Communications in Algebra®, 35: 1075–1086, 2007

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870500243072

QUOTIENTS OF INCIDENCE ALGEBRAS AND THE EULER CHARACTERISTIC

Ibrahim Assem
Département de Mathématiques, Université de Sherbrooke, Sherbrooke, Québec, Canada

Diane Castonguay and Eduardo N. Marcos
Departamento de Matemática, IME, Universidade de São Paulo, São Paulo, Brazil

Sonia Trepode
Departamento de Matemáticas, Universidad Nacional de Mar del Plata, Mar del Plata, Argentina

In this note, we show that, if $A \cong kQ_A/I_A$ is a schurian strongly simply connected algebra given by its normed presentation, and Σ is the unique poset whose Hasse quiver coincides with Q_A, then $A \cong k\Sigma$ if and only if I_A has a generating set consisting of exactly $\chi(Q_A)$ elements, where $\chi(Q_A)$ is the Euler characteristic of Q_A. We also prove that a quotient of an incidence algebra $A = k\Sigma/J$ is strongly simply connected if and only if A is simply connected and $k\Sigma$ is strongly simply connected.

Key Words: Euler characteristic; Incidence algebras; Strongly simply connected algebras.

2000 Mathematics Subject Classification: 16G20.

1. INTRODUCTION

In this note, we show how the Euler characteristic of a quiver can be used to study the strong simple connectedness of incidence algebras and their quotients.

Our motivation comes from the study of the finite dimensional algebras over an algebraically closed field k. For such an algebra A, there exists a quiver Q_A, and an ideal I of the path algebra kQ_A such that $A \cong kQ_A/I$. For each such pair (Q_A, I), called a presentation of A, one can define the fundamental group $\pi_1(Q_A, I)$, see for instance, Gabriel and Roiter (1992) and Martínez-Villa and de la Peña (1983). The algebra A is called simply connected if Q_A is acyclic and, for each presentation (Q_A, I), the group $\pi_1(Q_A, I)$ is trivial (Assem and Skowroński, 1988). If A is the incidence algebra of a (finite) poset Σ, then its quiver Q_A coincides with

Dedicated to Otto Kerner on his 60th birthday.

Received June 15, 2004; Revised September 15, 2004. Communicated by M. Ferrero.

Address correspondence to Eduardo N. Marcos, Departamento de Matemática, IME, Universidade de São Paulo, Caixa Postal 66281, São Paulo, Brazil; Fax: 55-11-818-618; E-mail: enmarcos@ime.usp.br

the Hasse quiver of Σ, see Gabriel and Roiter (1992). Moreover, all presentations of the incidence algebra A give rise to isomorphic fundamental groups Bardzell and Marcos (2002) and A is simply connected if and only if so is the associated simplicial complex (Bustamante, 2002; Reynaud, 2003). Simply connected representation-finite algebras were characterized in Bongartz and Gabriel (1981/82) and Bretscher and Gabriel (1983). Also, the simple connectedness of right multipeak algebras, a class containing that of incidence algebras, was studied in Kasjan (1999). While it is difficult to find criteria for an algebra to be simply connected (see, for instance, Assem et al., 2003), one subclass seems easier to handle: this is the class of strongly simply connected algebras, introduced by Skowroński in Skowroński (1993). An algebra A is called strongly simply connected if every full convex subcategory of A is simply connected. Characterizations of strongly simply connected algebras have been given in Assem and Liu (1998). In particular, an incidence algebra is strongly simply connected if and only if its quiver contains no crowns (Dräxler, 1994) or, equivalently, if and only if the corresponding poset is dismantlable (Rival, 1976).

It was already observed in Dräxler (1994) that, if A is a schurian strongly simply connected algebra, then there exists a unique poset Σ, whose Hasse quiver coincides with Q_A. As a consequence of this result and Assem and Liu (1998, 2.4), there exists a presentation of A, called normed presentation, so that A is the quotient of the incidence algebra $k\Sigma$ by an ideal generated by paths.

In this article, using properties of the Euler characteristic $\chi(Q_A)$ of the quiver Q_A, we first show that a quotient A of an incidence algebra $k\Sigma$ is strongly simply connected if and only if it is simply connected and $k\Sigma$ is strongly simply connected. We next prove that, if $A = kQ_A/I$ is a schurian strongly simply connected algebra, then the ideal I has at least $\chi(Q_A)$ generators and it reaches this number if and only if A is an incidence algebra. In a forthcoming article (Assem et al., in preparation), we give several new characterizations of schurian strongly simply connected algebras.

