4).

We end the paper with an application to unfoldings. This notion was introduced by S. Brenner in [**11**], where it is shown that a triangular finite dimensional algebra A over an algebraically closed field has an unfolding if and only if it is APR-iterated tilted and, if this is the case, then it allows one to determine the type of A. A slightly different interpretation of unfoldings using the homotopy category $\mathrm{K}^b(\mathrm{proj}\, A)$ of bounded complexes of finitely generated projective right A-modules was given in [**2**]. Since the inductive construction of unfoldings closely resembles that of postprojective components, W. Crawley-Boevey has conjectured (see [**11**]) that the unfolding is the postprojective component of the full subcategory of $\mathrm{K}^b(\mathrm{proj}\, A)$ consisting of the complexes having vanishing cohomology in the positive indices. This was shown to be true if A is iterated tilted of Dynkin type [**2**] (4.1). We prove here this conjecture in general (4.4).

For the sake of brevity, we refrain from stating the dual statements for left triangulated categories with left semi-equivalences.

1. Definition and first properties

1.1. Throughout this paper, we let k denote a commutative ring. We call a category **k-linear** if it is additive, and the morphism groups have a k-module structure with bilinear composition ; we call it **k-abelian** if it is moreover abelian. A functor between k-linear categories is called **k-linear** if it preserves the linear operations of the morphisms. For a category $\mathcal{C}$, we denote by $\mathcal{C}_0$ its class of objects.

DEFINITION . A **right triangulated** (or **suspended**) category is a triple $(\mathcal{C}, T, \Delta)$, where :

a) $\mathcal{C}$ is a k-linear category.
b) $T : \mathcal{C} \to \mathcal{C}$ is a k-linear functor, called the **shift** (of **suspension**) of $\mathcal{C}$.
c) Δ is a class of sequences of three morphisms of the form $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$, called **triangles**, and satisfying the following :

(RT 1) If $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ is a triangle, and $U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} TU'$ is a sequence of morphisms such that there exists a commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccc} U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & TU \\ f\downarrow\cong & & g\downarrow\cong & & h\downarrow\cong & & Tf\downarrow\cong \\ U' & \xrightarrow{u'} & V' & \xrightarrow{v'} & W' & \xrightarrow{w'} & TU' \end{array}$$

where f, g, h are isomorphisms, then the lower row is a triangle.

(RT 2) For every object U in $\mathcal{C}$, the sequence $0 \to U \xrightarrow{1} U \to 0$ is a triangle.

(RT 3) For every morphism $u : U \to V$ in $\mathcal{C}$, there exists a triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$.

(RT 4) If $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ is a triangle, then so is $V \xrightarrow{v} W \xrightarrow{w} TU \xrightarrow{-Tu} TV$ (called its **shift**).

(RT 5) For any two triangles $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ and $U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} TU'$ and any two morphisms $f : U \to U', g : V \to V'$ such that $gu = u'f$, there exists $h : W \to W'$ making the following diagram commutative

$$\begin{array}{ccccccc} U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & TU \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\downarrow \\ U' & \xrightarrow{u'} & V' & \xrightarrow{v'} & W' & \xrightarrow{w'} & TU' \end{array}$$

(RT 6) For any two triangles $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ and $U' \xrightarrow{u'} U \xrightarrow{v'} W' \xrightarrow{w'} TU'$, there exists a commutative diagram

$$\begin{array}{ccccccc} U' & \xrightarrow{u'} & U & \xrightarrow{v'} & W' & \xrightarrow{w'} & TU' \\ {\scriptstyle 1}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow \\ U' & \xrightarrow{uu'} & V & \xrightarrow{p} & V' & \xrightarrow{q} & TU' \\ {\scriptstyle u'}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle Tu'}\downarrow \\ U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & TU \\ & & & & {\scriptstyle Tv'\cdot w}\downarrow & & \\ & & & & TW' & & \end{array}$$

where the middle row and the column before the last are triangles.

Thus, if $(\mathcal{C}, T, \Delta)$ is a right triangulated category, with $T : \mathcal{C} \to \mathcal{C}$ an equivalence, then $(\mathcal{C}, T, \Delta)$ is a triangulated category [**22**].

1.2. We now outline a construction of [**8**], comprising most known examples of right triangulated categories. Let $\mathcal{A}$ be a k-linear category, a **k-linear subcategory** of $\mathcal{A}$ is a full subcategory closed under isomorphic images, finite direct sums and direct summands. If $\mathcal{X}$ is a k-linear subcategory of $\mathcal{A}$, a complex in $\mathcal{A}$

$$\cdots \longrightarrow U_{i+1} \xrightarrow{u_{i+1}} U_i \xrightarrow{u_i} U_{i-1} \longrightarrow \cdots$$

is called **$\mathcal{X}$-exact** if, for each $X \in \mathcal{X}_0$, the induced complex

$$\cdots \to \mathrm{Hom}_{\mathcal{A}}(U_{i-1}, X) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(u_i, X)} \mathrm{Hom}_{\mathcal{A}}(U_i, X) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(u_{i+1}, X)} \mathrm{Hom}_{\mathcal{A}}(U_{i+1}, X) \to \cdots$$

is exact for all i. A morphism $u : U \to V$ in $\mathcal{A}$ is an **$\mathcal{X}$-monic** if the complex $0 \to U \xrightarrow{u} V$ is $\mathcal{X}$-exact. Assume that $\mathcal{X}$ is covariantly finite in $\mathcal{A}$ (see [**5**], [**6**]) and that each $\mathcal{X}$-monic has a cokernel in $\mathcal{A}$, then, for each $U \in \mathcal{A}_0$, there exists a left $\mathcal{X}$-approximation $f_U : U \to \mathcal{X}(U)$, hence a sequence

$$U \xrightarrow{f_U} \mathcal{X}(U) \xrightarrow{g_U} T(U)$$

where g_U is a cokernel of f_U. By fixing such a sequence for each $U \in \mathcal{A}_0$, we make an $\mathcal{X}$-assignment for $\mathcal{A}$. Let $\mathcal{A}/\mathcal{X}$ be the quotient (stable) category, and denote by $\underline{U}$, $\underline{u}$, respectively, the image of an object U and a morphism u under the projection

$\mathcal{A} \to \mathcal{A}/\mathcal{X}$. For an $\mathcal{X}$-monic $u : U \to V$, let $v : V \to W$ be a cokernel of u, and $w : W \to T(U)$ be defined by the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{u} & V & \xrightarrow{v} & W \\ {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle w} \\ U & \xrightarrow{f_U} & \mathcal{X}(U) & \xrightarrow{g_U} & T(U) \end{array}$$

then the class $\Delta_{\mathcal{X}}$ of sequences of morphisms of $\mathcal{A}/\mathcal{X}$

$$\underline{U} \xrightarrow{\quad \underline{u} \quad} \underline{V} \xrightarrow{\quad \underline{v} \quad} \underline{W} \xrightarrow{\quad \underline{w} \quad} T(\underline{U})$$

defines a right triangulation on $\mathcal{A}/\mathcal{X}$, with shift T, see [**8**]. One shows that it depends only on $\mathcal{X}$, in the sense that any two $\mathcal{X}$-assignments yield equivalent right triangulated structures. In the sequel, this construction is called the **standard** right triangulation of $\mathcal{A}/\mathcal{X}$.

For instance, let A be an artin algebra with (artinian) centre k, or a locally bounded k-category (where k is an algebraically closed field), let $\mathcal{A} = \operatorname{mod} A$, and $\mathcal{X}$ consist of the injective A-modules, then the above construction yields a right triangulated structure on the stable category $\overline{\operatorname{mod} A}$.

1.3. Let $\mathcal{C}$ be a k-linear category. A **pseudokernel** of a morphism $v : V \to W$ in $\mathcal{C}$ is a morphism $u : U \to V$ such that $vu = 0$ and, if $u' : U' \to V$ is such that $vu' = 0$, there exists $f : U' \to U$ (not necessarily unique) such that $u' = uf$. **Pseudocokernels** are defined dually. From now on, let $(\mathcal{C}, T, \Delta)$ be a right triangulated category.

LEMMA . *Let $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ be a triangle, then :*

(a) *v is a pseudocokernel of u, and w is a pseudocokernel of v.*
(b) *If T is full and faithful, then u is a pseudokernel of v and v is a pseudokernel of w.*

PROOF. (a) That $vu = 0$ follows from the commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccc} U & \xrightarrow{1} & U & \longrightarrow & 0 & \longrightarrow & TU \\ {\scriptstyle 1}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} \\ U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & TU \end{array}$$

(where the first triangle exists by (RT 4) (RT 2)). Let $v' : V \to W'$ be such that $v'u = 0$. The existence of $f : W \to W'$ such that $v' = fv$ follows from the commutative diagram

$$\begin{array}{ccccccc} U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & TU \\ \downarrow & & {\scriptstyle v'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & W' & \xrightarrow{1} & W' & \longrightarrow & 0. \end{array}$$

This shows the first statement, the second follows upon applying it to the shifted triangle $V \xrightarrow{v} W \xrightarrow{w} TU \xrightarrow{-Tu} TV$.

