Mutation classes of skew-symmetric $3 \times 3-$matrices

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Dedicated to the memory of A.V.Roiter

Abstract

In this paper, we establish a bijection between the set of mutation classes of mutation-cyclic skew-symmetric integral $3 \times 3-$matrices and the set of triples of integers (a, b, c) such that $2 \leq a \leq b \leq c$ and $ab \geq c$. We also give an algorithm allowing to verify whether a matrix is mutation-cyclic or not. We prove that these two cases are not intertwined.

1 Introduction

Cluster algebras have been introduced and studied by Fomin and Zelevinsky in [4, 5, 2]. In particular, it was shown in [5] that every cluster algebra of finite type is acyclic, and corresponds to a Dynkin quiver. In [3], a fruitful connection between acyclic cluster algebras and representations of quivers has been established. In general, there is no known distinction between cyclic and acyclic cluster algebras. The objective of the present note is to study the first non-trivial situation, that of the (coefficient-free) cluster algebras of rank three which are given by a square skew-symmetric matrix.

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In general, for any square skew-symmetric integral matrix B, we denote by $\mathcal{A}(B)$ the associated coefficient-free cluster algebra, as in [4]. We say that $\mathcal{A}(B)$ has rank n if B is an $n \times n-$matrix. The algebra $\mathcal{A}(B)$ is constructed using mutations on B, thus depends not on B itself, but rather on its mutation class within the set $\mathrm{Skew}_n(\mathbb{Z})$ of the skew-symmetric integral $n \times n-$matrices. Every matrix $B = [b_{ij}]$ in $\mathrm{Skew}_n(\mathbb{Z})$ determines a quiver Q_B having $\{1, \ldots, n\}$ as set of points, and b_{ij} arrows from i to j whenever $b_{ij} > 0$. Thus, mutations on B can equivalently be expressed as mutations on the quiver Q_B (see (2.1)). The cluster algebra $\mathcal{A}(B)$ is called *acyclic* if there is a matrix in the mutation class of B whose quiver is acyclic, and otherwise it is called cyclic (see [5]). We say that a matrix $B \in \mathrm{Skew}_n(\mathbb{Z})$ is *cyclic* (or *acyclic*) if the quiver Q_B is so, and we say that B is *mutation-cyclic* (or *mutation-acyclic*) if the corresponding cluster-algebra $\mathcal{A}(B)$ is cyclic (or acyclic, respectively).

In this paper, we consider the case where $n = 3$: this corresponds to quivers with three points. As we see in (2.2) below, the mutation class of a connected quiver with three points always contains a cyclic representative which is determined by three positive integral parameters a, b, c corresponding to the number of arrows on each side. Up to orientation, we may suppose that $a \leq b \leq c$. Accordingly, the cyclic matrices $B \in \mathrm{Skew}_3(\mathbb{Z})$ are determined, up to transposition and simultaneous permutation of rows and columns, by the three parameters $a \leq b \leq c$. Our first theorem characterizes mutation-cyclic quivers (or, equivalently, matrices), in terms of these parameters. Note that we consider all matrices up to transposition and permutation.

Theorem 1.1 *There exists a bijection between the set of mutation classes of mutation-cyclic matrices in* $\mathrm{Skew}_3(\mathbb{Z})$ *and the set of triples* (a, b, c) *of integers such that* $2 \leq a \leq b \leq c$ *and* $ab \geq 2c$.

Our proof gives at the same time a handy algorithm allowing to verify whether a given matrix (or the corresponding quiver) is mutation-cyclic or not. In our second theorem, we show that the two cases (mutation-cyclic and mutation-acyclic) are not intertwined.

Theorem 1.2 *Let* $2 \leq a \leq b \leq c$. *Then there exists a unique integer* $c_0 \in [ab - b, ab - 1]$ *such that a cyclic matrix in* $\mathrm{Skew}_3(\mathbb{Z})$ *which is represented by the triple* (a, b, c) *is mutation-acyclic if and only if* $c > c_0$.