The article is organized as follows. Section 2 is devoted to fixing the notation and briefly recalling the necessary concepts and results. In Section 3, we show that, if $k\Sigma$ is a strongly simply connected incidence algebra, then $k\Sigma \cong kQ_\Sigma/I_\Sigma$, where I_Σ has a generating set consisting of exactly $\chi(Q_\Sigma)$ elements. We prove our main results in Section 4.

2. PRELIMINARIES

2.1. Notation. In this article, by algebra, we always mean a basic and connected finite dimensional algebra over an algebraically closed field k. Given a quiver Q, we denote by Q_0 its set of points and by Q_1 its set of arrows. A *relation* in Q from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each i, $\lambda_i \in k$ is nonzero and w_i is a path of length at least two from x to y. A relation ρ as before is called *monomial* if $m = 1$, *binomial* if $m = 2$, and a *commutativity relation* if it equals the difference of two paths. We denote by kQ the path algebra of Q and by $kQ(x, y)$ the k-vector space generated by all paths in Q from x to y. For an algebra A, we denote by Q_A its quiver. For every algebra A, there exists an ideal I in kQ_A, generated by a set of relations, such that $A \cong kQ_A/I$. The pair (Q_A, I) is called a *presentation* of A. An algebra $A = kQ/I$ can equivalently be considered as a k-category of which the object class A_0 is Q_0, and the set of morphisms $A(x, y)$ from

x to y is the quotient of $kQ(x, y)$ by the subspace $I(x, y) = I \cap kQ(x, y)$, see Bongartz and Gabriel (1981/82). A full subcategory B of A is called *convex* if any path in A with source and target in B lies entirely in B. An algebra A is called *triangular* if Q_A is acyclic, and it is called *schurian* if, for all $x, y \in A_0$, we have $\dim_k A(x, y) \leq 1$. In this article, we deal exclusively with schurian triangular algebras. For a point x in the quiver Q_A of an algebra A, we denote by e_x the corresponding idempotent.

2.2. Simple Connectedness

Let Q be a connected acyclic quiver and I be an ideal of kQ generated by relations. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is called *minimal* if $m \geq 2$ and, for every nonempty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. For an arrow α, we denote by α^{-1} its formal inverse. A *walk* in Q from x to y is a formal composition $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \cdots \alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{1, -1\}$ for all i) starting at x and ending at y. The *fundamental group* $\pi_1(Q, I)$ is the quotient of the fundamental group $\pi_1(Q)$ of Q by the normal subgroup generated by all elements of the form $[\gamma^{-1} u^{-1} v \gamma]$, where γ is a walk from the base point to x and u, v are paths from x to y such that there exists a minimal relation $\sum \lambda_i w_i \in I(x, y)$ with $u = w_1$ and $v = w_2$, see Martínez-Villa and de la Peña (1983). It is well-known that $\pi_1(Q)$ is the free group in $\chi(Q)$ generators, where $\chi(Q) = 1 - |Q_0| + |Q_1|$ is the *Euler characteristic* of Q, see, for instance, Massey (1967).

A triangular algebra A is called *simply connected* if, for any presentation (Q_A, I) of A, the group $\pi_1(Q_A, I)$ is trivial (Assem and Skowroński, 1988). It is called *strongly simply connected* if every full convex subcategory of A is simply connected (Skowroński, 1993).

It is shown in Bardzell and Marcos (2002) that, if an algebra $A \cong kQ_A/I$ is schurian, then the fundamental group $\pi_1(Q_A, I)$ does not depend on the presentation (Q_A, I) of A. Since in this article we study only schurian triangular algebras, we therefore use the unambiguous notation $\pi_1(A)$ to stand for $\pi_1(Q_A, I)$.

2.3. Strong Simple Connectedness

We need the following notions and results from Assem and Liu (1998). Let Q be a connected acyclic quiver. A *contour* (p, q) in Q from x to y is a pair of parallel paths of positive length from x to y. A contour (p, q) is called *interlaced* if p and q have a common point besides x and y. A contour (p, q) is called *irreducible* if there exists no sequence of paths $p = p_0, p_1, \ldots, p_m = q$ in Q from x to y such that, for each i, the contour (p_i, p_{i+1}) is interlaced. A cycle C in Q is called *irreducible* if, either C is an irreducible contour, or C is not a contour, but satisfies the following condition and its dual: for each source x of C, no proper successor of x in Q is also a source of C, and exactly two proper successors of x in Q are sinks of C. This is easily seen to be equivalent to the definition of irreducibility given in Assem and Liu (1998, 1.5). It is proven in Assem and Liu (1998, 2.1) that, if $A = kQ/I$ is any presentation of a schurian strongly simply connected algebra, and if (p, q) is a contour in Q, then the path p lies in I if and only if the path q lies in I. Moreover, we have the following theorem.

