RIGHT ADA ALGEBRAS

EDSON RIBEIRO ALVARES, IBRAHIM ASSEM, DIANE CASTONGUAY, AND ROSANA VARGAS

ABSTRACT. We introduce and study the class of right ada algebras. An artin algebra is right ada if every indecomposable projective module lies in the left or in the right part of its module category. We study the Auslander-Reiten components of a right ada algebra which is not quasi-tilted and prove that they are of three types: components of the left and of the right support, and transitional components each containing a right section.

CONTENTS

1. INTRODUCTION

Let A be an artin algebra. In order to study the representation theory of A, Happel, Reiten and Smalϕ have introduced in [20] the notion of left and right parts $\mathcal{L}_A$ and $\mathcal{R}_A$ of its module category. These parts turned out to be well-behaved and are by now largely understood, see, for instance, [20],[6], [5], [1]. Making hypotheses on these parts leads to define algebras whose representation theory is to a large extent predictable, such as the quasi-tilted algebras of [20], the shod algebras of [17], the weakly shod algebras of [16] or the laura algebras of [3]. A recent addition to this list is the class of ada algebras [2]. An algebra A is ada if every indecomposable projective and every indecomposable injective A-module lies in $\mathcal{L}_A$ or in $\mathcal{R}_A$. It was shown in [2] that the representation theory of an ada algebra is entirely determined by those of its left and right support algebras, both of which are tilted. We recall from [6] that the left support A_λ of an artin algebra A is the endomorphism algebra of a complete set of representatives of the isoclasses (isomorphism classes) of those indecomposable projective A-modules which lie in $\mathcal{L}_A$. One defines dually the right support algebra A_ρ.

One objective in the present paper is to prove a similar statement for a larger class of algebras, generalising ada algebras by relaxing its defining condition. We define an artin

Date: August 21, 2016.
The first named author acknowledges support from DMAT-UFPR and CNPq-Universal 477880/2012-6.
The second named author gratefully acknowledges partial support from the NSERC of Canada.
The fourth named author acknowledges partial support from FAPESP 2013/17903-3.

algebra A to be right ada if every indecomposable projective A-module belongs to $\mathcal{L}_A$ or to $\mathcal{R}_A$. This class shares some of the nicest properties of ada algebras: it behaves well with respect to taking full subcategories, split extensions and skew group algebras. Also, a right ada algebra has representation dimension at most 3 and global dimension at most 4 (further, any indecomposable module has projective dimension at most 2 or injective dimension at most 1). Moreover, if A is a finite dimensional right ada algebra over an algebraically closed field, then A is simply connected if and only if the first Hochschild cohomology group $HH^1(A)$ of A with coefficients in the bimodule ${}_AA_A$ vanishes. This answer positively for right ada algebras a well-known question of Skowroński [22]. We prove that an algebra A is right ada if and only if every indecomposable A-module is a module over its left support A_λ or lies in $\mathcal{R}_A$ (and then, it is an A_ρ-module). Using this result, we characterise the Auslander-Reiten components of a right ada algebra as in the following theorem.

Theorem. *Let A be a right ada algebra which is not quasi-tilted. Then there exist a finite family $(\Gamma_i)_{i=1}^t$ of connected components of the Auslander-Reiten quiver $\Gamma(mod\ A)$ of A containing right sections $(\Sigma_i)_{i=1}^t$ such that:*

- *(a) For each i, the full subcategory $(\Gamma_i)_{\geq \Sigma_i}$ of successors of Σ_i, inside Γ_i is directed, generalised standard and convex in $\operatorname{ind} A$. Also $(\Gamma_i)_{\geq \Sigma_i} = \Gamma_i \cap \mathcal{R}_A$.*
- *(b) The full subcategory $(\Gamma_i)_{\ngeq \Sigma_i}$ of non-successors of Σ_i inside Γ_i consists of A_λ-modules not in $\mathcal{R}_A$.*
- *(c) If Γ is a component of $\Gamma(mod\ A)$ distinct from the Γ_i, then either Γ is a component of $\Gamma(mod\ A_\lambda)$ or is entirely contained in $\mathcal{R}_A$ (and in this case is a component of $\Gamma(mod\ A_\rho)$.*
- *(d) If moreover $\operatorname{Hom}_A(\Gamma, \cup_{i=1}^t \Gamma_i) \neq 0$, then Γ is a component of $\Gamma(\operatorname{mod} A_\lambda)$.*
- *(e) Let M be an indecomposable A-module. Then $M \notin \mathcal{L}_A \cup \mathcal{R}_A$ if and only if there exist an indecomposable projective module $P \in \mathcal{R}_A$, an indecomposable injective A_λ-module and two paths $I \rightsquigarrow M$, $M \rightsquigarrow P$ which are not refinable to sectional paths.*

Because the study of right ada algebras closely resembles that of the (two-sided) ada algebras of [2], most of the techniques introduced there can be applied to our case yielding, however, weaker results, beacuse we are dealing with a much larger class.

The paper is organised as follows. After a short preliminary section 2, we define and study the most immediate properties of right ada algebras in section 3. We start describing their module categories in section 4 and prove our main theorem in section 5. The paper ends with an example.

2. Preliminaries

2.1. **Notation.** Let A be a basic connected artin algebra. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ a full subcategory consisting of one representative from each isoclass of indecomposable A-modules. When we speak about an A-module, or an indecomposable A-module, we always mean that it belongs to $\operatorname{mod} A$, or $\operatorname{ind} A$, respectively. All subcategories of $\operatorname{mod} A$ are full and so are identified with their object classes.