(b) We already know that $wv = 0$, $vu = 0$. Let $u' : U' \to V$ be such that $vu' = 0$, and consider the commutative diagram

$$\begin{array}{ccccccc} U' & \longrightarrow & 0 & \longrightarrow & TU' & \xrightarrow{1} & TU' \\ {\scriptstyle u'}\downarrow & & \downarrow & & {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle Tu'} \\ V & \xrightarrow{v} & W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV. \end{array}$$

There exists $\tilde{f} : TU' \to TU$ such that $Tu' = (-Tu)\tilde{f}$. Since T is full and faithful, there exists a unique $f : U' \to U$ such that $Tf = -\tilde{f}$. Hence $u' = uf$. This shows the first statement, the second follows upon shifting. □

1.4. Let $\operatorname{Mod} k$ denote the category of k-modules. A k-linear functor $F : \mathcal{C}^{op} \to \operatorname{Mod} k$ is called **cohomological** if, for any triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$, the induced sequence of k-modules

$$\cdots \xrightarrow{-FTv} FTV \xrightarrow{-FTu} FTU \xrightarrow{Fw} FW \xrightarrow{Fv} FV \xrightarrow{Fu} FU$$

is exact. Dually, a k-linear functor $F : \mathcal{C} \to \operatorname{Mod} k$ is called **homological** if, for any triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$, the induced sequence of k-modules

$$FU \xrightarrow{Fu} FV \xrightarrow{Fv} FW \xrightarrow{Fw} FTU \xrightarrow{-FTu} FTV \xrightarrow{-FTv} \cdots$$

is exact. The following corollary is immediate.

COROLLARY . (a) *For any object X in $\mathcal{C}$, the functor* $\operatorname{Hom}_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \to \operatorname{Mod} k$ *is cohomological.*

(b) *If T is full and faithful, and X is any object in $\mathcal{C}$, the functor* $\operatorname{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \operatorname{Mod} k$ *is homological.* □

1.5. We have the following corollary.

COROLLARY . *Let $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ and $U' \xrightarrow{u'} V' \xrightarrow{v'} W' \xrightarrow{w'} TU'$ be two triangles, and $f : U \to U'$, $g : V \to V'$ be two isomorphisms such that $gu = u'f$. There exists an isomorphism $h : W \to W'$ such that the following diagram is commutative*

$$\begin{array}{ccccccc} U & \xrightarrow{u} & V & \xrightarrow{v} & W' & \xrightarrow{w} & TU \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle Tf} \\ U' & \xrightarrow{u'} & V' & \xrightarrow{v'} & W' & \xrightarrow{w'} & TU'. \end{array}$$

PROOF. By (RT5), there exists a morphism $h : W \to W'$ making the diagram commutative. Applying the cohomological functor $\operatorname{Hom}_{\mathcal{C}}(-, X)$, then the Five Lemma, yields that $\operatorname{Hom}_{\mathcal{C}}(h, X) : \operatorname{Hom}_{\mathcal{C}}(W', X) \to \operatorname{Hom}_{\mathcal{C}}(W, X)$ is an isomorphism for each X. Thus $\operatorname{Hom}_{\mathcal{C}}(h, -) : \operatorname{Hom}_{\mathcal{C}}(W', -) \to (W, -)$ is a functorial isomorphism. By Yoneda's lemma, h is an isomorphism. □

1.6. We also deduce the following corollary.

COROLLARY . *Let T be full and faithful, and $TU \xrightarrow{\tilde{u}} TV \xrightarrow{\tilde{v}} TW \xrightarrow{\tilde{w}} T^2U$ be a triangle. There exists a triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ such that $\tilde{u} = -Tu$, $\tilde{v} = -Tv$ and $\tilde{w} = -Tw$.*

PROOF. There exists a unique $u : U \to V$ such that $\tilde{u} = -Tu$, thus a triangle $U \xrightarrow{u} V \xrightarrow{v'} W' \xrightarrow{w'} TU'$ and hence a commutative diagram

$$\begin{array}{ccccccc} TU & \xrightarrow{\tilde{u}=-Tu} & TV & \xrightarrow{-Tv'} & TW' & \xrightarrow{-Tw'} & T^2U \\ \downarrow 1 & & \downarrow 1 & & & & \downarrow 1 \\ TU & \xrightarrow{\tilde{u}} & TV & \xrightarrow{\tilde{v}} & TW & \xrightarrow{\tilde{w}} & T^2U. \end{array}$$

By (1.5), there exists an isomorphism $TW' \to TW$ making the diagram commute. Since T is full and faithful, $W' \cong W$. □

1.7. This leads to the following definition.

DEFINITION . Let $(\mathcal{C}, T, \Delta)$ be a right triangulated category. The shift T is called **right dense** (or, more precisely, **right Δ-dense**) if, for any triangle of the form $U \xrightarrow{u} TV \xrightarrow{v} W' \xrightarrow{w} TU$, there exists $W \in \mathcal{C}_0$ such that $W' \cong TW$. The shift T is called a **right semi-equivalence** (or, more precisely, **right Δ-semi-equivalence**) if it is full, faithful and right ($\Delta-$) dense.

COROLLARY . *Let T be a right semi-equivalence. For any morphism $u : U \to TV$, there exists a triangle $V \xrightarrow{v} W \xrightarrow{w} U \xrightarrow{u} TV$.*

PROOF. There exists a triangle $U \xrightarrow{u} TV \xrightarrow{v'} W' \xrightarrow{w'} TU$. Since T is right dense, there exists $W \in \mathcal{C}_0$ such that $W' \cong TW$. Shifting yields a triangle $TV \xrightarrow{v'} TW \xrightarrow{w'} TU \xrightarrow{-Tu} T^2V$. Applying (1.6) completes the proof. □

2. Auslander-Reiten Theory in a right triangulated category

2.1. We start with a characterisation of those triangles in a right triangulated category that correspond to split exact sequences.

LEMMA . *Let T be full and faithful, and $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ be a triangle. then :*

(a) *u is a section if and only if v is a retraction, if and only if $w = 0$.*
(b) *u is a retraction if and only if $v = 0$, if and only if w is a section.*
(c) *$u = 0$ if and only if v is a section, if and only if w is a retraction.*

PROOF. (a) If u is a section, so is Tu. Hence $Tu \cdot w = 0$ yields $w = 0$. Conversely, since Tu is a pseudocokernel of $w = 0$, there exists $\tilde{u}' : TV \to TU$ such that $\tilde{u}' \cdot Tu = 1$. Since T is full and faithful, there exists a unique $u' : V \to U$ such that $\tilde{u}' = Tu'$. Hence $u'u = 1$ and u is a section. The proof that v is a retraction if and only if $w = 0$ is similar.
(b) (c) follow from (a) upon shifting. □

2.2. We can define projective and injective objects in a right triangulated category as follows.

DEFINITION . An object $U \in \mathcal{C}_0$ is called **projective** (or **injective**) in $(\mathcal{C}, T, \Delta)$ if $\mathrm{Hom}_{\mathcal{C}}(U, T-) = 0$ (or $\mathrm{Hom}_{\mathcal{C}}(-, TU) = 0$, respectively).

Note that these notions are relative to the particular right triangulation Δ on $(\mathcal{C}, T)$. Also, U is injective if and only if $TU = 0$ so that, if T is faithful, U is injective if and only if $U = 0$.

LEMMA . *Let $P \in \mathcal{C}_0$ and consider the three conditions :*

(a) *P is projective.*

(b) *For every triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ and every morphism $f : P \to W$ there exists $g : P \to V$ such that $f = vg$.*

(c) *Every triangle $V \xrightarrow{v} W \xrightarrow{w} P \xrightarrow{u} TV$ satisfies $u = 0$.*

If T is full and faithful, (a) and (b) are equivalent and imply (c).
If, moreover, T is right dense, (c) also implies (a).

PROOF. (a) implies (b). Apply $\mathrm{Hom}_{\mathcal{C}}(P, -)$ to the given triangle, then use (1.4) and the projectivity of P.

(b) implies (a). Let $U \in \mathcal{C}_0$ and $f : P \to TU$ be a morphism. Applying (b) to the triangle $U \to 0 \to TU \to TU$ yields immediately $f = 0$.

(b) implies (c). Applying (b) to $1 : P \to P$ yields that w is a retraction. Hence $u = 0$, by (2.1).

(c) implies (a). Let $U \in \mathcal{C}_0$, and $f : P \to TU$ be a morphism. By (1.7). there exists a triangle $U \to V \to P \xrightarrow{f} TU$. The hypothesis gives $f = 0$. □

2.3. From now on, we assume that $(\mathcal{C}, T, \Delta)$ is such that $\mathcal{C}$ is a Krull-Schmidt category, and that T is a right semi-equivalence.

LEMMA . (see [4] (2.7)) *Let $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ be a triangle with no morphism equal to zero. Then u is irreducible if and only if, for every $f : X \to W$, there exists $g : X \to V$ such that $f = vg$ or $g : V \to X$ such that $v = fg$.*

PROOF. Necessity. Consider the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{wf} & TU & \xrightarrow{p'} & Y' & \xrightarrow{q'} & TX \\ {\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle h'}\downarrow & & {\scriptstyle Tf}\downarrow \\ W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV & \xrightarrow{-Tv} & TW. \end{array}$$

Since T is right dense, there exists $Y \in \mathcal{C}_0$ such that $Y' \cong TY$. Since T is full and faithful, the irreducibility of u implies that of Tu. Hence p' is a section or h' is a retraction. In the first case, $p'wf = 0$ gives $wf = 0$. By (1.3), there exists $g : X \to V$ such that $f = vg$. In the second case, let $h : Y \to V$ be such that $h' = Th$. Since h' is a retraction, so is h, that is, there exists $\ell : V \to Y$ such that $h\ell = 1$. Let $q : Y \to X$ be such that $q' = Tq$. Then $-Tv.Th = Tf.Tq$ yields $-vh = fq$ so that $v = f(-q\ell)$.