Theorem (Assem and Liu, 1998, 2.4). *An algebra A is schurian and strongly simply connected if and only if*

(a) *all irreducible cycles are irreducible contours, and*
(b) *there exists a presentation $A \cong kQ_A/I_A$ such that for each irreducible contour (p, q), we have $p, q \notin I_A$ but $p - q \in I_A$.* □

Such a presentation (in which every irreducible contour, and hence every contour, is commutative) is called a *normed presentation* of A.

3. QUOTIENTS OF INCIDENCE ALGEBRAS

3.1.

Let $(\Sigma, \leq)$ be a finite poset with n elements. The incidence algebra $k\Sigma$ of Σ is the subalgebra of the algebra $M_n(k)$ of all $n \times n$ matrices over k consisting of the matrices $[a_{ij}]$ satisfying $a_{ij} = 0$ if $j \not\leq i$. The quiver Q_Σ of $k\Sigma$ is the Hasse quiver of Σ, and $k\Sigma \cong kQ_\Sigma/I_\Sigma$, where I_Σ is generated by all differences $p - q$, for (p, q) a contour in Q_Σ.

In this article, a quotient of an incidence algebra means an algebra A which admits a presentation $A \simeq kQ_A/I$ such that there exists a poset Σ with $Q_\Sigma = Q_A$ and, furthermore, $I = I_\Sigma + J$, where J is an ideal of kQ_Σ generated by monomials in the radical square of the algebra. By abuse of language, we identify J with an ideal of $k\Sigma$ and write $A = k\Sigma/J$. Such an algebra $A = k\Sigma/J$ is schurian, triangular and satisfies the following condition: for any contour (p, q) in Q_A, we have $p \notin I_\Sigma + J$ if and only if $q \notin I_\Sigma + J$. In this case, we say that (p, q) is a nonzero contour in A. We also observe that such a presentation of $k\Sigma/J$, generated by monomials and differences of parallel paths, is a normed presentation in the sense of 2.3.

In the next subsection, we consider such an algebra $A = k\Sigma/J$ and construct inductively a set of generators of the ideal $I_\Sigma + J$.

3.2.

We start by constructing a set of contours, following an idea inspired from Assem et al. (2003, 5.2). Let x be a source in Q_A, and $x^{\rightarrow}$ be the set of all arrows of source x. Let $\approx$ be the least equivalence on $x^{\rightarrow}$ such that $\alpha \approx \beta$ if there exist paths u, v in Q_Σ such that $(\alpha u, \beta v)$ is a contour Assem and de la Peña (1996, 2.1). Let t be the cardinality of $[\alpha]$. We construct by induction on $s \leq t$ a family of $t - 1$ irreducible contours of source x.

If $s = 1$, we let $\alpha_1 \in [\alpha]$ be arbitrary. If $t = 1$, we have finished. Otherwise, we consider the set C_1 of all pairs (β, y), where $\beta \in x^{\rightarrow}$ and $y \in (Q_\Sigma)_0$ are such that there exists a nonzero non-interlaced contour $(\alpha_1 p, \beta q)$ from x to y. We let $M_1 = \{y | (\beta, y) \in C_1\}$. Since $t > 1$, we have $C_1 \neq \emptyset$. We thus choose a pair $(\alpha_2, y_2) \in C_1$ such that y_2 is maximal in the set M_1 and we also choose a nonzero non-interlaced contour $(\alpha_1 p_{1,2}, \alpha_2 q_{1,2})$ from x to y_2.

We claim that this contour $(\alpha_1 p_{1,2}, \alpha_2 q_{1,2})$ is in fact irreducible. Indeed, if this is not the case, then there exists a sequence $\alpha_1 p_{1,2} = \beta_1 w_1, \beta_2 w_2, \ldots, \beta_r w_r = \alpha_2 q_{1,2}$ with $r \geq 3$ such that each pair $(\beta_i w_i, \beta_{i+1} w_{i+1})$ is interlaced. Since $\alpha_1 \neq \alpha_2$, there exists a least i such that $\beta_i \neq \beta_{i+1}$. Also, there exists a point z on both $\beta_i w_i$ and $\beta_{i+1} w_{i+1}$ and a non-interlaced contour (u, v) from x to z such that $\beta_i w_i = u w'_i$, $\beta_{i+1} w_{i+1} = v w'_{i+1}$ for some paths w'_i, w'_{i+1}. It follows from the maximality of y_2 in M_1 that $A(z, y_2) = 0$ or else $u = 0$ in A (and this is equivalent to saying that $A(x, z) = 0$, because A is a quotient of an incidence algebra). In either case, $A(x, y_2) = 0$, a contradiction. This establishes our claim.

