On the Galois coverings of a cluster-tilted algebra *

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Abstract. We study the module category of a certain Galois covering of a cluster-tilted algebra which we call the cluster repetitive algebra. Our main result compares the module categories of the cluster repetitive algebra of a tilted algebra C and the repetitive algebra of C, in the sense of Hughes and Waschbüsch.

Keywords: Cluster-tilted algebra, Galois covering, repetitive algebra.

0. Introduction

The cluster category was introduced in (BMRRT, 2006) and also in (CCS1, 2006) for type $\mathbb{A}$, as a categorical model to understand better the cluster algebras of Fomin and Zelevinsky (FZ, 2002). It is a quotient of the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$ of the finitely generated modules over a finite dimensional hereditary algebra A. It was then natural to consider the endomorphism algebras of tilting objects in the cluster category. Such algebras are called cluster-tilted, and have been the subject of several investigations since their introduction in (BMR1, 2007; CCS1, 2006), see, for instance (BMR2, 2006; CCS2, 2006; KR, preprint; ABS1, preprint; ABS2, preprint; BFPT, preprint). In particular, it was shown in (ABS1, preprint) that the cluster-tilted algebras are trivial extensions of tilted algebras by a certain bimodule.

Now, the class of trivial extensions of tilted algebras by the minimal injective cogenerator has been extensively investigated. They play an important rôle in the classification results for self-injective algebras. In this study, one of the essential tools is the repetitive algebra, introduced by Hughes and Waschbüsch in (HW, 1983). In previous works (ABST1,

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2006; ABST2, to appear), we have related the cluster category and the m-cluster category to the repetitive algebra of a hereditary algebra.

Our initial motivation in this paper is different. Given a tilted algebra C, we wish to relate the trivial extension $T(C)$ of C by its minimal injective cogenerator DC and the corresponding cluster-tilted algebra $\tilde{C}$. Doing so has been difficult to achieve directly, so we decided to work instead with certain Galois coverings of these two algebras, the repetitive algebra $\hat{C}$ of C, which is a covering of $T(C)$, and the algebra $\check{C}$ constructed in a similar manner starting from $\tilde{C}$, which we call the cluster repetitive algebra.

Before stating our main theorem, we recall from (BMR1, 2007) that, if $\tilde{T}$ is a tilting object in the cluster category $\mathcal{C}_A$, and $\tilde{C} = \operatorname{End}_{\mathcal{C}_A}\tilde{T}$, then the functor $\operatorname{Hom}_{\mathcal{C}_A}(\tilde{T}, -) : \mathcal{C}_A \to \operatorname{mod}\tilde{C}$ induces an equivalence $\mathcal{C}_A/\operatorname{iadd}(\tau\tilde{T}) \cong \operatorname{mod}\tilde{C}$, where $\operatorname{iadd}(\tau\tilde{T})$ is the ideal consisting of all morphisms which factor through a direct sum of summands of the Auslander-Reiten translate $\tau\tilde{T}$ of $\tilde{T}$. Our main theorem says that this functor lifts to a functor $\operatorname{mod}\hat{C} \to \operatorname{mod}\check{C}$ which satisfies a similar condition. Namely, we give a different realisation of the cluster category, using only the tilted algebra C, which we denote as $\mathcal{C}_C$, then construct two functors $\phi : \operatorname{mod}\hat{C} \to \operatorname{mod}\check{C}$ and $\hat{\pi} : \operatorname{mod}\hat{C} \to \mathcal{C}_C$ as well as an ideal $\mathcal{J}$ of $\operatorname{mod}\hat{C}$ which satisfy the properties stated in the following theorem.

THEOREM 0.1. *Let C be a tilted algebra. Then there is a commutative diagram of dense functors*

$$\begin{array}{ccc} \operatorname{mod}\hat{C} & \xrightarrow{\ \phi\ } & \operatorname{mod}\check{C} \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle G_\lambda} \\ \mathcal{C}_C & \xrightarrow{\operatorname{Hom}_{\mathcal{C}_C}(\hat{\pi}C,-)} & \operatorname{mod}\tilde{C} \end{array}$$

where $G_\lambda : \operatorname{mod}\check{C} \to \operatorname{mod}\tilde{C}$ is the push-down functor associated to the covering $\check{C} \to \tilde{C}$. Moreover, ϕ is full and induces an equivalence of categories $\operatorname{mod}\hat{C}/\mathcal{J} \cong \operatorname{mod}\check{C}$.

Note that the functor G_λ is always dense: this is not true of the push-down functor $\operatorname{mod}\hat{C} \to \operatorname{mod}T(C)$ (see, for instance, (AS, 1993)).

As a consequence of this theorem, we are able to relate the Auslander-Reiten quivers of $\hat{C}$ and $\check{C}$, this yields the required relation between $T(C)$ and $\tilde{C}$.

The paper is organised as follows. After brief preliminaries, we start by introducing the notion of cluster repetitive algebra and study its most elementary properties in section 1. In section 2, we relate the module category of $\check{C}$ to the bounded derived category $\mathcal{D}^b(\operatorname{mod} A)$, and show that $\operatorname{mod}\check{C}$ is equivalent to a quotient of $\mathcal{D}^b(\operatorname{mod} A)$ by a certain ideal. Section 3 is devoted to the proof of our main theorem. Finally, in section 4, motivated by the need to bring down this information to $\operatorname{mod}\tilde{C}$, we compute a fundamental domain for $\operatorname{mod}\tilde{C}$ inside $\operatorname{mod}\check{C}$, and show that such a domain lies entirely inside a certain finite dimensional quotient of $\check{C}$, which we call the cluster duplicated algebra.

1. The cluster repetitive algebra

1.1. Notation

Throughout this paper, all algebras are basic locally finite dimensional algebras over an algebraically closed field k. For an algebra C, we denote by $\operatorname{mod} C$ the category of finitely generated right C-modules and by $\operatorname{ind} C$ a full subcategory of $\operatorname{mod} C$ consisting of exactly one representative from each isomorphism class of indecomposable modules. When we speak about a C-module (or an indecomposable C-module), we always mean implicitly that it belongs to $\operatorname{mod} C$ (or to $\operatorname{ind} C$, respectively). Also, all subcategories of $\operatorname{mod} C$ are full and so are identified with their object classes. Given a subcategory $\mathcal{C}$ of $\operatorname{mod} C$, we sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. We denote by $\operatorname{add}\mathcal{C}$ the full subcategory of $\operatorname{mod} C$ with objects the finite direct sums of modules in $\mathcal{C}$ and, if M is a module, we abbreviate $\operatorname{add}\{M\}$ as $\operatorname{add} M$.

