Canadian Mathematical Society
Conference Proceedings
Volume 14, 1993

Multicoil Algebras

IBRAHIM ASSEM and ANDRZEJ SKOWROŃSKI

ABSTRACT. Let A be a finite dimensional algebra over an algebraically closed field. We define a coil in the Auslander-Reiten quiver of A to be a component obtained from a stable tube by a sequence of operations that we shall call admissible. We shall show that, for every coil, there exists a triangular algebra having this coil as a standard component of its Auslander-Reiten quiver. We also give an axiomatic characterisation of coils. A multicoil consists of a finite number of coils glued together by some directed parts. A multicoil algebra is an algebra having the property that every cycle of non-zero non-isomorphisms between indecomposables lies in a standard coil of a multicoil. Multicoil algebras are cycle-finite and hence tame. We show here that a multicoil algebra is minimal representation-infinite if and only if it is tame concealed.

Let k be an algebraically closed field, and A be a finite dimensional associative k-algebra with an identity. We shall denote by mod A the category of finitely generated right A-modules. It follows from a well-known result of Drozd [12] that the representation theory of A belongs to one of two non-intersecting classes: "wild", which contain the classical problem of reducing a pair of matrices under simultaneous conjugations, and "tame" in which indecomposable finite dimensional modules occur, in each dimension $d \geq 1$, in a finite number of discrete and a finite number of one-parameter families. If there exists $m \in \mathbb{N}$ such that the least number of these one-parameter families is bounded, in each dimension d, by d^m, then A is said to be of polynomial growth [22]. This class of tame algebras is at present believed to be the most accessible, and has been the subject of intensive research over the last few years

1991 Mathematics Subject Classification 16G70, 16G60, 16G20
The first author gratefully acknowledges support from the NSERC of Canada and the University of Sherbrooke as well as the hospitality of Nicholas Copernicus University. The second author gratefully acknowledges support from the Polish scientific Grant KBN No. 1222/2/9 and the hospitality of the University of Sherbrooke.
This paper is in final form and no version of it will be submitted for publication elsewhere.

© 1993 American Mathematical Society
0731-1036/93 $1.00 + $.25 per page

[1, 2, 3, 6, 16, 17, 18, 21, 22, 23, 24, 25]. All these results have pointed out the importance of the notion of tube [13, 21] for the representation theory of such algebras: in fact, it is shown in [11] that for a representation-infinite tame algebra, all but finitely many non-isomorphic indecomposable modules of dimension d belong to homogeneous tubes. One of the main purposes of the present paper is to introduce the notion of admissible operations, and a component obtained from a stable tube by a sequence of admissible operations will be called a coil (observe that this use of the term coil deviates from its use in [2]). We shall show that, for any coil, there exists a triangular algebra (that is, an algebra having no oriented cycle in its ordinary quiver) having this coil as a standard component of its Auslander-Reiten quiver. We shall also characterise coils axiomatically. Namely, we shall describe a topological space, called a crowned cylinder, to which the underlying topological space (as defined in [19]) of a coil, modulo some projective-injective points, is homeomorphic. Then we shall prove:

THEOREM (A). Let Γ be a translation quiver without multiple arrows and containing a cyclical path. Then Γ is a coil if and only if the following conditions are satisfied:

(1) Let Γ' denote the full translation subquiver of Γ consisting of all points except those which are projective-injective middle terms of a mesh in Γ with three middle terms. Then the underlying topological space of Γ' is homeomorphic to a crowned cylinder.

(2) For any mesh with three middle terms, none of which is projective-injective, two of the middle terms lie on the mouth of Γ.

(3) For any projective $p \in \Gamma_0$, or injective $q \in \Gamma_0$, and any $x \in \Gamma_0$, we have $\dim_k \mathrm{Hom}_{k(\Gamma)}(p, x) \le 1$, or $\dim_k \mathrm{Hom}_{k(\Gamma)}(x, q) \le 1$, respectively.

(4) For any projective $p \in \Gamma_0$, or injective $q \in \Gamma_0$, there exists a ray starting at p, or a coray ending in q, respectively.

(5) The τ-orbit of any projective, or injective, contains a point which belongs to a cyclical path.

(6) There exists a length function on Γ.

We shall then introduce the notion of a multicoil: it consists of a finite set of coils glued together by some directed part. An algebra A will be called a multicoil algebra if any cycle in mod A (that is, any oriented cycle of non-zero non-isomorphisms between indecomposable modules) belongs to one standard coil of a multicoil in the Auslander-Reiten quiver of A. Clearly, for such a cycle, no morphism on the cycle lies in the infinite power of the radical of mod A and consequently, by [2] (1.4), any multicoil algebra is tame. In fact, it is shown in [3] (4.6) that multicoil algebras are of polynomial growth. The class of multicoil algebras contains all the best understood examples of algebras of polynomial growth and finite global dimension, and it seems to be of fundamental interest for studying the simply connected algebras of polynomial growth. In fact, it is shown in [25] that, if A is such that every full convex subcategory is simply connected, then A is of polynomial growth if and only if A is a multicoil algebra. Further, it is proved in [26] that, if A has directing indecomposable projective modules, than A is tame if and only if A is a multicoil algebra. The present paper is devoted to providing a setting for the study of multicoil algebras. We shall show that multicoil algebras are triangular, then prove that a full convex subcategory of a multicoil algebra is a multicoil algebra. This will allow us to show that the minimal representation-infinite multicoil algebras coincide with the tame concealed algebras of [14,21] (note that this characterisation is similar to the one obtained in [2] for a different concept of coil).

THEOREM (B). Let A be a basic and connected finite dimensional algebra over an algebraically closed field. The following conditions are equivalent:

(i) A is a tame concealed algebra,

(ii) A is a representation-infinite multicoil algebra and, for every $0 \neq e^2 = e \in A$, A/AeA is representation-finite,

(iii) A is a representation-infinite multicoil algebra and every proper full convex subcategory of A is representation-finite.

This implies that any representation-infinite multicoil algebra contains a tame concealed algebra as a full convex subcategory. Applying the above results in [3], we show that indecomposable modules lying in a stable tube of a multicoil algebra have as their support a tame concealed or a tubular algebra (in the sense of [21]). Thus the structure of such indecomposables is completely described. Here, we shall prove the following result on the structure of non-stable coils over multicoil algebras.

THEOREM (C). Let A be a multicoil algebra and Γ be a non stable coil in the Auslander-Reiten quiver Γ_A of A. Then there exists a tame concealed full convex subcategory C of A and a stable tube T of Γ_C such that Γ is obtained from T by a sequence of admissible operations and the support algebra Supp Γ of Γ is obtained from C = Supp T by the corresponding sequence of one-point extensions and coextensions.

In forthcoming papers [4, 5], using the above theorem, we shall describe the structure of indecomposable modules lying in a cyclical path of a coil in a multicoil algebra.