Sufficiency. By (2.1), u is neither a section nor a retraction. Assume $u = h\ell$ and consider the commutative diagram

$$\begin{array}{ccccccc} U & \xrightarrow{\ell} & X & \xrightarrow{p} & Y & \xrightarrow{q} & TU \\ {\scriptstyle 1}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow \\ U & \xrightarrow{u} & V & \xrightarrow{v} & W & \xrightarrow{w} & TU. \end{array}$$

If there exists $g : Y \to V$ such that $f = vg$, then $q = wf = wvg = 0$ and, by (2.1), ℓ is a section. If there exists $g : V \to Y$ such that $v = fg$, then $qg = wfg = 0$ gives $h' : V \to X$ such that $g = ph'$. Now $v = fg = fph' = vhh'$ gives $v(1 - hh') = 0$, so there exists $u' : V \to U$ such that $1 - hh' = uu'$, that is, $1 = hh' + uu' = h(h' + \ell u')$ and h is a retraction. □

2.4. We can now prove the analog of Auslander-Reiten's characterisation of almost split sequences [**4**] (2.14), (2.15).

THEOREM (Auslander-Reiten). *The following conditions are equivalent for a triangle* $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$:

(a) u *is left minimal almost split.*
(b) v *is right minimal almost split.*
(c) u *is left almost split and* W *is indecomposable.*
(d) v *is right almost split and* U *is indecomposable.*
(e) u, v *are irreducible and* U, W *are indecomposable.*

PROOF. (a) implies (b). Since u is not a section, v is not a retraction. Let $f : X \to W$ be a non-retraction. If $wf = 0$, we apply (1.5). If $wf \neq 0$, let $g : V \to X$ be such that $v = fg$ and consider the commutative diagram

$$\begin{array}{ccccccc}
V & \xrightarrow{g} & X & \xrightarrow{h} & Y & \xrightarrow{\ell} & TV \\
{\scriptstyle 1}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle r}\downarrow & & {\scriptstyle 1}\downarrow \\
V & \xrightarrow{v} & W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV \\
{\scriptstyle g}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle s}\downarrow & & {\scriptstyle Tg}\downarrow \\
X & \xrightarrow{f} & W & \xrightarrow{p} & Z & \xrightarrow{q} & TX \\
 & & & & {\scriptstyle Th.q}\downarrow & & \\
 & & & & TY & &
\end{array}$$

obtained from (RT 6), using that the middle row is uniquely determined by (1.6). Then s is not a section (for, otherwise, $r = 0$ yields $wf = 0$, a contradiction) and there exists by hypothesis $s' : TV \to Z$ such that $s = -s'.Tu$. Hence $p = sw = -s'.Tu.w = 0$ so that f is a retraction, a contradiction. Thus v' is right almost split.

To show that v is right minimal, let $f : V \to V$ be such that $vf = v$. Then the commutative diagram

$$\begin{array}{ccccccc}
V & \xrightarrow{v} & W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV \\
{\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle Tf}\downarrow \\
V & \xrightarrow{v} & W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV
\end{array}$$

yields $g : TU \to TU$ such that $gw = w$. Since u is left minimal almost split, U is indecomposable. Consequently, $\mathrm{End}_{\mathcal{C}}\, TU$ is local and g or $(1-g)$ is an isomorphism. If $(1-g)$ is an isomorphism, then $(1-g)w = 0$ yields $w = 0$ so that v is a retraction, a contradiction. Hence g is an isomorphism. By (1.5), Tf is an isomorphism. Hence so is f.

(b) implies (a). The proof is similar.

(b) implies (d). Since (b) implies (a), W is indecomposable.

(d) implies (b). We need to show that v is right minimal. Let $v : V \to V$ be such that $vf = v$, then the commutative diagram

$$\begin{array}{ccccccc} V & \xrightarrow{v} & W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV \\ {\scriptstyle f}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle Tf}\downarrow \\ V & \xrightarrow{v} & W & \xrightarrow{w} & TU & \xrightarrow{-Tu} & TV \end{array}$$

yields $g : TU \to TU$ with $gw = w$. As above, g is an isomorphism. Hence so is f.

(a) is equivalent to (c). The proof is similar.

(a) implies (e). This is trivial.

(e) implies (a). By hypothesis, U is indecomposable and v is not a retraction. Assume that $f : X \to W$ is not a retraction. We may suppose that X is indecomposable (replacing it, if necessary, by one of its indecomposable summands). Since u is irreducible, (2.3) gives $g : X \to V$ such that $f = vg$ (and then we are done) or $g : V \to X$ such that $v = gf$. In this second case, g is a section, because v is irreducible, and f is not a retraction. Since X is indecomposable, g is an isomorphism. But then $f = vg^{-1}$.

□

DEFINITION . A triangle $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} TU$ is called **almost split** if it satisfies the equivalent conditions of the above theorem.

One proves exactly as for almost split sequences that an almost split triangle is uniquely determined (up to isomorphism) by its first, or its third, term.

2.5. We have the following lemma.

LEMMA . *Let $\mathcal{K}$ be a Krull-Schmidt category such that any morphism has a pseudokernel. A morphism $v : V \to W$ is right minimal if and only if, for every pseudokernel $u : U \to V$ of v, we have $u \in \operatorname{rad}(U, V)$.*

PROOF. Necessity. Let $f : V \to U$ be arbitrary, then $v(1 - uf) = v$ implies that $1 - uf$ is an isomorphism.

Sufficiency. If v satisfies $vf = v$, then $v(1 - f) = 0$ implies the existence of u' such that $1 - f = uu'$. Then $f = 1 - uu'$ is an isomorphism, because $u \in \operatorname{rad}(U, V)$. □

2.6. We say that a Krull-Schmidt category $\mathcal{K}$ **has right** (or **right minimal**) **almost split morphisms** if each indecomposable object in $\mathcal{K}$ is the target of a right (or right minimal, respectively) almost split morphism. We define dually what it means to have left (or left minimal) almost split morphisms.

LEMMA . *Let $\mathcal{K}$ be a Krull-Schmidt category having right almost split morphisms, and such that any morphism has a pseudokernel. Then $\mathcal{K}$ has right minimal almost split morphisms.*

PROOF. Let $W \in \mathcal{K}_0$ be indecomposable, and $v : V \to W$ be a right almost split morphism with the property that the number n of indecomposable summands of V is minimal. We claim that v is right minimal almost split. Indeed, if $n = 1$, then V is indecomposable, so that, if $f : V \to V$ is such that $vf = v$, then f or $1 - f$ is invertible (because $\operatorname{End}_{\mathcal{K}} V$ is local) but, if $1 - f$ were invertible, we would get a contradiction to $v(1-f)=0$, hence f is invertible and v is right minimal. Assume $n>1$, and that v is not right minimal. By (2.5), there exists a pseudokernel $u: U \to V$ of v such that $u \notin \mathrm{rad}(U,V)$. Writing $U = \bigoplus_{i=1}^{m} U_i$, $V = \bigoplus_{j=1}^{n} V_j$ with the U_i, V_j indecomposable, $v = [v_1, \dots v_n] : V \to W$ and $u = [u_{ji} : U_i \to V_j]_{1\le i\le m, 1\le j\le n}$, this means that (rearranging the indices if necessary) $u_{11} \notin \mathrm{rad}(U_1, V_1)$, that is, u_{11} is an isomorphism. Let $V' = \bigoplus_{j>1} V_j$. We claim that $v' = [v_2 \dots v_n] : V' \to W$ is right almost split, and this will complete the proof, since it contradicts the minimality of n. Observe first that $vu = 0$ implies $\sum_{j=1}^{n} v_j u_{j1} = 0$, so that $v_1 u_{11} = -\sum_{j>1} v_j u_{j1}$ and $v_1 = -\sum_{j>1} v_j u_{j1} u_{11}^{-1}$.

(a) Assume that v' is a retraction. Then there exists $v'' = [v_2'' \dots v_n'']^t : W \to V'$ such that $v'v'' = 1$. Let $v^\star = [0, v_2'' \dots v_n'']^t : W \to V$, then, clearly, $vv^\star = v'v'' = 1$, and this contradicts the fact that v is not a retraction.

(b) Assume that $f: X \to W$ is not a retraction. There exists $g = [g_1, \dots g_n]^t : X \to V$ such that $f = vg$. But then

$$f = \sum_{j=1}^{n} v_j g_j = \sum_{j>1} v_j g_j + (-\sum_{j>1} v_j u_{j1} u_{11}^{-1}) g_1 = \sum_{j>1} v_j [g_j - u_{j1} u_{11}^{-1} g_1]$$

that is, f factors through v'. We are done.