The proofs of these theorems are purely combinatorial. The problem of characterizing the mutation-cyclic skew-symmetric $3 \times 3-$matrices is also considered in [1] from the geometrical point of view. In [6], the wild cluster-tilted algebras having three isomorphism classes of simple modules are studied. The quivers of those algebras yield the mutation-acyclic skew-symmetric $3 \times 3-$matrices which are mutation-infinite.
The paper is organized as follows. In section 2, after a brief preliminary discussion, we prove Theorem 1.1 and state our algorithm. Section 3 contains the proof of Theorem 1.2 and ends with some examples.

2 Mutation classes of quivers with three points

2.1 Preliminaries

Since our intuition is graphical, we work with quivers rather than with matrices. For any quiver Q, we denote by $|Q_1|$ the number of arrows of Q. The map $B \mapsto Q_B$ yields a bijection between $\mathrm{Skew}_n(\mathbb{Z})$ and the set $\mathcal{Q}_n$ of quivers with n points having neither loops nor cycles of length two. We begin by discussing the effect of certain matrix operations on the corresponding quivers.
When constructing the algebra $\mathcal{A}(B)$, the matrix B is given only up to simultaneous row and column permutations. This corresponds to considering the quivers Q_B up to isomorphism. The transposition $B \mapsto B^T$ of the matrix B corresponds to forming the opposite quiver. To formulate our results in a more concise way, we always consider quivers up to isomorphism and change of orientation. The mutation at k of a matrix B as defined in [4] yields a mutation of a quiver $Q \in \mathcal{Q}_n$ at the point k as follows:

(1) All arrows passing through k are reversed.

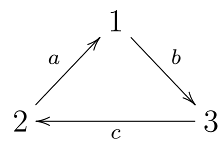
(2) If Q has r_{ij} paths of length two from i to j passing through k, then we add r_{ij} arrows from i to j.

(3) We delete all pairs of arrows which form cycles of length two.

Every mutation is an involution on $\mathcal{Q}_n$. Taking the reflexive and transitive closure yields an equivalence relation denoted by $\sim$. If Q' is obtained from Q by a mutation at the point k, we also write $Q \overset{k}{\sim} Q'$.

2.2

When we deal with a cyclic quiver $Q \in \mathcal{Q}_3$, we always represent it as

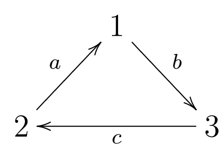


where a, b, c (called the *parameters* of Q) represent the number of arrows in the shown direction. Up to isomorphism of quivers and replacing Q by its opposite quiver, we can suppose that $0 < a \leq b \leq c$.

Lemma 2.1 *Let Q be a connected quiver in $\mathcal{Q}_3$, then*

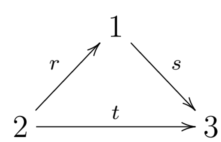
(1) Q is mutation-equivalent to a cyclic quiver.

(2) Q is mutation-acyclic if and only if Q is mutation-equivalent to the quiver Q'

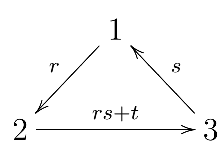


where $0 < a \leq b \leq c$ and $ab \leq c$.

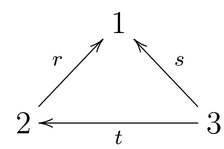
Proof. (1) If Q is acyclic, it is of the form



where r, s, t represent the numbers of arrows in the shown direction. Since Q is connected, we may assume that at most one of r, s, t is zero. If r and s are non-zero, then mutation at 1 yields the quiver Q''



which is cyclic because $rs + t \geq rs > 0$.
If r and t are non-zero, then mutation at 3 yields the quiver Q'''



and thus, as before, mutation at 2 yields a cyclic quiver. The case where s and t are non-zero is dual.