Following (BG, 1981), we sometimes consider equivalently an algebra C as a locally bounded k-category, in which the object class C_0 is a complete set $\{e_i\}_i$ of primitive orthogonal idempotents of C, and the group of morphisms from e_i to e_j is $e_i C e_j$. We denote the projective (or the injective) dimension of a module M as $\operatorname{pd} M$ (or $\operatorname{id} M$, respectively). The global dimension of C is denoted by gl.dim.C. Finally, we denote by $\Gamma(\operatorname{mod} C)$ the Auslander-Reiten quiver of an algebra C, and by $\tau_C = D\,Tr$, $\tau_C^{-1} = Tr\,D$ its Auslander-Reiten translations. For further definitions and facts needed on $\operatorname{mod} C$ or $\Gamma(\operatorname{mod} C)$, we refer the reader to (ASS, 2006; ARS, 1995).

1.2. Cluster-tilted algebras

Let A be a finite dimensional hereditary algebra. The *cluster category* $\mathcal{C}_A$ of A is defined as follows. Let F be the automorphism of $\mathcal{D}^b(\operatorname{mod} A)$ defined as the composition $\tau^{-1}_{\mathcal{D}^b(\operatorname{mod} A)}[1]$, where $\tau^{-1}_{\mathcal{D}^b(\operatorname{mod} A)}$ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} A)$ and [1] is the shift functor. Then $\mathcal{C}_A$ is the orbit category $\mathcal{D}^b(\mathrm{mod}\,A)/F$, that is, the objects of $\mathcal{C}_A$ are the F-orbits $\tilde{X} = (F^i X)_{i\in\mathbf{Z}}$, where $X \in \mathcal{D}^b(\mathrm{mod}\,A)$, and the set of morphisms from $\tilde{X} = (F^i X)_{i\in\mathbf{Z}}$ to $\tilde{Y} = (F^i Y)_{i\in\mathbf{Z}}$ is

$$\mathrm{Hom}_{\,\mathcal{C}_A}(\tilde{X}, \tilde{Y}) = \bigoplus_{i\in\mathbf{Z}} \mathrm{Hom}_{\,\mathcal{D}^b(\mathrm{mod}\,A)}(X, F^i Y).$$

It is shown in (BMRRT, 2006; K, 2005) that $\mathcal{C}_A$ is a triangulated category with almost split triangles. Furthermore, the projection $\pi : \mathcal{D}^b(\mathrm{mod}\,A) \to \mathcal{C}_A$ is a functor of triangulated categories and commutes with the Auslander-Reiten translations, see (BMRRT, 2006).

An object $\tilde{T}$ in $\mathcal{C}_A$ is called a *tilting object* provided $\mathrm{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals the rank of the Grothendieck group $\mathrm{K}_0(A)$. The endomorphism algebra $B = \mathrm{End}_{\,\mathcal{C}_A}(\tilde{T})$ is then called a *cluster-tilted algebra*. The functor $\mathrm{Hom}_{\,\mathcal{C}_A}(\tilde{T}, -) : \mathcal{C}_A \to \mathrm{mod}\,B$ induces an equivalence

$$\mathcal{C}_A/\mathrm{iadd}\,(\tau_{\mathcal{C}_A}\tilde{T}) \cong \mathrm{mod}\,B,$$

where $\tau_{\mathcal{C}_A}$ is the Auslander-Reiten translation in $\mathcal{C}_A$ and $\mathrm{iadd}\,(\tau_{\mathcal{C}_A}\tilde{T})$ is the ideal of $\mathcal{C}_A$ consisting of all morphisms which factor through objects of $\mathrm{add}\,(\tau_{\mathcal{C}_A}\tilde{T})$. Also, the above equivalence commutes with the Auslander-Reiten translations in both categories, see (BMR1, 2007).

Let $B = \mathrm{End}_{\,\mathcal{C}_A}\tilde{T}$ be a cluster-tilted algebra. It is shown in (BMRRT, 2006) that we may suppose without loss of generality that the object $\tilde{T} = (F^i T)_{i\in\mathbb{Z}}$ is such that $T \in \mathcal{D}^b(\mathrm{mod}\,A)$ is an A-module. In this case, the algebra $C = \mathrm{End}_A T$ is tilted, the trivial extension $\tilde{C} = C \ltimes \mathrm{Ext}^2_C(DC, C)$ is cluster-tilted and, conversely, any cluster-tilted algebra is of this form, see (ABS1, preprint). We also need the following easy lemma.

LEMMA 1.1. *Let T be an A-module such that $\tilde{T} = (F^i T)_{i\in\mathbf{Z}}$ is a tilting object in $\mathcal{C}_A$, then*

$$\mathrm{Hom}_{\,\mathcal{D}^b(\mathrm{mod}\,A)}(T, \tau F^i T) = 0,$$

for all $i \in \mathbb{Z}$.

Proof. This follows from

$$\oplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\,\mathcal{D}^b(\mathrm{mod}\,A)}(T, \tau F^i T) \cong \mathrm{Hom}_{\,\mathcal{C}_A}(\pi T, \tau\pi T) = 0,$$

because $\tilde{T} = \pi T$ is tilting in $\mathcal{C}_A$. □

1.3. The cluster repetitive algebra

Let C be a tilted algebra. We define the *cluster repetitive algebra* to be the following locally finite dimensional algebra without identity

$$\check{C} = \begin{bmatrix} \ddots & & & 0 & \\ & C_{-1} & & & \\ & E_0 & C_0 & & \\ & & E_1 & C_1 & \\ & 0 & & & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero entries, $C_i = C$ and $E_i = \mathrm{Ext}^2_C(DC, C)$ for all $i \in \mathbb{Z}$, all the remaining entries are zero and the multiplication is induced from that of C, the C-C-bimodule structure of $\mathrm{Ext}^2_C(DC, C)$ and the zero map $\mathrm{Ext}^2_C(DC, C) \otimes_C \mathrm{Ext}^2_C(DC, C) \to 0$. The identity maps $C_i \to C_{i-1}$, $E_i \to E_{i-1}$ induce an automorphism φ of $\check{C}$. The orbit category $\check{C}/ < \varphi >$ inherits from $\check{C}$ the structure of a k-algebra and is easily seen to be isomorphic to the cluster-tilted algebra $\tilde{C} = C \ltimes \mathrm{Ext}^2_C(DC, C)$. The projection functor $G : \check{C} \to \tilde{C}$ is thus a Galois covering with infinite cyclic group generated by φ, see (G, 1981). We denote be $G_\lambda : \mathrm{mod}\, \check{C} \to \mathrm{mod}\, \tilde{C}$ the associated push-down functor. We need another description of $\check{C}$.