□

2.7. We have the following easy lemma.

LEMMA . *Let $\mathcal{K}$ be a Krull-Schmidt category, and $\mathcal{L}$ be a contravariantly finite k-linear subcategory of $\mathcal{K}$, then :*

(a) *If $\mathcal{K}$ has right almost split morphisms, so does $\mathcal{L}$.*

(b) *If any morphism in $\mathcal{K}$ has a pseudokernel, any morphism in $\mathcal{L}$ has a pseudokernel (in $\mathcal{L}$).*

PROOF. (a) Let $W \in \mathcal{L}_0$ be indecomposable. Since $\mathcal{L}$ is closed under direct summands, W is indecomposable in $\mathcal{K}$, hence there exists a right almost split morphism $v: V \to W$ in $\mathcal{K}$. Let $f_V : \mathcal{L}(V) \to V$ be a right $\mathcal{L}$-approximation. Then $v' = v f_V : \mathcal{L}(V) \to W$ is right almost split in $\mathcal{L}$.

(b) Let $v: V \to W$ be a morphism in $\mathcal{L}$, and $u: U \to V$ be a pseudokernel of v in $\mathcal{K}$. Let $f_U : \mathcal{L}(U) \to U$ be a right $\mathcal{L}$-approximation. Then $u' = u f_U : \mathcal{L}(U) \to V$ is a pseudokernel of v in $\mathcal{L}$.

□

2.8. We deduce a condition for the existence of relative (minimal) almost split morphisms.

THEOREM . *Let $\mathcal{C}$ be a right triangulated Krull-Schmidt category having left almost split morphisms, and $\mathcal{D}$ be a k-linear subcategory.*

(a) *$\mathcal{C}$ has left minimal almost split morphisms. If $\mathcal{D}$ is covariantly finite, then also $\mathcal{D}$ has left minimal almost split morphisms.*

(b) *Suppose that T is a right semi-equivalence.*

(i) *If $\mathcal{C}$ has right almost split morphisms, then $\mathcal{C}$ has right minimal almost split morphisms. If $\mathcal{D}$ is contravariantly finite, then $\mathcal{D}$ has right minimal almost split morphisms.*

(ii) *If $\mathcal{C}$ has right almost split morphisms and $\mathcal{D}$ is functorially finite, then both $\mathcal{C}$ and $\mathcal{D}$ have left and right minimal almost split morphisms.*

PROOF. Any right triangulated category has pseudocokernels by (1.3), hence the first statement of (a) follows from the dual of (2.6), while the second follows from the duals of (2.6) and (2.7). If T is a right semi-equivalence, then, by (1.7) and (1.3), any morphism has a pseudokernel, hence the first statement of (b) (i) follows from (2.6) and the second from (2.6) and (2.7). Finally, (b) (ii) follows from (a) and (b)(i). □

3. Right semi-equivalences in stable categories

3.1. Our objective is now to find a necessary and sufficient condition for the standard shift of a stable category (1.2) to be a right semi-equivalence. Since this condition is expressed in terms of relative homological algebra as developed, for instance, in [**7**], [**12**], [**9**] we start by recalling those concepts that will be needed. Throughout, we let $\mathcal{C}$ be a k-abelian category, and $\mathcal{A}, \mathcal{X}$ be two k-linear subcategories of $\mathcal{C}$. Then $\mathcal{A}$ is said to be **$\mathcal{X}$-coresolving** [**7**] if :

(a) $\mathcal{A}$ contains $\mathcal{X}$.
(b) If $0 \to U \to V \to W \to 0$ is $\mathcal{X}$-exact, with $U, V \in \mathcal{A}_0$, then $W \in \mathcal{A}_0$.
(c) If $0 \to U \to V \to W \to 0$ is $\mathcal{X}$-exact, with $U, W \in \mathcal{A}_0$, then $V \in \mathcal{A}_0$.

Assume that $\mathcal{A}$ is $\mathcal{X}$-coresolving and that each $\mathcal{X}$-monic in $\mathcal{A}$ is a monomorphism, then the stable category $\mathcal{A}/\mathcal{X}$ satisfies the conditions of (1.2), hence has a standard right triangulated structure. We also remark that $\mathcal{A}$ is an exact category in the sense of Quillen [**20**], having the $\mathcal{X}$-monics as admissible monomorphisms.

Assume that $\mathcal{X}$ is moreover a covariantly finite subcategory of $\mathcal{A}$. For any $U \in \mathcal{A}_0$, we can define $\mathcal{X}$-resolutions of objects in $\mathcal{A}$ and hence the relative extension functors $\mathrm{Ext}^i_{\mathcal{X}}(U, -) : \mathcal{A} \to \mathrm{Mod}\, k$ for all $i > 0$ as follows. Given $V \in \mathcal{A}_0$, there exists an $\mathcal{X}$-exact and exact sequence

$$0 \to V \to X^0 \to X^1 \to \cdots \to X^j \to \cdots$$

(with $X^j \in \mathcal{X}_0$ for all $j \geq 0$) where we use the facts that each $\mathcal{X}$-monic is a monomorphism, that the cokernel of a morphism in $\mathcal{A}(\supseteq \mathcal{X})$ lies in $\mathcal{A}$, and that $\mathcal{X}$ is covariantly finite in $\mathcal{A}$. We then let $\mathrm{Ext}^i_{\mathcal{X}}(U, V)$ be the i^{th} cohomology module of the complex obtained from this sequence by deleting V, then applying $\mathrm{Hom}_{\mathcal{C}}(U, -)$. It is easily seen that the functors $\mathrm{Ext}^i_{\mathcal{X}}(U, -)$ are well-defined and, if $0 \to V' \to V \to V'' \to 0$ is an $\mathcal{X}$-exact sequence in $\mathcal{A}$, then we have a long exact cohomology sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(U, V') \to \mathrm{Hom}_{\mathcal{C}}(U, V) \to \mathrm{Hom}_{\mathcal{C}}(U, V'') \to \mathrm{Ext}^1_{\mathcal{X}}(U, V') \to \\ \mathrm{Ext}^1_{\mathcal{X}}(U, V) \to \mathrm{Ext}^1_{\mathcal{X}}(U, V'') \to \mathrm{Ext}^2_{\mathcal{X}}(U, V') \to \cdots$$

Also, $\mathrm{Ext}^1_{\mathcal{X}}(U, V)$ coincides with the class of all $\mathcal{X}$-exact and exact sequences $0 \to V \to E \to U \to 0$ in $\mathcal{A}$, so that $\mathrm{Ext}^1_{\mathcal{X}}(-, -)$ is a k-linear subfunctor of the restriction to $\mathcal{A}$ of the functor $\mathrm{Ext}^1_{\mathcal{C}}(-, -)$. An object $U \in \mathcal{A}_0$ is called **$\mathcal{X}$-projective** (or **$\mathcal{X}$-injective**) if $\mathrm{Ext}^1_{\mathcal{X}}(U, -) = 0$ (or $\mathrm{Ext}^1_{\mathcal{X}}(-U) = 0$, respectively).

Under the stated hypotheses, the $\mathcal{X}$-injectives coincide with the objects in $\mathcal{X}$. Indeed, it is clear that any object in $\mathcal{X}$ is $\mathcal{X}$-injective. Conversely, let $I \in \mathcal{A}_0$ be $\mathcal{X}$-injective. The left $\mathcal{X}$-approximation $f_I : I \to \mathcal{X}(I)$ is an $\mathcal{X}$-monic, hence a monomorphism. Since Coker $f_I \in \mathcal{A}_0$, and I is $\mathcal{X}$-injective, then f_I is a section. Since $\mathcal{X}(I) \in \mathcal{X}_0$, and $\mathcal{X}$ is closed under direct summands, we infer that $I \in \mathcal{X}_0$.

3.2. The following lemma generalises properties of the stable module category of an artin algebra. Only its necessity part is needed for the proof of our main theorem (3.3), but we include the sufficiency for completeness. Here, and in the sequel, we use the notation of (1.2).

LEMMA . *Assume that $\mathcal{A}$ is $\mathcal{X}$-coresolving, that each $\mathcal{X}$-monic in $\mathcal{A}$ is a monomorphism, and that $\mathcal{X}$ is covariantly finite in $\mathcal{A}$. Let $u : U \to V$ be a morphism in $\mathcal{A}$.*

(a) $\underline{u} = 0$ *in* $\mathcal{A}/\mathcal{X}$ *if and only if* $\mathrm{Ext}^1_{\mathcal{X}}(W,u) = 0$ *for each* $W \in \mathcal{A}_0$.

(b) $\underline{u}$ *is an isomorphism in* $\mathcal{A}/\mathcal{X}$ *if and only if* $\mathrm{Ext}^1_{\mathcal{X}}(W,u)$ *is an isomorphism for each* $W \in \mathcal{A}_0$.

PROOF. The necessity of the conditions in (a) and (b) follows from the fact that, since the objects of $\mathcal{X}$ are $\mathcal{X}$-injective, then the functor $\mathrm{Ext}^1_{\mathcal{X}}(W,-) : \mathcal{A} \to \mathrm{Mod}\, k$ induces a functor $\mathcal{A}/\mathcal{X} \to \mathrm{Mod}\, k$. We thus just have to prove the sufficiency.