Following [15], we equivalently consider an algebra A as a k-category, whose object class $A_0 = \{1, \cdots, n\}$ is in bijection with a complete set $\{e_1, \cdots, e_n\}$ of primitive orthogonal idempotents and the space of morphisms from i to j is e_iAe_j. An algebra B is a *full subcategory* of A if there exists $e \in A$, sum of some of the e_i, such that $B = eAe$. It is *convex* if, for any sequence $i_0, \cdots, i_t$ in A_0 such that $e_{i_k}Ae_{i_{k+1}} \neq 0$ for all k with

$0 \leq k < t$ and $i_0, i_t \in B_0$ then $i_k \in B_0$ for all k. We say that A is *triangular* if there is no sequence of distinct objects $i_0, \cdots, i_t = i_0$ with $t \geq 1$ such that $e_{i_k} A e_{i_{k+1}} \neq 0$ for all k. We denote by e_x, P_x, I_x, S_x respectively the primitive idempotent, the indecomposable projective, the indecomposable injective and the simple module corresponding to $x \in A_0$.

Given a subcategory $\mathcal{C}$ of $\mathrm{mod}\ A$, we write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. We denote by $\mathrm{add}\ \mathcal{C}$ the subcategory of $\mathrm{mod}\ A$ consisting of the direct sums of direct summands of objects in $\mathcal{C}$. If $\mathcal{C}, \mathcal{D}$ are two subcategories of $\mathrm{mod}\ A$, $\mathrm{Hom}_A(\mathcal{C}, \mathcal{D}) = 0$ expresses that $\mathrm{Hom}_A(M, N) = 0$ for all $M \in \mathcal{C}, N \in \mathcal{D}$. If $\mathcal{C}$ is a subcategory of $\mathrm{mod}\ A$ closed under extensions, then $M \in \mathcal{C}$ is called *Ext-projective* in $\mathcal{C}$ if $\mathrm{Ext}^1_A(M, -)|_{\mathcal{C}} = 0$. It is shown in [12] (3.4) that M is Ext-projective in $\mathcal{C}$ if and only if $\tau M \notin \mathcal{C}$. One defines and characterises dually *Ext-injective* in $\mathcal{C}$.

Given an A-module M, we denote by $\mathrm{pd}\ M$ and $\mathrm{id}\ M$, respectively, its projective and injective dimensions. The global dimension of A is denoted by $\mathrm{gl.dim}\ A$. For further definitions and results on the representation theory of A, we refer the reader to [14], [10].

2.2. **Paths, left and right parts.** . Let A be an algebra. Given $M, N \in \mathrm{ind}\ A$, a *path* from M to N (denoted by $M \rightsquigarrow N$) is a sequence of nonzero morphisms

$$(*)\ \ M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_t} M_t = N$$

with $M_i \in \mathrm{ind}\ A$ for all i. Then N is called *successor* of M and M *predecessor* of N.

A path $(*)$ from M to M where at least one of the f_i is not an isomorphism is a *cycle*. A module $M \in \mathrm{ind}\ A$ is *directed* if it lies on no cycle. If each f_i is irreducible in $(*)$, then $(*)$ is a *path of irreducible morphisms*. If moreover, $\tau M_{i+1} \neq M_{i-1}$, for each i with $0 < i < t$, then $(*)$ is a *sectional* path. A *refinement* of $(*)$ is a path

$$M = M'_0 \to M'_1 \to \cdots \to M'_s = N$$

with $s \geq t$ and an order-preserving map $\{1, \cdots, t-1\} \to \{1, \cdots, s-1\}$ such that $M_i \simeq M'_{\sigma(i)}$ for all i. A path $(*)$ is *refinable to a sectional path* if it has a refinement which is sectional.

Following [19], the *left part* $\mathcal{L}_A$ of $\mathrm{mod}\ A$ is the full subcategory whose objects are those $M \in \mathrm{ind}\ A$ such that every predecessor of M has projective dimension at most one. Clearly $\mathcal{L}_A$ is closed under predecessors. The *right part* $\mathcal{R}_A$ is defined dually and has dual properties.

3. Definition and first properties

We recall that an artin algebra A is *ada* if every indecomposable projective and every indecomposable injective A-modules lies in $\mathcal{L}_A$ or in $\mathcal{R}_A$, that is, if $A \oplus DA \in \mathrm{add}\ (\mathcal{L}_A \cup \mathcal{R}_A)$, see [2]. We define right ada algebras by asking this condition only of projectives.

Definition 3.1. An artin algebra A is called a *right ada* algebra if $A_A \in \mathrm{add}\ (\mathcal{L}_A \cup \mathcal{R}_A)$.

This is clearly equivalent to requiring that, for every $x \in A_0$, we have $P_x \in \mathcal{L}_A \cup \mathcal{R}_A$.

Dually, A is *left ada* if $DA_A \in \mathrm{add}\ (\mathcal{L}_A \cup \mathcal{R}_A)$ or, equivalently, every indecomposable injective module lies in $\mathcal{L}_A \cup \mathcal{R}_A$. Clearly A is right ada if and only if its opposite algebra A^{op} is left ada. Finally, A is ada if it is both left and right ada.

Example 3.2.

(a) An algebra A is quasi-tilted if and only if $A_A \in \mathrm{add}\ \mathcal{L}_A$ if and only if $DA_A \in \mathrm{add}\ \mathcal{R}_A$, see [20] (II. 1.4). Then, quasi-tilted algebras are right and left ada. A right or left ada algebra which is not quasi-tilted is called *strict*.

(b) Let A be a shod algebras, then $\operatorname{ind} A = \mathcal{L}_A \cup \mathcal{R}_A$, see [17] or [5](6.1). Thus, shod algebras are right and left ada.

(c) The following is an example of a right ada algebra which is not left ada (and thus not ada). Let A be given by the quiver

$$1 \xleftarrow{\epsilon} 2 \xleftarrow{\delta} 3 \xleftarrow{\gamma} 4 \xleftarrow{\beta} 5 \xleftarrow{\alpha} 6$$

bounded by $\alpha\beta = 0, \beta\gamma = 0, \delta\epsilon = 0$.

We now study elementary properties of right ada algebras. We start by proving that a full subcategory of a right ada algebra is right ada.

