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On tame repetitive algebras

by

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Abstract. Let A be a finite dimensional algebra over an algebraically closed field, and denote by $T(A)$ (respectively, $\widehat{A}$) the trivial extension of A by its minimal injective cogenerator bimodule (respectively, the repetitive algebra of A). We characterise the algebras A such that $\widehat{A}$ is tame and exhaustive, that is, the push-down functor $\operatorname{mod}\widehat{A} \to \operatorname{mod} T(A)$ associated with the covering functor $\widehat{A} \to T(A) \xrightarrow{\sim} \widehat{A}/(\nu_A)$ is dense. We show that, if $\widehat{A}$ is tame and exhaustive, then A is simply connected if and only if A is not an iterated tilted algebra of type $\widetilde{\mathbb{A}}_m$. Then we prove that $\widehat{A}$ is tame and exhaustive if and only if A is tilting-cotilting equivalent to an algebra which is either hereditary of Dynkin or Euclidean type or is tubular canonical.

Introduction. Let k denote an algebraically closed field, and A be a finite dimensional k-algebra (associative, with an identity) which we shall moreover assume to be basic and connected. We shall denote by $\operatorname{mod} A$ the category of finite dimensional right A-modules, and by $D = \operatorname{Hom}_k(-, k)$ the standard duality in $\operatorname{mod} A$. The *repetitive algebra* $\widehat{A}$ of A is the self-injective, locally finite dimensional matrix algebra without identity defined by

$$\widehat{A} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & A_{i-1} & & & \\ & Q_{i-1} & A_i & & \\ & & Q_i & A_{i+1} & \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero coefficients, $A_i = A$, $Q_i = {}_A DA_A$ for all $i \in \mathbb{Z}$, all the remaining coefficients are zero, and the multiplication is induced from the canonical bimodule structure of DA and the zero morphism $DA \otimes_A DA \to 0$ (see [27]). The identity morphisms $A_i \to A_{i-1}$,

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$Q_i \to Q_{i-1}$ induce an automorphism ν_A of $\widehat{A}$ (called the *Nakayama automorphism*) and the orbit space $\widehat{A}/(\nu_A)$ inherits the structure of a finite dimensional k-algebra, isomorphic to the *trivial extension* $T(A) = A \ltimes DA$ of A by its minimal injective cogenerator bimodule ${}_ADA_A$. This is the algebra whose additive structure is that of the group $A \oplus DA$, and whose multiplication is defined by

$$(a, f)(b, g) = (ab, ag + fb)$$

for $a, b \in A$ and $f, g \in {}_ADA_A$. Thus $\widehat{A}$ is a Galois covering (in the sense of [19]) of the self-injective algebra $T(A)$ with the infinite cyclic group (ν_A) generated by ν_A, and the category $\operatorname{mod} \widehat{A}$ can then be regarded as the category of finite dimensional $\mathbb{Z}$-graded $T(A)$-modules.

Due to this connexion with trivial extension algebras, repetitive algebras played a crucial rôle in the classification of the representation-finite and polynomial growth self-injective algebras [27], [12], [3], [4], [30], [35], [38], [29]. Also, it was shown by Happel [21] that, if $\operatorname{gl.dim} A < \infty$, then the stable module category $\underline{\operatorname{mod}}\, \widehat{A}$ is equivalent (as a triangulated category) to the derived category of bounded complexes over $\operatorname{mod} A$ (in the sense of Verdier [43]). This result led to the description of the derived categories for various classes of algebras [21], [25], [33], [7]. Further, Wakamatsu has proved [44], [41] (see also [21]) that, if A is a finite dimensional algebra, T_A a tilting module (in the sense of [23]) and $B = \operatorname{End} T_A$, then $\underline{\operatorname{mod}}\, \widehat{A} \xrightarrow{\sim} \underline{\operatorname{mod}}\, \widehat{B}$. Thus, repetitive algebras were also useful in classifying the iterated tilted algebras (in the sense of [2]) of Dynkin and Euclidean type [2], [5]–[8], as well as the representation-infinite algebras which are tilting-cotilting equivalent (in the sense of [5]) to the tubular canonical algebras of [32] (see [7]).

Of particular interest in all these results is the case where the repetitive algebra $\widehat{A}$ is *locally support-finite* [15], that is, if, for each indecomposable projective $\widehat{A}$-module P, the set of isomorphism classes of indecomposable projective $\widehat{A}$-modules P' such that there exists an indecomposable finite dimensional $\widehat{A}$-module M with both $\operatorname{Hom}_{\widehat{A}}(P, M) \neq 0$ and $\operatorname{Hom}_{\widehat{A}}(P', M) \neq 0$, is finite. It is shown in [36] that $\widehat{A}$ is locally support-finite if and only if the strong global dimension of A (in the sense of Ringel) is finite, that is, the indecomposable complexes in the derived category have bounded length. The aim of the present article is to present a classification of the tame repetitive algebras which are locally support-finite.

We shall need the following definitions. Recall from [16] that the repetitive algebra $\widehat{A}$ is called (ν_A)-*exhaustive* (or, more briefly, *exhaustive*) whenever the push-down functor $F_\lambda : \operatorname{mod} \widehat{A} \to \operatorname{mod} T(A)$ associated with the covering functor $\widehat{A} \to T(A)$ (see [10]) is dense (equivalently, every finite dimensional $T(A)$-module is $\mathbb{Z}$-gradable). It is shown in [15] that, if $\widehat{A}$ is locally support-finite, then $\widehat{A}$ is exhaustive. oreover, if $\widehat{A}$ is locally support-finite, then A is *triangular*, that is, its ordinary quiver has no oriented cycles [35]. Finally, recall from [6] that a triangular algebra A is called *simply connected* if, for any presentation of A as a bound quiver algebra, the fundamental group of its bound quiver (in the sense of [28]) is trivial. Equivalently, a triangular algebra is simply connected if and only if it has no proper Galois covering [39]. Our first theorem is a criterion for the simple connectedness of the algebras having a tame and exhaustive repetitive algebra:

THEOREM (A). *Let k be an algebraically closed field, and A be a finite dimensional, basic and connected k-algebra. Assume that $\widehat{A}$ is tame and exhaustive. Then A is simply connected if and only if A is not an iterated tilted algebra of type $\widetilde{\mathbb{A}}_m$ $(m \geq 1)$.*

Observe that in [6], we have obtained the same criterion for the simple connectedness of the algebras A such that $\underline{\mathrm{mod}}\,\widehat{A}$ is *cycle-finite*, that is, for any chain $M_0 \to M_1 \to \ldots \to M_t = M_0$ of non-zero non-isomorphisms between indecomposable objects of $\underline{\mathrm{mod}}\,\widehat{A}$, all the M_i lie in one standard tube (in the sense of [32]) of the stable Auslander–Reiten quiver of $\widehat{A}$. Our second theorem shows that this is not a coincidence.

THEOREM (B). *Let k be an algebraically closed field, and A be a finite dimensional, basic and connected k-algebra. The following conditions are equivalent*:

(i) *$\widehat{A}$ is tame and exhaustive.*

(ii) *$\widehat{A}$ is tame and locally support-finite.*

(iii) *There exists an algebra B which is either tilted of Dynkin type, or representation-infinite tilted of Euclidean type, or tubular, such that $\widehat{A} \xrightarrow{\sim} \widehat{B}$.*

(iv) *There exists an algebra C which is either hereditary of Dynkin or Euclidean type, or tubular canonical, such that A and C are tilting-cotilting equivalent.*

(v) *$\underline{\mathrm{mod}}\,\widehat{A}$ is cycle-finite.*

(vi) *There exists an algebra C which is either hereditary of Dynkin or Euclidean type, or tubular canonical, such that $\underline{\mathrm{mod}}\,\widehat{A} \xrightarrow{\sim} \underline{\mathrm{mod}}\,\widehat{C}$.*

The equivalence of the last three conditions follows directly from [7]. On the other hand, the tilted algebras of Dynkin type were completely classified in [1], [24], [13], [20], [8], while the representation-infinite tilted algebras of Euclidean type and the tubular algebras were described in [32]. Thus, our results provide a covering-theoretical interpretation of the results in [6], [7]. They may also be interpreted in terms of trivial extension algebras.