(a) Assume that $\mathrm{Ext}^1_{\mathcal{X}}(W,u) = 0$ for each $W \in \mathcal{A}_0$ and consider the exact and $\mathcal{X}$-exact sequence

$$0 \longrightarrow U \xrightarrow{\quad f_U \quad} \mathcal{X}(U) \xrightarrow{\quad g_U \quad} T(U) \longrightarrow 0$$

Then $\mathrm{Ext}^1_{\mathcal{X}}(T(U),u) = 0$ gives that the lower sequence in the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{f_U} & \mathcal{X}(U) & \xrightarrow{g_U} & T(U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & V & \longrightarrow & E & \longrightarrow & T(U) & \longrightarrow & 0 \end{array}$$

splits, hence there exists $u' : \mathcal{X}(U) \to V$ such that $u = u' f_U$. Thus $\underline{u} = 0$.

(b) Assume that $\mathrm{Ext}^1_{\mathcal{X}}(W,u)$ is an isomorphism for each $W \in \mathcal{A}_0$. The morphism $\begin{bmatrix} u \\ f_U \end{bmatrix} : U \to V \oplus \mathcal{X}(U)$ is an $\mathcal{X}$-monic, hence a monomorphism, so that we have an exact and $\mathcal{X}$-exact sequence

$$0 \longrightarrow U \xrightarrow{\begin{bmatrix} u \\ f_U \end{bmatrix}} V \oplus \mathcal{X}(U) \longrightarrow U' \longrightarrow 0$$

where $U' = \mathrm{Coker} \begin{bmatrix} u \\ f_U \end{bmatrix}$ belongs to $\mathcal{A}$. Then, for each $W \in \mathcal{A}_0$, we have a long exact cohomology sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{C}}(W,U) \to \mathrm{Hom}_{\mathcal{C}}(W, V \oplus \mathcal{X}(U)) \to \\ \to \mathrm{Hom}_{\mathcal{C}}(W,U') \to \mathrm{Ext}^1_{\mathcal{X}}(W,U) \to \mathrm{Ext}^1_{\mathcal{X}}(W, V \oplus \mathcal{X}(U)) \\ \to \mathrm{Ext}^1_{\mathcal{X}}(W,U') \to \mathrm{Ext}^2_{\mathcal{X}}(W,U) \to \cdots \end{aligned}$$

Since $\mathrm{Ext}^1_{\mathcal{X}}(W, f_U) = 0$ by (a), and $\mathrm{Ext}^1_{\mathcal{X}}(W, u)$ is an isomorphism, we infer that $\mathrm{Ext}^1_{\mathcal{X}}\left(W, \begin{bmatrix} u \\ f_U \end{bmatrix}\right)$ is an isomorphism. Setting $W = U'$, we deduce that the original short exact sequence splits, so that $\begin{bmatrix} u \\ f_U \end{bmatrix}$ is a section. Consequently, $\underline{u} : \underline{U} \to \underline{V}$ is a section. In order to show that $\underline{u}$ is actually an isomorphism, it suffices to prove that $\mathrm{Ext}^1_{\mathcal{X}}(W, U') = 0$. For, if this is the case, the arbitrariness of W yields that U' is $\mathcal{X}$-injective, that is, by (3.1), $U' \in \mathcal{X}_0$. Thus, the original short exact sequence induces a triangle in $\mathcal{A}/\mathcal{X}$ of the form

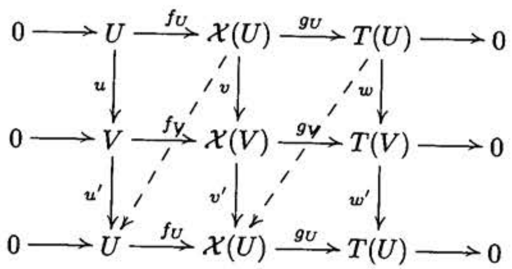
$$\underline{U} \xrightarrow{\underline{u}} \underline{V} \to 0 \to T(\underline{U})$$

By (2.1), $\underline{u}$ is a retraction and we are done.

Now, $\mathrm{Ext}^1_{\mathcal{X}}(W, U') = 0$ whenever $\mathrm{Ext}^2_{\mathcal{X}}\left(W, \begin{bmatrix} u \\ f_U \end{bmatrix}\right)$ or, equivalently, $\mathrm{Ext}^2_{\mathcal{X}}(W, u)$, is a monomorphism. Considering left $\mathcal{X}$-approximations of U, V, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{f_U} & \mathcal{X}(U) & \xrightarrow{g_U} & T(U) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & V & \xrightarrow{f_V} & \mathcal{X}(V) & \xrightarrow{g_V} & T(V) & \longrightarrow & 0
\end{array}$$

By dimension shifting, it suffices to show that $\mathrm{Ext}^1_{\mathcal{X}}(W, w)$ is a monomorphism. Since $\underline{u}$ is a section, there exist u', f' such that $u'u + f'f_U = 1$. We thus have a commutative diagram with exact rows



Since $f_U f' f_U = f_U - f_U u' u = f_U - v' v f_U$, we have $(1 - v'v - f_U f') f_U = 0$. Hence there exists $g' : T(U) \to \mathcal{X}(U)$ such that $1 - v'v - f_U f' = g' g_U$. Thus $g_U = g_U v' v + g_U f_U f' + g_U g' g_U = w' w g_U + g_U g' g_U$. Since g_U is an epimorphism, we have $1 = w'w + g_U g'$. Hence $\underline{1} = \underline{w'}\,\underline{w}$. By (a), $\underline{1} - \underline{w'}\,\underline{w} = 0$ yields $\mathrm{Ext}^1_{\mathcal{X}}(W, w')\,\mathrm{Ext}^1_{\mathcal{X}}(W, w) = 1$. Hence $\mathrm{Ext}^1_{\mathcal{X}}(W, w)$ is a section and, in particular, a monomorphism.

□

3.3. We may now state, and prove, our main theorem.

THEOREM . *Let $\mathcal{C}$ be a k-abelian category, and $\mathcal{X}$, $\mathcal{A}$ be two k-linear subcategories of $\mathcal{C}$. Suppose that $\mathcal{A}$ is $\mathcal{X}$-coresolving, that each $\mathcal{X}$-monic in $\mathcal{A}$ is a monomorphism and that $\mathcal{X}$ is covariantly finite in $\mathcal{A}$. Then :*

(a) *The standard right triangulation in $\mathcal{A}/\mathcal{X}$ has a right dense shift functor T.*
(b) *T is a right semi-equivalence if and only if every object in $\mathcal{X}$ is $\mathcal{X}$-projective.*

PROOF. (a) Let $(\mathcal{A}/\mathcal{X}, T, \Delta)$ be the standard right triangulated structure on $\mathcal{A}/\mathcal{X}$. We must show that T is right dense. Assume a triangle in Δ of the form

$$\underline{U} \xrightarrow{\underline{u}} T(\underline{V}) \xrightarrow{\underline{v}} \underline{W'} \xrightarrow{\underline{w}} T(\underline{U}).$$

There exists an exact and $\mathcal{X}$-exact sequence in $\mathcal{A}$

$$0 \to V \xrightarrow{f_V} \mathcal{X}(V) \xrightarrow{g_V} T(V) \to 0.$$

There also exists an exact sequence

$$0 \to U \xrightarrow{u} T(V) \xrightarrow{v} W' \to 0$$

where $v = \operatorname{coker} u$, and u is a monomorphism. We claim that there exists a commutative diagram in $\mathcal{C}$ with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & V & \xrightarrow{1} & V & & & & \\
 & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f_V} & & & & \\
0 & \longrightarrow & W & \xrightarrow{h} & \mathcal{X}(V) & \xrightarrow{vg_V} & W' & \longrightarrow & 0 \\
 & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g_V} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & U & \xrightarrow{u} & T(V) & \xrightarrow{v} & W' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $h = \ker g_V$. Indeed, $u = \ker v$, hence $p : W \to U$ exists and we set $q = \ker p$ (it is easily seen that $q : V \to W$). Since q is a kernel, it is a monomorphism, while p is an epimorphism (by the amalgamated sum diagram in the centre). In particular, the sequence

$$0 \to V \xrightarrow{q} W \xrightarrow{p} U \to 0$$

is exact in $\mathcal{C}$. It is also $\mathcal{X}$-exact : indeed, we need to show that q is $\mathcal{X}$-monic, that is , for each $X \in \mathcal{X}_0$, the morphism $\operatorname{Hom}_{\mathcal{C}}(q, X)$ is an epimorphism. But $hq = f_V$ yields $\operatorname{Hom}_{\mathcal{C}}(q, X)\operatorname{Hom}_{\mathcal{C}}(h, X) = \operatorname{Hom}_{\mathcal{C}}(f_V, X)$ and the result follows because f_V is $\mathcal{X}$-monic. Since $\mathcal{A}$ is $\mathcal{X}$-coresolving, $U, V \in \mathcal{A}_0$ imply $W \in \mathcal{A}_0$.