Assume inductively that we have a sequence $\{\alpha_1, \ldots, \alpha_s\}$ of arrows of $[\alpha]$ as well as $s - 1$ nonzero non-interlaced contours starting at these arrows. If $s = t$, we have finished. Otherwise, we let C_s be the set of all pairs (β, y), with $\beta \in x^{\rightarrow}$ and $y \in (Q_\Sigma)_0$ such that $\beta \notin \{\alpha_1, \ldots, \alpha_s\}$ and there exists an $i \in \{1, \ldots, s\}$ and a nonzero non-interlaced contour $(\alpha_i p, \beta q)$ from x to y. We let $M_s = \{y | (\beta, y) \in C_s\}$. Since $t > s$, we have $C_s \neq \emptyset$. We thus choose $(\alpha_{s+1}, y_{s+1}) \in C_s$ such that y_{s+1} is maximal in M_s, and a nonzero non-interlaced contour $(\alpha_{i_{s+1}} p_{i_{s+1}, s+1}, \alpha_{s+1} q_{i_{s+1}, s+1})$ from x to y_{s+1}, where $1 \leq i_{s+1} \leq s$.

As before, the maximality of the points y_{s+1} in the sets M_s implies that, in fact, each of the contours chosen is irreducible.

We denote by $I_{[\alpha]}$ the subideal of $I_\Sigma + J$ generated by all elements of the form $\alpha_{i_s} p_{i_s, s} - \alpha_s q_{i_s, s}$, where $(\alpha_{i_s} p_{i_s, s}, \alpha_s q_{i_s, s})$ is a contour of the above set. Thus $I_{[\alpha]}$ is generated by exactly $t - 1$ elements and each of these generators is the commutativity relation of an irreducible contour.

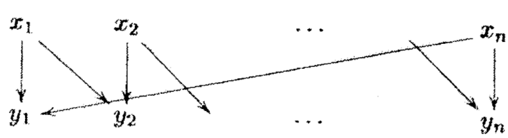
Lemma. *Let x be a source in A, and $(\alpha p, \beta q)$ be a nonzero irreducible contour of source x. Then there exists a set of generators of $I_{[\alpha]}$, constructed as before, containing the element $\alpha p - \beta q$.*

Proof. In the above construction, let $\alpha_1 = \alpha$. Then $\beta \approx \alpha$. Since $(\alpha p, \beta q)$ is an irreducible contour from x to y (say), it is not interlaced, so that $(\beta, y) \in C_1$ and y is a maximal element in M_1. We may thus take $(\alpha_2, y_2) = (\beta, y)$ and $(\alpha_1 p_{1,2}, \alpha_2 q_{1,2}) = (\alpha p, \beta q)$. □

3.3.

Let A be a schurian triangular algebra, the *interval* $[x, y]$ between x and y is the full subcategory of A generated by all points $z \in A_0$ which lie on a nonzero path from x to y, that is, such that $A(x, z)A(z, y) \neq 0$. Clearly, if all paths from x to y in A are nonzero, then $[x, y]$ coincides with the full subcategory of A generated by the convex hull of x and y. This is the case, for instance, whenever A is an incidence algebra.

Generalizing Assem et al. (2003, 3.1) (see also Assem et al., in preparation), we define a notion of a full weak crown in a quotient of an incidence algebra A. Let C be a full subcategory of A generated by $2n$ points $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, with $n \geq 2$, and of the form:



We say that C is a *full weak crown* in A if:

(a) $[x_i, y_j] \cap [x_h, y_l] \neq \emptyset$ if and only if $j = i$ and $(h, l) \in \{(i, i), (i-1, i), (i, i+1)\}$ or $j = i+1$ and $(h, l) \in \{(i, i+1), (i, i), (i+1, i+1)\}$ (we agree to set $y_{n+1} = y_1$ and $x_0 = x_n$), and

(b) the intersection of three distinct $[x_h, y_l]$ is empty.

A full weak crown C as above is called a *full crown* if, for each i, $[x_i, y_i] \cap [x_i, y_{i+1}] = \{x_i\}$ and $[x_i, y_i] \cap [x_{i-1}, y_i] = \{y_i\}$.

As in Assem et al. (2003, 3.2) (see also Assem et al., in preparation), we easily prove that the convex hull of each full weak crown C contains a full crown as a full subcategory.

A point $x \in A_0$ is said to *top* a full weak crown C if x is a direct predecessor of each of the x_i.