COROLLARY. *Let k be an algebraically closed field. A finite dimensional, basic and connected k-algebra A satisfies the equivalent conditions of The-*

orem (B) *if and only if its trivial extension* $T(A)$ *is tame and every finite dimensional* $T(A)$*-module is* $\mathbb{Z}$*-gradable. If moreover* A *is simply connected*, *then* A *satisfies the equivalent conditions of Theorem* (B) *if and only if* $T(A)$ *is tame of polynomial growth.*

Indeed, the first assertion follows from [15], and the second from [3], [4] in the domestic case, from [30] in the non-domestic. The hypothesis that A is simply connected cannot be dropped from the second assertion: let $k[X]$ denote the polynomial algebra in one variable X; then $A = k[X]/(X^2)$ does not satisfy the conditions of Theorem (B), but $T(A)$ is of polynomial growth.

The article is organised as follows. In a preliminary section 1, we shall recall some concepts and results necessary for the proofs of the theorems. Section 2 will consist of some preparatory lemmas and Section 3 of the proofs of our theorems. Whenever one of our proofs can be taken verbatim from [6] or [7], it will simply be referred.

1. Preliminaries

1.1. For a (locally) finite dimensional algebra A, we shall denote by Q_A its ordinary quiver. For a point i in Q_A, we denote by e_i the corresponding primitive idempotent of A, by $S(i)$ the corresponding simple A-module and by $P(i)$ the projective cover (respectively, by $I(i)$ the injective envelope) of $S(i)$. We shall denote by $A[M]$ (respectively, $[M]A$) the one-point extension (respectively, coextension) of A by the module M_A. Following [10], we shall sometimes equivalently consider the algebra A as a locally bounded k-category. We shall use freely properties of the Auslander–Reiten translations $\tau = D\operatorname{Tr}$ and $\tau^{-1} = \operatorname{Tr} D$, and the Auslander–Reiten quiver of A, for which we refer to [9], [18], [32]. For tilting theory, we refer the reader to [23], [32]. For the notions of branch and truncated branch enlargements, we refer the reader to [7], [8] or [4]. Recall that these enlargements allow to classify the representation-infinite tilted or iterated tilted algebras of Euclidean type and the algebras tilting-cotilting equivalent (in the sense of [5]) to the tubular canonical algebras (see [32, 4.9] and [7, 2.5]).

1.2. Let A be a triangular algebra and $i\in(Q_A)_0$ be a sink. The *reflection* S_i^+A of A at i is defined to be the quotient of the one-point extension $A[I(i)]$ by the two-sided ideal generated by the idempotent e_i (see [27]). Dually, starting with a source j, we define the reflection S_j^-A. Clearly, the repetitive algebras (and also the trivial extensions) of A and S_i^+A are isomorphic. Also, A and S_i^+A are tilting-cotilting equivalent [42]. Note that the sink i of Q_A is replaced in the quiver of S_i^+A by a source denoted by i'. A (ν_A)-*reflection sequence* of sinks $i_1,\ldots,i_t$ is a sequence of points of Q_A such that i_s is a sink in the quiver of $S_{i_{s-1}}^+\ldots S_{i_1}^+A$ for $1\le s\le t$, where $S_{i_0}^+A = A$.

1.3. Recall that a locally bounded k-category A is called *biserial* if the radical of any non-uniserial left or right projective A-module is a sum of two non-uniserial submodules whose intersection is simple or zero. It is called *special biserial* [40] if it has a presentation $A \xrightarrow{\sim} kQ/I$ such that:

(1) The number of arrows with a prescribed source or target is at most two.

(2) For any $\alpha \in Q_1$, there is at most one arrow β and one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ are not in I.

A special biserial k-category is biserial [40]. A triangular special biserial k-category $A \xrightarrow{\sim} kQ/I$ is called *gentle* [5] if it satisfies the following conditions:

(3) I is generated by a set of paths of length two.

(4) For any $\alpha \in Q_1$, there is at most one arrow λ and one arrow μ such that $\alpha\lambda$ and $\mu\alpha$ are in I.

Let C be a gentle cycle without double arrows. We say that C satisfies the *clock condition* whenever the number of the clockwise oriented zero-relations on C equals the number of the counterclockwise oriented ones (see [34], [35]). It is shown in [5] that the class of iterated tilted algebras of type $\widetilde{\mathbb{A}}_m$ coincides with the class of gentle algebras having a unique cycle satisfying the clock condition.

1.4. We shall need the following result from [15] which relates the representation types of $\widehat{A}$ and $T(A)$. For the definitions of tame, domestic and polynomial growth, we refer the reader respectively to [17], [31] and [37] (see also [38]).

PROPOSITION. *If $\widehat{A}$ is locally support-finite, then the push-down functor $F_\lambda : \operatorname{mod} \widehat{A} \to \operatorname{mod} T(A)$ induces a bijection between the (ν_A)-orbits of isomorphism classes of indecomposable $\widehat{A}$-modules and the isomorphism classes of indecomposable $T(A)$-modules. In particular, $\widehat{A}$ is tame* (*respectively, domestic, of polynomial growth*) *if and only if $T(A)$ is tame* (*respectively, domestic, of polynomial growth*).

2. Preparatory lemmas

2.1. Our first lemma is a criterion for the exhaustibility of the repetitive algebra. Recall that an $\widehat{A}$-module M is called *locally finite dimensional* if, for each indecomposable projective $\widehat{A}$-module P, $\operatorname{Hom}_{\widehat{A}}(P, M)$ is finite dimensional.

LEMMA. *Let A be an algebra. Then $\widehat{A}$ is exhaustive if and only if (ν_A) acts freely on the isomorphism classes of indecomposable locally finite dimensional $\widehat{A}$-modules.*

Proof. [16, 2.4].

In particular, in order to show that $\widehat{A}$ is not exhaustive, it suffices to construct an indecomposable locally finite dimensional $\widehat{A}$-module M such that its stabiliser

$$(\nu_A)_M = \{g \in (\nu_A) \mid {}^gM \xrightarrow{\sim} M\}$$

is non-trivial.

2.2. Lemma. *Let A be an algebra, T_A be a tilting module, and $B = \operatorname{End} T_A$. Then $\widehat{A}$ is exhaustive if and only if $\widehat{B}$ is exhaustive.*

Proof. It follows from the proof in [44], [41] that there is a commutative diagram

$$\begin{array}{ccc} \underline{\operatorname{mod}}\,\widehat{A} & \xrightarrow{\sim} & \underline{\operatorname{mod}}\,\widehat{B} \\ {\scriptstyle F_\lambda^A}\big\downarrow & & \big\downarrow{\scriptstyle F_\lambda^B} \\ \underline{\operatorname{mod}}\,T(A) & \xrightarrow{\sim} & \underline{\operatorname{mod}}\,T(B) \end{array}$$

where F_λ^A and F_λ^B denote the corresponding push-down functors. The statement follows at once.

2.3. Lemma. *Let $B = C[M]$ be a one-point extension of a hereditary algebra C of type $\widetilde{\mathbb{A}}_m$ by an indecomposable regular module of regular length 2 lying in a tube of rank at least 2. Then $\widehat{B}$ is not exhaustive.*

Proof. We first show that C may be assumed to have the orientation in the Tables of [14]: indeed, let U_C be the slice module of a complete slice in the preprojective component of C such that $H = \operatorname{End} U_C$ has the required orientation, and let P_B denote the remaining indecomposable projective B-module such that $\operatorname{rad} P = M$. Then $T_B = U \oplus P$ is a tilting B-module, and its endomorphism algebra $E = \operatorname{End} T_B$ has the form

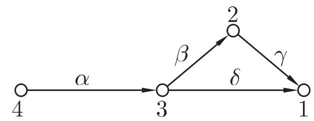
$$E = \begin{bmatrix} H & 0 \\ \operatorname{Hom}_C(U,M) & k \end{bmatrix} = H[\operatorname{Hom}_C(U,M)]\,.$$

Clearly, $\operatorname{Hom}_C(U,M)$ is a regular indecomposable H-module of regular length at least 2 lying in a tube of rank at least 2. By 2.2, $\widehat{B}$ is not exhaustive if and only if $\widehat{E}$ is not. We may thus assume from the start that C is of this form.