LEMMA 1.2. *Let T be a tilting A-module, and $C = \mathrm{End}_A T$. Then*

$$\check{C} \cong \mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i \in \mathbb{Z}} F^i T).$$

Proof. As a k-vector space, we have

$$\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i \in \mathbb{Z}} F^i T) = \oplus_{i,j \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(F^i T, F^j T).$$

But $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(F^i T, F^j T) = 0$ unless $i = j$ or $i = j - 1$ since $T \in \mathrm{mod}\, A$. Moreover, $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(F^i T, F^i T) = \mathrm{Hom}_A(T, T) = C$ and $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(F^i T, F^{i+1} T) = \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(T, FT) = \mathrm{Ext}^2_C(DC, C)$, where the last isomorphism follows from (ABS1, preprint, Theorem 3.4). □

1.4. The quiver of $\check{C}$

The quiver $Q_{\check{C}}$ of $\check{C}$ is easily deduced from the quiver Q_C of C. Let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of C, then $\{e_{\ell,i} \mid 1 \le \ell \le n, i \in \mathbb{Z}\}$ is a complete set of primitive orthogonal idempotents of $\check{C}$. We have moreover

$$e_{\ell,i}\ \check{C}\ e_{h,j} \cong \begin{cases} e_\ell\ C\ e_h & \text{if } i = j \\ e_\ell\ \mathrm{Ext}^2_C(DC, C)\ e_h & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

We now recall that a *system of relations* R for $C = kQ_C/I$ is a subset R of $\cup_{\ell,h=1}^{n} e_\ell I e_h$, such that R, but no proper subset of R, generates I as an ideal of kQ_C.

LEMMA 1.3. *Let C be a tilted algebra and R be a system of relations for $C = kQ_C/I$. The quiver $Q_{\check{C}}$ of the cluster repetitive algebra $\check{C}$ is constructed as follows:*

(a) $(Q_{\check{C}})_0 = \{(\ell, i) \mid 1 \leq \ell \leq n, i \in \mathbb{Z}\}$.

(b) For $(\ell, i), (h, j) \in (Q_{\check{C}})_0$, the set of arrows from (ℓ, i) to (h, j) equals

(i) The set of arrows from ℓ to h in Q_C if $i = j$,

(ii) $\mathrm{Card}\,(R \cap e_h I e_\ell)$ *arrows if $i = j + 1$,*

and there are no other arrows.

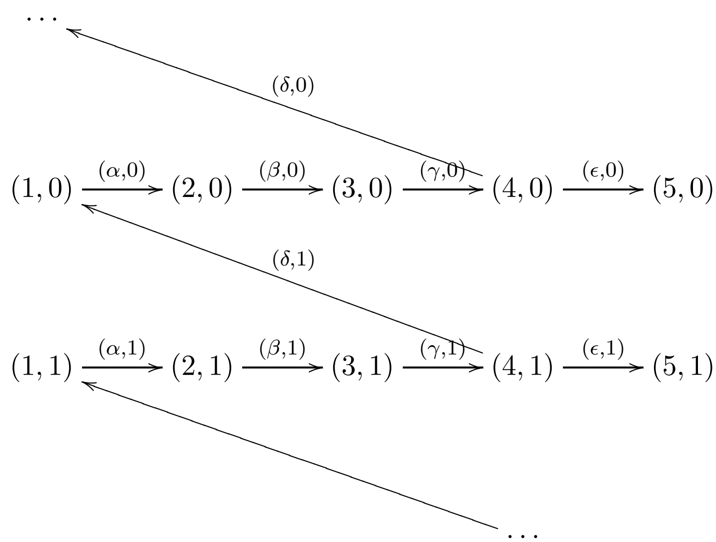
Proof. This follows at once from the above comments and (ABS1, preprint, Theorem 2.6). □

Thus the quiver of $\check{C}$ can be thought of as consisting of infinitely many copies $(Q_{C_i})_{i \in \mathbb{Z}}$ of the quiver of C, joined together by additional arrows from $Q_{C_{i+1}}$ to Q_{C_i}, corresponding to $\mathrm{Ext}_C^2(DC, C)$. In particular, the quiver $Q_{\check{C}}$ is connected if and only if the tilted algebra C is not hereditary.

EXAMPLE 1.4. *Let C be given by the quiver*

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4 \longrightarrow 5$$

bound by the relation $\alpha\beta\gamma = 0$. Then $\check{C}$ is given by the infinite quiver



bound by the relations

$$(\alpha, i)(\beta, i)(\gamma, i) = 0, \qquad (\delta, i+1)(\alpha, i)(\beta, i) = 0,$$
$$(\gamma, i+1)(\delta, i+1)(\alpha, i) = 0, \qquad (\beta, i)(\gamma, i)(\delta, i) = 0,$$

for all $i \in \mathbb{Z}$.

2. The relation with the derived category

2.1.

Throughout this paper, we let A be a finite dimensional hereditary algebra, T be a tilting A-module and $C = \mathrm{End}_A T$ be the corresponding tilted algebra. By Lemma 1.2, the natural functor

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^i T, -)$$

carries $\mathcal{D}^b(\mathrm{mod}\, A)$ into the category $\mathrm{Mod}\, \check{C}$ of (not necessarily finitely generated) $\check{C}$-modules. Since, for every indecomposable object X in $\mathcal{D}^b(\mathrm{mod}\, A)$, we have $\dim_k \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^i T, X) < \infty$, then its image lies in $\mathrm{mod}\, \check{C}$.

PROPOSITION 2.1. *The functor*

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^i T, -) : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathrm{mod}\, \check{C}$$

is full and dense and it induces an equivalence of categories

$$\mathcal{D}^b(\mathrm{mod}\,A)/\mathrm{iadd}\,\{\tau F^iT\}_{i\in\mathbb{Z}} \xrightarrow{\cong} \mathrm{mod}\,\check{C},$$

where $\mathrm{iadd}\,\{\tau F^iT\}_{i\in\mathbb{Z}}$ *denotes the ideal of* $\mathcal{D}^b(\mathrm{mod}\,A)$ *consisting of all morphisms which factor through* $\mathrm{add}\,\{\tau F^iT\}_{i\in\mathbb{Z}}$.

Proof. We first claim that

$$\mathrm{Ker}\,\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT,-) = \mathrm{iadd}\,\{\tau F^iT\}_{i\in\mathbb{Z}}.$$

Indeed, let $f : X \to Y$ be a morphism in $\mathcal{D}^b(\mathrm{mod}\,A)$ such that

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, f) = 0.$$

By definition of the cluster category, this means that the induced morphism

$$\mathrm{Hom}_{\mathcal{C}_A}(\pi T, \pi f) : \mathrm{Hom}_{\mathcal{C}_A}(\pi T, \pi X) \to \mathrm{Hom}_{\mathcal{C}_A}(\pi T, \pi Y)$$

is zero. Therefore πf lies in the kernel of $\mathrm{Hom}_{\mathcal{C}_A}(\pi T, -)$, that is, πf factors through an object of $\mathrm{add}\,\pi(\tau T)$. But this implies that f factors through $\mathrm{add}\,\{\tau F^iT\}_{i\in\mathbb{Z}}$.