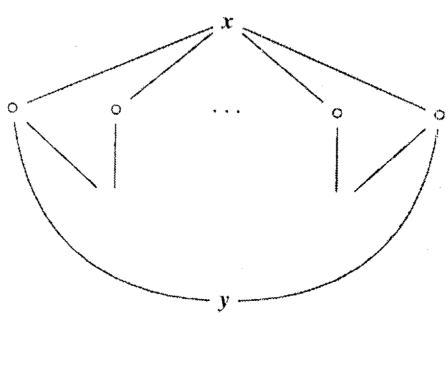
Let now $A = k\Sigma/J$ and x be a source in A. Let $B = A/\langle e_x \rangle$. We observe that, if $A = kQ_A/I_A$ is a normed presentation of A, and $B \cong kQ_B/I_B$ is the induced presentation of B (that is, the one such that $I_B = I_A \cap kQ_B$), then the latter is a normed presentation of B. Finally, we recall that the source x is separating if the number of indecomposable summands of $\text{rad}(e_xA)$ equals the number of connected components of B.

Lemma. *Let $A = kQ_A/I_A$ be a quotient of an incidence algebra, given its normed presentation, x be a separating source in A, topping no full weak crown, and $B = A/\langle e_x \rangle$. If $B \cong kQ_B/I_B$ is the induced presentation, then*

$$I_A = I_B + \sum_{[\alpha]} I_{[\alpha]} + \langle m_1, \ldots, m_r \rangle$$

where the m_i are paths starting at x, and the sum is taken over all equivalence classes of arrows in $x^{\rightarrow}$.

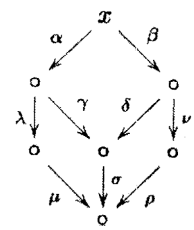
Proof. We may clearly assume that $x^{\rightarrow}$ contains only one equivalence class and that $|A_0| \geq 2$. Let $t = |x^{\rightarrow}|$. If $t = 1$, $I_{[\alpha]} = 0$ and $I_A = I_B + \langle m_1, \ldots, m_r \rangle$ for some paths m_i starting at x. Assume $t > 1$, and let (p, q) be a nonzero irreducible contour starting at x. There exist i, j such that $1 \leq i, j \leq t$ and paths u, v such that $p = \alpha_i u$ and $q = \alpha_j v$. By the construction in 3.2, there exists a unique sequence $\{\alpha_i = \alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_s} = \alpha_j\}$ such that each pair $(\alpha_{i_h}, \alpha_{i_{h+1}})$ corresponds to a nonzero irreducible contour of target $y_{i_{h+1}}$ and this contour is one of the generators of $I_{[\alpha]}$. We have the following situation:



If there is no nonzero path between y and one of the y_{i_h}, then either there exists in A a full weak crown topped by x, or else, one of the y_{i_h} is not maximal, a contradiction in either case. Thus, there is a nonzero path between y and each of the y_{i_h}. Since the y_{i_h} are maximal points in the respective sets M_{i_h}, we deduce that $A(y_{i_h}, y) \neq 0$ for all h. This implies that the relation $p - q$ is induced from relations in $I_{[\alpha]}$ and relations in I_B.

Finally, if (p, q) is a zero contour, then we assume that p and q are among the m_i. This completes the proof. □

3.4. Example. The lemma above does not hold true if A is a schurian algebra which is not a quotient of an incidence algebra. Let A be given by the quiver



bound by $\gamma\sigma = 0$, $\delta\sigma = 0$, $\alpha\gamma = \beta\delta$ and $\alpha\lambda\mu = \beta\nu\rho$. Clearly, the relation $\alpha\lambda\mu = \beta\nu\rho$ is not induced from relations in I_B and relations in $I_{[\alpha]}$.

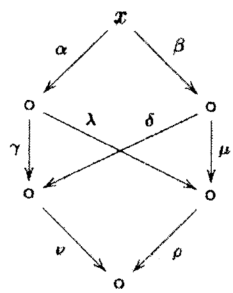
3.5. Corollary. *Let Σ be a poset, x be a separating source in $k\Sigma$ and $\Sigma' = \Sigma\backslash\{x\}$. Assume x tops no weak crown. Then $I_\Sigma = I_{\Sigma'} + \sum_{[\alpha]} I_{[\alpha]}$.* □

3.6. Proposition. *Let $k\Sigma \cong kQ_\Sigma/I_\Sigma$ be a strongly simply connected incidence algebra, and let (p, q) be a given irreducible contour in Q_Σ. Then there exists a set of generators of I_Σ, of cardinality $\chi(Q_\Sigma)$, consisting of the commutativity relations of irreducible contours and such that $p - q$ is one of these generators.*

Proof. By induction on $|\Sigma|$. If $|\Sigma| = 1$, then there is nothing to prove. Assume that $|\Sigma| > 1$ and that the result holds for all posets Ω such that $|\Omega| < |\Sigma|$. Let x be a maximal element of Σ and $\Sigma' = \Sigma\backslash\{x\}$. Without loss of generality, we may assume that Σ' is connected. Since $k\Sigma$ is strongly simply connected, the source x is separating, by Assem and de la Peña (1996, 2.6). Therefore all arrows of $x^{\rightarrow}$ are equivalent, by Assem and de la Peña (1996, 2.2). Moreover, Σ contains no full crown, by Dräxler (1994, 3.3). Hence, by Assem et al. (2003, 3.2), the point x tops no full weak crown. On the other hand, since $k\Sigma'$ is a full convex subcategory of $k\Sigma$, we have that $k\Sigma'$ is strongly simply connected.