Next, we claim that h is $\mathcal{X}$-monic. Indeed, let $X \in \mathcal{X}_0$ and assume $\ell.\operatorname{Hom}_{\mathcal{C}}(h, X) = 0$ for some morphism ℓ in $\operatorname{Mod} k$. Then $0 = \ell.\operatorname{Hom}_{\mathcal{C}}(h, X)$ $\operatorname{Hom}_{\mathcal{C}}(g_V, X) = \ell.\operatorname{Hom}_{\mathcal{C}}(p, X)\operatorname{Hom}_{\mathcal{C}}(u, X)$ implies $\ell\operatorname{Hom}_{\mathcal{C}}(p, X) = 0$ because u is $\mathcal{X}$-monic. On the other hand, we have an exact sequence in $\operatorname{Mod} k$

$$0 \to \operatorname{Hom}_{\mathcal{C}}(U, X) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(p,X)} \operatorname{Hom}_{\mathcal{C}}(W, X) \xrightarrow{\operatorname{Hom}_{\mathcal{C}}(q,X)} \operatorname{Hom}_{\mathcal{C}}(V, X) \to 0$$

hence there exists a morphism ℓ' such that $\ell = \ell'.\operatorname{Hom}_{\mathcal{C}}(q, X)$. Consequently, $0 = \ell.\operatorname{Hom}_{\mathcal{C}}(h, X) = \ell'.\operatorname{Hom}_{\mathcal{C}}(q, X)\operatorname{Hom}_{\mathcal{C}}(h, X) = \ell'\operatorname{Hom}_{\mathcal{C}}(f_V, X)$

which yields $\ell' = 0$. This establishes our claim which implies that the exact sequence

$$0 \to W \xrightarrow{h} \mathcal{X}(V) \xrightarrow{vg_V} W' \to o$$

(which lies in $\mathcal{A}$) is $\mathcal{X}$-exact, with $\mathcal{X}(V) \in \mathcal{X}_0$. But then $\underline{W'} \cong T(\underline{W})$ and we are done.

(b) Assume that T is a right semi-equivalence. Let $X \in \mathcal{X}_0$ and $U \in \mathcal{A}_0$. The exact and $\mathcal{X}$-exact sequence in $\mathcal{A}$

$$0 \to U \xrightarrow{f_U} \mathcal{X}(U) \xrightarrow{g_U} T(U) \to 0$$

yields an exact cohomology sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X,U) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,f_U)} \mathrm{Hom}_{\mathcal{C}}(X,\mathcal{X}(U)) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,g_U)}$$
$$\to \mathrm{Hom}_{\mathcal{C}}(X,T(U)) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{X}}(X,U) \to 0$$

(because $\mathcal{X}(U)$ is $\mathcal{X}$-injective). Let $w : X \to T(U)$ be a morphism in $\mathcal{A}$, and consider its image $\partial(w)$ in $\mathrm{Ext}^1_{\mathcal{X}}(X,U)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{u} & V & \xrightarrow{v} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & U & \xrightarrow{f_U} & \mathcal{X}(U) & \xrightarrow{g_U} & T(U) & \longrightarrow & 0. \end{array}$$

Since $f_U = w'u$ is $\mathcal{X}$-monic, so is u. Also, $U, X \in \mathcal{A}_0$ imply $V \in \mathcal{A}_0$ so that $\partial(w)$ induces a triangle in $\mathcal{A}/\mathcal{X}$

$$\underline{U} \xrightarrow{\underline{u}} \underline{V} \xrightarrow{\underline{v}} \underline{X} \xrightarrow{\underline{w}} T(\underline{U}).$$

Since $\underline{X} = 0$, then $T(\underline{u})$ is an isomorphism (by (2.1)). Since T is full and faithful, $\underline{u}$ is an isomorphism. By (3.2), $\mathrm{Ext}^1_{\mathcal{X}}(W,u)$ is an isomorphism for each $W \in \mathcal{A}_0$. Applying $\mathrm{Hom}_{\mathcal{C}}(X,-)$ to the $\mathcal{X}$-exact sequence $\partial(w)$ yields an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X,U) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,u)} \mathrm{Hom}_{\mathcal{C}}(X,V) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(X,v)}$$
$$\to \mathrm{Hom}_{\mathcal{C}}(X,X) \xrightarrow{\partial} \mathrm{Ext}^1_{\mathcal{X}}(X,U) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{X}}(X,V) \to 0.$$

Hence $\mathrm{Hom}_{\mathcal{C}}(X,v)$ is an epimorphism : there exists $v' : X \to V$ such that $vv' = 1$. But then $wv = g_U w'$ implies $w = g_U w' v' = \mathrm{Hom}_{\mathcal{C}}(X,g_U)\ (w'v')$, that is, $\mathrm{Hom}_{\mathcal{C}}(X,g_U)$ is an epimorphism. By the first cohomology sequence above, this implies $\mathrm{Ext}^1_{\mathcal{X}}(X,U) = 0$ for all $X \in \mathcal{X}_0, U \in \mathcal{A}_0$, that is, each object in $\mathcal{X}$ is $\mathcal{X}$-projective.

Conversely, assume that $\mathrm{Ext}^1_{\mathcal{X}}(X,U) = 0$ for all $X \in \mathcal{X}_0, U \in \mathcal{A}_0$. Let $\underline{u} : \underline{U} \to \underline{V}$ be a morphism in $\mathcal{A}/\mathcal{X}$ with $T(\underline{u}) = 0$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{f_U} & \mathcal{X}(U) & \xrightarrow{g_U} & T(U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''=T(u)} & & \\ 0 & \longrightarrow & V & \xrightarrow{f_V} & \mathcal{X}(V) & \xrightarrow{g_V} & T(V) & \longrightarrow & 0. \end{array}$$

Since $\underline{u}'' = 0$, there exists $v : \mathcal{X}(T(U)) \to T(V)$ such that $u'' = v f_{T(U)}$. We consider the following fibered product in $\mathcal{C}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{p} & E & \xrightarrow{q} & \mathcal{X}(T(U)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & V & \xrightarrow{f_V} & \mathcal{X}(V) & \xrightarrow{g_V} & T(V) & \longrightarrow & 0. \end{array}$$

Then p is $\mathcal{X}$-monic, because $f_V = wp$ is. Hence $V, \mathcal{X}(T(U)) \in \mathcal{A}_0$ imply $E \in \mathcal{A}_0$. By hypothesis, $\mathrm{Ext}^1_{\mathcal{X}}(\mathcal{X}(T(U)), V) = 0$ hence there exists $w' : \mathcal{X}(T(U)) \to \mathcal{X}(V)$ such that $v = g_V w'$. Therefore $u'' = v f_{T(U)} = g_V w' f_{T(U)}$ and u'' factors through g_V. Consequently, u factors through f_U so that $\underline{u} = 0$. Thus, T is faithful.

To prove that T is full, let $\underline{u}'' : T(\underline{U}) \to T(\underline{V})$ be a morphism in $\mathcal{A}/\mathcal{X}$. We have in $\mathcal{A}$ a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{f_U} & \mathcal{X}(U) & \xrightarrow{g_U} & T(U) & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle u''} & & \\ 0 & \longrightarrow & V & \xrightarrow{f_V} & \mathcal{X}(V) & \xrightarrow{g_V} & T(V) & \longrightarrow & 0. \end{array}$$

We form the fibered product

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{p} & E & \xrightarrow{q} & \mathcal{X}(U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u''g_U} & & \\ 0 & \longrightarrow & V & \xrightarrow{f_V} & \mathcal{X}(V) & \xrightarrow{g_V} & T(V) & \longrightarrow & 0. \end{array}$$

Since $V, \mathcal{X}(U) \in \mathcal{A}_0$ and p is $\mathcal{X}$-monic (because $f_V = wp$ is), we have $E \in \mathcal{A}_0$. By hypothesis, $\mathrm{Ext}^1_{\mathcal{X}}(\mathcal{X}(U), V) = 0$ hence there exists $u' : \mathcal{X}(U) \to \mathcal{X}(V)$ such that $g_V u' = u'' g_U$. By passing to the kernels, there exists a unique $u : U \to V$ such that $f_V u = u' f_U$. By construction, $\underline{u}'' = T(\underline{u})$.

□

REMARK . The necessary and sufficient condition of (b) can be interpreted in terms of the **right $\mathcal{X}$-orthogonal** of $\mathcal{X}$: this is the k-linear subcategory $\mathcal{X}^{\perp}$ of $\mathcal{A}$ consisting of those $U \in \mathcal{A}_0$ such that $\mathrm{Ext}^i_{\mathcal{X}}(X, U) = 0$ for all $i \geq 1$ (see [7]). Then every object in $\mathcal{X}$ is $\mathcal{X}$-projective if and only if $\mathcal{A} = \mathcal{X}^{\perp}$. Indeed, the sufficiency is obvious and, for the necessity, let $U \in \mathcal{A}_0$, then there exists an $\mathcal{X}$-exact and exact sequence

$$0 \longrightarrow U \xrightarrow{u^0} X^0 \xrightarrow{u^1} X^1 \xrightarrow{u^2} X^2 \longrightarrow \cdots$$

with $X^j \in \mathcal{X}_0$ for all $j \geq 0$. Setting $U^j = \mathrm{Coker}\, u^j$ for $j \geq 0$, we have $U^j \in \mathcal{A}_0$ because $\mathcal{A}$ is $\mathcal{X}$-coresolving. Dimension shifting yields $\mathrm{Ext}^{i+1}_{\mathcal{X}}(X, U) \cong \mathrm{Ext}^1_{\mathcal{X}}(X, U^i) = 0$ for each $i \geq 1$.