Conversely, we prove that any morphism which factors through $\mathrm{add}\,\{\tau F^iT\}_{i\in\mathbb{Z}}$ has a zero image. For this, it suffices to show that the image of any object of the form τF^jT (with $j \in \mathbb{Z}$) is zero. But now

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, \tau F^jT) = \oplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T, F^{j-i}\tau T) = 0$$

because of Lemma 1.1.

We now claim that the functor $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, -)$ induces an equivalence between $\mathrm{add}\,\{F^iT\}_{i\in\mathbb{Z}}$ and the subcategory $\mathrm{proj}\,\check{C}$ of $\mathrm{mod}\,\check{C}$ consisting of the projective $\check{C}$-modules. Indeed, by Lemma 1.2, the restriction of $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, -)$ to $\mathrm{add}\,\{F^iT\}_{i\in\mathbb{Z}}$ maps into $\mathrm{proj}\,\check{C}$. Since, conversely, an indecomposable projective $\check{C}$-module $\check{P}_0$ is an indecomposable summand of $\check{C}_{\check{C}} = \mathrm{End}\,(\oplus_{i\in\mathbb{Z}}F^iT)$, then there exists an indecomposable object $T_0 \in \mathrm{add}\,\{F^iT\}_{i\in\mathbb{Z}}$ such that $\check{P}_0 \cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, T_0)$, that is, the functor is dense. By Yoneda's lemma, it is full. Let thus $f : T_1 \to T_0$ be a morphism in $\mathrm{add}\,\{F^iT\}_{i\in\mathbb{Z}}$ such that $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, f) = 0$. Then f factors through an object of $\mathrm{add}\,\{\tau F^iT\}_{i\in\mathbb{Z}}$. Therefore, by Lemma 1.1, $f = 0$. Thus the functor is faithful and our claim is established.

It remains to show that the functor $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT, -) : \mathcal{D}^b(\mathrm{mod}\,A) \to \mathrm{mod}\,\check{C}$ is full and dense. Let $L \in \mathrm{mod}\,\check{C}$ and consider the minimal projective presentation

$$\check{P}_1 \xrightarrow{\ u\ } \check{P}_0 \longrightarrow L \longrightarrow 0$$

in $\operatorname{mod}\check{C}$. By our claim above, there exist $T_0, T_1 \in \operatorname{add}\{F^iT\}_{i\in\mathbb{Z}}$ and a morphism $v : T_1 \to T_0$ such that $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT, v) = u$. Applying the functor $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT, -)$ to the triangle

$$T_1 \xrightarrow{\ v\ } T_0 \longrightarrow \overline{L} \longrightarrow T_1[1]$$

and using that $T_1[1] \in \operatorname{add}\{\tau F^iT\}_{i\in\mathbb{Z}}$ (because $T_1 \in \operatorname{add}\{F^iT\}_{i\in\mathbb{Z}}$ and $F = \tau^{-1}[1]$) yields an exact sequence

$$\check{P}_1 \xrightarrow{\ u\ } \check{P}_0 \longrightarrow \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT, \overline{L}) \longrightarrow 0$$

in $\operatorname{mod}\tilde{C}$. Therefore, $L \cong \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT, \overline{L})$ and the functor is dense.

Finally, we show that it is full. Let $f : L \to M$ be a morphism in $\operatorname{mod}\tilde{C}$. Taking minimal projective presentations of L and M, we deduce a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\check{P}_1 & \xrightarrow{\ u\ } & \check{P}_0 & \longrightarrow & L & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\check{P}'_1 & \xrightarrow{\ u'\ } & \check{P}'_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

in $\operatorname{mod}\check{C}$. Considering the morphisms $v : T_1 \to T_0$ and $v' : T'_1 \to T'_0$ in $\operatorname{add}\{F^iT\}_{i\in\mathbb{Z}}$ corresponding to u, u', respectively, we find a diagram in $\mathcal{D}^b(\operatorname{mod} A)$ where the rows are triangles

$$\begin{array}{ccccccc}
T_1 & \xrightarrow{\ v\ } & T_0 & \longrightarrow & \overline{L} & \longrightarrow & T_1[1] \\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \vdots{\scriptstyle g} & & \downarrow{\scriptstyle g_1[1]} \\
T'_1 & \xrightarrow{\ v'\ } & T'_0 & \longrightarrow & \overline{M} & \longrightarrow & T'_1[1],
\end{array}$$

that is, there exists $g : \overline{L} \to \overline{M}$ such that the above diagram commutes. Consequently, $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT, g) = f$ and the proof is complete. □

2.2.

It is well-known (see (KR, preprint)) that the cluster-tilted algebra $\tilde{C}$ is 1-Gorenstein, that is, such that for every injective $\tilde{C}$-module I, we have $\operatorname{pd} I \leq 1$ and, for every projective $\tilde{C}$-module P, we have $\operatorname{id} P \leq 1$. This property clearly lifts to its Galois covering $\check{C}$. This also follows from Proposition 2.1.

COROLLARY 2.2. *The cluster repetitive algebra* $\check{C}$ *is 1-Gorenstein. In particular,* gl.dim.$\tilde{C} \in \{1, \infty\}$.

Proof. By (ASS, 2006, (IV.2.7) p.115), we need to prove that

$$\mathrm{Hom}_{\check{C}}(D\check{C}, \tau_{\check{C}} I) = 0,$$

for every injective $\check{C}$-module I. Now under the equivalence of Proposition 2.1, every injective $\check{C}$-module is the image of an object of the form $\tau^2 T_0 \in \mathcal{D}^b(\mathrm{mod}\, A)$, where $T_0 \in \mathrm{add}\, \{F^i T\}_{i\in\mathbb{Z}}$. It thus suffices to show that

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}}\, \tau^2 F^i T, \tau^3 T_0) = 0.$$

But this follows from the fact that τ is an equivalence in $\mathcal{D}^b(\mathrm{mod}\, A)$ and from Lemma 1.1. Thus, $\check{C}$ is 1-Gorenstein. The proof of the second statement is standard (see, for instance, (KR, preprint)). □

2.3.

The following Lemma is a “derived” version of the projectivisation procedure of (ARS, 1995, II.2.1).

LEMMA 2.3. *Let* $T_0 \in \mathrm{add}\, \{F^i T\}_{i\in\mathbb{Z}}$ *and* $X \in \mathcal{D}^b(\mathrm{mod}\, A)$*, then the map* $f \mapsto \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^i T, f)$ *induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(T_0, X) \cong \mathrm{Hom}_{\check{C}}(\mathrm{Hom}\,(\oplus F^i T, T_0), \mathrm{Hom}\,(\oplus F^i T, X)).$$

Proof. Since the surjectivity follows from the fact that the functor $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^i T, -)$ is full (see Proposition 2.1), we prove the injectivity. Assume $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^i T, f) = 0$, then f factors through an object of $\mathrm{add}\, \{\tau F^i T\}_{i\in\mathbb{Z}}$. We then infer from Lemma 1.1 that $f = 0$. □

2.4.