Assume first that x is not the source of (p, q). Then we may, by the induction hypothesis, assume that $p - q$ belongs to a set of generators of $I_{\Sigma'}$, of cardinality $\chi(Q_{\Sigma'})$, and consisting of commutativity relations of irreducible contours lying entirely in Σ'. On the other hand, if x is a source of (p, q), we may, by the lemma in 3.2, assume that $p - q$ belongs to a set of generators of $I_{[\alpha]}$, of cardinality $|x^{\rightarrow}| - 1$ (recall that $x^{\rightarrow} = [\alpha]$). By 3.5, we have $I_\Sigma = I_{\Sigma'} + I_{[\alpha]}$ where $I_{\Sigma'}$ (or $I_{[\alpha]}$) is generated by the commutativity relations of irreducible contours lying entirely in Σ' (or of irreducible contours starting at x, respectively). Since $\chi(Q_{\Sigma'}) + |x^{\rightarrow}| - 1 = \chi(Q_{\Sigma})$, this completes the proof. □

3.7. Example. If $k\Sigma$ is an incidence algebra which is not strongly simply connected, then the conclusion of 3.6 may, or may not, hold true. Let Σ be the poset with quiver



Then $\chi(Q_{\Sigma}) = 1 - 6 + 8 = 3$, while it is easily seen that a set of generators of least cardinality for I_{Σ} is the set $\{\alpha\gamma - \beta\delta, \alpha\lambda - \beta\mu, \gamma\nu - \lambda\rho, \delta\nu - \mu\rho\}$, which has 4 elements. Note that $k\Sigma$ is simply connected, and even *separated and coseparated*. Thus, the conclusion of 3.6 does not hold true in this case.

On the other hand, let $\Sigma' = \Sigma \backslash \{x\}$. Then $\chi(Q_{\Sigma'}) = 1 - 5 + 6 = 2$, while $I_{\Sigma'}$ is generated by the set $\{\alpha\gamma - \beta\delta, \alpha\lambda - \beta\mu\}$, which has 2 elements. Note that $k\Sigma'$ is simply connected and separated (but not coseparated). Thus, the conclusion of 3.6 holds true in this case.

4. THE MAIN RESULTS

4.1. Lemma. *Let G be a free group in n generators, and S be a subset of G such that $|S| < n$. Then the normal subgroup H generated by S is a proper subgroup of G.*

Proof. Let G' be the derived subgroup of G, and assume $G = H$. Then $H/G' = G/G'$ is free abelian of rank n. However, since H/G' is abelian, it is generated by the cosets sG', with $s \in S$. Hence the rank of H/G' is at most $|S| < n$, and this yields a contradiction. □

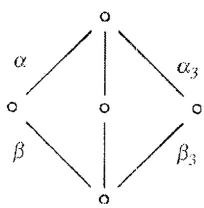
4.2.

Before proving the following lemma, we need a simple, but useful observation. Assume $A = kQ_A/I$, where the ideal I is generated by a set of relations of the form $S \cup M$, where M consists of monomials, while S consists of minimal relations. As seen in 2.2, $\pi_1(A) = \pi_1(Q_A)/N$, where N is the normal subgroup of $\pi_1(Q_A)$ generated by all uv^{-1}, with (u, v) a contour in Q_A. It is easily seen that in fact, N is the normal subgroup generated by all uv^{-1}, where (u, v) occurs in a minimal relation belonging to the set S (see Farkas et al., 2000).

Lemma. *Assume that $A = kQ_A/I$ is a schurian simply connected algebra. Let S be a set of minimal relations and M be a set of monomials such that I is generated by $S \cup M$. Then $|S| \geq \chi(Q_A)$.*

Proof. Since A is schurian, each minimal relation is binomial. Let N be the normal subgroup of $\pi_1(Q_A)$ generated by all uv^{-1}, with u, v are paths occurring in a binomial relation in S. It follows from 2.2 that $\pi_1(A) = \pi_1(Q_A)/N$. Now, the hypothesis implies that $\pi_1(Q_A) = N$. Since $\pi_1(Q_A)$ is free in $\chi(Q_A)$ generators, the result follows from 4.1. □

4.3. Example. The statement of the lemma is not true if A is not schurian. For instance, the algebra A given by the quiver Q_A

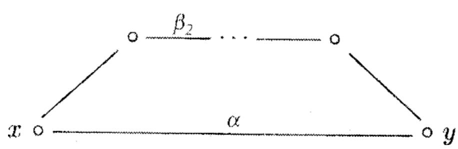


bound by $\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$ is such that $|S| = 1$, but $\chi(Q_A) = 2$.