3.4. We apply our theorem to the stable module category of an artin algebra.

COROLLARY . *Let A be an artin algebra with centre k. The following conditions are equivalent for the standard shift T in $\underline{\mathrm{mod}}A$:*

(a) *T is a right semi-equivalence.*

(b) *T is an equivalence.*
(c) *A is a self-injective algebra.*

PROOF. Indeed, if A is an artin algebra, there exists a bijection between the (finite) sets of isomorphism classes of indecomposable projective and indecomposable injective A-modules. Applying (3.3) to $\mathcal{A} = \mathcal{C} = \overline{\mathrm{mod}}A$ and observing that the $\mathcal{X}$-projectives (or $\mathcal{X}$-injectives) coincide with the projective (or injective, respectively) A-modules yield that (a) implies (c). Since (c) implies (b) by **[15]** (I.2.2) and (b) implies (a) trivially, we are done. □

REMARK . The above statement shows that the study of right semi-equivalences for $\overline{\mathrm{mod}}A$, with A an artin algebra, reduces to the study when $\overline{\mathrm{mod}}A$ is triangulated. This is not the case if we consider the stable module category $\overline{\mathrm{mod}}\Lambda$, where Λ is a locally bounded k-category, with k an algebraically closed field. Indeed, as before, the standard shift on $\overline{\mathrm{mod}}\Lambda$ is a right semi-equivalence if and only if every injective Λ-module is projective. But this does not imply that Λ is self-injective. The following is an example of a non-self-injective locally bounded k-category Λ such that the standard shift on $\overline{\mathrm{mod}}\Lambda$ is a right semi-equivalence. Let A be a finite dimensional k-algebra. We define the right repetitive category of A to be the k-category defined by the algebra of lower triangular matrices

$$\Lambda \quad = \quad \begin{bmatrix} A_0 & & 0 & \\ Q_0 & A_1 & & \\ & Q_1 & A_2 & \\ 0 & & \ddots & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero coefficients, $A_i = A, Q_i = \mathrm{D}A = \mathrm{Hom}_k(A, k)$ (with its canonical $A - A$-bimodule structure) for all $i \geq 0$, addition is the usual addition of matrices and multiplication is induced from the bimodule structure of $\mathrm{D}A$ and the zero morphism $\mathrm{D}A \otimes_A \mathrm{D}A \to 0$. The right repetitive category was shown to be useful in the covering theory of representation-finite and polynomial growth self-injective algebras **[1]**, **[3]**.

4. Complexes and unfoldings

4.1. The aim of this section is to prove a conjecture of W. Crawley-Boevey which states that, if A is an APR-iterated tilted algebra over an algebraically closed field, then the unfolding of A is the postprojective component of the full subcategory $\mathcal{H} = \mathcal{H}^-(A)$ of $\mathrm{K}^b(\mathrm{proj}\, A)$ consisting of the complexes having vanishing cohomology in the positive indices. Throughout, we assume that k is an algebraically closed field, and A is a triangular finite dimensional k-algebra. For unfoldings, we use the notation and results of **[2]**. We recall that if A is a triangular algebra then it has finite global dimension, hence $\mathrm{K}^b(\mathrm{proj}\, A)$ is equivalent, as a triangulated category, to the derived category $\mathrm{D}^b(\mathrm{mod}\, A)$ of bounded complexes of A-modules, and has almost split triangles **[15]**.

LEMMA . *$\mathcal{H} = \mathcal{H}^-(A)$ is a right triangulated subcategory of* $\mathrm{K}^b(\mathrm{proj}\, A)$*, whose shift is a right semi-equivalence.*

PROOF. Let $\mathrm{C}^{[b,0]}(\mathrm{proj}\, A)$ be the category of bounded complexes of projective A-modules $P^\bullet$ such that $P^i = 0$ for all $i > 0$, and $\mathcal{X}$ be the k-linear subcategory of $\mathrm{C}^{[b,0]}(\operatorname{proj} A)$ generated by those complexes of the form $0 \to P \xrightarrow{1} P \to 0$, where $P \in (\operatorname{proj} A)_0$. We consider the quotient category $\mathrm{K}^{[b,0]}(\operatorname{proj} A) = \mathrm{C}^{[b,0]}(\operatorname{proj} A)/\mathcal{X}$.

We claim that $\mathcal{H} \cong \mathrm{K}^{[b,0]}(\operatorname{proj} A)$. Indeed, there clearly exists a full and faithful embedding $\mathrm{K}^{[b,0]}(\operatorname{proj} A) \hookrightarrow \mathcal{H}$. It suffices to show that each object $U^\bullet = (U^i, d^i_{U^\bullet}) \in \mathcal{H}_0$ is isomorphic (inside $\mathrm{K}^b(\operatorname{proj} A) \cong \mathrm{D}^b(\operatorname{mod} A)$) to a complex in $\mathrm{K}^{[b,0]}(\operatorname{proj} A)$. Consider the complex $K^\bullet$ defined by $K^i = 0$ if $i > 0$, $K^0 = \operatorname{Ker} d^0_{U^\bullet}$ and $K^i = U^i$ if $i < 0$. Then there exists a morphism of complexes $g^\bullet : K^\bullet \to U^\bullet$ defined by $g^i = 0$ if $i < 0$, g^0 equal to the inclusion and $g^i = 1$ if $i > 0$. Clearly, $H^i(g^\bullet)$ is an isomorphism for each i, so that $U^\bullet \cong K^\bullet$ inside $K^b(\operatorname{proj} A)$.

Now, we want to apply (3.3) to our situation. We notice that $\mathrm{C}^{[b,0]}(\operatorname{proj} A)$ is a full subcategory of the abelian category $\mathrm{C}^b(\operatorname{mod} A)$, and is closed under extensions. We claim that $\mathcal{X}$ is covariantly finite in $\mathrm{C}^{[b,0]}(\operatorname{proj} A)$. Let $U^\bullet \in (C^{[b,0]}(\operatorname{proj} A))_0$ be arbitrary, and consider the complex $(\mathcal{X}(U^\bullet)^\bullet$ defined by $\mathcal{X}(U^\bullet)^i = 0$ for $i > 0$, $\mathcal{X}(U^\bullet)^0 = U^0$ and $\mathcal{X}(U^\bullet)^i = U^i \oplus U^{i+1}$ for $i < 0$ with differential $d^\bullet_{\mathcal{X}(U^\bullet)^\bullet}$ defined by $d^i_{\mathcal{X}(U^\bullet)^\bullet} = 0$ for $i \geq 0$, $d^{-1}_{\mathcal{X}(U^\bullet)^\bullet} = [0, 1]$ and $d^i_{\mathcal{X}(U^\bullet)^\bullet} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for $i \leq -2$. Then $\mathcal{X}(U^\bullet)^\bullet \in \mathcal{X}_0$ and there is a morphism $f^\bullet_{U^\bullet} : U^\bullet \to \mathcal{X}(U^\bullet)^\bullet$ defined by $f^i_{U^\bullet} = 0$ if $i > 0$, $f^0_{U^\bullet} = 1$ and $f^i_{U^\bullet} = \begin{bmatrix} 1 \\ d^i_{U^\bullet} \end{bmatrix}$ if $i < 0$. Then $f^\bullet_{U^\bullet}$ is a left $\mathcal{X}$-approximation. For, let $X^\bullet \in \mathcal{X}_0$ and $f^\bullet : U^\bullet \to X^\bullet$ be a non-zero morphism. We can assume that $X^\bullet$ is of the form $0 \to X \xrightarrow{1} X \to 0$, with $X \in (\operatorname{proj} A)_0$. There exists $i < 0$ such that the only non-zero components of $f^\bullet$ are $f^i : U^i \to X$ and $f^{i+1} : U^{i+1} \to X$ which further satisfy $f^i d^{i+1}_{U^\bullet} = f^{i+1}$. But then the morphism $g^\bullet : \mathcal{X}(U^\bullet) \to X^\bullet$ such that $g^i = [f^i \ 0]$, $g^{i+1} = [0 \ f^i]$ and $g^j = 0$ for $j \neq i, i+1$ satisfies $g^\bullet f^\bullet_{U^\bullet} = f^\bullet$.