We now prove the main result of this section.

THEOREM 2.4. *There exists a commutative diagram of dense functors*

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{mod} A) & \xrightarrow{\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT,-)} & \operatorname{mod} \check{C} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle G_\lambda} \\ \mathcal{C}_A & \xrightarrow{\operatorname{Hom}(\pi T,-)} & \operatorname{mod} \tilde{C} \end{array}$$

where G_λ is the push-down functor associated to the Galois covering $G : \check{C} \to \tilde{C}$.

Proof. Since $\pi(\tau F^i T) = \tau_{\mathcal{C}_A}(\pi T)$ for each i, we have

$$\pi(\operatorname{iadd}\{\tau F^i T\}_{i\in\mathbb{Z}}) = \operatorname{iadd} \tau_{\mathcal{C}_A}(\pi T).$$

Therefore, using Proposition 2.1, the functor π induces a functor $H : \operatorname{mod} \check{C} \to \operatorname{mod} \tilde{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{mod} A) & \xrightarrow{\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^iT,-)} & \operatorname{mod} \check{C} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle H} \\ \mathcal{C}_A & \xrightarrow{\operatorname{Hom}(\pi T,-)} & \operatorname{mod} \tilde{C} \end{array}$$

We must show that $H = G_\lambda$. Let M be a $\check{C}$-module and set $\tilde{M} = H(M)$. We must prove that, for every $a \in \tilde{C}_0$, we have

$$\tilde{M}(a) = \oplus_{x/a} M(x),$$

where the sum is taken over all $x \in \check{C}_0$ in the fibre $G^{-1}(a)$ of a. We use the following notation: for $x \in \check{C}_0$, we denote by $\tilde{P}_x$ (or $\check{P}_x$) the corresponding indecomposable projective $\tilde{C}$-module (or $\check{C}$-module, respectively) and by $\tilde{T}_x$ the corresponding summand of πT.

By Proposition 2.1, there exists an object $\overline{M} \in \mathcal{D}^b(\operatorname{mod} A)$ such that $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^i T, \overline{M}) = M$, thus we have

$$\begin{aligned} \tilde{M}(a) &\cong \operatorname{Hom}_{\tilde{C}}(\tilde{P}_a, \tilde{M}) \\ &\cong \operatorname{Hom}_{\mathcal{C}_A}(\tilde{T}_a, \pi\overline{M}), \end{aligned}$$

because no morphism from $\tilde{T}_a$ to $\pi\overline{M}$ factors through $\operatorname{add}(\tau\pi T)$. Let thus $T_x \in \mathcal{D}^b(\operatorname{mod} A)$ be such that $\pi T_x = \tilde{T}_x$. Using Lemma 1.2, we have

$$\begin{aligned} \tilde{M}(a) &\cong \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(F^i T_a, \overline{M}) \\ &\cong \oplus_{x/a} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(T_x, \overline{M}), \end{aligned}$$

and by Lemma 2.3, this is isomorphic to

$\oplus_{x/a} \operatorname{Hom}_{\check{C}}(\operatorname{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^iT, T_x), \operatorname{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^iT, \overline{M}))$.

We obtain

$$\begin{aligned}\tilde{M}(a) &\cong \oplus_{x/a} \operatorname{Hom}_{\check{C}}(\check{P}_x, M)\\ &\cong \oplus_{x/a} M(x).\end{aligned}$$

This completes the proof that $H = G_\lambda$. Finally, G_λ is dense because so is the composition $\operatorname{Hom}(\pi T, -) \circ \pi$.

□

2.5.

We deduce the relations between the Auslander-Reiten quivers of $\check{C}$ and $\tilde{C}$.

COROLLARY 2.5.

(a) *The push-down of an almost split sequence of* $\mathrm{mod}\, \check{C}$ *is an almost split sequence of* $\mathrm{mod}\, \tilde{C}$.

(b) *The push-down functor induces an isomorphism of the quotient* $\Gamma(\mathrm{mod}\, \check{C})/\mathbb{Z}$ *of the Auslander-Reiten quiver of* $\check{C}$ *onto the Auslander-Reiten quiver of* $\tilde{C}$.

Proof. This follows from (G, 1981, 3.6) using the density of the push-down functor. □

2.6.

Finally, the following proposition is an analog of (BMR1, 2007, 3.2), and the proof can be easily adapted from there. We include it here for convenience.

PROPOSITION 2.6. *The almost split sequences in* $\mathrm{mod}\, \check{C}$ *are induced by the almost split triangles in* $\mathcal{D}^b(\mathrm{mod}\, A)$.

Proof. By (AR, 1977), the image under $\operatorname{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^iT, -)$ of a left (or right) minimal almost split morphism is left (or right, respectively) minimal almost split. Let $u : E \to M$ be a right minimal almost split epimorphism in $\mathrm{mod}\, \check{C}$. Then there exists a right minimal almost split morphism $g : Y \to Z$ in $\mathcal{D}^b(\mathrm{mod}\, A)$ such that $\operatorname{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(\oplus_{i\in\mathbb{Z}} F^iT, g) = u$. We have an almost split triangle

$$\tau Z = X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow X[1].$$

Applying $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT,-)$, we get an exact sequence

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT,X) \xrightarrow{f^*} E \xrightarrow{u} M \longrightarrow 0.$$

Since f is minimal left almost split, then so is f^*. In particular, f^* is irreducible. Since $u \neq 0$, f^* is not an epimorphism, thus it is a monomorphism. Therefore f^* factors through τM. That is $\tau M \cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT,X)$, because f^* is irreducible. □

3. The relation with the repetitive algebra

3.1.

We recall from (HW, 1983) that the *repetitive algebra* $\hat{C}$ of a finite dimensional algebra C is the self-injective locally finite dimensional algebra without identity

$$\hat{C} = \begin{bmatrix} \ddots & & & 0 & \\ & C_{i-1} & & & \\ & Q_i & C_i & & \\ & & Q_{i+1} & C_{i+1} & \\ & 0 & & & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero entries, $C_i = C$ and $Q_i = DC$ for all $i \in \mathbb{Z}$, all the remaining entries are zero, addition is the usual addition of matrices and the multiplication is induced from that of C, the C-C-bimodule structure of DC and the zero maps $DC \otimes_C DC \to 0$. The identity maps $C_i \to C_{i-1}$, $Q_i \to Q_{i-1}$ induce the so-called Nakayama automorphism ν of $\hat{C}$. The orbit category $\hat{C}/<\nu>$ is then isomorphic to the trivial extension $T(C) = C \ltimes DC$ of C by its minimal injective cogenerator DC.