The present paper is organised as follows. After a brief introductory section (1), in which we fix the notations and recall the basic definitions, section (2) is devoted to the description of admissible operations on finite dimensional algebras. In section (3), we describe the corresponding admissible operations on translation quivers and introduce the notions of coils and multicoil algebras. Section (4) is devoted to the proof of theorem (A) and finally section (5) is devoted to the proofs of theorems (B) and (C).

1. Notation and preliminary definitions.

1.1 Throughout this paper, k will denote a fixed algebraically closed field. An algebra A will always mean an associative finite dimensional k-algebra with an identity, and will be usually assumed to be basic and connected. For such an algebra A, there exists a connected bound quiver (Q_A,I) and an isomorphism $A \xrightarrow{\sim} kQ_A/I$. Equivalently, $A = kQ_A/I$ may be considered as a k-linear category, of which the object class A_0 is the set $(Q_A)_0$ of points of Q_A, and the set of morphisms $A(x, y)$ from x to y is the quotient of the k-vector space $kQ_A(x, y)$ having as basis the set of paths in Q_A from x to y by the subspace $I(x, y) = I \cap kQ_A(x, y)$, see [10]. A full subcategory C of A will be called <u>convex</u> (in A) if any path in A with source and target in C lies entirely in C. It is called <u>triangular</u> if Q_C contains no oriented cycle.

By an A-module is always meant a finitely generated right A-module, and we shall denote their category by mod A. For an A-module M, we denote by add M the additive full subcategory of mod A consisting of the direct sums of indecomposable summands of M. A <u>cycle</u> in mod A is a sequence of non-zero non-isomorphisms $M_0 \to M_1 \to \ldots \to M_t = M_0$ where the M_i are indecomposable. A module M is called <u>directing</u> if it lies on no cycle in mod A. For $i \in (Q_A)_0$, we denote by S(i) the corresponding simple A-module and by P(i) (respectively, I(i)) the projective cover (respectively, the injective envelope) of S(i). The <u>dimension-vector</u> of a module M is the vector <u>dim</u> $M = (\dim_k \mathrm{Hom}_A(P(i), M))_{i \in (Q_A)_0}$. The <u>support</u> Supp (<u>d</u>) of a dimension-vector $\underline{d} = (d_i)_{i \in (Q_A)_0}$ is the full subcategory of A with object class $\{i \in (Q_A)_0 \mid d_i \neq 0\}$. The support of a module M is by definition the support of its dimension-vector.

1.2 We shall use freely properties of the Auslander-Reiten translations $\tau_A = DTr$ and $\tau_A^{-1} = TrD$ (which we shall denote respectively by τ or τ^{-1}, if no confusion is possible) and the Auslander-Reiten quiver Γ_A of A, for which we refer to [7, 21]. We shall agree to identify the points in Γ_A with the corresponding indecomposable A-modules. Let Γ be a translation subquiver of Γ_A. We shall denote by ind Γ the full subcategory of mod A determined by one representative of each point in Γ. We shall say that Γ is <u>standard</u> if ind $\Gamma \simeq k(\Gamma)$, where the latter denotes the mesh category of Γ, see [10]. The support Supp Γ of Γ is defined to be the full subcategory of A with object class $\{i \in (Q_A)_0 \mid (\underline{\dim}\, M)_i \neq 0 \text{ for some } M \in \Gamma_0\}$.

A translation quiver Γ is called a <u>tube</u> [13,21] if it contains a cyclical path and if its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$ (when S^1 is the unit circle, and $\mathbb{R}^+$ the set of non-negative real numbers). A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. This also applies to sectional paths, this is, paths $x_0 \to x_1 \to \ldots \to x_m$ in Γ such that $x_{i-1} \neq \tau x_{i+1}$ for all $0 < i < m$. An infinite sectional path consisting of arrows pointing to infinity (respectively, to the mouth) is called a <u>ray</u> (respectively, a <u>coray</u>). It is not hard to show that the composition of the morphisms of mod A determined by a sectional path in a tube of Γ_A is non-zero. Tubes containing no projectives or injectives are called <u>stable</u>. In this paper, all tubes are assumed to be coherent with length functions [13] (3) that is, are obtained from stable tubes by a finite number of ray or coray insertions.

1.3 The one-point extension of the algebra A by the A-module X_A is the algebra

$$A[X] = \begin{bmatrix} A & 0 \\ X & k \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of A [X] contains Q_A as a full subquiver and there is an additional (extension) point which is a source. The A [X]-modules are usually identified with the triples (V, M, φ), where V is a k-vector space, M an A-module and $\varphi: V \to \mathrm{Hom}_A(X, M)$ is a k-linear map. An A [X]-linear map (V, M, φ) $\to$(V', M', φ') is thus a pair (f, g), where f: $V \to V'$ is k-linear and g: $M \to M'$ is A-linear such that $\varphi' f = \mathrm{Hom}_A(X,g)\varphi$. One defines dually the one-point coextension [X]A of A by X.

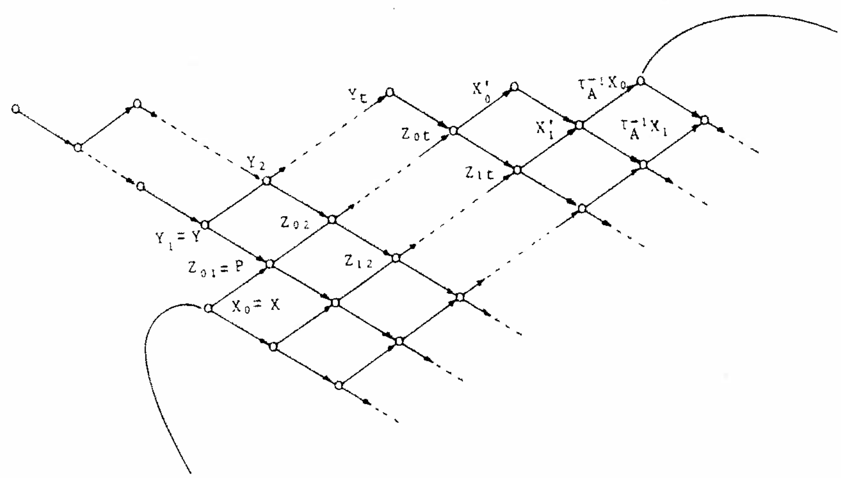
2. Construction of standard components.