Lemma 3.3. *Let A be a right ada algebra, and $e \in A$ be an idempotent. Then $B = eAe$ is right ada.*

Proof. Let $x \in B_0$ and $P_x = e_x B$. Then $P_x \otimes_B eA \cong e_x A$, because $e_x e = e_x$. Therefore, $P_x \otimes_B eA \in \mathcal{L}_A \cup \mathcal{R}_A$. Because of [4](Corollary 2.3), we have $\operatorname{Hom}_A(eA, P_x \otimes_B eA) \in \mathcal{L}_B \cup \mathcal{R}_B$. But $\operatorname{Hom}_A(eA, P_x \otimes_B eA) \cong P_x$, because of [4](Lemma 2.1). Hence $P_x \in \mathcal{L}_B \cup \mathcal{R}_B$ and B is right ada. □

For split-by-nilpotent extensions, we refer the reader to [11].

Lemma 3.4. *Let R be a split extension of A by a nilpotent bimodule. If R is right ada, then so is A.*

Proof. Let $x \in A_0$. Then $e_x R_R \cong e_x A \otimes_A R_R$. The statement then follows immediately from [11](2.3)(b). □

For skew group algebras, we refer to [21, 9].

Proposition 3.5. *Let A be an artin algebra, and G a group acting on A with $|G|$ invertible in A. Then the basic algebra $R = A[G]^b$ associated to the skew group algebra is right ada if and only if A is right ada.*

Proof. Suppose that A is right ada, and let $\overline{P}$ be an indecomposable projective R-module. Because of [9](4.3), there exists an indecomposable projective summand P_A of $\operatorname{Hom}_R(R, \overline{P})$ such that $\overline{P}_R$ is a direct summand of $P \otimes_A R$.

Suppose $P \in \mathcal{L}_A$. Because of [9](5.2)(a), we have $P \otimes_A R \in \operatorname{add} \mathcal{L}_R$. Therefore $\overline{P} \in \mathcal{L}_R$. Suppose on the other hand that $P \in \mathcal{R}_A$. Let X be an indecomposable R-module such that $\operatorname{Hom}_R(\overline{P}, X) \neq 0$. We claim that $\operatorname{id} X \leq 1$. Because of [9](4.6), there exist $\sigma \in G$ and an indecomposable summand M of $\operatorname{Hom}_R(R, X)$ such that X is a direct summand of ${}^\sigma M \otimes_A R$ and $\operatorname{Hom}_A(P, {}^\sigma M) \neq 0$. Because $P \in \mathcal{R}_A$, we get ${}^\sigma M \in \mathcal{R}_A$. In particular, $\operatorname{id} {}^\sigma M \leq 1$. Now the functor $- \otimes_A R : \operatorname{mod} A \to \operatorname{mod} R$ is exact and sends injectives to injectives, we get $\operatorname{id} ({}^\sigma M \otimes_A R) \leq 1$. Therefore $\operatorname{id} X \leq 1$, as required. Because of [9](1.1) this yields $\overline{P} \in \mathcal{R}_R$. We have proved that $P \in \mathcal{L}_A \cup \mathcal{R}_A$ implies $\overline{P} \in \mathcal{L}_R \cup \mathcal{R}_R$. Then A right ada implies R right ada.

Conversely, assume that R is right ada, and let P_A be an indecomposable projective A-module. Then there exists an indecomposable projective summand $\overline{P}$ of $P \otimes_A R$ such that P is a direct summand of $\operatorname{Hom}_R(R, \overline{P})$.

Suppose $\overline{P} \in \mathcal{L}_R$. Because of [9](5.2)(b), we have $\operatorname{Hom}_A(R, \overline{P}) \in \operatorname{add} \mathcal{L}_A$. Therefore $P \in \mathcal{L}_A$. Suppose now that $\overline{P} \in \mathcal{R}_R$, and let M be an indecomposable A-module such that $\operatorname{Hom}_A(P, M) \neq 0$. We claim that $\operatorname{id} M \leq 1$. Because of [9](4.4)(a), we have $\operatorname{Hom}_R(\overline{P}, M \otimes_A R) \neq 0$. Because of [21] (1.1, 1.8), there exists an indecomposable decomposition $M \otimes_A R = \oplus_{i=1}^m X_i$ such that $\operatorname{Hom}_R(R, X_i) = \oplus_{\sigma \in H_i} {}^\sigma M$ for some

$H_i \subseteq G$. Hence there exists i with $1 \le i \le m$ and $\mathrm{Hom}_R(\overline{P}, X_i) \neq 0$. Because $\overline{P} \in \mathcal{R}_R$, we get $X_i \in \mathcal{R}_R$ and so id $X_i \le 1$. Therefore, for every $\sigma \in H_i$, we have id ${}^{\sigma}M \le 1$. Thus id $M \le 1$, as required. Therefore $P \in \mathcal{R}_A$. This proves that $\overline{P} \in \mathcal{L}_R \cup \mathcal{R}_R$ implies $P \in \mathcal{L}_A \cup \mathcal{R}_A$. □

As immediate corollaries of (3.3), (3.4), (3.5) and their duals we obtain [2] (2.8),(2.9),(2.10).

Corollary 3.6. *(a) Let A be an ada algebra and $e \in A$ be an idempotent, then eAe is ada.*
(b) Let R be a split extension of A by a nilpotent bimodule. If R is ada, then A is ada.
(c) Let A be an artin algebra and G a group acting on A with $|G|$ invertible in A. Then the basic algebra $R = A[G]^b$ associated to the skew group algebra is ada if and only if A is ada. □

We now study homological dimensions of a right ada algebra. It is shown in [2] that an ada algebra has global dimension at most 4 and for every indecomposable A-module M, one has pd $M \le 2$ or id $M \le 1$. We now prove that the same statement holds true for right ada algebras, but we obtain it as a result of more general considerations.

Given $m, n \ge 1$, we define two full subcategories $\mathcal{L}_m$ and $\mathcal{R}_n$ of $\mathrm{ind}A$ as follows: $\mathcal{L}_m = \{M \in \mathrm{ind}A \mid$ If $L \rightsquigarrow M, L \in \mathrm{ind}\ A$, then $\mathrm{pd}_A\ L \le m\}$ and $\mathcal{R}_n = \{M \in \mathrm{ind}A \mid$ If $M \rightsquigarrow N, N \in \mathrm{ind}\ A$, then $\mathrm{id}_A\ N \le n\}$.