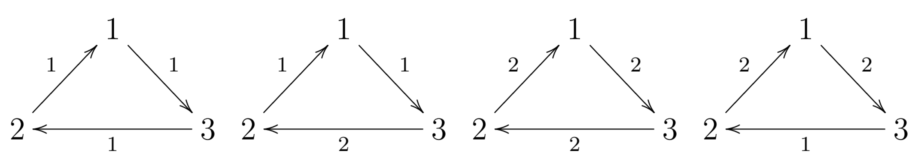
(2) The quiver Q' from (2) is clearly mutation-acyclic, since mutating at 1 yields an acyclic quiver when $ab \leq c$. Conversely, starting with an acyclic quiver Q, we obtain in the proof of (1) the quivers Q'' and Q'''. Then Q'' satisfies the required condition, as seen by setting $a = s, b = r$ and $c = rs + t$. Similarly for Q'''. □

2.3 Mutation-finite quivers

A quiver in $\mathcal{Q}_n$ is called *mutation-finite* if its mutation class is a finite set. The results in this subsection can also be derived from [7] and [8].

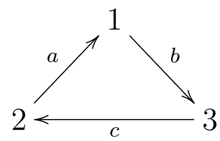
Lemma 2.2 *Let Q be a connected quiver in $\mathcal{Q}_3$, then Q is mutation-finite if and only if any cyclic quiver Q' in its mutation class has at most two parallel arrows.*

Proof. Sufficiency. There are only four connected cyclic quivers in $\mathcal{Q}_3$ with at most two parallel arrows:



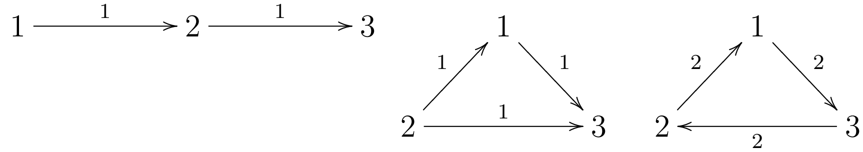
By [8], the first three quivers are 4-bounded, hence mutation-finite. The last one does not satisfy the hypothesis, because mutation at 1 yields a cyclic quiver with 3 arrows from 3 to 2.

Necessity. Assume $Q' \sim Q$ with Q' of the form



where $0 < a \leq b \leq c$ and $c \geq 3$. Mutating at 3 yields a quiver Q'' having $bc - a$ arrows from 1 to 2. Since $c \geq 3$ and $b \geq a$, then $bc - a \geq 2a > a$. Thus $|Q_1''| > |Q_1'|$. Induction shows that we may repeat this procedure infinitely many times leading to quivers with more and more arrows. Therefore Q is not mutation-finite. □

Corollary 2.3 *There are exactly three mutation classes of mutation-finite connected quivers in $\mathcal{Q}_3$, given by the following representatives:*

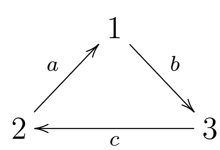


Of these, only the last quiver is mutation-cyclic. Moreover, it is the only representative in its mutation class.

2.4

Consider a cyclic quiver given by parameters $0 < a \leq b \leq c$. Then Lemma 2.1, (2) implies that the quiver is mutation-acyclic if $a = 1$. Thus, since we are interested in mutation-cyclic quivers, we may assume that $(a, b, c) \in \mathbb{Z}^3$ satisfy $2 \leq a \leq b \leq c$. The following lemma shows when such a quiver stays cyclic under one mutation. It also describes how the parameters change.

Lemma 2.4 *Let Q be the cyclic quiver*



with $2 \leq a \leq b \leq c$, and Q' be obtained from Q by a mutation at a point $k \in \{1, 2, 3\}$.

(1) If $k = 1$, then Q' is cyclic with $|Q'_1| \geq |Q_1|$ if and only if $ab \geq 2c$.

(2) If $k = 2$, then Q' is cyclic with the parameters satisfying $2 \leq a \leq c \leq ac - b$.

(3) If $k = 3$, then Q' is cyclic with the parameters satisfying $2 \leq b \leq c \leq bc - a$.