2.1 In this section, we shall introduce admissible operations and show that, under reasonable assumptions, these preserve the standardness of components. These admissible operations will in turn motivate the introduction of coils and of multicoil algebras. Throughout this section, let A be an algebra, and Γ be a standard component of Γ_A. For an indecomposable A-module X in Γ, called the pivot, we shall define admissible operations depending on the shape of the support of the functor $\mathrm{Hom}_A(X,-)|_{\mathrm{ind}\,\Gamma}$. This is, by definition, the subcategory of ind Γ consisting of the modules M such that $\mathrm{Hom}_A(X, M) \neq 0$ and whose non-zero morphisms are the morphisms $f: M \to N$ such that $\mathrm{Hom}_A(X, f) \neq 0$.

ad1) Assume that Supp $\mathrm{Hom}_A(X,-)|_{\mathrm{ind}\Gamma}$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \ldots$$

Let D denote the full txt lower triangular matrix algebra (with $t \geq 1$) and Y denote the unique indecomposable projective-injective D-module. We define the modified algebra A' of A to be the one-point extension $A' = (A \times D)[X \oplus Y]$, and the modified component Γ' of Γ to be:



where $Z_{ij} = \left(k, X_i \oplus Y_j, \begin{pmatrix}1\\1\end{pmatrix}\right)$ for $i \geq 0$, $1 \leq j \leq t$.

$X'_i = (k, X_i, 1)$

and the morphisms are the obvious ones. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{0j} = Y_{j-1}$ if $j \geq 2$, $Z_{01} = P$ is projective, $\tau' X'_0 = Y_t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 1$, $\tau'(\tau_A^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining points of Γ (respectively, Γ_D), τ' coincides with τ_A (respectively, τ_D).

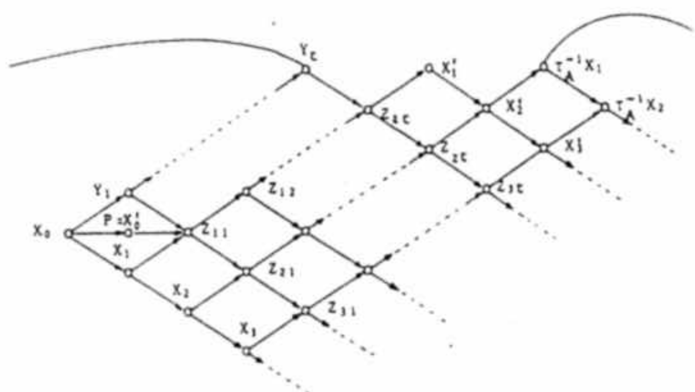
The support of the functor $\mathrm{Hom}_{k(\Gamma')}(P, -)$ will be called the <u>rectangle determined by P</u>: it equals the full subquiver of Γ' consisting of the points Z_{ij} and X'_i. We shall say that Γ' is obtained from Γ (and Γ_D) by inserting this rectangle.

If $t = 0$, we define the modified algebra A' to be the one-point extension $A' = A[X]$. In this case, the rectangle determined by P consists solely of the ray formed by the X'_i 's: thus the modified component Γ' is obtained from Γ by a single ray insertion.

ad2) Assume that $\mathrm{Supp}\ \mathrm{Hom}_A(X,-)|_{\mathrm{ind}\ \Gamma}$ consists of two sectional paths starting at X, one infinite and the other finite with at least one arrow

$$Y_t \leftarrow \ldots\leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \ldots$$

with $t \geq 1$, so that, in particular, X is injective. We define the modified algebra A' of A to be the one-point extension $A' = A\,[X]$ and the modified component Γ' of Γ to be

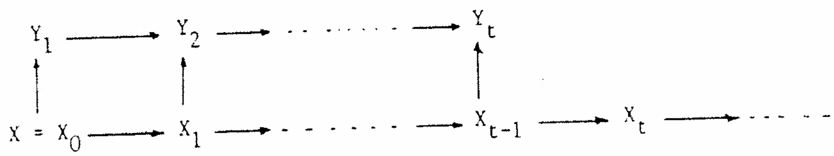


where $\quad Z_{ij} = \left(k,\ X_i \oplus Y_j, \begin{pmatrix}1\\1\end{pmatrix}\right) \qquad$ for $i \geq 1,\ 1 \leq j \leq t$

$\qquad\quad X'_i = (k, X_i, 1) \qquad$ for $i \geq 1$

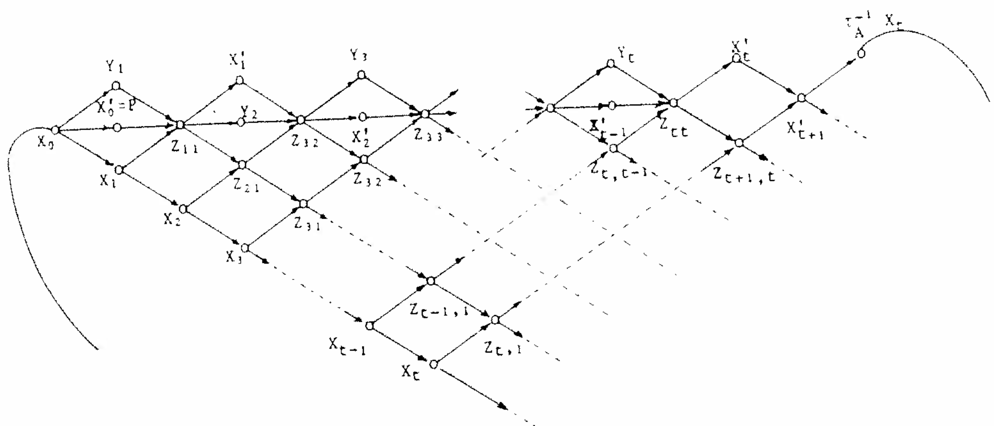
and the morphisms are the obvious ones. The translation τ' of Γ' is defined as follows: $P = X'_0$ is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' Z_{1j} = Y_{j-1}$ if $j \geq 2$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq 2$, $\tau' X'_i = Y_t$, $\tau'(\tau_A^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining points of Γ', τ' coincides with the translation τ of Γ. The support of the functor $\mathrm{Hom}_{k(\Gamma')}(P,-)$ will be called the <u>rectangle determined by P</u>: it equals the full subquiver of Γ' consisting of the points Z_{ij} and X'_i. We shall say that Γ' is obtained from Γ by inserting this rectangle.

ad3) Assume that $\mathrm{Supp\,Hom}_A(X,-)|_{\mathrm{ind}\Gamma}$ consists of two parallel sectional paths, the first infinite and starting at X, the second finite with at least one arrow and starting at a point Y_1, such that there is an arrow $X \to Y_1$, not lying on the first path,

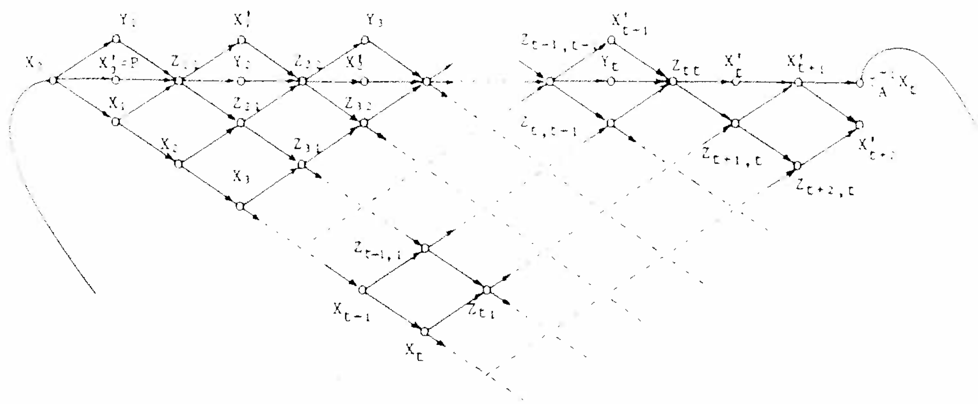


where $t \geq 2$, so that, in particular, X_{t-1} is injective. We define the modified algebra A' of A to be the one-point extension A' = A [X] and the modified component Γ' of Γ to be:

— If t is odd:



— If t is even:



when $Z_{ij} = \left(k, X_i \oplus Y_j, \binom{1}{1}\right)$ for $i \geq 1$, $1 \leq j \leq i$

$X'_i = (k, X_i, 1)$

and the morphisms are the obvious ones. The translation τ' is defined as follows: $P = X'_0$ is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2$, $2 \leq j \leq i$, $\tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X'_i = Y_i$ if $1 \leq i \leq t$, $\tau' X'_i = Z_{i-1,t}$ if $i > t$, $\tau' Y_j = X'_{j-2}$ if $2 \leq j \leq t$, $\tau'(\tau_A^{-1} X_i) = X'_i$ if $i \geq t$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. We note that X'_{t-1} is injective. For the remaining points of Γ', τ' coincides with the translation τ_A of Γ. The support of the functor $\mathrm{Hom}_{k(\Gamma')}(P,-)$ will be called the rectangle determined by P: it equals the full subquiver of Γ' consisting of the points Z_{ij} and X'_i. We shall say that Γ' is obtained from Γ by inserting this rectangle.

The respective duals of ad1), ad2), ad3) will be denoted by ad1*) ad2*), ad3*). These six operations will be called admissible. In the following lemmas, we shall prove that the component of the Auslander-Reiten quiver of the modified algebra containing the pivot is equal to the modified component and, under suitable assumptions, is standard.

2.2 LEMMA. In the case ad1), the component of $\Gamma_{A'}$ containing X (considered as an A'-module) is equal to Γ' and is standard.

Proof. By construction, P is the only indecomposable projective A'-module which is neither an indecomposable projective A-module, nor an indecomposable projective D-module. Also, there are inclusion morphisms of X and Y as summands of rad P, which are therefore irreducible in mod A'. Recall that, by [21] 2.5(5) p. 87, if $g: M \to N$ is right minimal almost split in mod A, then $(0, g)$: $(\mathrm{Ker\,Hom}_{A\times D}(X \oplus Y, g), M, u) \to (0, N, 0)$ (where u denotes the canonical inclusion) is right minimal almost split in mod A'.

Let $i \geq 0$. The right minimal almost split morphism in mod A' ending in $(0, X_i, 0)$ is given by $(0, g)$ with $g: M \to X_i$ right minimal almost split in mod A. Clearly, $M = X_{i-1} \oplus M'$ where we agree that $X_{-1} = 0$ and, for $i \geq 1$, we have $X_{i-1} \notin$ add M'. Since M is an A-module, $\mathrm{Hom}_{A \times D}(Y, M) = 0$ so $\mathrm{Ker}\ \mathrm{Hom}_{A \times D}(X \oplus Y, g) = \mathrm{Ker}\ \mathrm{Hom}_A(X, g)$ where $\mathrm{Hom}_A(X, g)$: $\mathrm{Hom}_A(X, X_{i-1} \oplus M') \to \mathrm{Hom}_A(X, X_i)$. By our assumption on $\mathrm{Supp}\ \mathrm{Hom}_A(X, -)|_{\mathrm{ind}\Gamma}$, we have $\mathrm{Hom}_A(X, M') = 0$ while $\mathrm{Hom}_A(X, g)$: $\mathrm{Hom}_A(X, X_{i-1}) \to \mathrm{Hom}_A(X, X_i)$ is injective and so our right minimal almost split morphism is $(0, g)$: $(0, M, 0) \to (0, X_i, 0)$. In particular, the morphisms $X_i \to X_{i+1}$ remain irreducible in mod A'. Moreover, there exists no irreducible morphism $X_i \to U$ in mod A' with $U \not\cong Z_{i1}, X_{i+1}$: indeed, otherwise, $U_{A'}$ cannot be projective, hence there exists an irreducible morphism $\tau_{A'} U \to X_i$ in mod A', which contradicts the above description of the right minimal almost split morphism of mod A' ending in X_i.

Applying again [21] 2.5 (5) p.87, we see that all irreducible morphisms in mod D remain so in mod A'. A straightforward induction on the construction of the cokernel term in the respective sequences shows that we have indeed the almost split sequences ending at the modules in the rectangle determined by P. There remains to compute the almost split sequences starting at the X'_i. Assume there exists an irreducible morphism $X_i \to V$ in mod A, with V indecomposable. By [9] and our assumption on $\mathrm{Supp}\ \mathrm{Hom}_A(X,-)|_{\mathrm{ind}\Gamma}$, we must have $V \xrightarrow{\sim} X_{i+1}$ or $V \xrightarrow{\sim} \tau_A^{-1} X_{i+1}$. The left minimal almost split morphism starting at X_i in mod A is thus $f: X_i \to X_{i+1} \oplus \tau_A^{-1} X_{i-1}$ from which we deduce a left minimal almost split morphism: $(1, f): X'_i \to \left(\mathrm{Hom}_{A\times D}(X \oplus Y, X_i), X_{i+1} \oplus \tau_A^{-1} X_{i-1}, \mathrm{Hom}_{A\times D}(X \oplus Y, f)\right)$ $= (k, X_{i+1}, 1) \oplus \left(0, \tau_A^{-1} X_{i-1}, 0\right)$. This completes the proof that Γ' is the component of $\Gamma_{A'}$ containing X.

In order to show the standardness of Γ', let $\Phi: k(\Gamma) \to \mathrm{ind}\Gamma$ and $\Phi': k(\Gamma') \to \mathrm{ind}\Gamma'$ denote the canonical functors. By hypothesis, Φ is an equivalence, and we want to show that so is Φ'. Since Φ' is clearly dense, we must prove it is full and faithful, that is, for all $M, N \in \mathrm{ind}\ \Gamma$, the functor Φ' induces an isomorphism $\mathrm{Hom}_{k(\Gamma')}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{A'}(M, N)$.

Let $F: k(\Gamma) \to k(\Gamma')$ denote the k-linear embedding which is the identity on all objects and all arrows except arrows of the form $X_i \to \tau_A^{-1} X_{i-1}$, the image of which is the corresponding sectional path. Let $F': \mathrm{ind}\ \Gamma \to \mathrm{ind}\ \Gamma'$ be the functor induced by F. We have a commutative diagram

$$\begin{array}{ccc} k(\Gamma) & \xrightarrow{\quad F \quad} & k(\Gamma') \\ \Phi \downarrow & & \downarrow \Phi' \\ \text{ind } \Gamma & \xrightarrow{\quad F' \quad} & \text{ind } \Gamma' \end{array}$$

In particular, M, N $\in$ ind Γ implies $\mathrm{Hom}_{k(\Gamma')}$ (M, N) = $\mathrm{Hom}_{A'}$ (M, N).