4.4. Lemma. *Let A be a schurian strongly simply connected algebra. Then*:

(a) *There exists a unique poset $\Sigma = \Sigma(A)$ such that $Q_\Sigma = Q_A$.*
(b) *The incidence algebra $k\Sigma$ is strongly simply connected.*

Proof. (a) It suffices (see, for instance, Gatica and Redondo, 2001) to prove that Q_A contains no subquiver of the form



with α, β_1, ..., β_s arrows of Q_A. Assume that this is the case, and let $A \cong kQ_A/I$ be an arbitrary presentation of A. Since $\alpha \notin I$, we have that $\beta_1 \cdots \beta_s \notin I$, by 2.3. Since $I \subseteq \mathrm{rad}^2 kQ_A$, there is no relation involving α and $\beta_1 \cdots \beta_s$ and of the form $\lambda\alpha + \mu\beta_1 \cdots \beta_s = 0$ (where λ $\mu \in k$ are nonzero). Consequently, $\dim_k A(x, y) \geq 2$, a contradiction to A being schurian.

(b) Since $Q_\Sigma = Q_A$ and A is schurian strongly simply connected, every irreducible cycle in Q_Σ is an irreducible contour in Q_Σ (this property, indeed, depends on the quiver only). Since any contour in $k\Sigma$ is commutative, we deduce from 2.3 that $k\Sigma$ is strongly simply connected. □

4.5.

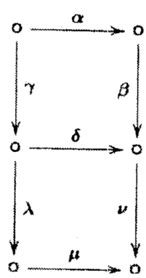
One consequence of this lemma is that every schurian strongly simply connected algebra is in fact a quotient of an incidence algebra, a fact already shown in Dräxler (1994, 2.7). In fact, we have the following stronger result.

Corollary. *An algebra A is schurian and strongly simply connected if and only if there exists a poset* Σ *such that* $k\Sigma$ *is strongly simply connected and* $A \cong k\Sigma/J$, *where J is an ideal of* $k\Sigma$ *generated by paths which are not entirely contained in irreducible contours.*

Proof. This follows easily from 4.4 and 2.3. □

Remark. If, in particular, $A = k\Sigma/J$ is a quotient of an incidence algebra, with $k\Sigma$ strongly simply connected, then A is strongly simply connected if and only if J is generated by paths which are not entirely contained in irreducible contours.

4.6. Example. Let A be given by the quiver



bounded by $\alpha\beta = \gamma\delta$, $\delta\nu = \lambda\mu$, $\gamma\lambda = 0$ and $\beta\nu = 0$. Here, $A \cong k\Sigma/J$, where $\Sigma = \Sigma(A)$ and J is generated by the paths $\beta\nu$ and $\gamma\lambda$. These paths are entirely contained in the contour $(\alpha\beta\nu, \gamma\lambda\mu)$. However, this contour is not irreducible. Accordingly, A is schurian strongly simply connected.

4.7.

As a first consequence, we have the following characterization of strongly simply connected quotient of an incidence algebras. We recall that the strong simple connectedness of incidence algebras is completely characterized by the absence of full crowns (see Dräxler, 1994, 3.3).

Theorem. *Let* $A = k\Sigma/J$ *be a quotient of an incidence algebra. Then A is strongly simply connected if and only if A is simply connected and* $k\Sigma$ *is strongly simply connected.*

Proof. Since the necessity follows from 4.4, we only need to prove the sufficiency. Assume that A is not strongly simply connected. By 4.5, there exists an irreducible contour (p, q) in Q_A such that $p \in J$ (or, equivalently, $q \in J$). By 3.6, there exists a set $\{\gamma_1, \ldots, \gamma_{\chi(Q_A)}\}$ of generators of I_Σ such that $\gamma_1 = p - q$. Let S be the subset of $\{\gamma_1, \ldots, \gamma_{\chi(Q_A)}\}$ corresponding to those contours in Q_A which are nonzero in A. Thus, $|S| < \chi(Q_A)$ since $\gamma_1 \notin S$. Hence, $I_\Sigma + J$ is generated by S and a set $\{m_1, \ldots, m_s\}$ of monomials, where $s \geq 2$ and m_1, m_2 are respectively subpaths of p, q. Now, $\pi_1(A) \cong \pi_1(Q_A)/N$, where N is a normal subgroup of $\pi_1(Q_A)$ generated by $|S|$ elements (see the remark in 4.2). Since $|S| < \chi(Q_A)$, the statement of 4.2 yields a contradiction to the simple connectedness of A. □

4.8.