The repetitive algebra is closely related to the derived category: if gl.dim.$C < \infty$, then $\mathcal{D}^b(\mathrm{mod}\,C)$ is equivalent, as a triangulated category, to the stable module category $\underline{\mathrm{mod}}\,\hat{C}$, see (H, 1988, II.4.9).

Let now, as in section 2, A be a finite dimensional hereditary algebra, T be a tilting A-module and $C = \mathrm{End}_A T$. We denote by Ω^i the i-th syzygy of a module. Also, we identify the $\hat{C}$-modules C_0 and C.

LEMMA 3.1. *The functor* $\underline{\mathrm{Hom}}_{\hat{C}}(\oplus_{i\in\mathbb{Z}}\tau^{-i}\Omega^{-i}C,-)$ *maps* $\underline{\mathrm{mod}}\,\hat{C}$ *into* $\mathrm{mod}\,\check{C}$ *and induces an equivalence*

$$\underline{\mathrm{mod}}\,\hat{C}/\mathrm{iadd}\,\{\tau^{1-i}\Omega^{-i}C\}_{i\in\mathbb{Z}} \cong \mathrm{mod}\,\check{C}$$

where $\mathrm{iadd}\,\{\tau^{1-i}\Omega^{-i}C\}_{i\in\mathbb{Z}}$ *denotes the ideal of* $\underline{\mathrm{mod}}\,\hat{C}$ *consisting of all morphisms which factor through an object of* $\mathrm{add}\,\{\tau^{1-i}\Omega^{-i}C\}_{i\in\mathbb{Z}}$.

Proof. By Proposition 2.1, the functor $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT,-)$ induces an equivalence between $\mathcal{D}^b(\mathrm{mod}\,A)/\mathrm{iadd}\,\{\tau F^iT\}_{i\in\mathbb{Z}}$ and $\mathrm{mod}\,\check{C}$. By (H, 1988, III.2.10 and II.4.9), we have

$$\mathcal{D}^b(\mathrm{mod}\,A)\cong\mathcal{D}^b(\mathrm{mod}\,C)\cong\underline{\mathrm{mod}}\,\hat{C}.$$

Also, under these equivalences, we have

$$\check{C}=\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT)\cong\underline{\mathrm{End}}_{\hat{C}}(\oplus_{i\in\mathbb{Z}}\,\tau^{-i}\Omega^{-i}C),$$

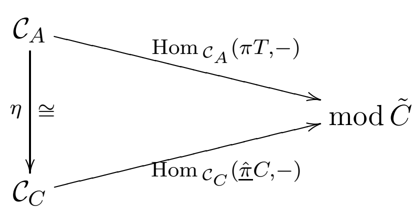
the image of τF^iT is $\tau^{1-i}\Omega^{-i}C$ (for any $i\in\mathbb{Z}$) and also the functor $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(\oplus_{i\in\mathbb{Z}}F^iT,-)$ becomes $\underline{\mathrm{Hom}}_{\hat{C}}(\oplus_{i\in\mathbb{Z}}\tau^{-i}\Omega^{-i}C,-)$. This implies the statement. □

3.2.

We now wish to introduce a different realisation of the cluster category. Let C be a tilted algebra, then there exists an automorphism $F_C:\underline{\mathrm{mod}}\,\hat{C}\to\underline{\mathrm{mod}}\,\hat{C}$ defined by $F_C=\tau^{-1}\Omega^{-1}$. We define $\mathcal{C}_C$ to be the orbit category of $\underline{\mathrm{mod}}\,\hat{C}$ under the action of F_C, that is, the objects of $\mathcal{C}_C$ are the orbits $(F_C^iX)_{i\in\mathbb{Z}}$ of the objects X of $\underline{\mathrm{mod}}\,\hat{C}$, and the morphism set from $(F_C^iX)_{i\in\mathbb{Z}}$ to $(F_C^iY)_{i\in\mathbb{Z}}$ is $\oplus_{i\in\mathbb{Z}}\,\underline{\mathrm{Hom}}_{\hat{C}}(X,F_C^iY)$.

We denote by $\underline{\hat{\pi}}:\underline{\mathrm{mod}}\,\hat{C}\to\mathcal{C}_C$ the projection functor.

LEMMA 3.2. *Let A be a finite dimensional hereditary algebra, T be a tilting A-module and $C=\mathrm{End}_AT$. Then there exists an equivalence $\eta:\mathcal{C}_A\to\mathcal{C}_C$ such that the following diagram commutes.*



Furthermore, $\mathrm{Hom}_{\mathcal{C}_C}(\underline{\hat{\pi}}C,-)$ *is full and dense and induces an equivalence of categories* $\mathcal{C}_C/\mathrm{add}\,(\tau\underline{\hat{\pi}}C)\cong\mathrm{mod}\,\tilde{C}$.

Proof. By (H, 1988, III.2.10 and II.4.9), we have an equivalence of triangulated categories

$$\mathcal{D}^b(\mathrm{mod}\,A)\cong\mathcal{D}^b(\mathrm{mod}\,C)\cong\underline{\mathrm{mod}}\,\hat{C}.$$

Under these equivalences, the automorphism F goes to F_C, and the object T onto the $\hat{C}$-module C. Therefore there is an equivalence $\eta : \mathcal{C}_A \to \mathcal{C}_C$ making the shown diagram commute. The last statement follows from (BMR1, 2007). □

3.3.

Let $p : \operatorname{mod} \hat{C} \to \underline{\operatorname{mod}}\, \hat{C}$ denote the canonical projection. We define the functor $\phi : \operatorname{mod} \hat{C} \to \operatorname{mod} \check{C}$ to be the composition

$$\operatorname{mod} \hat{C} \xrightarrow{\quad p \quad} \underline{\operatorname{mod}}\, \hat{C} \xrightarrow{\underline{\operatorname{Hom}}_{\hat{C}}(\oplus\tau^{-i}\Omega^{-i}C,-)} \operatorname{mod} \check{C}.$$

Also, we denote by $\hat{P}_x$ the indecomposable projective $\hat{C}$-module corresponding to an object $x \in \hat{C}_0$.

LEMMA 3.3. *The kernel $\mathcal{J}$ of ϕ consists of all morphisms factoring through an object of* $\operatorname{add}\{\hat{P}_x \oplus \tau^{1-i}\Omega^{-i}C\}_{x\in\hat{C}_0, i\in\mathbb{Z}}$.