Proof. Assume first $k = 1$.

i) If $c \geq ab$, then $|Q'_1| = a + b + (c - ab) = |Q_1| - ab < |Q_1|$.

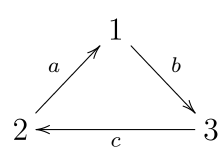
ii) If $c < ab$, then $|Q'_1| = a + b + (ab - c) = |Q_1| + (ab - 2c)$.

Thus, Q' is cyclic when $ab > c$, and $|Q'_1| \geq |Q_1|$ if and only if $ab \geq 2c$.

Assume now $k = 2$. Then the numbers of arrows of Q' are a, c and $ac - b$, respectively. Clearly $2 \leq a \leq c$. Moreover, since $a \geq 2$ and $c \geq b$ we have $ac \geq c + c \geq b + c$, thus $ac - b \geq c$ which is the inequality we wanted. The proof is similar for $k = 3$. □

Remark. Parts (2) and (3) of the preceding lemma show that mutation in a point opposite to one of the parameters a or b necessarily yields a new maximal parameter. This new parameter is strictly greater than the two other parameters unless $a = 2$ and $b = c$ in (2) or $a = b = c = 2$ in case (3).

Proposition 2.5 *Let Q be the cyclic quiver*

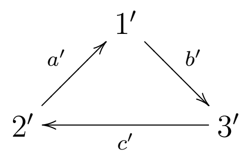


with $2 \leq a \leq b \leq c$. If $ab \geq 2c$, then Q is mutation-cyclic and, for any $Q' \sim Q$, we have $|Q'_1| \geq |Q_1|$.

Proof. If $Q' \sim Q$, there exists a sequence of mutations

$$Q = Q^{(0)} \overset{k_1}{\sim} Q^{(1)} \overset{k_2}{\sim} \cdots \overset{k_n}{\sim} Q^{(n)} = Q'$$

Without loss of generality, we may assume this sequence to be reduced, that is, two consecutive mutations in this sequence are not inverse to each other (thus, $k_i \neq k_{i+1}$ for all i). We show by induction on n that the quiver $Q^{(n)}$ is cyclic (and so are all quivers $Q^{(i)}$ in the sequence) and that $|Q^{(n)}| \geq |Q^{(n-1)}|$. Lemma 2.4 shows that $Q^{(1)}$ is cyclic with $|Q_1^{(1)}| \geq |Q_1|$, thus we consider the induction step. Suppose $n \geq 2$, and let $Q^{(n-1)}$ be the quiver



which is cyclic by induction. Up to duality, we can suppose $a' \leq b' \leq c'$, and by induction, the numbers of arrows cannot decrease by mutations, thus $2 \leq a'$. The quiver $Q^{(n)}$ is obtained from $Q^{(n-1)}$ by mutation at k_n. If $k_n \neq 1'$ then the statement follows by lemma 2.4. Assume now that $k_n = 1'$. Since the sequence above is supposed to be minimal, the mutation preceding k_n satisfies $k_{n-1} \neq 1'$. This means that one of the values a' or b' has been changed when going from $Q^{(n-2)}$ to $Q^{(n-1)}$. But we know that $a' \leq b' \leq c'$, therefore the remark following lemma 2.4 shows that we are in one of the cases $a' = 2$ and $b' = c'$ or $a' = b' = c' = 2$. If $a' = b' = c' = 2$, then all the quivers $Q^{(i)}$ are isomorphic, and the statement holds. If $a' = 2$ and $b' = c'$, then mutation at $1'$ transforms the number c' in $Q^{(n-1)}$ into $a'b' - c' = 2b' - b' = b' = c'$, thus it stays the same, which implies that $Q^{(n)}$ is cyclic and $|Q_1'| \geq |Q_1|$, which we wanted to show. □

2.5

In the sequel, we call *root* (of its mutation class) a cyclic quiver $Q \in \mathcal{Q}_3$ whose parameters (a, b, c) satisfy $2 \leq a \leq b \leq c$ and $ab \geq 2c$.