Clearly, $\mathcal{L}_1 = \mathcal{L}_A$, $\mathcal{R}_1 = \mathcal{R}_A$. Also, $\mathcal{L}_m$ is closed under predecessors in $\mathrm{ind}\ A$, while $\mathcal{R}_n$ is closed under successors.

Proposition 3.7. *Let A be an artin algebra such that $A_A \in \mathrm{add}\,(\mathcal{L}_m \cup \mathcal{R}_n)$. Then:*
(a) For any $M \in \mathrm{ind}\ A$, we have $\mathrm{pd}_A\ M \le m+1$ or $\mathrm{id}_A\ M \le n$.
(b) gl.dim $A \le m+n+2$.

Proof. (a) Let $f : P_0 \to M$ be a projective cover. If some indecomposable summand P_0' of P_0 belongs to $\mathcal{L}_m$, then any indecomposable summand of the kernel of f also lies in $\mathcal{L}_m$. Therefore pd $M \le m+1$. Otherwise, if no indecomposable summand of P_0 lies in $\mathcal{L}_m$, then $P_0 \in \mathrm{add}\ \mathcal{R}_n$. Hence $M \in \mathcal{R}_n$ and id $M \le n$.

(b) Let $M \in \mathrm{ind}\ A$ and suppose that pd $M \ge n+1$. Then we have the start of a minimal projective resolution

$$0 \to K_n \to P_n \to \cdots \to P_0 \to M \to 0.$$

Also, for every indecomposable summand X of K_n, we have $\mathrm{Ext}_A^{n+1}(M, X) \neq 0$. But this implies id $X \ge n+1$. Because of (a), we have pd $X \le m+1$. Therefore pd $K_n \le m+1$ and so pd $M \le (m+1) + (n+1) = m+n+2$. □

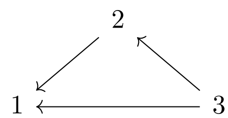
Corollary 3.8. *Let A be a right ada algebra. Then:*
(a) For every indecomposable module M, we have pd $M \le 2$ *or* id $M \le 1$
(b) gl.dim $A \le 4$.

Proof. This follows immediately from (3.7). □

Examples 3.9. (a) The bound obtained in (b) above is sharp: it is attained in the case of the radical square zero algebra with quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

(b) The conditions of (3.8) do not suffice to characterise right ada algebras. Indeed, let A be the radical square zero algebra given by the quiver



Then it is easily seen that for every indecomposable module M, we have $\mathrm{pd}_A M \leq 2$. Hence $\mathrm{gl.dim}\ A = 2$. On the other hand, $\mathcal{L}_A = \{P_1, P_2\}$ and $\mathcal{R}_A = \{I_2, I_3\}$. Thus A satisfies the conditions of (3.8), but it is not right ada, because $P_3 \notin \mathcal{L}_A \cup \mathcal{R}_A$.

If A is ada, then, by symmetry, we also have that for every indecomposable module M, one has $\mathrm{pd}\ M \leq 1$ or $\mathrm{id}\ M \leq 2$. This is usually not true for right ada algebras.

The most recent and perhaps the most intriguing homological dimension is the representation dimension, introduced by Auslander in [13] and denoted as $\mathrm{rep.dim}\ A$, for an algebra A.

Proposition 3.10. *Let A be a right ada algebra. Then* $\mathrm{rep.dim}\ A \leq 3$.

Proof. This is proved in [7] (5.2). □

We recall definitions from [8]. Given a strict right ada algebra A, let $\mathcal{P}_A$ denote the set of indecomposable projectives lying in $\mathcal{R}_A$. Because each such projective is directed [6] (1.6), we can define a partial order by setting $P \leq P'$ if and only if there exists a path $P \rightsquigarrow P'$ in $\mathrm{ind}\ A$. Because $\mathcal{P}_A$ is a finite set, it contains maximal elements. Such a maximal element $P = eA$ is called *maximal projective*. Setting $M = \mathrm{rad}\ P$ and $B = (1-e)A(1-e)$, we have $A = B[M]$: we say that A is a *maximal extension*.

Lemma 3.11. *Let A be a strict right ada algebra, then there exists a sequence of right ada algebras of the form*

$$A_\lambda = A_0 \subsetneqq A_1 \subsetneqq \cdots \subsetneqq A_t = A$$

sucht that, for each i with $1 \leq i < t$, there exists an A_i-module M_i such that $A_{i+1} = A_i[M_i]$ is a maximal extension and is also right ada.

Proof. Because A is strict, there exists an indecomposable projective module in $\mathcal{R}_A \setminus \mathcal{L}_A$ which we may assume to be maximal $P_t = e_t A$. Setting $M_{t-1} = \mathrm{rad}\ P_{t-1}$ and $A_{t-1} = (1-e_t)A(1-e_t)$, we get that $A = A_{t-1}[M_{t-1}]$ is a maximal extension. Because of (3.3), A_{t-1} is also right ada. If it is not strict, then every indecomposable projective A_{t-1}-module lies in $\mathcal{L}_{A_{t-1}} = \mathcal{L}_A \cap \mathrm{ind}\ A_{t-1}$ and so $A_{t-1} = A_\lambda$. Otherwise, we apply induction. □

Let A be a finite dimensional algebra over an algebraically closed field k. We denote by $HH^i(A)$ the i^{th} Hochschild cohomology group of A with coefficients in the bimodule ${}_AA_A$ and by $HH^*(A)$ the Hochschild cohomology ring, see [18]. Also, A is called *simply connected* if for every presentation $A = kQ/I$ of A as a bound quiver algebra, we have $\Pi_1(Q, I) = 0$, see [22]. It was asked in [22] for which algebras is simple connectedness equivalent to the vanishing of the first Hochschild cohomology group. This is answered positively for right ada algebras.

Corollary 3.12. *Let A be a right ada algebra. The following conditions are equivalent:*

(a) A is simply connected.
(b) $HH^1(A) = 0$.
(c) $HH^*(A) = 0$.