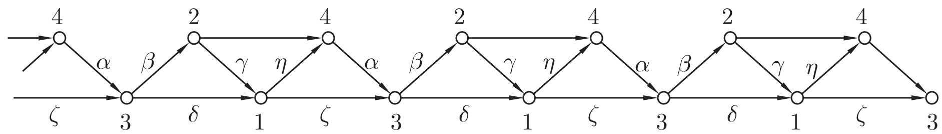
In order to show that $\widehat{B}$ is not exhaustive, it suffices, by 2.1, to construct an indecomposable locally finite dimensional $\widehat{B}$-module with non-trivial stabiliser. On the other hand, if B contains a full subcategory K such that $\widehat{K}$ is not exhaustive, then, clearly, $\widehat{B}$ itself is not exhaustive. We have three cases to consider (see [14, Tables]).

Case 1: If $\underline{\dim} = 1\begin{smallmatrix} 1\,1\ldots 1 \\ 1\,0\ldots 0\end{smallmatrix}1$, then B contains a full subcategory

K given by the quiver



bound by $\alpha\beta\gamma = 0$. Then the quiver of $\widehat{K}$ is



We define M by $M_1 = M_2 = M_3 = M_4 = k^2$ and

$$M(\alpha) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad M(\beta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\delta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

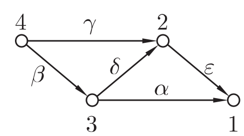
$$M(\xi) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\eta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\zeta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Clearly, M is a locally finite dimensional $\widehat{K}$-module with non-trivial stabiliser. In order to show that M is indecomposable, we shall prove that $E = \operatorname{End} M$ is local. Let thus $f \in \operatorname{End} M$. Then $f = (f_i)_{1 \le i \le 4}$, where $f_i \in \operatorname{End} M_i$ is a 2×2 matrix such that, for any arrow $\lambda : i \to j$, we have $f_j M(\lambda) = M(\lambda) f_i$. A straightforward calculation shows that

$$f_1 = \begin{bmatrix} t & a \\ 0 & t \end{bmatrix}, \quad f_2 = \begin{bmatrix} t & b \\ 0 & t \end{bmatrix}, \quad f_3 = \begin{bmatrix} t & c \\ 0 & t \end{bmatrix}, \quad f_4 = \begin{bmatrix} t & d \\ 0 & t \end{bmatrix}$$

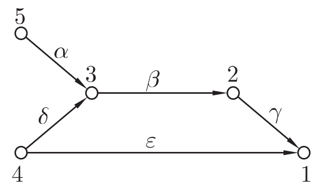
with $t, a, b, c, d \in k$. Thus, $f = t \cdot 1_M + g$ where $t \in k$ and $g \in \operatorname{End} M$ is nilpotent. Let now $I = \{f \in E \mid t = 0\}$. Then I is a nilpotent ideal of E. Since moreover $E/I \xrightarrow{\sim} k$, I is a maximal ideal of E. Therefore $I = \operatorname{rad} E$ and E is local.

Case 2: If $\underline{\dim} = 1\,\begin{smallmatrix} 1\,1 \dots 1 \\ 0\,0 \dots 1 \end{smallmatrix}\,1$, then B contains a full subcategory K' given by the quiver



bound by $\beta\delta = 0$, $\beta\alpha = \gamma\varepsilon$. Then, clearly, $S_2^+ S_1^+ K \xrightarrow{\sim} K'$ and thus $\widehat{K} \xrightarrow{\sim} \widehat{K'}$. Therefore $\widehat{K'}$ is not exhaustive.

Case 3: If $\underline{\dim} = {}_0\begin{smallmatrix} 0 \dots 0\,1\,1\,0 \dots 0 \\ 0 \dots\dots\dots\dots 0 \end{smallmatrix}{}_0$, then B contains a full subcategory K'' given by the quiver



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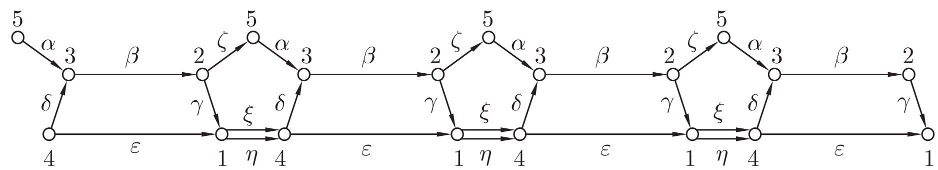
bound by $\alpha\beta\gamma = 0$, and the quiver of $\widehat{K''}$ is



We define M by $M_1 = M_2 = M_3 = M_4 = k^3$, $M_5 = k^2$ and

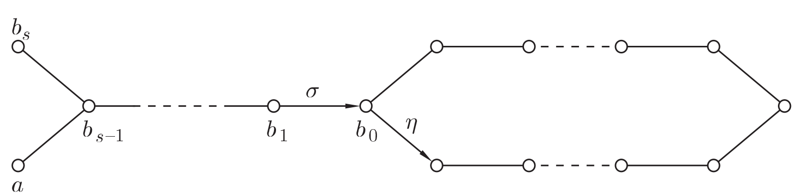
$$M(\alpha) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M(\beta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M(\delta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\varepsilon) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M(\eta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M(\zeta) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

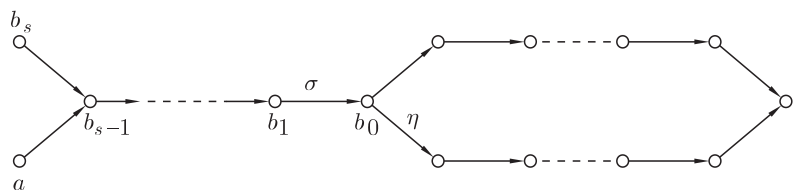
As before, any $f \in \operatorname{End} M$ can be written in the form $f = t \cdot 1_M + g$, where $t \in k$ and $g \in \operatorname{End} M$ is nilpotent. Therefore, M is indecomposable and clearly its stabiliser is non-trivial. This completes the proof.

2.4. Lemma. *Let B be given by the quiver*

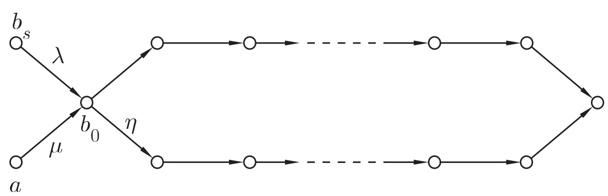


(*where* $s > 1$ *and unoriented edges may be oriented arbitrarily*) *bound only by* $\sigma\eta = 0$. *Then* $\widehat{B}$ *is not exhaustive.*

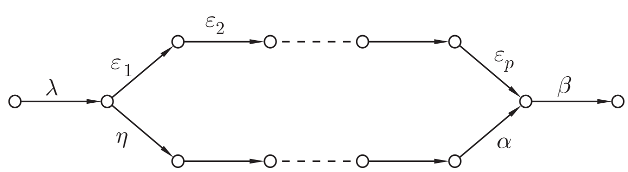
Proof. It suffices, by 2.2 and 2.3, to prove that B is tilting-cotilting equivalent to a one-point extension of a hereditary algebra of type $\widetilde{\mathbb{A}}_m$ by an indecomposable regular module of regular length 2 lying in a tube of rank at least 2. First, observe that we can assume the cycle in B to be given the orientation in the Tables of [14] such that b_0 is a source on the cycle. Indeed, if this is not the case, we may apply to B a tilting module with summand the slice module of a complete slice in the preprojective component whose endomorphism ring has the required property. Moreover, applying if necessary a sequence of APR-tilts and cotilts to the vertices of B not lying on the cycle, we may suppose B to be given by the quiver



bound by $\sigma\eta = 0$. Then B contains a full subcategory C given by the quiver