If M is a D-module and $\mathrm{Hom}_{A'}$ (M, N) $\neq 0$, then N is a D-module or of the form Z_{ij}. Similarly, if N is a D-module and $\mathrm{Hom}_{A'}$ (M, N) $\neq 0$, then M is a D-module. Hence, if M or N is a D-module, then Φ' induces the required isomorphism $\mathrm{Hom}_{k(\Gamma')}$ (M, N) $\xrightarrow{\sim}$ $\mathrm{Hom}_{A'}$ (M, N). We may thus assume that neither M nor N is a D-module.

We note that the morphisms $Z_{ij} \to X'_i$ in mod A' induced by the corresponding sectional paths in Γ' are surjective. Moreover, if $\tau_A^{-1} X_{i-1} \neq 0$, our assumption on Supp $\mathrm{Hom}_A(X,-)|_{\text{ind } \Gamma}$ implies that the irreducible morphism $X'_i \to \tau_A^{-1} X_{i-1}$ in mod A is surjective and hence so is the irreducible morphism $X'_i \to \tau_A^{-1} X_{i-1}$ in mod A'. Let thus N $\in$ ind Γ and M $\notin$ ind Γ. Then M = Z_{ij} or X'_i for some i, j. A non-zero morphism f: M$\to$ N in mod A can always be written as f = gv, where v: M $\to \tau_A^{-1} X_{i-1}$ is induced by the corresponding sectional path in Γ'. Since v belongs to the image of Φ', and so does g (by commutativity of the above diagram), Φ' induces a surjection $\mathrm{Hom}_{k(\Gamma')}$ (M, N) $\to$ $\mathrm{Hom}_{A'}$ (M, N). On the other hand, v is an epimorphism in mod A' (by the above observations) and F' is faithful. Consequently the above surjection is an isomorphism.

Dually, if f: M $\to$ N is a non-zero morphism in mod A' with M $\in$ ind Γ, N $\notin$ ind Γ, then f can be written as f = uh, for some h: M $\to X_i$ and u: $X_i \to$ N induced by the corresponding sectional path. Since u is a monomorphism, it follows from the commutativity of the above diagram that Φ' induces the required isomorphism $\mathrm{Hom}_{k(\Gamma')}$ (M, N) $\xrightarrow{\sim}$ $\mathrm{Hom}_{A'}$ (M, N).

There remains to consider the case where both M and N belong to the rectangle. In this case, a non-zero morphism f: M $\to$ N is mod A' can be written as f= ugv + h, where u: $X_r \to$ N and v: M $\to \tau_A^{-1} X_{s-1}$ are induced by the corresponding sectional paths, g:$\tau_A^{-1} X_{s-1} \to X_r$ and h is zero or a composition of irreducible morphisms corresponding to arrows in the rectangle. Since h, u, v belong to the image of Φ', and so does g (by the previous considerations), Φ' induces a surjection $\mathrm{Hom}_{k(\Gamma')}$ (M, N) $\to$ $\mathrm{Hom}_{A'}$ (M, N). Now h is non-zero in mod A' if and only if it is non-zero in $k(\Gamma')$. Similarly, since u is injective and v is surjective in mod A' and since F is faithful then ugv is non-zero in mod A' if and only if it is non-zero in $k(\Gamma')$. Now, any non-zero morphism $f: M \to N$ in $k(\Gamma')$ can be written as $f = ugv + h$ with u, g, v, h as above. Thus $\Phi'(f) = 0$ implies $0 \neq \Phi'(h) = -\Phi(ugv)$. But h does not factor through modules in Γ, while g does. This contradiction shows that Φ' induces an isomorphism $\mathrm{Hom}_{k(\Gamma')}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{A'}(M, N)$. The proof of the lemma is now complete.

REMARK. The above argument provides an alternative proof of [13] (2.3).

2.3. LEMMA. In the case ad2), the component of $\Gamma_{A'}$ containing X (considered as an A'-module) is equal to Γ'. Further, if the subquiver of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$ is such that its connected component Γ^* containing X does not contain any of the $\tau_A^{-1} Y_{i-1}$, then Γ' is standard.

Proof. As in (2.2), the irreducible morphisms $X_0 \to Y_1 \to .. \to Y_t$ and $X_0 \to X_1 \to X_2 \to ...$ in mod A remain so in mod A'. We clearly have an almost split sequence

$$0 \to X_0 \to P \oplus X_1 \oplus Y_1 \to Z_{11} \to 0$$

in mod A'. Also the right minimal almost split morphisms ending at the X_i and Y_j in mod A remain so in mod A'. It follows from our hypothesis that Y_j $(j \geq 2)$ has at most two direct successors in mod A, namely Y_{j+1} and $\tau_A^{-1} Y_{j-1}$. But this implies that Y_j has at most two non-injective direct predecessors in mod A (thus in mod A', by what has been said above). Therefore Y_j has at most two non-projective direct successors in mod A', hence at most two direct successors in mod A'. Computing inductively almost split sequences, we prove, as in (2.2), that Γ' is indeed the component of $\Gamma_{A'}$ containing X.

Now for the proof of standardness, let Γ^* be as in the statement: there exists a full embedding $F: k(\Gamma^*) \to k(\Gamma')$ inducing a commutative diagram

$$\begin{array}{ccc} k(\Gamma^*) & \xrightarrow{\quad F \quad} & k(\Gamma') \\ \Big\downarrow \Phi & & \Big\downarrow \Phi' \\ \text{ind } \Gamma^* & \xrightarrow{\quad F' \quad} & \text{ind } \Gamma' \end{array}$$

where the standardness of Γ and the property assumed imply that Φ is an equivalence. The proof proceeds as in (2.2), replacing Γ by Γ^*.

2.4. LEMMA. In the case ad3), the component of Γ_A containing X (considered as an A'-module) is equal to Γ'. Further, if the subquiver of Γ obtained by deleting the arrows $Y_i \to \tau_A^{-1} Y_{i-1}$ is such that its connected component Γ^* containing X does not contain any of the $\tau_A^{-1} Y_{i-1}$, then Γ' is standard.