Proof. The proof in [2] applies word to word. □

4. Support Algebras of Right Ada Algebras

We recall the definition. Let P denote the direct sum of a complete set of representatives of the isoclasses of indecomposable projective modules lying in $\mathcal{L}_A$. Then $A_\lambda = \text{End } P$ is called the *left support* of A (see [6]). Then A_λ is a full convex subcategory of A, closed under successors and $\mathcal{L}_A \subseteq \text{ind } A_\lambda$. Moreover, it is shown in [6] (2.3) that A_λ is a direct product of quasi-tilted algebras. The *right support* algebra A_ρ of A is defined dually and has the dual properties.

Lemma 4.1. *Let A be a right ada algebra. Then:*

(1) A is triangular.
(2) $A = A_\lambda \cup A_\rho$

Proof. (1) Because of [6](2.2)(a), A can be written in triangular matrix form $A = \begin{bmatrix} A_\lambda & 0 \\ M & B \end{bmatrix}$. Because A_λ is a direct product of quasi-tilted algebras, it is triangular. Let $x \in B_0$, then the indecomposable projective A-module P_x does not lie in $\mathcal{L}_A$. Therefore $P_x \in \mathcal{R}_A$. Now, projectives in $\mathcal{R}_A$ are directed [1](6.4). In particular, B is triangular. Hence so is A. (2) Let $x \in A_0$. Then $P_x \in \mathcal{L}_A \cup \mathcal{R}_A$. If $P_x \in \mathcal{L}_A$, then $x \in (A_\lambda)_0$. If $P_x \in \mathcal{R}_A$, then $\text{Hom}_A(P_x, I_x) \neq 0$ implies that $I_x \in \mathcal{R}_A$ and so $x \in (A_\rho)_0$. □

Theorem 4.2. *An algebra A is right ada if and only if* $\text{ind } A = \text{ind } A_\lambda \cup \mathcal{R}_A$.

Proof. Assume that A is right ada, and let $M \in \text{ind } A$ be such that $M \notin \text{ind } A_\lambda$. Then there exists an indecomposable projective A-module $P_x \notin \mathcal{L}_A$, such that $\text{Hom}_A(P_x, M) \neq 0$. Then $P_x \in \mathcal{R}_A$ and so $M \in \mathcal{R}_A$.

Conversely, assume that $\text{ind}A = \text{ind } A_\lambda \cup \mathcal{R}_A$. Let P be an indecomposable projective A-module. Then P lies in $\text{ind}A_\lambda$ and so $P \in \mathcal{L}_A$ (because the indecomposable projective A_λ-modules are the projective modules lying in $\mathcal{L}_A$), or else $P \in \mathcal{R}_A$. □

Corollary 4.3. *(a) If A is right ada, then* $\text{ind } A = \text{ind } A_\lambda \cup \text{ind } A_\rho$.
(b) An algebra A is ada if and only if $\text{ind } A = \text{ind } A_\lambda \cup \mathcal{R}_A = \text{ind } A_\rho \cup \mathcal{L}_A$

Proof. (a) Follows from the theorem and from $\mathcal{R}_A \subseteq \text{ind } A_\rho$.
(b) Follows from the theorem and its dual. □

Notice that (b) above is [2] (2.5).

Theorem (4.2) will be our chief tool for proving our main theorem in section 5 below. For now, we apply it to obtain a characterisation of right ada algebras which are laura or weakly shod. Recall that an algebra A is *laura* if $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\text{ind } A$, see [3] and it is *weakly shod* if there exists a bound on the length of any path of non-isomorphisms in $\text{ind } A$ from an indecomposable injective to an indecomposable projective module [16].

Example 4.4. In general, right ada algebras are not laura. Here is an example. Let A be the radical square zero algebra given by the quiver

$$1 \leftleftarrows 2 \leftleftarrows 3 \leftleftarrows 4$$

Lemma 4.5. *Let A be a right ada algebra. If A is laura, then it is a weakly shod algebra.*

Proof. We may assume that A is a strict. Because A is laura, $\Gamma(\text{mod } A)$ admits a unique faithful non-semiregular component Γ. In order to prove that A is weakly shod, it suffices to prove that Γ contains no oriented cycles, see [5] (5.12). Let M be an indecomposable lying in a cycle in Γ. Because of [3] (1.5), we have $M \notin \mathcal{L}_A$ and $M \notin \mathcal{R}_A$. In particular,

$M \in \operatorname{ind} A_\lambda$ because of (4.2). But then the whole cycle lies in $\operatorname{ind} A_\lambda$. Because A_λ is a direct product of quasi-tilted algebras, then the cycle lies in a tube in $\Gamma(\operatorname{mod} A_\lambda)$. But then the cycle in Γ can be made arbitrarily large, using more modules from the tube. Therefore we get arbitrarily many indecomposables lying in $\operatorname{ind} A_\lambda \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$, a contradiction to A being laura. □

Corollary 4.6. *If A is a representation-finite right ada algebra, then A is weakly shod.*

5. The Auslander-Reiten Components

Let A be a strict right ada algebra. Our objective now is to describe the components of $\Gamma(\operatorname{mod} A)$. Because A is strict, there exists $x \in A_0$ such that $P_x \notin \mathcal{L}_A$. Then $P_x \in \mathcal{R}_A$ and so is Ext-projective in $\operatorname{add}\ \mathcal{R}_A$. Therefore the set Σ of Ext-projectives in $\operatorname{add} \mathcal{R}_A$ is not empty. Decompose Σ as $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_m$ where we assume that each Σ_i is the non-empty set of Ext-projectives in $\operatorname{add} \mathcal{R}_A$ lying in the same connected component Γ_i of $\Gamma(\operatorname{mod} A)$. Observe that in general Σ_i is not connected.

Because of [1] (6.7), each Σ_i is a right section and $A/\operatorname{Ann}\Sigma_i$ is a product of tilted algebras admmiting Σ_i as disjoint union of complete slices.