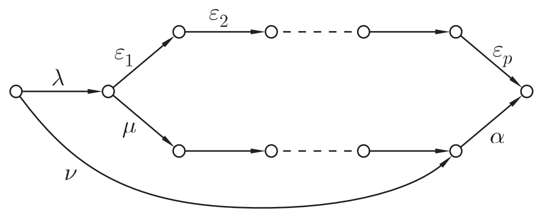


bound by $\lambda\eta = 0$, $\mu\eta = 0$. Let $D = S_a^- C$. Then D is given by the quiver



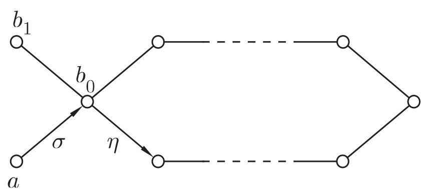
bound by $\lambda\eta = 0$, $\alpha\beta = 0$, $\lambda\varepsilon_1 \dots \varepsilon_p\beta = 0$. Let E be the full subcategory of D consisting of all its objects except the unique source. Then E is a tilted algebra of type $\widetilde{\mathbb{A}}_{m+s}$ having a complete slice in its preprojective component.

Let V be the slice module of a complete slice in the preprojective component of E, considered as a D-module. Then $V \oplus P(a)$ is a tilting module with endomorphism algebra given by the quiver



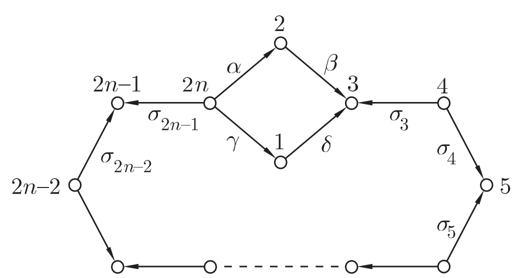
bound by $\lambda\mu = 0$ and $\nu\alpha = \lambda\varepsilon_1 \ldots \varepsilon_p$. This is a one-point extension of a hereditary algebra of type $\widetilde{\mathbb{A}}_{m+s}$ by an indecomposable regular module of regular length 2 lying in a tube of rank at least 2. The proof is complete.

Remark. We shall also need to consider in the case $s = 1$ the algebra B given by the quiver



bound by $\sigma\eta = 0$. In this case, B and $\widehat{B}$ are wild.

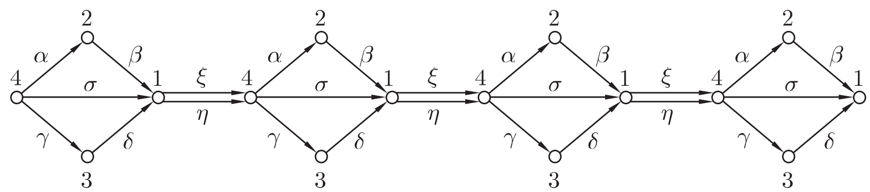
2.5. Lemma. *Let B be given by the quiver*



bound by $\alpha\beta = \gamma\delta$, and these are the only paths of length at least 2. Then $\widehat{B}$ is not exhaustive.

Proof. As in 2.3, we shall construct an indecomposable locally finite dimensional $\widehat{B}$-module M with non-trivial stabiliser. We shall consider two cases:

1) B has four objects: in this case, $\widehat{B}$ is given by the quiver



We define M by $M_2 = M_3 = k^2$, $M_1 = M_4 = k^3$ and

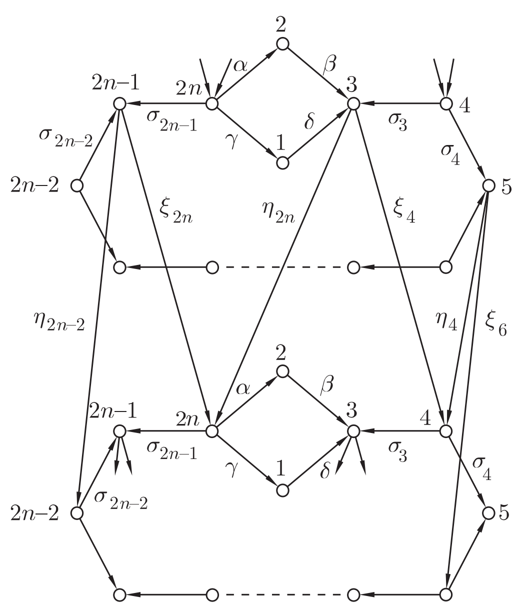
$$M(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\beta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\sigma) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M(\xi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M(\eta) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

A straightforward calculation shows that $\operatorname{End} M$ is local. On the other hand, it is clear that M has a non-trivial stabiliser.

2) B has more than four objects: in this case, $\widehat{B}$ is given by the quiver



We define M by $M_3 = M_{2n} = k^3$, $M_1 = M_2 = M_4 = \ldots = M_{2n-1} = k^2$ and

$$M(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\beta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\sigma_3) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M(\sigma_{2n-1}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M(\sigma_j) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (4 \le j \le 2n-2),$$

$$M(\eta_{2n}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad M(\xi_4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M(\xi_{2n}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M(\eta_{2n}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$M(\eta_{2j}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (3 \le j < n).$$

Again, it is clear that M has a non-trivial stabiliser. It is not hard to prove that M is indecomposable.

2.6. Lemma. *Let $B = C[M]$ be a one-point extension of a tame concealed algebra C of type $\widetilde{\mathbb{D}}_n$ $(n \ge 4)$ by an indecomposable regular module of regular length 2 lying in a tube of rank $n-2$. Then $\widehat{B}$ is not exhaustive.*

Proof. It follows from 2.2 that we can assume B to be a one-point extension of a hereditary algebra H of type $\widetilde{\mathbb{D}}_n$ by an indecomposable regular H-module M of regular length 2 lying in a tube of rank $n-2$. Further, we can assume that the quiver of H has the orientation given in [14, Tables]. We shall consider two cases:

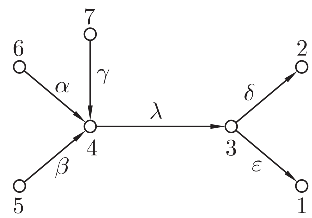
a) Suppose first that $n \ge 5$. Then the dimension-vector of M is equal to

$$\begin{smallmatrix}1\\1\end{smallmatrix} 2\ 1\ \ldots\ 1 \begin{smallmatrix}1\\1\end{smallmatrix}, \quad \begin{smallmatrix}1\\1\end{smallmatrix} 1\ \ldots\ 1\ 2 \begin{smallmatrix}1\\1\end{smallmatrix}, \quad \text{or} \quad \begin{smallmatrix}0\\0\end{smallmatrix} 0\ \ldots\ 0\ 1\ 1\ 0 \ldots\ 0 \begin{smallmatrix}0\\0\end{smallmatrix}.$$

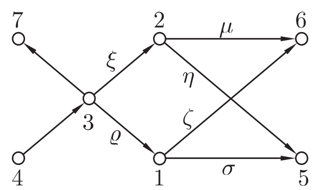
If $\underline{\dim}\, M = \begin{smallmatrix}1\\1\end{smallmatrix} 2\ 1\ \ldots\ 1 \begin{smallmatrix}1\\1\end{smallmatrix}$, then B contains a full subcategory of the type studied in 2.5. Therefore $\widehat{B}$ is not exhaustive.

If $\underline{\dim}\, M = \begin{smallmatrix}1\\1\end{smallmatrix} 1\ \ldots\ 1\ 2 \begin{smallmatrix}1\\1\end{smallmatrix}$, let i denote the extension point. Then $S_i^- B$ contains a full subcategory of the type studied in 2.5, and therefore $\widehat{B} \xrightarrow{\sim} \widehat{S_i^- B}$ is not exhaustive.