Proof. By hypothesis, the only irreducible morphism is mod A starting at Y_1 is $Y_1 \to Y_2$. By [21] 2.5 (5) (6) p. 87 and 88, we have almost split sequences in mod A'

$$0 \to X_0 \to P \oplus X_1 \oplus Y_1 \to Z_{11} \to 0$$
$$0 \to P = X'_0 \to Z_{11} \to Y_2 \to 0$$
$$0 \to X_i \to X_{i+1} \oplus Z_{i1} \to Z_{i+1,1} \to 0$$

for $i \geq 1$. Also, $Y_1 \to Z_{11}$ is left minimal almost split so that we also have an almost split sequence

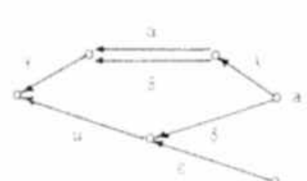
$$0 \to Y_1 \to Z_{11} \to X'_1 \to 0$$

Since no other arrow starts at Y_1, we also have almost split sequences

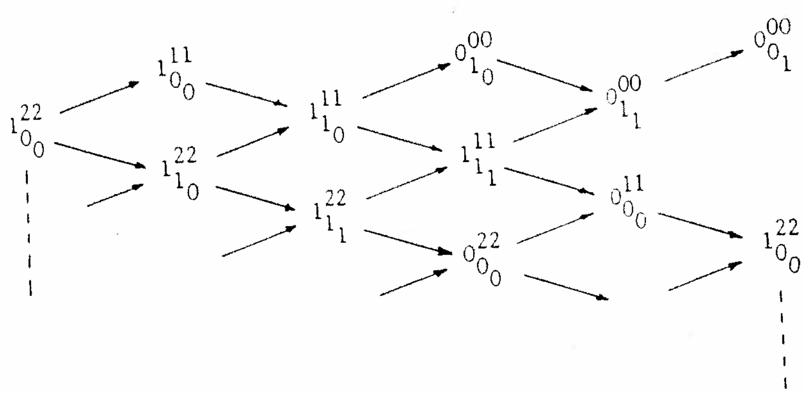
$$0 \to Z_{11} \to X'_1 \oplus Y_2 \oplus Z_{21} \to Z_{22} \to 0$$
$$0 \to Z_{i1} \to Z_{i2} \oplus Z_{i+1,1} \to Z_{i+1,2} \to 0$$

for $i \geq 1$. The proof that the component of $\Gamma_{A'}$ containing X is Γ' is finished by induction, and standardness is proved as in (2.2) and (2.3).

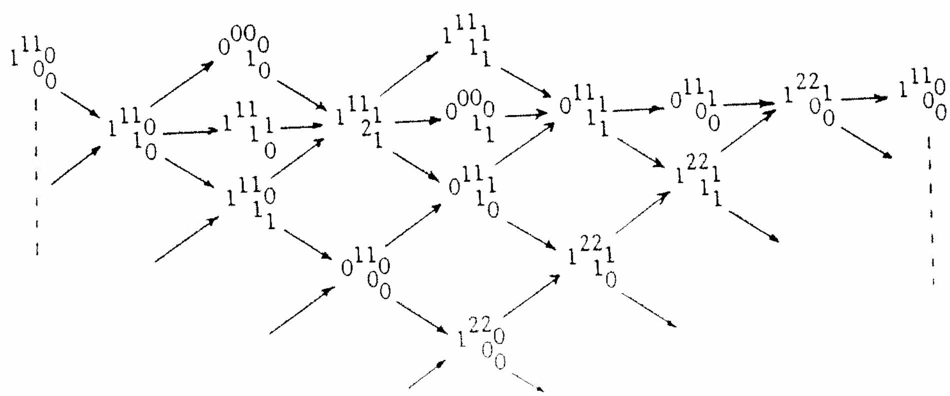
2.5. EXAMPLE. Let A be given by the quiver



bound by $\lambda\alpha = 0$, $\alpha\gamma = 0$ $\lambda\beta\gamma = \delta\mu$. Let B be the full convex subcategory of A consisting of all points except a. Then Γ_B has as component a standard tube Γ of the form



where indecomposables are represented by their dimension-vectors and one identifies along the dotted lines. Then A is a one-point extension of B by the indecomposable in Γ of dimension-vector $1{}^{11}_{1_0}$. The corresponding modified component Γ' is of the form



3. Coils and multicoil algebras.

3.1. The admissible operations of (2.1) can be regarded as operations on translation quivers rather than on Auslander-Reiten components. The definitions of the admissible operations ad1) ad2) ad3) ad1*) ad2*) ad3*) for translation quivers are done in the obvious manner. We refer the reader to [3] for the details. Let us recall

DEFINITION. A translation quiver is called a <u>coil</u> if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, ...\Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each $0 \leq i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation.

Thus, any stable tube is (trivially) a coil. A tube is a coil having the property that each admissible operation in the sequence defining it is of the form ad1) or ad1*). A coil without injectives (or without projectives) is a tube. A quasi-tube (in the sense of [24]) is a coil having the property that each admissible operation in the sequence defining it is of the form ad1), ad1*), ad2) or ad2*). For examples of coils which are not tubes, we refer to example (2.5) above (or to [3] (2.1)).

3.2. PROPOSITION. Let Γ be a coil. There exists a triangular algebra A such that Γ is a standard component of Γ_A.

<u>Proof</u>. Let $\Gamma_0, \Gamma_1, ...\Gamma_m = \Gamma$ be a sequence of translation quivers as in the definition (3.1). Clearly, there exists a tame hereditary algebra C having the stable tube Γ_0 as a standard component. Inductively, we construct a sequence of algebras $C = A_0, A_1, ...A_m = A$ such that A_{i+1} is obtained from A_i by the admissible operation with pivot in Γ_i such that the component of $\Gamma_{A_{i+1}}$ containing the pivot is Γ_{i+1}. It is easily seen that the conditions for standardness in (2.3) and (2.4) are satisfied at each step. This shows that Γ is a standard component of Γ_A. The triangularity of A follows from the fact that A is obtained from a tame hereditary algebra by a sequence of one-point extensions and coextensions.

3.3. It follows from the definition that coils share many properties with tubes. For instance, all but finitely many points belong to a cyclical path. A point x in a coil Γ will be said to belong to the <u>mouth</u> of Γ if x is the starting, or ending, point of a mesh in Γ with a unique middle term. Also, Γ contains a (maximal) tube as a cofinite full translation subquiver. Arrows of this tube may thus be subdivided into two classes: arrows pointing to the mouth and arrows pointing to infinity. An infinite sectional path in Γ

$$x = x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \to x_i \xrightarrow{\alpha_i} x_{i+1} \to$$

such that, for some $i_0 \geq 1$, we have all α_i with $i \geq i_0$ pointing to infinity will be called a ray starting at x, and denoted by $[x, \infty[$. Dually an infinite sectional path in Γ

$$\cdots \to y_{i+1} \xrightarrow{\beta_i} y_1 \to \cdots \xrightarrow{\beta_2} y_2 \xrightarrow{\beta_1} y_1;$$

such that, for some $i_0 \geq 1$, we have all β_i with $i \geq i_0$ pointing to the mouth will be called a coray ending with y, and denoted by $]\infty, y]$. For example, it follows directly from the definition of a coil that if $p \in \Gamma_0$ is projective (respectively, $q \in \Gamma_0$ is injective), there exists a ray $[p, \infty[$ (respectively, a coray $]\infty, q]$).