We denote by $(\Gamma_i)_{\geq \Sigma_i}$ the full subquiver of Γ_i consisting of the successors of Σ_i and by $(\Gamma_i)_{\ngeq \Sigma_i}$ the full subquiver consisting of the non-successors. Because of the definition of Σ_i, all its successors lie in $\mathcal{R}_A$, thus are indecomposable A_ρ-modules. Our first lemma is as follows.

Lemma 5.1. *Let A be a right ada algebra.*

(a) If $P_x \in \Sigma_i$ is projective, then every projective successor of P_x lies in the same connected component of Σ_i.
(b) $(\Gamma_i)_{\geq \Sigma_i} = \Gamma_i \cap \mathcal{R}_A$.
(c) Let M be a proper predecessor of Σ. Then M is an A_λ-module and $M_A \notin \mathcal{R}_A$.

Proof. (a) Assume that we have a path $P_x \rightsquigarrow P_y$ with P_y projective. Because $P_x \in \mathcal{R}_A$, we have also $P_y \in \mathcal{R}_A$. Then [1](6.3), gives that the path $P_x \rightsquigarrow P_y$ can be refined to a path sectional path of irreducible morphisms. This implies the statement.
(b) Assume $M \in (\Gamma_i)_{\geq \Sigma_i}$. We have already pointed out that $M \in \mathcal{R}_A$. Clearly $M \in \Gamma_i$ so that $M \in \Gamma_i \cap \mathcal{R}_A$.

Conversely, if $M \in \Gamma_i \cap \mathcal{R}_A$ then, by [1](6.6), there exists $m \geq 0$ such that $\tau_A^m M \in \Sigma_i$. Therefore, $M \in (\Gamma_i)_{\geq \Sigma_i}$.
(c) Let M be a proper predecessor of Σ. Then $M \notin \mathcal{R}_A$, because of (b) above. Let $x \in A_0$ be such that $\operatorname{Hom}_A(P_x, M) \neq 0$. We claim that $P_x \in \mathcal{L}_A$. For this, it suffices to prove that $P_x \notin \mathcal{R}_A$. However, if $P_x \in \mathcal{R}_A$ then $\operatorname{Hom}_A(P_x, M) \neq 0$ and the existence of a nontrivial path $M \rightsquigarrow N$, with $N \in \Sigma$ yields a composed path $P_x \to M \rightsquigarrow N$. But $P_x \in \mathcal{R}_A$ and projective give $P_x \in \Sigma$. Therefore $M \in \Sigma$ because of [1](6.3). This is a contradiction. Therefore $P_x \notin \mathcal{R}_A$. □

We are now in position to prove the main theorem. By component of $\Gamma(\operatorname{mod} A)$, we mean a connected component.

Theorem 5.2. *Let A be a strict right ada algebras. There exists a finite family $(\Gamma_i)_{i=1}^t$ of components of $\Gamma(\operatorname{mod} A)$ containing right sections $(\Sigma_i)_{i=1}^t$, respectively, such that:*

(a) *Each $(\Gamma_i)_{\geq \Sigma_i} = \Gamma_i \cap \mathcal{R}_A$ is directed, generalised standard and convex in $\operatorname{ind} A$.*
(b) *For each i, $(\Gamma_i)_{\ngeq \Sigma_i} \subseteq \operatorname{ind} A_\lambda \setminus \mathcal{R}_A$.*

(c) *If* Γ *is a component of* $\Gamma(\mathrm{mod}\, A)$ *distinct from the* Γ_i*, then either* Γ *is a component of* $\Gamma(\mathrm{mod} A_\lambda)$*, or is entirely contained in* $\mathcal{R}_A$ *(and in this case is a component of* $\Gamma(\mathrm{mod} A_\rho)$*).*
(d) *If moreover* $\mathrm{Hom}_A(\Gamma, \cup_i \Gamma_i) \neq 0$*, then* Γ *is a component of* $\Gamma(\mathrm{mod} A_\lambda)$*.*
(e) *Let* M *be an indecomposable* A*-module. Then* $M \notin \mathcal{L}_A \cup \mathcal{R}_A$ *if and only if there exist an indecomposable projective module* $P_A \in \Sigma$*, an indecomposable injective module* $I_A \in \mathrm{ind} A_\lambda$ *and two paths* $I \rightsquigarrow M$*,* $M \rightsquigarrow P$ *which are not refinable to sectional paths.*

Proof. (a) Because Σ_i is a right section in Γ_i, then $(\Gamma_i)_{\geq \Sigma_i}$ is directed and generalised standard, see [1](2.2) and (2.3). We have already shown that $(\Gamma_i)_{\geq \Sigma_i} = \Gamma_i \cap \mathcal{R}_A$. There remains to prove the convexity of $(\Gamma_i)_{\geq \Sigma_i}$. Assume that we have a path in $\mathrm{ind} A$:

$$M = M_0 \xrightarrow{f_1} M_1 \to \dots \xrightarrow{f_t} M_t = N$$

with $M, N \in (\Gamma_i)_{\geq \Sigma_i}$ but $M_1, \dots, M_{t-1} \notin (\Gamma_i)_{\geq \Sigma_i}$. Because $M \in \mathcal{R}_A$ then $M_{t-1} \in \mathcal{R}_A$. Because of (5.1) (b), we must have $M_{t-1} \notin \Gamma_i$. Therefore $f_t \in \mathrm{rad}^\infty_A(M_{t-1}, M_t)$. Then, for any $s \geq 0$, we have a path $M_{t-1} \xrightarrow{h_s} N_s \xrightarrow{g_s} \dots \to N_1 \xrightarrow{g_1} N_0 = N$ in $\mathrm{ind} A$, with $g_1, \dots, g_s$ irreducible and $h_s \in \mathrm{rad}^\infty_A(M_{t-1}, N_s)$ such that $g_1 \cdots g_s h_s \neq 0$. Because $N \in (\Gamma_i)_{\geq \Sigma_i}$ there exists s such that $N_s \notin (\Gamma_i)_{\geq \Sigma_i}$. In particular, $N_s \in \Gamma_i$. Then (5.1) (b) yields $N_s \notin \mathcal{R}_A$. But $M_{t-1} \in \mathcal{R}_A$, and this is a contradiction.