Next, any $\mathcal{X}$-monic $f^\bullet : U^\bullet \to V^\bullet$ in $\mathrm{C}^{[b,0]}(\operatorname{proj} A)$ is a monomorphism. This will follow from the fact that $f^\bullet : U^\bullet \to V^\bullet$ is an $\mathcal{X}$-monic if and only if, for each i, $f^i : U^i \to V^i$ is a section. Indeed, if $f^\bullet : U^\bullet \to V^\bullet$ is an $\mathcal{X}$-monic, then $\operatorname{Hom}(f^\bullet, \mathcal{X}(U^\bullet)^\bullet) : \operatorname{Hom}(V^\bullet, \mathcal{X}(U^\bullet)^\bullet) \to \operatorname{Hom}(U^\bullet, \mathcal{X}(U^\bullet)^\bullet)$ is an epimorphism, hence there exists $g^\bullet : V^\bullet \to \mathcal{X}(U^\bullet)^\bullet$ such that $g^\bullet f^\bullet = f^\bullet_{U^\bullet}$. But this means that, for each i, we have $\begin{bmatrix} g^i \\ g'^i \end{bmatrix} : V^i \to U^i \oplus U^{i+1}$ such that $\begin{bmatrix} g^i \\ g'^i \end{bmatrix} f^i = \begin{bmatrix} 1 \\ d^i_{U^\bullet} \end{bmatrix}$. In particular, $g^i f^i = 1$. Conversely, if f^i a section for all i, and $u^\bullet : U^\bullet \to X^\bullet$ is a non-zero morphism, with $X^\bullet \in \mathcal{X}_0$ (and assumed of the form $0 \to X \xrightarrow{1} X \to 0$ with $X \in (\operatorname{proj} A)_0$), then there exists $v^\bullet : V^\bullet \to X^\bullet$ such that $v^\bullet f^\bullet = u^\bullet$. For, there exists $i < 0$ such that the only non-zero components of $u^\bullet$ are $u^i : U^i \to X$ and $u^{i+1} : U^{i+1} \to X$ which further satisfy $u^i d^{i+1}_{U^\bullet} = u^{i+1}$. Also, there exists $g^i : V^i \to U^i$ such that $g^i f^i = 1$. We then define $v^\bullet$ by $v^i = u^i g^i$, $v^{i+1} = u^i g^i d^{i+1}_{V^\bullet}$ and $v^j = 0$ for $j \neq i, i+1$

This also implies that $\mathrm{C}^{[b,0]}(\operatorname{proj} A)$ is $\mathcal{X}$-coresolving. Indeed, it contains $\mathcal{X}$, is closed under extensions, and the above statement implies that it is also closed under cokernels of $\mathcal{X}$-monics. By (3.3), $\mathcal{H} = \mathrm{K}^{[b,0]}(\operatorname{proj} A)$ is a right triangulated category with right dense shift.

Moreover, the shift T in $\mathcal{H}$ is the restriction of the usual shift in $\mathrm{K}^b(\operatorname{proj} A)$ (given by $T(U^\bullet) = V^\bullet$, where $V^i = U^{i+1}$, $d^i_{V^\bullet} = d^{i+1}_{U^\bullet}$). Indeed, for $U^\bullet \in \mathcal{H}_0$, $T(U^\bullet)$ is given by the cokernel of the left approximation $f^\bullet_{U^\bullet} : U^\bullet \to \mathcal{X}(U^\bullet)^\bullet$ and it is easily verified that this gives the restriction of the usual shift in $\mathrm{K}^b(\operatorname{proj} A)$. In particular, T is full and faithful, hence is a right semi-equivalence. $\square$

REMARK . It was already observed by Dowbor and Meltzer [13] that $\mathcal{H}$, equipped with the restriction of the usual shift in $K^b(\operatorname{proj} A)$, is right triangulated. We have shown that this right right triangulated structure is the standard one (1.2), and that the shift is a right semi-equivalence.

4.2. We now characterise indecomposable projective objects in $\mathcal{H}$.

LEMMA . *An object $P^\bullet \in \mathcal{H}_0$ is indecomposable projective if and only if $P^\bullet$ is a stalk complex with stalk concentrated in degree zero and equal to an indecomposable projective A-module.*

PROOF. Let $P^\bullet$ be the given form. For any $X^\bullet \in \mathcal{H}_0$, we have $(TX^\bullet)^0 = 0$, by definition of T. Hence $\operatorname{Hom}(P^\bullet, TX^\bullet) = 0$ and $P^\bullet$ is projective in $\mathcal{H}$. Obviously, $P^\bullet$ is indecomposable.

Conversely, let $P^\bullet \in \mathcal{H}_0$ be indecomposable projective, and view $\mathcal{H}$ as a full subcategory of the derived category $D^b(\operatorname{mod} A)$. For each injective A-module I, and each $n \in \mathbb{Z}$, there exists a canonical isomorphism $\operatorname{Hom}_{D^b(\operatorname{mod} A)}(P^\bullet, T^n I) \cong \operatorname{Hom}_A(H^{-n}P^\bullet, I)$ (see [16]). These k-spaces vanish for $n > 0$ because $P^\bullet \in \mathcal{H}_0$. Therefore $P^\bullet$ is isomorphic, in $D^b(\operatorname{mod} A)$, to a stalk complex with stalk equal to the A-module M, say. Now, for each A-module N, and each $n > 0$, there exists an isomorphism $\operatorname{Hom}_{D^b(\operatorname{mod} A)}(M, T^n N) \cong \operatorname{Ext}^n_A(M, N)$ (see [15]). As above, these k-spaces vanish because $P^\bullet \in \mathcal{H}_0$, so that M must be a projective A-module. Since $\operatorname{mod} A$ is a full subcategory of $D^b(\operatorname{mod} A)$, it is clear that M is indecomposable. □

4.3. We deduce the existence of minimal almost split morphisms.

LEMMA . *$\mathcal{H}$ is functorially finite in* $K^b(\operatorname{proj} A)$. *In particular, $\mathcal{H}$ has left and right minimal almost split morphisms.*

PROOF. First, $\mathcal{H}$ is covariantly finite. Let $U^\bullet$ be a complex in $K^b(\operatorname{proj} A)$ and $U^\bullet{}_-$ be its truncation defined by $U^i_- = U^i$ if $i \leq 0$, $U^i_- = 0$ if $i > 0$, then $U^\bullet{}_- \in (K^b(\operatorname{proj} A))_0$ and there exists a morphism of complexes $t^\bullet : U^\bullet \to U^\bullet{}_-$ defined by $t^i = 1$ if $i \leq 0$, $t^i = 0$ if $i > 0$. Then, clearly, $t^\bullet$ is a left $\mathcal{H}$-approximation.

Next, $\mathcal{H}$ is also contravariantly finite. Here we view again $\mathcal{H}$ as a full subcategory of $D^b(\operatorname{mod} A)$. Let $U^\bullet$ be a complex in $K^b(\operatorname{proj} A)$ and $K^\bullet$ be the complex defined by $K^i = 0$ if $i > 0, K^0 = \operatorname{Ker} d^0_{U^\bullet}$ and $K^i = U^i$ if $i < 0$. As seen in (4.1), there exists a quasi-isomorphism $g^\bullet : K^\bullet \to U^\bullet$ and $K^\bullet$ belongs to the essential image of the embedding of $\mathcal{H}$ in $D^b(\operatorname{mod} A)$. Thus $g^\bullet$ is a right $\mathcal{H}$-approximation.

The last statement follows from (2.8). □

4.4. We finally prove the main result of this section.

THEOREM . *Let A be APR-iterated tilted. The quiver $\Gamma(\mathcal{H})$ of $\mathcal{H}$ has a unique postprojective component, equal to the quiver of the unfolding $\mathcal{K}(A)$ of A.*

PROOF. If A is APR-iterated tilted of type Σ, say, it has a simple Σ-unfolding $\mathcal{K}(A)$, and $\mathcal{K}(A)$ contains (by construction and (4.2)) all the indecomposable projective objects in $\mathcal{H}$. Hence, if $\mathcal{K}(A)$ has a postprojective component, it has exactly one. Since the quiver $\Gamma(\mathcal{K}(A))$ of $\mathcal{K}(A)$ is a postprojective translation quiver, it suffices to show that $\Gamma(\mathcal{K}(A))$ is a component of $\Gamma(\mathcal{H})$. This is done by descending induction on a fixed sequence of APR-tilts $A_0 = A$, $A_1, \dots A_n = k\Sigma$. For $i = n$, there is nothing to show, since the unfolding of the hereditary algebra $k\Sigma$ is just its postprojective component by [2] and [15]. We assume the statement for $B = A_{i+1}$

and prove it for $C = A_i$. There exists a simple projective C-module $P_C = eC$ such that the APR-tilting complex $T^\bullet$ defined as the projective resolution of the APR-tilting module $\tau_C^{-1}(eC) \oplus (1-e)C$ (see[**2**] (3.1)) satisfies $\operatorname{End} T^\bullet = B$. Let $\mathcal{K}(B)$ and $\mathcal{K}(C)$ denote the unfoldings of B and C, respectively. Since, by Rickard's theorem [**21**], $\mathrm{K}^b(\operatorname{proj} B)$ and $\mathrm{K}^b(\operatorname{proj} C)$ are equivalent as triangulated categories, and the images of the summands of $T^\bullet$ under this equivalence are just the projective B-modules, then the full subcategory of $\mathcal{K}(C)$ consisting of the (non-necessarily proper) successors of the summands of $T^\bullet$ corresponds under this equivalence to $\mathcal{K}(B)$. By the induction hypothesis, $\mathcal{K}(B)$ is a component of the category $\mathcal{H}^-(B)$ of those objects in $\mathrm{K}^b(\operatorname{proj} B)$ with vanishing cohomology in the positive indices. Hence any point among the sucessors of $T^\bullet$ in $\Gamma(\mathcal{K}(C))$ represents an indecomposable object in $\mathcal{H}^-(C)$ and any arrow between two successors of summands of $T^\bullet$ represents an irreducible morphism in $\mathcal{H}^-(C)$. Finally, the morphisms of source $P = eC$ in $\Gamma(\mathcal{K}(C))$ are inclusions of P as a radical summand of other projectives, hence are also irreducible. □

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