The previous results yield the following algorithm which decides whether a given connected quiver $Q \in \mathcal{Q}_3$ is mutation-cyclic or not:

(1) If Q is acyclic, stop.

(2) Otherwise, Q is a cyclic quiver with ordered parameters $0 \leq a \leq b \leq c$. If $ab \leq c$, then Q is mutation-acyclic by lemma 2.1, stop.

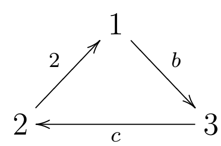
(3) Perform a mutation at the point opposite to c. If the number of arrows has decreased, go back to step (1). Otherwise, Q is a root by lemma 2.4, and thus Q is mutation-cyclic.

This procedure must clearly stop after finitely many steps, since we deal with strictly decreasing sequences of natural numbers.

Proof of theorem 1.1. We have shown in proposition 2.5 that every root is mutation-cyclic. Conversely, let Q be a mutation-cyclic quiver. Applying the algorithm above, we find a root in the mutation class of Q. Up to duality, this root is uniquely described by its parameters $a \le b \le c$. $\square$

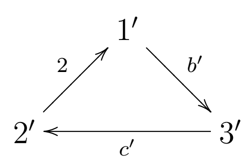
3 Separating the cyclic case from the acyclic

Lemma 3.1 *The quiver Q*



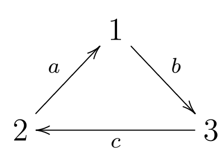
with $2 \le b \le c$ is mutation-cyclic if and only if $b = c$.

Proof. Sufficiency follows from proposition 2.5, so we only show necessity. Asssume that Q is mutation-cyclic. By applying a sequence of mutations at the point opposed to the maximum, we reach a root Q'



with $2 \le b' \le c'$ and $2b' \ge 2c'$ (because Q' is a root). Thus $b' = c'$. But mutations at $1'$ or $2'$ do not change the quiver Q'. This implies that $Q' = Q$, and so $b = b' = c' = c$. $\square$

Lemma 3.2 *Let Q be the quiver*



with $2 \le a \le b \le c$.

(1) If $c \leq ab - b$, then Q is mutation-cyclic.

(2) If $c \geq ab - 1$, then Q is mutation-acyclic.

Proof. Mutating at 1 yields a quiver Q' with $ab - c$ arrows between 2 and 3.

(1) If $ab - c \geq b$ (or, equivalently, $c \leq ab - b$), then the new maximal number is $ab - c$. Since the maximum did not change its position, it follows from lemma 2.4 that Q or Q' is a root. In particular, Q is mutation-cyclic.

(2) If $ab - c \leq 1$ (or, equivalently, $c \geq ab - 1$), then Q is mutation-acyclic by lemma 2.1, (2).

□

Lemma 3.3 *Consider the homogeneous difference equation*

$$S_{n+2} = aS_{n+1} - S_n$$

(with $a \geq 2, S_0 = 0$ and $S_1 = 1$), then

(1) $S_n = \begin{cases} \frac{1}{2^n\sqrt{a^2-4}}\left[(a+\sqrt{a^2-4})^n - (a-\sqrt{a^2-4})^n\right] & \text{if } a \geq 3 \\ n & \text{if } a = 2 \end{cases}$

(2) The sequence $(S_n)_{n\geq 0}$ is strictly increasing

(3) $S_n \geq a$ for any $n \geq 2$.

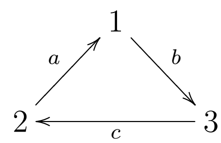
Proof.

(1) This is straightforward and left to the reader.

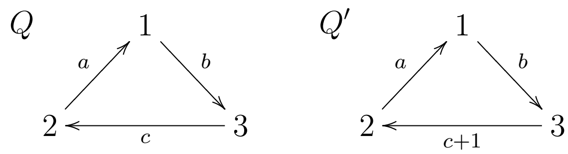
(2) Use induction on n: clearly $S_0 < S_1$. Assume $S_n < S_{n+1}$, then $S_{n+2} = aS_{n+1} - S_n > aS_{n+1} - S_{n+1} \geq S_{n+1}$ because $a \geq 2$.