Proof. Clearly, all such morphisms lie in the kernel of ϕ. Conversely, let $f : X \to Y$ be a morphism in $\operatorname{mod} \hat{C}$ such that $\phi(f) = 0$. Then $p(f)$ factors through an object of $\operatorname{add}\{\tau^{1-i}\Omega^{-i}C\}_{i\in\mathbb{Z}}$, that is, there exist $Z \in \operatorname{add}\{\tau^{1-i}\Omega^{-i}C\}_{i\in\mathbb{Z}}$ and morphisms $f_2 : X \to Z$, $f_1 : Z \to Y$ such that $p(f) = p(f_1)p(f_2)$. Thus $f - f_1f_2 \in \ker p$, that is, $f - f_1f_2$ factors through a projective-injective $\hat{C}$-module $\hat{P}$. Thus there exist morphisms $g_2 : X \to \hat{P}$, $g_1 : \hat{P} \to Y$ such that $f - f_1f_2 = g_1g_2$. Therefore $f = [f_1 g_1] \begin{bmatrix} f_2 \\ g_2 \end{bmatrix}$ factors through $Z \oplus \hat{P}$. □

3.4.

Let now $\hat{\pi}$ denote the composition

$$\operatorname{mod} \hat{C} \xrightarrow{\quad p \quad} \underline{\operatorname{mod}}\, \hat{C} \xrightarrow{\quad \underline{\hat{\pi}} \quad} \mathcal{C}_C.$$

We prove finally our main theorem.

THEOREM 3.4. *There is a commutative diagram of dense functors*

$$\begin{array}{ccc} \operatorname{mod} \hat{C} & \xrightarrow{\quad \phi \quad} & \operatorname{mod} \check{C} \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle G_\lambda} \\ \mathcal{C}_C & \xrightarrow{\operatorname{Hom}(\hat{\pi}C,-)} & \operatorname{mod} \tilde{C} \end{array}$$

Moreover, ϕ is full and induces an equivalence of categories $\operatorname{mod}\hat{C}/\mathcal{J} \cong \operatorname{mod}\check{C}$.

Proof. The commutativity of the diagram follows from Theorem 2.4 and Lemma 3.2, where we use the fact that $\underline{\hat{\pi}}C = \hat{\pi}C$. The functor $\hat{\pi}$ is dense, since it is the composition of two dense functors and, similarly, ϕ is full and dense, since it is the composition of two full and dense functors. Finally, the stated equivalence follows from Lemma 3.3 □

3.5.

The relation between the Auslander-Reiten quivers of $\hat{C}$ and $\check{C}$ follows from the next statement.

PROPOSITION 3.5. *The almost split sequences in* $\operatorname{mod}\check{C}$ *are induced from the almost split triangles in* $\operatorname{mod}\hat{C}$.

Proof. Similar to the proof of Proposition 2.6. □

EXAMPLE 3.6. *Let C be the tilted algebra of example 1.4. We illustrate the Auslander-Reiten quivers of $\hat{C}$ and $\check{C}$ in Figure 1. In the Auslander-Reiten quiver of $\hat{C}$, the positions of the projective-injective modules are marked by diamonds and the positions of the indecomposable summands of $\oplus_{i\in\mathbb{Z}}\tau^{1-i}\Omega^{-i}C$ are marked by circles. As we see, removing the points corresponding to those modules in the Auslander-Reiten quiver of $\hat{C}$ yields exactly the Auslander-Reiten quiver of $\check{C}$.*

4. Fundamental domains

4.1.

Let C be a tilted algebra. We define the *cluster duplicated algebra* $\overline{C}$ of C to be the (finite dimensional) matrix algebra

$$\overline{C} = \begin{bmatrix} C_0 & 0 \\ E & C_1 \end{bmatrix},$$

where $C_0 = C_1 = C$ and $E = \operatorname{Ext}^2_C(DC, C)$, endowed with the ordinary matrix addition, and the multiplication induced from that of C and from the C-C-bimodule structure of $\operatorname{Ext}^2_C(DC, C)$.

Clearly, $\overline{C}$ is identified to the quotient algebra of $\check{C}$ defined by the surjection

$$\check{C} \longrightarrow \begin{bmatrix} C_0 & 0 \\ E_1 & C_1 \end{bmatrix},$$

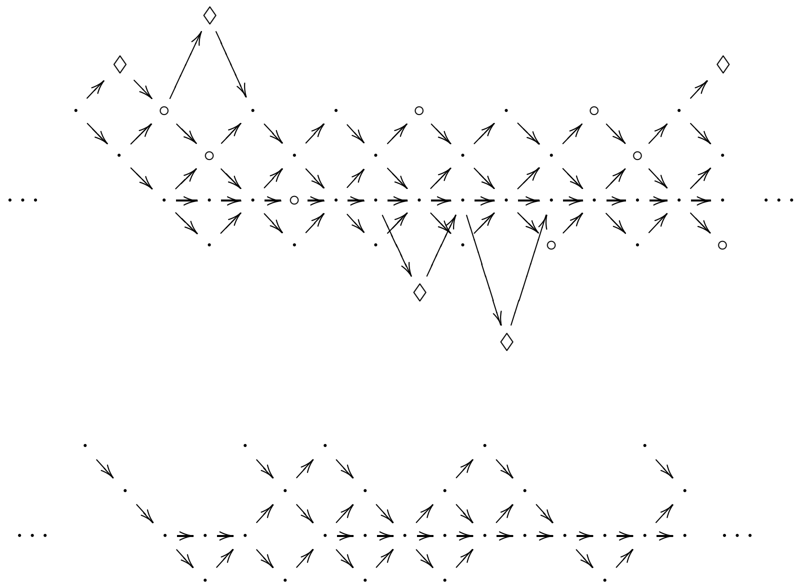


Figure 1. Auslander-Reiten quivers of $\hat{C}$ and $\check{C}$

in the notation of section 1.3. In particular, the quiver $Q_{\overline{C}}$ of $\overline{C}$ is identified to the full subquiver of $Q_{\check{C}}$ defined by the points

$$\{(h,0) \mid h \in (Q_C)_0\} \quad \bigcup \quad \{(h,1) \mid h \in (Q_C)_0\}.$$

Thus, $Q_{\overline{C}}$ is connected if and only if C is not hereditary.

Since the trivial extension $\tilde{C} = C \ltimes \mathrm{Ext}^2_C(DC, C)$ is a subalgebra of $\overline{C}$, the inclusion map $\tilde{C} \to \overline{C}$ defines a functor $\zeta : \mathrm{mod}\,\overline{C} \to \mathrm{mod}\,\tilde{C}$ (by restriction of scalars).

First, we recall that, denoting be e_0 and e_1 the matrices

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad e_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then any $\overline{C}$-module can be written in the form $M = (U, V, \mu)$ where $U = Me_0$, $V = Me_1$ are C-modules, and $\mu : U \otimes_C E \to V$ is the multiplication map $u \otimes x \mapsto ux$ ($u \in U, x \in E$).