(b) This follows from (5.1)(b). Indeed, $(\Gamma_i)_{\ngeq \Sigma_i} \subseteq \mathrm{ind}\, A \setminus \mathcal{R}_A$. Because of (4.2), we deduce $(\Gamma_i)_{\geq \Sigma_i} \subseteq \mathrm{ind}\, A_\lambda$. Therefore $(\Gamma_i)_{\geq \Sigma_i} \subseteq \mathrm{ind}\, A_\lambda \setminus \mathcal{R}_A$.

(c) Let Γ be a component of $\Gamma(\mathrm{mod}\ A)$, distinct from $(\Gamma_i)_{i=0}^t$. Then $\Gamma \cap \Sigma = \emptyset$. Because of [1](Theorem B), we have either $\Gamma \subseteq \mathcal{R}_A$ or $\Gamma \cap \mathcal{R}_A = \emptyset$. In the first case, Γ is a component of $\Gamma(\mathrm{mod}\ A_\rho)$ contained in $\mathcal{R}_A$. In the second case, because of (4.2), every $M \in \Gamma$ is an A_λ-module so Γ is a component of $\Gamma(\mathrm{mod}\ A_\lambda)$.

(d) Assume now that Γ satisfies $\mathrm{Hom}_A(\Gamma, \cup_{i=1}^t \Gamma_i) \neq 0$. If Γ is not a component of $\Gamma(\mathrm{mod} A_\lambda)$ then, because of (4.2), Γ contains an indecomposable module $M \in \mathcal{R}_A$. Therefore $\Gamma \cap \mathcal{R}_A \neq \emptyset$. Because of [1](Theorem B), then $\Gamma \subseteq \mathcal{R}_A$.

Because $\mathrm{Hom}_A(\Gamma, \cup_{i=1}^t \Gamma_i) \neq 0$, there exist $M \in \Gamma$, an index i with $1 \leq i \leq t$ and $N \in \Gamma_i$ such that $\mathrm{Hom}_A(M, N) \neq 0$. Because of (5.1) (b), we have $N \in (\Gamma_i)_{\geq \Sigma_i}$. Let $f : M \to N$ be a nonzero morphism. Because $\Gamma \neq \Gamma_i$, we have $f \in \mathrm{rad}\,{}^\infty_A(M, N)$. For any $s \geq 0$, there exists a path in $\mathrm{ind} A$

$$M \xrightarrow{h_s} N_s \xrightarrow{g_s} \dots \to N_1 \xrightarrow{g_1} N_0 = N$$

with $g_1, \dots, g_s$ irreducible and $h_s \in \mathrm{rad}\,{}^\infty_A(M, N_s)$ such that $g_1 \dots g_s h_s \neq 0$. Then there exists $s \geq 0$ such that $N_s \in (\Gamma_i)_{\ngeq \Sigma_i}$. Then (5.1) (b) gives $N_s \notin \mathcal{R}_A$, a contradiction to the fact that $M \in \mathcal{R}_A$.

(e) Assume $M \notin \mathcal{L}_A \cup \mathcal{R}_A$. Because $M \notin \mathcal{R}_A$, it has a successor N such that $\mathrm{id}\, N \geq 1$. Because of [10] (IV.2.7), there exist an indecomposable projective A-module P and a path

$$M \rightsquigarrow N \to * \to \tau_A^{-1} N \to P.$$

Because $M \notin \mathcal{L}_A$, then $P \notin \mathcal{L}_A$. Because A is right ada, then $P \in \mathcal{R}_A$ and so $P \in \Sigma$ (because it is necessarily Ext-projective in $\mathcal{R}_A$).

Because $M \notin \mathcal{L}_A$, there exist, similarly, a predecessor L of M and an indecomposable injective A-module I such that we have a path

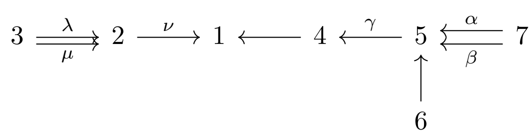
$$I \to \tau_A L \to * \to L \rightsquigarrow M.$$

Because $N \notin \mathcal{R}_A$ then $I \notin \mathcal{R}_A$. Therefore I is an A_λ-module.

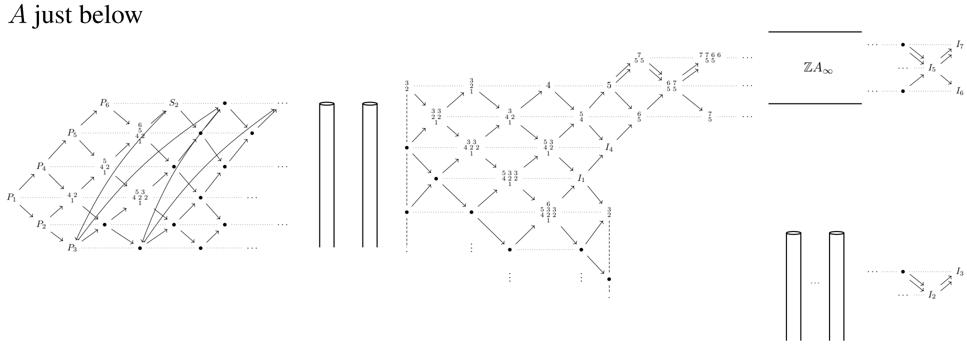
Conversely, assume that two paths as in the statement exists. Because $P \in \Sigma$, it follows from [1] (6.3) that $M \notin \mathcal{R}_A$: indeed, if $M \in \mathcal{R}_A$, then every path from M to P is refinable to a sectional path, a contradiction. On the other hand, if $M \in \mathcal{L}_A$ then $I \in \mathcal{L}_A$. But I, being injective is Ext-injective in $\mathcal{L}_A$. Hence every path $I \rightsquigarrow M$ is refinable to a sectional path, a contradiction. Therefore $M \notin \mathcal{L}_A$. □

Thus, the Auslander-Reiten components of a right ada algebra can be divided into three types: those which are components of the lef support algebra, those which are components of the right support algebra and the finite family of components containing the right section Σ_i. Because the left and the right supports are direct products of quasi-tilted algebras, they can be considered as known. One nice consequence of the theorem is the fact that right ada algebras always admit postprojective components: this indeed follows from the fact that A_λ always admits postprojective components.