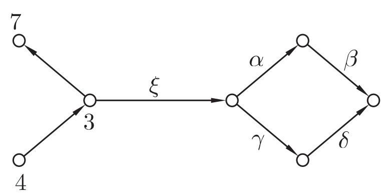
If $\underline{\dim}\, M = \begin{smallmatrix}0\\0\end{smallmatrix}\,0\ \ldots\ 0\ 1\ 1\ 0 \ldots\ 0\,\begin{smallmatrix}0\\0\end{smallmatrix}$, then B contains a full subcategory K given by the quiver



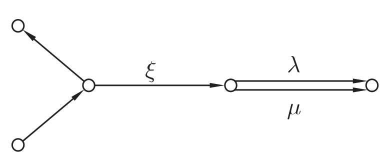
bound by $\gamma\lambda\delta = 0$, $\gamma\lambda\varepsilon = 0$. We shall show that $\widehat{K}$ is not exhaustive. First, observe that $\widehat{K} \xrightarrow{\sim} \widehat{D}$ for $D = S_4^+ S_3^+ S_2^+ S_1^+ K$ given by the quiver



bound by $\varrho\zeta = \xi\mu$, $\xi\eta = \varrho\sigma$. The full subcategory of D formed by the objects 1, 2, 5, 6 is hereditary of type $\widetilde{\mathbb{A}}_{2,2}$. Then there exists a tilting module T_D with endomorphism algebra E given by the quiver



bound by $\xi\alpha\beta = \xi\gamma\delta$. Finally, E contains a full subcategory isomorphic to the category F given by the quiver



bound by $\xi\mu = 0$. By 2.4, $\widehat{F}$ is not exhaustive. Consequently, by 2.2, $\widehat{K} \xrightarrow{\sim} \widehat{D}$ is not exhaustive.

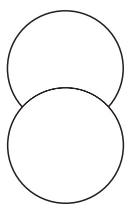
b) Suppose now $n = 4$. Then H has three tubes of rank 2.

If M lies in the tube containing the unique simple H-module that is regular, then either B or $S_i^- B$ (with i the extension point) contains a full subcategory of the type studied in 2.5. If this is not the case, then B is, up to symmetry, given by a quiver containing two sinks i, j such that the full subcategory of $S_i^+ B$ consisting of all objects except j is not exhaustive by 2.3. The proof is now complete.

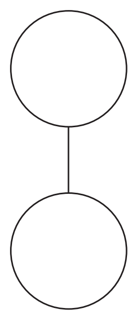
2.7. LEMMA. *Let B be a gentle algebra with at least two cycles. Then $\widehat{B}$ is not exhaustive.*

Proof. The proof will be done in two steps. We shall first reduce the general case to four simple cases, then in each of these construct an indecomposable locally finite dimensional $\widehat{B}$-module with non-trivial stabiliser. Let B be a gentle algebra with at least two cycles. Replacing, if necessary, B by a full subcategory, we may assume that B is of one of the following forms:

(i)



(ii)



Moreover, by [35], B is triangular and any cycle in B satisfies the clock condition.

We shall first show that the only relations on a cycle in B may be assumed to occur at the points of intersection, that is, to have as midpoints those points of B having exactly three neighbours. Indeed, if this is not the case, then, by the clock condition, there exist on a cycle C in B two consecutive zero relations of opposite directions such that the midpoint of at least one of these relations is not a point of intersection. We may further assume (by duality) that two relations point towards each other. Let a_1 and a_2 denote the midpoints of these relations, and let w denote the walk on C between a_1 and a_2 containing the sinks of these zero relations. Then w contains a sink a (on C). We claim that a may be assumed to be a sink in B as well. Indeed, suppose that this is not the case. Since B is triangular, there exists a path in B from a to a sink b in B:

$$a = b_0 \to b_1 \to \ldots \to b_t = b$$

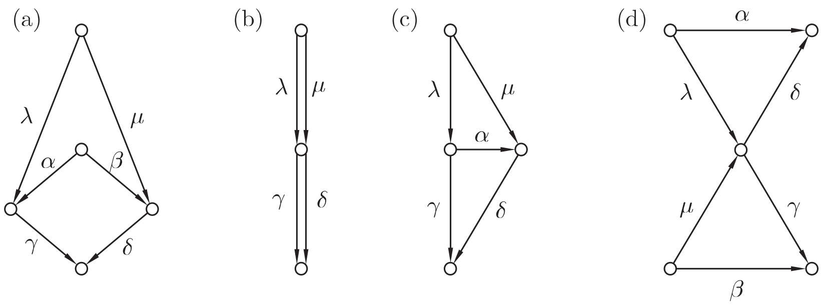
$(t \geq 1)$. Let $B' = S_b^+ B$. Then $\widehat{B'} \xrightarrow{\sim} \widehat{B}$. Since B is gentle, $\widehat{B}$ is special biserial and consequently B' is gentle (by [5, 1.3]). Replacing, if necessary, B' by a full subcategory, we see that B' is of the same form as B, thus is gentle with two cycles. Also, the above procedure does not create new zero relations whose midpoints are not points of intersection. Moreover, there exists a path in B' from a to a sink b' in B:

$$a = b'_0 \to b'_1 \to \ldots \to b'_s = b'$$

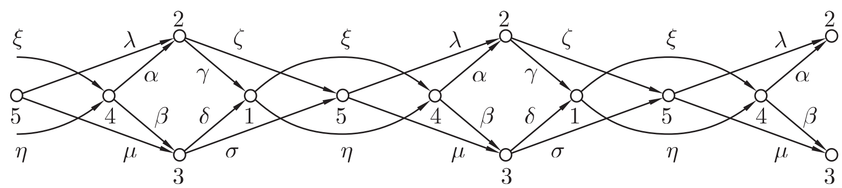
with $s < t$. Applying inductively this procedure, we can assume that a is a sink in B. We can also suppose that $B(a_1, a) \neq 0$, $B(a_2, a) \neq 0$. For, if this is not the case, then we apply a sequence of APR-tilts and cotilts corresponding to the points of w and, if B'' denotes the endomorphism algebra of an APR-tilting or cotilting module, then B'' is again gentle with two cycles and moreover, $\widehat{B''}$ is exhaustive if and only if $\widehat{B}$ is exhaustive (by 2.2). Let now $B^* = S_a^+ B$. Then $\widehat{B^*} \xrightarrow{\sim} \widehat{B}$. oreover, since B is gentle, so is $\widehat{B}$ and consequently so is B^* (by [5, 1.3]). Replacing, if necessary, B^* by a full subcategory, we see that B^* is of the same form as B, thus B^* is gentle with two cycles, one of which, C', say, containing a', a_1 and a_2. Now C' has at least one zero relation less than C, namely any relation whose midpoint is not a point of intersection is erased, and no new relation whose midpoint is not a point of intersection appears as a result of the above procedure. By induction, we can thus suppose that the only relations on a cycle in B have as midpoints the points of intersection.

Next, we shall show that we can assume B to be of the form (i). Indeed, suppose that B is of the form (ii) and let w denote the walk linking the two cycles. Assume that w is bound. If w contains two consecutive zero relations of opposite directions, we can, as above, erase both of these relations. Thus we can suppose that all relations on w point in the same direction. Assume that they point towards the cycle C. Let K denote the full subcategory of B consisting of C together with all points x on w such that there is a non-zero path from x to C. Then, by [32, 4.9], K is a tilted algebra of type $\widetilde{\mathbb{A}}_m$ having a complete slice in its preinjective component. Let U be the slice module of a complete slice in $\operatorname{mod} K$, considered as a B-module, and put $T_B = U_B \oplus (\bigoplus_{y \notin K_0} P(y))$. Then T_B is a tilting module and $\operatorname{End} T_B$ is of the same form as B, but the walk of $\operatorname{End} T_B$ corresponding to w has one zero relation less than w. We may thus assume that w is not bound. Applying a sequence of APR-tilts and cotilts, and passing to a full subcategory, we arrive at the case where w is of length zero, that is, where the two cycles intersect, so that B is of the form (i).