We then define $\xi : \mathrm{mod}\,\overline{C} \to \mathrm{mod}\,\tilde{C}$ as follows. For a $\overline{C}$-module (U, V, μ), the $\tilde{C}$-module $\xi(U, V, \mu)$ has the C-module structure of $U \oplus V$ and the multiplication of $(u, v) \in U \oplus V$ by $x \in E$ is given by

$$(u, v)\, x = (0\,,\ \mu(u \otimes x)).$$

Thus, for $(u, v) \in M$ and $\begin{bmatrix} c & 0 \\ x & c \end{bmatrix} \in \tilde{C} = C \ltimes E$,

$$(u, v) \begin{bmatrix} c & 0 \\ x & c \end{bmatrix} = (uc, vc + \mu(u \otimes x)).$$

We define in the same way the action of ξ on the morphisms: if $(g, h) : (U, v, \mu) \to (U', V', \mu')$ is a $\overline{C}$-linear map, we put $\xi(g, h) = g \oplus h : U \oplus V \to U' \oplus V'$ as a C-linear map, the compatibility of this definition with the multiplication by elements of E follows from the fact that $h\mu = \mu'(g \otimes 1)$.

We now give another description of the functor ξ. Let ξ be the canonical embedding functor of $\operatorname{mod} \overline{C}$ into $\operatorname{mod} \check{C}$ (which is obtained by "extending by zeros"): it is full, exact, preserves indecomposable modules and their composition lengths. We have the following easy lemma.

LEMMA 4.1. $\xi = G_\lambda \circ \zeta$.

Proof. This is a straightforward calculation. □

4.2.

We have the following remark about the global dimension of $\overline{C}$.

LEMMA 4.2. gl.dim. $\overline{C} \leq 5$.

Proof. This follows from (PR, 1973, Corollary 4'). □

Easy examples show that this is a strict bound (take for instance C given by the quiver $\bullet \xleftarrow{\beta} \bullet \xleftarrow{\alpha} \bullet$ bound by $\alpha\beta = 0$).

4.3.

Before stating the main result of this section, we need the following notation. Let Λ be any finite dimensional k-algebra and M, N be two indecomposable Λ-modules. A *path* from M to N in $\operatorname{ind} \Lambda$ is a sequence of non-zero morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N$$

with all M_i in $\operatorname{ind} \Lambda$. In this situation we say that M is a *predecessor* of N and write $M \leq N$ and that N is a *successor* of M.

If S_1 and S_2 are two sets of modules, we write $S_1 \leq S_2$ if every module in S_2 has a predecessor in S_1, every module in S_1 has a successor in S_2, no module in S_2 has a successor in S_1 and no module in S_1 has a predecessor in S_2. The notation $S_1 < S_2$ stands for $S_1 \leq S_2$ and $S_1 \cap S_2 = \emptyset$.

We define a *fundamental domain* for the functor G_λ to be a full convex subcategory Ω of $\operatorname{mod} \check{C}$ such that the restriction

$$G_\lambda : \Omega \longrightarrow \operatorname{ind} \tilde{C}$$

is bijective on objects, faithful, preserves irreducible morphisms and almost split sequences.

Let now Σ be a complete slice in $\operatorname{mod} C$. We denote by Σ_i the images of Σ in $\operatorname{mod} C_i$ under the isomorphisms $C_i \cong C$, $i \in \mathbb{Z}$.

THEOREM 4.3. *Let Σ be a complete slice in* $\operatorname{mod} C$. *Then*

$$\Omega = \{M \in \operatorname{ind} \overline{C} \mid \Sigma_0 \leq M < \Sigma_1\}$$

is a fundamental domain for the functor G_λ.

Proof. Without loss of generality, we may assume that T is an A-module and that $\Sigma = \operatorname{Hom}_A(T, DA)$. Let

$$\Omega_{\mathcal{D}} = \{X \in \operatorname{ind} \mathcal{D}^b(\operatorname{mod} A) \mid DA \leq X < F\,DA\}.$$

By (BMRRT, 2006), $\Omega_{\mathcal{D}}$ is a fundamental domain for the functor $\pi : \mathcal{D}^b(\operatorname{mod} A) \to \mathcal{C}_A$.

We first claim that the image $\check{\Omega}$ of Ω under the functor

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^i T, -)$$

is equal to the full subcategory of $\operatorname{ind} \check{C}$ defined by

$$\check{\Omega} = \{\check{M} \in \operatorname{ind} \check{C} \mid \Sigma_0 \leq \check{M} < \Sigma_1\}.$$

We have $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(F^i T, DA) = 0$ unless $i = 0$, since T is an A-module. Hence

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^i T, DA) &= \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(T, DA)\\ &= \operatorname{Hom}_A(T, DA)\\ &= \Sigma_0\end{aligned}$$

Similarly,

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}} F^i T, F DA) = \Sigma_1.$$

By Proposition 2.1, this shows our claim.

Now, note that $\check{\Omega}$ is a fundamental domain for the functor G_λ. This indeed follows from Theorem 2.4, because $\Omega_{\mathcal{D}}$ is a fundamental domain for π and from Corollary 2.5.

Finally, we prove that $\check{\Omega} = \Omega$. For this it suffices to prove that $\check{\Omega} \subset \operatorname{mod}\overline{C}$. Slices are sincere, thus every simple C_0 (or C_1)-module occurs as a simple composition factor in $\operatorname{add}\Sigma_0$ (or $\operatorname{add}\Sigma_1$, respectively). Let e be the sum of all primitive idempotents of $\check{C}$ corresponding to the simple modules in C_0 and C_1. We have just shown that $e\check{C}e = \overline{C}$ and $\overline{C} \subset \operatorname{Supp}\check{\Omega}$, where $\operatorname{Supp}\check{\Omega}$ is the support of $\check{\Omega}$, that is, the full subcategory of $\check{C}$ generated by all the points $x \in \check{C}_0$ such that $Me_x \neq 0$ for some $M \in \check{\Omega}$.

Now we show that $\overline{C} = \operatorname{Supp}\check{\Omega}$. Suppose there is some $M \in \check{\Omega}$ having a composition factor S_x with x not in C_1 or C_2. Assume first that x lies in C_i, where $i \geq 2$. Then there is a nonzero morphism $f : M \to I_x$, where I_x is the indecomposable injective $\check{C}$-module corresponding to x. Since I_x is a successor of Σ_2 and M is a predecessor of Σ_1, lifting this map to the derived category yields a nonzero morphism from a predecessor of $F\,DA$ to a successor of F^2DA, which is impossible (we have used the fact that the functor $\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(\oplus_{i\in\mathbb{Z}}F^iT, -)$is full, by Proposition 2.1). The proof is entirely similar in case $i \leq -1$.

We have shown that the indecomposable objects in Ω and $\check{\Omega}$ coincide. Let now $X \to Y$ be an indecomposable morphism in $\check{\Omega}$. Since X, Y are both $\overline{C}$-modules, then this is an irreducible morphism in $\operatorname{mod}\overline{C}$, hence in Ω. This shows that $\overline{C} = \operatorname{Supp}\check{\Omega}$, and the theorem follows. □

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