Our second main result characterizes the incidence algebras among the schurian strongly simply connected algebras.

Theorem. *Let $A = kQ_A/I_A$ be a schurian strongly simply connected algebra, given its normed presentation, and Σ be the associated poset. Then $A \cong k\Sigma$ if and only if I_A has a generating set consisting of exactly $\chi(Q_A)$ elements.*

Proof. Since the necessity follows at once from 3.6, let us show the sufficiency. Assume that A is schurian and strongly simply connected, and that (Q_A, I_A) is a normed presentation of A. We further assume that I_A has a generating set consisting of exactly $\chi(Q_A)$ relations (such a set is then, by 4.2, of least possible cardinality). Let $\Sigma = \Sigma(A)$ be as in 4.4. It follows from the preceding discussion that $I_A = J + I_\Sigma$, where J is generated by paths in I_A and I_Σ is generated by all possible relations of the form $p - q$, with (p, q) an irreducible contour. Clearly, these two sets of generators are disjoint. Moreover, we get from 3.6 that I_Σ has a generating set consisting of $\chi(Q_\Sigma) = \chi(Q_A)$ elements, all of which are commutativity relations of irreducible contours. This set has as many elements as the given generating set of I_A. Since any generating set of I_A must involve the commutativity of all irreducible contours, we deduce that $J = 0$, that is, $A \cong k\Sigma$. □

ACKNOWLEDGMENTS

The first, second and third authors thank respectively NSERC of Canada, FAPESP and CNPq of Brazil. This work was done during several visits of the authors at the Universidade de São Paulo and the Université de Sherbrooke, they are grateful to each other and to their colleagues in these institutions for the hospitality.

REFERENCES

Assem, I., de la Peña, J. A. (1996). The fundamental groups of a triangular algebra. *Comm. Algebra* 24(1):187–208.

Assem, I., Skowroński, A. (1988). On some class of simply connected algebras. *Proc. London Math. Soc.* 56(3):417–450.

Assem, I., Liu, S. (1998). Strongly simply connected algebras. *J. Algebra* 207:449–477.

Assem, I., Castonguay, D., Marcos, E. N., Trepode, S. Schurian strongly simply connected algebras. In preparation.

Assem, I., Platzeck, M. I., Redondo, M. J., Trepode, S. (2003). Simply connected incidence algebras. *Discrete Maths* 269:333–355.

Bardzell, M. J., Marcos, E. N. (2002). $H^1(\Lambda)$ and presentations of finite dimensional algebras. *Representations of Algebras.* Lecture Notes in Pure Applied Math. Vol. 224. pp. 31–38.

Bongartz, K., Gabriel, P. (1981/82). Covering spaces in representation theory. *Invent. Math.* 65(3):331–378.

Bretscher, O., Gabriel, P. (1983). The standard form of a representation finite algebra. *Bull. Soc. Math. France* 111:21–40.

Bustamante, J. C. (2002). On the fundamental group of a schurian algebra. *Comm. Algebra* 30(11):5307–5329.

Dräxler, P. (1994). Completely separating algebras. *J. Algebra* 165(2)550–565.
Farkas, D., Geiss, C., Green, E., Marcos, E. (2000). Diagonalizable derivations of finite-dimensional algebras I. *Israel J. Math.* 117:157–183.
Gabriel, P., Roiter, A. V. (1992). Representations of finite-dimensional algebras. In: Keller, B., ed. *Encyclopaedia Math. Sci. 73, Algebra, VIII*. Berlin: Springer.
Gatica, M. A., Redondo, M. J. (2001). Hochschild cohomology and fundamental groups of incidence algebras. *Comm. Algebra* 29(5):2269–2283.
Kasjan, S. (1999). Simply connected right multipeak algebras and the separation property. *Colloq. Math.* 82:137–153.
Martínez-Villa, R., de la Peña, J. A. (1983). The universal cover of a quiver with relations. *J. Pure and Applied Algebra* 30(3):277–292.
Massey, W. S. (1967). *Algebraic Topology, an Introduction.* Berlin-Heidelberg-New York: Springer-Verlag.
Reynaud, E. (2003). Algebraic fundamental group and simplicial complexes. *J. Pure Applied Algebra* 177(2):203–214.
Rival, I. (1976). A fixed point theorem for finite partially ordered sets. *J. Comb. Th. Ser. A* 21(3):309–318.
Skowroński, A. (1993). Simply connected algebras and Hochschild cohomologies. *Representations of Algebras*. Ottawa, 1992, CMS Conf. Proc. 14. Providence, RI: Amer. Math. Soc., pp. 431–447.