Applying now to B a sequence of APR-tilts and cotilts, we may assume that both cycles in B have the orientation in [14, Tables]. Then B contains a full subcategory of one of the following four forms:



bound by $\lambda\gamma = 0$, $\mu\delta = 0$ in each case. (a) In this case, $\widehat{B}$ is given by the quiver



Then the locally finite dimensional $\widehat{B}$-module M defined by $M_1 = M_4 = k^3$, $M_2 = M_3 = M_5 = k^2$ and

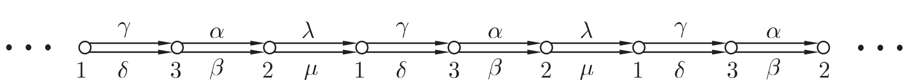
$$M(\alpha) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M(\beta) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$M(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M(\lambda) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\mu) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\zeta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$M(\sigma) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\eta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has a non-trivial stabiliser. It is not hard to prove that M is indecomposable.

(b) In this case, $\widehat{B}$ is given by the quiver



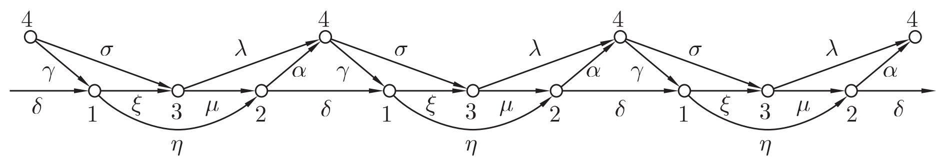
We define a locally finite dimensional module M by $M_1 = M_2 = M_3 = k^3$

and

$$M(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\mu) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$M(\delta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M(\beta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Clearly, M has a non-trivial stabiliser. It is not hard to prove that M is indecomposable.

(c) In this case, $\widehat{B}$ is given by the quiver

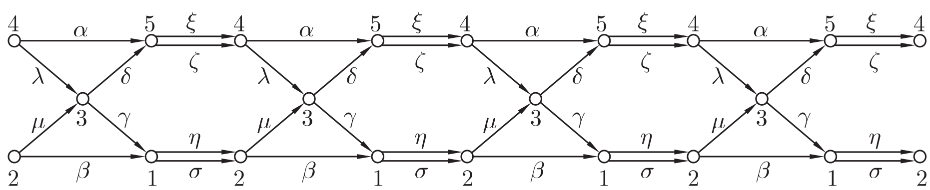


We define a locally finite dimensional module M by $M_1 = M_2 = M_3 = M_4 = k^2$ and

$$M(\alpha) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\gamma) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\delta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\lambda) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$
$$M(\mu) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\xi) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\eta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(\sigma) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly, M has a non-trivial stabiliser. It is not hard to prove that M is indecomposable.

(d) In this case, $\widehat{B}$ is given by the quiver



We define a locally finite dimensional module M by

$$M_1 = M_2 = k\,, \quad M_3 = M_4 = M_5 = k^2\,,$$
$$M(\alpha) = 0\,, \quad M(\beta) = 1\,, \quad M(\gamma) = [1\ 0]\,, \quad M(\eta) = 0\,, \quad M(\sigma) = 0\,,$$
$$M(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M(\mu) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$M(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M(\zeta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly, M has a non-trivial stabiliser. It is not hard to prove that M is indecomposable. The proof is now complete.

2.8. Lemma. *Let B be a cycle bound by zero relations and such that $\widehat{B}$ is tame and exhaustive. Then B is gentle and satisfies the clock condition.*

Proof. To prove the gentleness of B, we repeat the proof of [6, 4.5], using 2.3. The clock condition then follows from [5, 1.3].

2.9. Lemma. *Let B be an algebra given by a gentle cycle, and let $D = B[M]$ be a one-point extension of B by a B-module M such that all the simple direct summands of the top of M are isomorphic. Assume that D is not simply connected and that $\widehat{D}$ is tame and exhaustive. Then D is gentle and the extension point is connected to B by exactly one arrow.*

Proof. We repeat the proof of [6, 4.6], using 2.3, 2.8 and [5, 1.3].

3. Proofs of our theorems

3.1. Proof of Theorem (A). If A is an iterated tilted algebra of type $\widetilde{\mathbb{A}}_m$ then, by [5], A is not simply connected. On the other hand, by [7], $\widehat{A}$ is tame and locally support-finite. Consequently, $\widehat{A}$ is exhaustive. Now assume conversely that A is not simply connected, and that $\widehat{A}$ is tame and exhaustive. By [35, Theorem 1], A is triangular. There exists a presentation $A \xrightarrow{\sim} kQ/I$ such that the fundamental group of the bound quiver (Q, I) is not trivial. Then (Q, I) contains a closed walk w which is not contractible and is minimal with respect to the number of edges. In particular, it contains no pairs of the forms $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ ($\alpha \in Q_1$). Moreover, if w has more than two vertices, then 2.7 and the tameness of $\widehat{A}$ imply that w does not contain double arrows. Let C denote the subcategory of A consisting of the objects and morphisms of w. Then we prove exactly as in [6, §5] using 2.1 to 2.9 and [35, Theorem 1] that C is a full convex subcategory of A and that A is a gentle category with a unique cycle C satisfying the clock condition. Consequently, by [5], A is an iterated tilted algebra of type $\widetilde{\mathbb{A}}_m$.

3.2. Proof of Theorem (B). (iv)⇒(iii). If A is tilting-cotilting equivalent to a hereditary algebra of Dynkin type $\vec{\Delta}$ then, by [22], it is iterated tilted of Dynkin type $\vec{\Delta}$. Consequently, by [3, 3.2], there exist a tilted algebra B of Dynkin type $\vec{\Delta}$ and a (ν_B)-reflection sequence of sinks $i_1, \ldots, i_t$ such that $A \xrightarrow{\sim} S^+_{i_t} \ldots S^+_{i_1} B$. In particular, $\widehat{A} \xrightarrow{\sim} \widehat{B}$. Assume now that A is tilting-cotilting equivalent to a hereditary algebra of Euclidean type (respectively, to a tubular canonical algebra). If A is representation-finite and is not simply connected then, by [6], it is iterated tilted of type $\widetilde{\mathbb{A}}_m$ $(m \geq 1)$. It then follows from [5] that, by applying a sequence of reflections to the sources and sinks of the unique cycle, there exists a representation-infinite iterated tilted algebra A' of the same type such that $\widehat{A} \xrightarrow{\sim} \widehat{A'}$. On the other hand, if A is representation-finite and simply connected, then, by [4, 3.4], there exists a representation-infinite simply connected algebra A' obtained from A by a sequence of reflections, such that $\widehat{A} \xrightarrow{\sim} \widehat{A'}$. We may thus assume from the start that A is representation-infinite. It then follows from [7, 2.5] that A is a domestic (respectively, tubular) branch enlargement of a tame concealed algebra. Also, by [4, 2.6], there exists a truncated branch extension, that is, a tilted algebra of Euclidean type (respectively, a tubular algebra) which is obtained from A by a sequence of reflections. Therefore $\widehat{A} \xrightarrow{\sim} \widehat{B}$ and we are done.

(iii)$\Rightarrow$(ii). If B is a tilted algebra of Dynkin type then, by [3], $\widehat{B}$ is locally representation-finite and hence tame and locally support-finite. If B is a tilted algebra of Euclidean type, then, by [4], $\widehat{B}$ is domestic (hence tame) and locally support-finite. Finally, if B is a tubular algebra, then, by [30], $\widehat{B}$ is of polynomial growth (hence tame) and locally support-finite.

(ii)$\Rightarrow$(i). The local support-finiteness of a repetitive algebra implies its exhaustibility.