LEMMA. Let Γ be a coil. The full subquiver Γ_γ of Γ consisting of those points which belong to a cyclical path is also a coil.

Proof. Since ad2) ad2*) ad3) ad3*) only create points belonging to cyclical paths, the points in $\Gamma \setminus \Gamma_\gamma$ may only arise from ad1) or ad1*). Also, $\Gamma \setminus \Gamma_\gamma$ is the disjoint union of translation quivers containing no cyclical path. Let Δ be a connected component of $\Gamma \setminus \Gamma_\gamma$, considered as embedded in Γ. By duality, we may assume that there exists a projective point $p \in (\Gamma_\gamma)_0$ and a sectional path σ: $p = a_0 \to a_1 \to \dots \to a_t$ in Γ_γ from p to the mouth of Γ such that Δ consists of predecessors of σ. We must show that $\Gamma^* = \Gamma \setminus \Delta$ is a coil. Let $\mathcal{R}$ denote the full translation subquiver of Γ^* consisting of those points x such that there exists a ray $[a_i, \infty[$ (for some i) passing through x. Clearly Γ is obtained from $\Gamma^* \setminus \mathcal{R}$ by applications of ad1). On the other hand, Γ^* may equivalently be obtained from $\Gamma^* \setminus \mathcal{R}$ by successively inserting rays $[a_0, \infty[$, $[a_1, \infty[$, $\dots [a_t, \infty[$. Repeating this procedure for the other connected components of $\Gamma \setminus \Gamma_\gamma$, the lemma follows.

DEFINITION. A coil Γ is called proper if each of its points belongs to a cyclical path (that is, $\Gamma = \Gamma_\gamma$).

Thus, for an arbitrary coil Γ, Γ_γ is the unique maximal proper subcoil of Γ. It has the property that, for any two points $x, y \in (\Gamma_\gamma)_0$, there always exists a path from x to y in Γ_γ. Also, if x belongs to the mouth of a proper coil, there exists a unique ray $[x, \infty[$ and a unique coray $]\infty, x]$ passing through x.

3.4. DEFINITION. A translation quiver (Γ, τ) is said to be a multicoil if it contains a full translation subquiver Γ' such that:

(i) Γ' is a disjoint union of coils.

(ii) No point in $\Gamma \setminus \Gamma'$ belongs to a cyclical path.

It follows from (3.3) that Γ' may actually be assumed to be a disjoint union of proper coils.

DEFINITION. An algebra A is said to be a multicoil algebra if, for any cycle $M_0 \to M_1 \to \dots \to M_t = M_0$ in mod A, the indecomposable modules M_i belong to one standard coil of a multicoil in Γ_A.

Thus, a representation-finite algebra is a multicoil algebra if and only if it has a directed module category. All the best understood examples of tame algebras of finite global dimension are multicoil algebras: algebras tilting-cotilting equivalent to a tame hereditary algebra or to a tubular algebra [1], tame tilted algebras of wild type [15], iterated tubular algebras [18], or coil algebras in the sense of [2]. We shall give in (3.6) below an example of a multicoil algebra having a non-trivial multicoil as a connected component of its Auslander-Reiten quiver (see also [3] (2.2)). By definition of a multicoil algebra A, for any cycle in mod A, no morphism on the cycle lies in the infinite power of the radical of mod A (that is, A is cycle-finite in the terminology of [2]). Consequently, by [2] (1.4):

PROPOSITION. Let A be a multicoil algebra. Then A is tame.

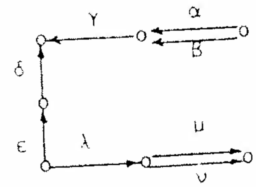
REMARKS. (a) Actually, any multicoil algebra is of polynomial growth. This will be shown in [3](4.6), using the results of the present paper.

(b) If a multicoil is a component of the Auslander-Reiten quiver of an algebra, then it contains finitely many coils. From now on, if A is a multicoil algebra, we shall briefly say "a coil in Γ_A" instead of " a coil in a multicoil in Γ_A".

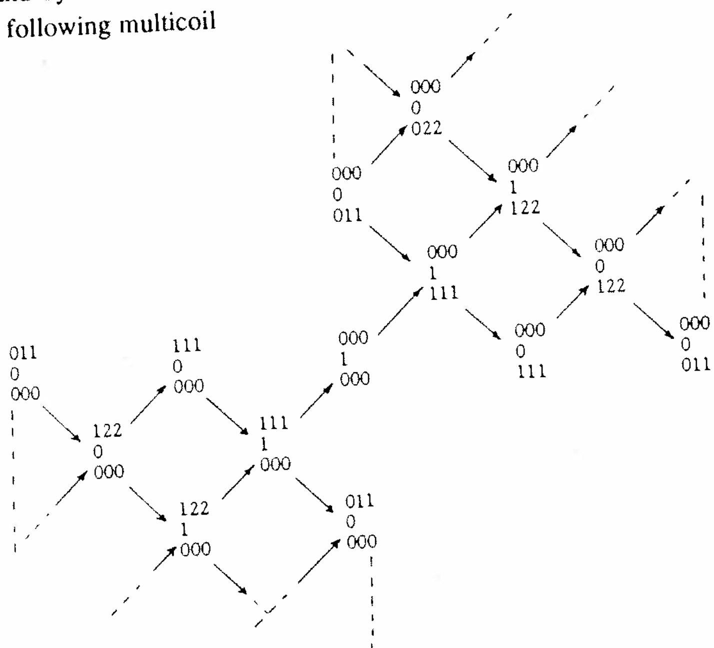
3.5. PROPOSITION. Let A be a multicoil algebra. Then A is triangular (thus gl.dim.A< ∞).

Proof. If this is not the case, there exists a cycle in mod A which consists of projective modules, and this cycle lies in a standard coil Γ (which may be assumed to be proper) in Γ_A. Since Γ is standard, there exists a cycle of projective objects in k(Γ). By (3.2), there exists a triangular algebra B and a component Γ' in Γ_B such that $k(\Gamma) \xrightarrow{\sim} k(\Gamma')$. In particular, we obtain a contradiction since this implies that the triangular algebra B contains a cycle of projective B-modules.

3.6. EXAMPLE. Let A be given by the quiver



bound by $\varepsilon\delta = 0$, $\lambda\nu = 0$, $\alpha\gamma = 0$. There A is a multicoil algebra and Γ_A contains the following multicoil

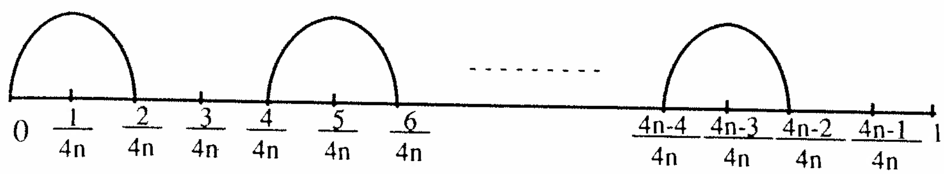


where we identify along the dotted lines and indecomposable modules are represented by their dimension-vectors.