(3) Since $S_2 = a$, this follows from (2). □

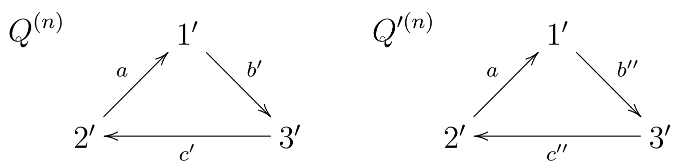
Proof of theorem 1.2. Let Q be the quiver



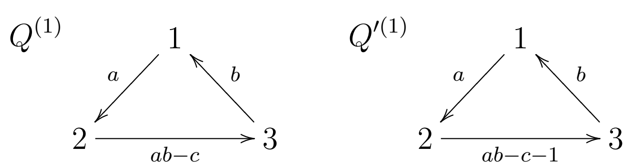
with $2 \leq a \leq b \leq c$. By lemma 3.1, the statement holds if $a = 2$ (with $c_0 = b$). Assume $a \geq 3$ and that such a c_0 does not exist. Then there exist quivers Q, Q'



with $3 \leq a \leq b \leq c$ such that Q is mutation-acyclic and Q' is mutation-cyclic. We show by induction that the quiver Q never reaches an acyclic representative. This will yield a contradiction which implies our statement. More precisely, we prove by induction on $n \geq 1$ that, after n mutations at the point opposed to the maximum, we obtain respectively the quivers $Q^{(n)}$ and $Q'^{(n)}$



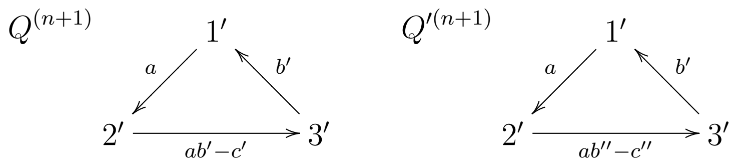
with $S_n = b' - b'', S_{n-1} = c' - c'', \max(a, b', c') = c'$ and $\max(a, b'', c'') = c''$ where S_n is as in lemma 3.3 and $Q^{(n)}$ is still a cyclic quiver (note that the parameter a does not change).
Assume first $n = 1$. After one mutation at the point opposed to the maximum, we obtain respectively the quivers



If $Q^{(1)}$ is not cyclic, then $ab - c \leq 0$, hence $ab - c - 1 < 0$ and so $Q'^{(1)}$ is not cyclic either. This is a contradiction to the hypothesis that Q' is mutation-cyclic, hence $Q^{(1)}$ is a cycle. In order to pursue the algorithm, we must have

$b > ab - c$ (otherwise, the next mutation yields $Q^{(2)} = Q$ and so Q or $Q^{(1)}$ is a root, contradicting the hypothesis that Q is mutation-acyclic). Therefore, $b > ab - c - 1$. Thus, in $Q^{(1)}$ and $Q'^{(1)}$, the maxima correspond to the same sides. Moreover, $S_1 = (ab - c) - (ab - c - 1) = 1$ and $S_0 = b - b = 0$.

Assume the statement holds for n. Mutating at $1'$ (which is opposed to the respective maxima c' and c'') yields respectively the quivers



We first note that $(ab'-c')-(ab''-c'') = a(b'-b'')-(c'-c'') = aS_n - S_{n-1} = S_{n+1}$. Also, $Q^{(n+1)}$ is a cycle. Indeed, if this is not the case, then $ab'-c' \leq 0$. Since $S_{n+1} \geq 0$ (by lemma 3.3), we have $ab'' - c'' \leq 0$, contradicting the hypothesis that Q' is mutation-cyclic.