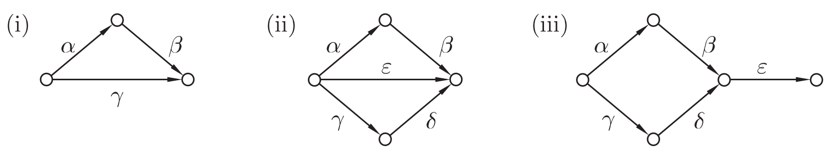
(i)$\Rightarrow$(iv). Let A be such that $\widehat{A}$ is tame and exhaustive. If A is not simply connected then, by Theorem (A), A is iterated tilted of type $\widetilde{\mathbb{A}}_m$. We may thus assume that A is simply connected. If A is iterated tilted of Dynkin type, we are done. Otherwise, if A is representation-finite, there exists a (ν_A)-reflection sequence of sinks $i_1, \ldots, i_t$ such that $S^+_{i_{t-1}} \ldots S^+_{i_1} A$ is representation-finite, but $B = S^+_{i_t} \ldots S^+_{i_1} A$ is representation-infinite and simply connected, by [4, 3.4]. Thus $\widehat{B} \xrightarrow{\sim} \widehat{A}$ and A and B are tilting-cotilting equivalent. We may thus assume from the start that A is representation-infinite and simply connected. We shall prove that A either is iterated tilted of type $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$ or $\widetilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ or is tilting-cotilting equivalent to a tubular canonical algebra, by showing that A is either a domestic or a tubular branch enlargement of a tame concealed algebra. A direct application of [7, 2.5] will then complete the proof. We shall proceed as in [7, §4], applying 2.1 to 2.9.

3.3. Lemma. *A contains a tame concealed convex full subcategory C.*

Proof. As [7, 4.3], using 2.4, 2.8 and the tameness of $\widehat{A}$.

3.4. Lemma. *Let $B = C[M]$ be a one-point extension of C which is a full subcategory of A. Then M is a simple regular C-module.*

P r o o f. Since $\widehat{B}$ is a full subcategory of $\widehat{A}$, it is tame and exhaustive. Then, as in [7, 4.4], we show that M is regular of regular length at most two with non-isomorphic regular composition factors. By 2.1, we may assume that C is hereditary having the orientation in [14, Tables]. If C is of type $\widetilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ then M is simple regular (because $\widehat{B}$ is tame). If C is of type $\widetilde{\mathbb{A}}_m$ $(m \geq 1)$, then M is simple regular by 2.3 and 2.7. Assume thus that C is of type $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$. It follows from 2.6 that M is not regular indecomposable of regular length two lying in a tube of rank $n-2$. Further, since $\widehat{B}$ is tame, M is not indecomposable of regular length two lying in a tube of rank two, nor is it the direct sum of two non-isomorphic simple regular modules, at least one of which lies in a tube of rank two. Therefore, we only need to consider the case where M is the direct sum of two non-isomorphic simple regular modules lying in the tube of rank $n - 2$. We shall show that in this case, $\widehat{B}$ is not exhaustive. It follows from [14, Tables] that B contains a full subcategory D of one of the forms:



bound respectively by $\alpha\beta = 0$, $\alpha\beta = \gamma\delta$ and $\alpha\beta\varepsilon = 0$ (the last case appears only if $n = 4$). In the first two cases, it follows respectively from 2.8 and 2.5 that $\widehat{B}$ is not exhaustive. In the third, let i denote the sink of D. Then $E = S_i^+ D$ is a one-point extension of a hereditary algebra of type $\widetilde{\mathbb{A}}_m$ by an indecomposable regular module of regular length two, and $\widehat{E} \xrightarrow{\sim} \widehat{D}$ is a full subcategory of $\widehat{B}$. Then 2.3 yields a contradiction to the exhaustibility of $\widehat{B}$. Hence M is simple regular.

3.5. LEMMA. *Let $B = C[M]$ be a one-point extension of C by a simple regular C-module M. Then $D = B[M]$ is not a full subcategory of $\widehat{A}$.*

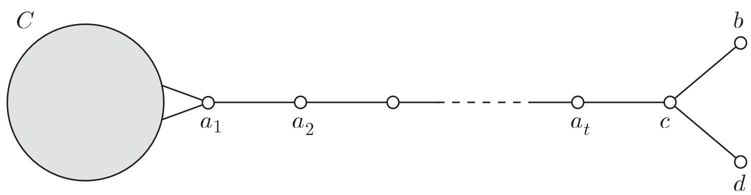
P r o o f. Suppose that D is a full subcategory of $\widehat{A}$. We may, as in 3.4, assume that C is hereditary having the orientation given in [14, Tables]. Since $\widehat{D}$ is tame, it follows from [31, 3.5] that C is either of type $\widetilde{\mathbb{A}}_m$ $(m \geq 1)$ or $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$ and in the latter case, M belongs to a tube of rank $n - 2$.

Let a denote the extension point of C in $B = C[M]$ and b the extension point of B in $D = B[M]$. Then $E = S_a^- D = ([M]C)[M]$ is also a full subcategory of $\widehat{A}$. Moreover, $B' = [M]C$ is, by [32, 4.9], a tilted algebra of Euclidean type $\widetilde{\mathbb{A}}_m$ or $\widetilde{\mathbb{D}}_n$, respectively, having a complete slice in its preprojective component. Let U be the slice module of such a complete slice, considered as an E-module, and $H = \operatorname{End} U$. Then $T_E = U \oplus P(b)$ is a tilting module with endomorphism algebra $F = H[\mathrm{Hom}_{B'}(U, M)]$. Moreover, since $\widehat{E} \xrightarrow{\sim} \widehat{D}$ is exhaustive, then so is $\widehat{F}$, by 2.2. It is not hard to see that $\mathrm{Hom}_{B'}(U, M)$ is a regular indecomposable H-module of regular length two. Moreover, if H is of type $\widetilde{\mathbb{D}}_{n+1}$, then $\mathrm{Hom}_{B'}(U, \ )$ lies in the tube of rank $n-1$. We thus obtain a contradiction by 2.3 and 2.6, respectively. Therefore D is not a full subcategory of $\widehat{A}$.

3.6. LEMMA. *Let $B = C[M]$ be a one-point extension of C by a simple regular C-module M, with extension point a, and let $D = B[X]$ be a one-point extension of B with extension point b. Suppose that D is a full subcategory of A and let N be an indecomposable direct summand of X containing $S(a)$ in its top. Then either $N \xrightarrow{\sim} P(a)_B$ or $N \xrightarrow{\sim} S(a)_B$.*

Proof. Repeat the proof of [7, 4.6] using 3.4.

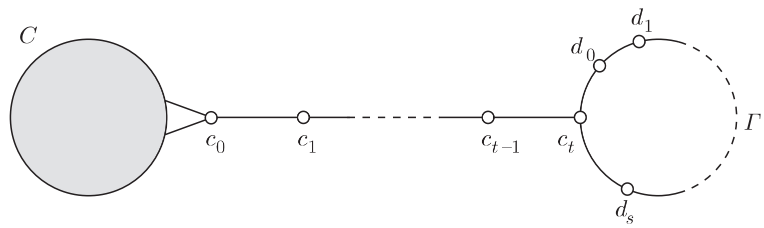
3.7. LEMMA. *A does not contain a full subcategory K of the form*



where the full subcategory formed by the objects a_t, b, c, d is hereditary.