Example 5.3. Let A be the following finite dimensional right ada algebra given by the quiver



bounded by $\lambda\nu = 0$, $\mu\nu = 0$, $\alpha\gamma = 0$, $\beta\gamma = 0$. We show the Auslander-Reiten quiver of A just below



In this example, we see that $\mathrm{ind} A_\lambda \cap \mathcal{R}_A \neq \varnothing$ because $I_4 \in \mathrm{ind} A_\lambda \cap \mathcal{R}_A$. Here A_λ is the algebra generated by $\{1, 2, 3, 4, 5, 6\}$ while A_ρ is the direct product of the algebras generated by $\{2, 3\}$ and $\{4, 5, 6, 7\}$. The reader will notice a component obtained by gluing a coray tube of A_λ with a posprojective component of A_ρ.

REFERENCES

[1] Ibrahim Assem. Left sections and the left part of an Artin algebra. *Colloq. Math.*, 116(2):273–300, 2009.
[2] Ibrahim Assem, Diane Castonguay, Marcelo Lanzilotta, and Rosana R. S. Vargas. Algebras determined by their supports. *J. Pure Appl. Algebra*, 216(5):1134–1145, 2012.
[3] Ibrahim Assem and Flávio U. Coelho. Two-sided gluings of tilted algebras. *J. Algebra*, 269(2):456–479, 2003.
[4] Ibrahim Assem and Flávio U. Coelho. Endomorphism algebras of projective modules over laura algebras. *J. Algebra Appl.*, 3(1):49–60, 2004.
[5] Ibrahim Assem, Flávio U. Coelho, Marcelo Lanzilotta, David Smith, and Sonia Trepode. Algebras determined by their left and right parts. In *Algebraic structures and their representations. Proceedings of 'XV coloquio Latinoamericano de álgebra', Cocoyoc, Morelos, México, July 20–26, 2003.*, pages 13–47. Providence, RI: American Mathematical Society (AMS), 2005.
[6] Ibrahim Assem, Flávio U. Coelho, and Sonia Trepode. The left and the right parts of a module category. *J. Algebra*, 281(2):518–534, 2004.

[7] Ibrahim Assem, Flávio U. Coelho, and Heily Wagner. On subcategories closed under predecessors and the representation dimension. *J. Algebra*, 418:174–196, 2014.

[8] Ibrahim Assem and Marcelo Lanzilotta. The simple connectedness of a tame weakly shod algebra. *Comm. Algebra*, 32(9):3685–3701, 2004.

[9] Ibrahim Assem, Marcelo Lanzilotta, and María Julia Redondo. Laura skew group algebras. *Comm. Algebra*, 35(7):2241–2257, 2007.

[10] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[11] Ibrahim Assem and Dan Zacharia. On split-by-nilpotent extensions. *Colloq. Math.*, 98(2):259–275, 2003.

[12] M. Auslander and Sverre O. Smalø. Almost split sequences in subcategories. *J. Algebra*, 69:426–454, 1981, Addendum 71 (1981) 592-594.

[13] Maurice Auslander. Representation dimension of Artin algebras. With the assistance of Bernice Auslander. Queen Mary College Mathematics Notes. London: Queen Mary College. 179 p. (1971)., 1971.

[14] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[15] K. Bongartz and P. Gabriel. Covering spaces in representation-theory. *Invent. Math.*, 65:331–378, 1982.

[16] Flávio U. Coelho and Marcelo A. Lanzilotta. Weakly shod algebras. *J. Algebra*, 265(1):379–403, 2003.

[17] Flávio Ulhoa Coelho and Marcelo Américo Lanzilotta. Algebras with small homological dimensions. *Manuscripta Math.*, 100(1):1–11, 1999.

[18] Dieter Happel. Hochschild cohomology of finite-dimensional algebras. Séminaire d'algèbre P. Dubreil et M.-P. Malliavin, Proc., Paris/Fr. 1987/88, Lect. Notes Math. 1404, 108-126 (1989)., 1989.

[19] Dieter Happel and Idun Reiten. Hereditary abelian categories with tilting object over arbitrary base fields. *J. Algebra*, 256(2):414–432, 2002.

[20] Dieter Happel, Idun Reiten, and Sverre O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.*, 120(575):viii+ 88, 1996.

[21] Idun Reiten and Christine Riedtmann. Skew group algebras in the representation theory of Artin algebras. *J. Algebra*, 92(1):224–282, 1985.

[22] Andrzej Skowroński. Simply connected algebras and Hochschild cohomologies. In *Representations of algebras. Proceedings of the sixth international conference on representations of algebras, Carleton University, Ottawa, Ontario, Canada, August 19-22, 1992*, pages 431–447. Providence, RI: American Mathematical Society, 1993.

(Edson Ribeiro Alvares) Centro Politécnico, Departamento de Matemática, Universidade Federal do Paraná, CP019081, Jardim das Americas, Curitiba-PR, 81531-990, Brazil
E-mail address: rolo1rolo@gmail.com, rolo@ufpr.br

(Ibrahim Assem) Département de Mathématiques, Université de Sherbrooke, Sherbrooke (Québec), J1K 2R1, Canada
E-mail address: ibrahim.assem@usherbrooke.ca

(Diane Castonguay) Instituto de Informática, Universidade Federal de Goiás, Campus II - Samambaia, CEP: 74001-970, Goiânia, Brazil
E-mail address: diane@inf.ufg.br

(Rosana Vargas) Rosana Vargas, Escola de Artes, Ciências e Humanidades (EACH), Universidade de São Paulo, CEP: 03828-000, São Paulo, Brazil
E-mail address: rosanav@usp.br