We now determine the maximal parameter in $Q^{(n+1)}$.
1) Assume $\max(a, b', ab' - c') = a$. By lemma 3.3, we have

$$(ab' - c') - (ab'' - c'') = S_{n+1} \geq a$$

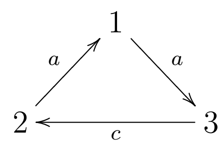
hence $ab'' - c'' \leq (ab' - c') - a \leq 0$, contradicting the hypotesis that Q' is mutation-cyclic.
2) Assume $\max(a, b', ab' - c') = ab' - c'$. Then, mutating at the point opposed to the maximum yields $Q^{(n+2)} = Q^{(n)}$, thus Q is mutation-cyclic, a contradiction.
Therefore $\max(a, b', ab'-c') = b'$. We claim that also $\max(a, b'', ab''-c'') = b''$. Assume first that $a > b''$. Since $ab'' - c'' \geq 2$, we have

$$\begin{array}{rcl} a(ab' - c') = a[(ab'' - c'') + S_{n+1}] & = & a(ab'' - c'') + aS_{n+1} \\ & \geq & 2a + aS_{n+1} \\ & > & 2b'' + aS_n \\ & > & 2b'' + 2S_n = 2b'. \end{array}$$

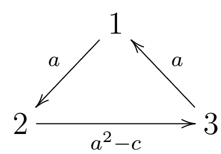
On the other hand, $S_{n+1} = (ab' - c') - (ab'' - c'') \geq 0$ yields $ab' - c' \geq ab'' - c'' \geq 2$. These inequalities show, by Theorem 1.1, that $Q^{(n+1)}$ is a root. Thus Q is mutation-cyclic, a contradiction. Therefore $a \leq b''$.
Since $b' > ab' - c'$, we have $b'' = b' - S_n > (ab' - c') - S_{n+1} = ab'' - c''$.

This shows that $b'' = \max(a, b'', ab''-c'')$, completes the proof of the induction statements and thus establishes the theorem. $\square$

Example 1. Let Q be the quiver

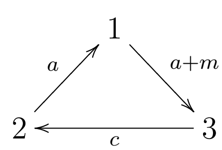


with $2 \leq a \leq c$. Then we claim that $c_0 = a^2 - 2$.
Indeed, applying a mutation at 1 yields the quiver Q'

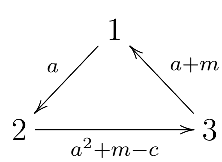


1) Suppose $c \leq a^2 - 2$. If $a^2 - c$ is the maximum, then the next mutation at the point opposed to the maximum yields Q so that Q or Q' is a root and Q is mutation-cyclic. If not, then a is the maximum and moreover $a(a^2 - c) \geq 2a$ so that Q' is a root, whence Q is again mutation-cyclic.
2) Suppose $c > a^2 - 2$. Then $a^2 - c \leq 1$ and so Q is mutation-acyclic by lemma 2.1, (2).

Example 2. Let Q be the quiver



where $2 < a \leq c$ and $1 \leq m \leq 4$. Then we claim that $c_0 = a^2 + am - 3$.
Indeed, applying a mutation at 1 yields the quiver Q'



1) Assume $c \leq a^2 + am - 3$. If $a^2 + am - c$ is the maximum, then the next mutation at the point opposed to the maximum yields Q again, which is

then mutation-cyclic. If not, then $a+m$ is the maximum. If $a \geq 2m$, then $c \leq a^2+am-3$ yields $a(a^2+am-c) \geq 3a = 2a+a \geq 2(a+m)$. Thus, Q' is a root and Q is mutation-cyclic. On the other hand, if $a < 2m$, then we have

$$3 \leq a < 2m \leq 8$$

$$a < a+m \leq a+4$$

$$a+m \leq c \leq a^2+am-c.$$

There are only finitely many quivers verifying these inequalities. A straightforward verification shows that in each case Q is mutation-cyclic, as desired.
2) Assume $c > a^2+am-3$. If $a^2+am-c \leq 1$, then by lemma 2.1, (2) Q is mutation-acyclic. Otherwise, $a^2+am-c=2$ and then the same conclusion follows from (3.1).

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