Proof. Suppose that A contains such a full subcategory K. Replacing, if necessary, K by a full subcategory, we can assume that the walk $a_1 — a_2 — \ldots — a_t — c$ has radical square zero, the restriction of $\mathrm{rad}\, P(a_2)$ to C is zero, and finally, by duality, that a_1 is an extension point. By 3.4, the largest C-submodule M of $P(a_1)_K$ is a simple regular C-module. Hence $\widehat{A}$ contains a full subcategory $\widehat{L}$, where L has the same form as K but in which the radical square zero walk $a_1 — a_2 — \ldots — a_t — c$ is not bound and the edge $a_1 — a_2$ is oriented as $a_1 \to a_2$. Since $\widehat{A}$ is tame and exhaustive, L is tame and $\widehat{L}$ is exhaustive. We may also assume that the edge $c — d$ is oriented as $c \leftarrow d$ (for, if this is not the case, we apply an APR-tilting module to the vertex d, and the endomorphism algebra L' is again tame with $\widehat{L'}$ exhaustive, by 2.2). Since L is tame, it follows, as in 3.5, that C is either of type $\widetilde{\mathbb{A}}_m$ or of type $\widetilde{\mathbb{D}}_n$. If C is of type $\widetilde{\mathbb{A}}_m$, then 2.4 yields a contradiction to the exhaustibility of $\widehat{L}$. If C is of type $\widetilde{\mathbb{D}}_n$, then the simple regular module M lies in a tube of rank $n-2$. Now the full subcategory E of L consisting of all objects of L except d is such that $L = E[P(c)]$. Let $F = S_d^- L$. Then $F = [P(c)]E$. Now, by [32, 4.9], E is a tilted algebra of type $\widetilde{\mathbb{D}}_n$ having a complete slice in its preprojective component. Let U be the slice module of such a complete slice, considered as an F-module, and $H = \operatorname{End} U$. Then $T_F = U \oplus I(d')$ is a cotilting module with endomorphism algebra $F' = \operatorname{End} T_F = [\operatorname{Hom}_E(P(c), U)]H$. Again, by 2.2, $\widehat{F'}$ is exhaustive. However, H is hereditary of type $\widetilde{\mathbb{D}}_q$, $q > n$, and the coextension module $\operatorname{Hom}_E(P(c), U)$ is an indecomposable regular H-module of regular length two lying in the tube of rank $q-2$. This contradicts 2.6.

3.8. Lemma. *A does not contain a full subcategory B of the form*

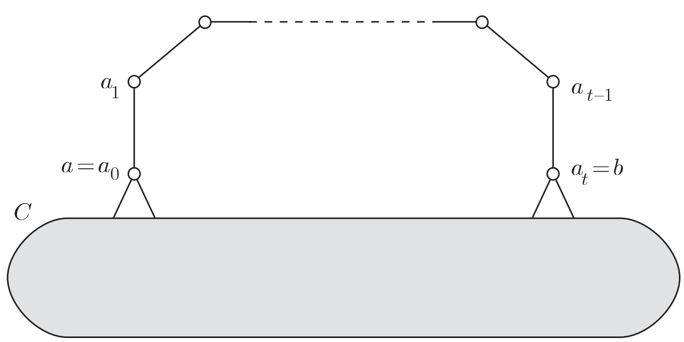


where Γ is a non-commutative cycle.

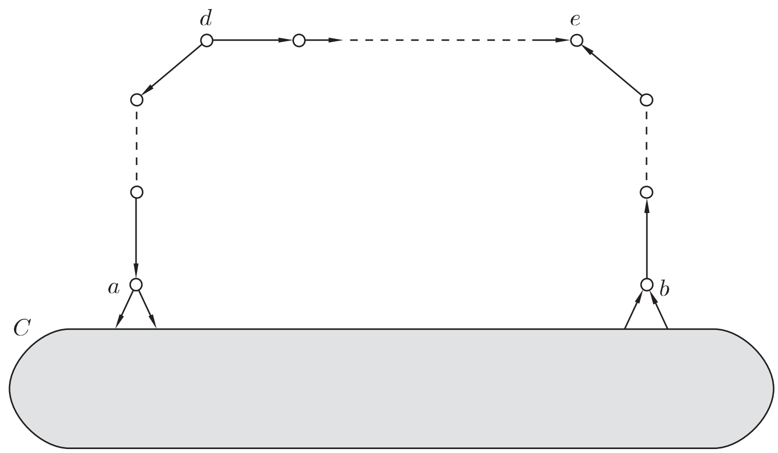
Proof. Suppose that B is a full subcategory of A. Then $\widehat{B}$ is tame and exhaustive. Moreover, by 3.7, the full subcategory of B consisting of all objects outside C is bound only by zero relations. Thus B is not simply connected. Since, by [5], B is clearly not iterated tilted of type $\widetilde{\mathbb{A}}_m$, we obtain a contradiction to Theorem (A).

3.9. Lemma. *Let a and b be two objects of A outside C, each of them connected to C by an edge. Then any walk in A connecting a and b must intersect C.*

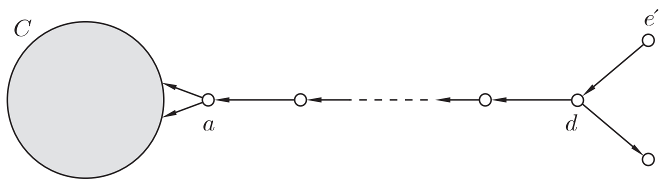
Proof. Suppose there is a walk $a = c_0 — c_1 — \ldots — c_s = b$ in A which does not intersect C. We shall deduce a contradiction to our hypotheses on $\widehat{A}$. As in [7, 4.9], using 3.8 above, we may assume that A contains a full subcategory B of the form



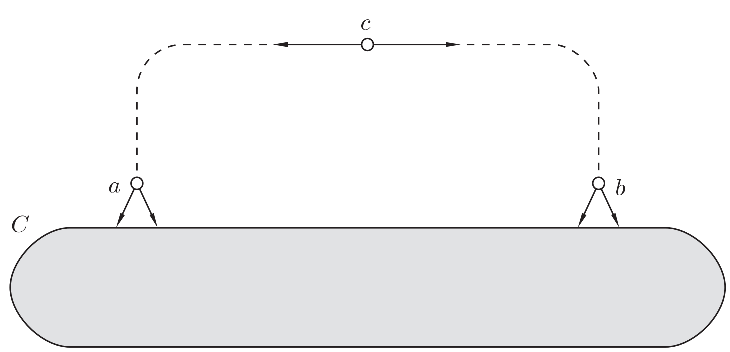
We claim that B is not simply connected. Indeed, since C is connected, there exists a walk w inside C linking $a = a_0$ to $b = a_t$. We shall show that the closed walk consisting of w and the walk $a_0 — a_1 — \ldots — a_{t-1} — a_t$ is not contractible. First observe that B is not of the form



with a commutativity relation from d to e. Indeed, in this case, $S_e^+ B$ contains a full subcategory of the form



hence a contradiction to 3.7. Thus, if B is simply connected, then it is of the form



where both paths from c to C are non-zero, and the largest C-submodule

X of $P(c)$ is indecomposable. Let M (respectively, N) be the largest C-submodule of $P(a)$ (respectively, $P(b)$). Then, by 3.4 and 3.5, M and N are non-isomorphic simple regular C-modules. Since $\mathrm{Hom}_B(P(a), P(c)) \neq 0$ and $\mathrm{Hom}_B(P(b), P(c)) \neq 0$ we have $\mathrm{Hom}_C(M, X) \neq 0$ and $\mathrm{Hom}_C(N, X) \neq 0$. Therefore X is a preinjective C-module. Hence B contains a full subcategory of the form $C[X]$, with X_C preinjective, and this contradicts 3.4. Therefore B is not simply connected. Since B is not iterated tilted of type $\widetilde{\mathbb{A}}_m$, by [5], we obtain a contradiction to Theorem (A). This completes the proof.

Remark. Using the same arguments as in 3.7 and 3.8, one obtains shorter alternate proofs to [7, 4.8 and 4.9].

3.10. Lemma. *A is a branch enlargement of C.*

Proof. Repeat the proof of [7, 4.10] using 3.8, 3.9, 3.5 and 3.7.

3.11. Lemma. *A is either a domestic or a tubular branch enlargement of C.*

Proof. It follows from [4, 2.6] that there exists a truncated branch extension B of C such that and $\widehat{B} \xrightarrow{\sim} \widehat{A}$ and $n_B = n_A$. oreover, by [4, 2.3] and [30, Lemma 2.1], $\widehat{B}$ is tame if and only if n_B is either domestic or tubular. Therefore n_A is either domestic or tubular. Consequently, again by [7, 2.5], A is either a domestic or a tubular branch enlargement of C.

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