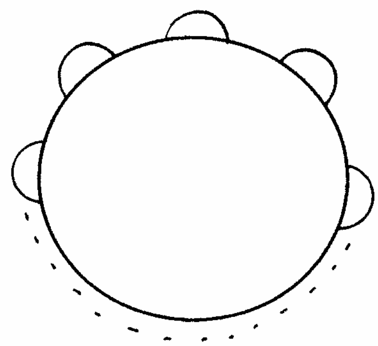
4. An axiomatic description of coils.

4.1. In this section, we shall characterise coils by means of a set of axioms rather than the inductive construction of section (3). We start by describing a topological space, which we call a crowned cylinder.

For $n \geq 1$, we partition the unit interval $[0, 1]$ into $4n$ equal subintervals. For each subinterval of the form $\left[\frac{4i}{4n}, \frac{4i+2}{4n}\right]$, consider the midpoint $c_i = \frac{4i+1}{4n}$, construct a semi-circle contained in the upper half-plane with centre c_i and radius $\frac{1}{4n}$. This yields the subspace X of the upper half-plane with the usual topology consisting of the points (x, y) with $x \in [0, 1]$ such that, if $\frac{4i}{4n} \leq x \leq \frac{4i+2}{4n}$ for some $0 \leq i < n$, $(x - c_i)^2 + y^2 \leq \left(\frac{1}{4n}\right)^2$ while if not, then $y = 0$.



Identifying the points 0 and 1, we obtain a topological space of the form

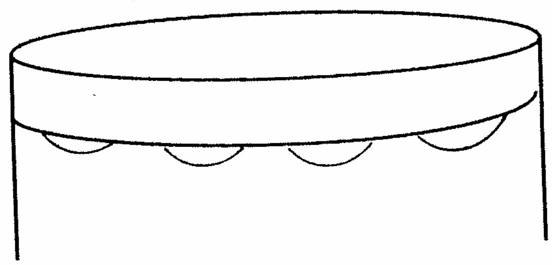


Such a space X will be called a crown. It is considered as a subspace of $\mathbb{R}^2$. Clearly, the unit circle S^1 is homeomorphic to the subspace X_0 of the crown X which is the image of the unit interval. We shall agree, by abuse of language, to consider the unit circle itself as a crown (corresponding to $n = 0$).

A crowned cylinder E is, by definition, the quotient space obtained from the topological sum of $S^1 \times \mathbb{R}^+$ and a crown X by identifying $(x, 1) \in S^1 \times \mathbb{R}^+$ with $h(x) \in X_0$ (where $h: S^1 \to X_0$ denotes a homeomorphism). In other words, E is the following amalgamated sum ("pushout")

$$\begin{array}{ccc} S^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ S^1 \times \mathbb{R}^+ & \longrightarrow & E \end{array}$$

Thus, a crowned cylinder is a subspace of $\mathbb{R}^3$ of the form



The reason for the introduction of crowned cylinders is the following. Let Γ be a coil, and Γ' be the full translation subquiver of Γ consisting of all points except those which are projective-injective middle terms of a mesh with three middle terms. Then the underlying topological space $|\Gamma'|$ of Γ' is homeomorphic to a crowned cylinder.

4.2. Let now Γ be a translation quiver without multiple arrows, containing a cyclical path, and let Γ' denote the full translation subquiver of Γ consisting of all points except those which are projective-injective middle terms of a mesh with three middle terms. Assume that $|\Gamma'|$ is homeomorphic to a crowned cylinder. Then, clearly, a mesh in Γ has at most three middle terms. A mesh with exactly three middle terms will be called <u>exceptional</u>, and a projective middle term of an exceptional mesh will be called <u>exceptional projective</u>. Other meshes and projectives will be called <u>ordinary</u>. The set of points which are the starting, or ending, point of a mesh in Γ with a unique middle term will be called the <u>mouth</u> of Γ. Thus an exceptional mesh must have one of its middle terms on the mouth or projective-injective. On the other hand, Γ contains a (maximal) tube as a cofinite full translation subquiver. We may thus define rays and corays in Γ exactly as in (3.3).

Finally, we shall say that a function $\ell\colon \Gamma_0 \to \mathbb{N}$ is a <u>length function</u> (see [13]) on Γ if:

(i) For any x which is not projective
$$\ell(x) + \ell(\tau x) = \sum_{y \in x^-} \ell(y)$$

(ii) For any x which is projective
$$\ell(x) = 1 + \sum_{y \in x^-} \ell(y)$$

where, as usual, x^- denotes the set of immediate predecessors of x in Γ. Our objective in this section is to prove the following:

THEOREM. Let Γ be a translation quiver without multiple arrows and containing a cyclical path. Then Γ is a coil if and only if it satisfies the following conditions:

(C1) Let Γ' denote the full translation subquiver of Γ consisting of all points except those which are projective-injective middle terms of a mesh with three middle terms. Then the underlying topological space of Γ' is homeomorphic to a crowned cylinder.

(C2) For any mesh with three middle terms, none of which is projective-injective, two of the middle terms lie on the mouth of Γ.

(C3) For any projective $p \in \Gamma_0$, or injective $q \in \Gamma_0$, and any $x \in \Gamma_0$, we have $\dim_k \mathrm{Hom}_{k(\Gamma)}(p, x) \leq 1$, or $\dim_k \mathrm{Hom}_{k(\Gamma)}(x, q) \leq 1$ respectively.

(C4) For any projective $p \in \Gamma_0$, or injective $q \in \Gamma_0$, there exists a ray $[p, \infty[$, or a coray $]\infty, q]$, respectively.

(C5) The τ-orbit of any projective, or injective, contains a point which belongs to a cyclical path.

(C6) There exists a length function ℓ on Γ.

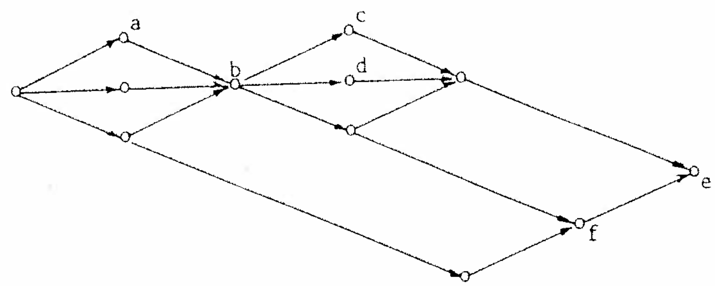
While the necessity follows by an easy induction on the number of admissible operations (using (3.2) and the fact that all the axioms are trivially satisfied for stable tubes), the sufficiency requires more work. In the remaining part of this section, Γ denotes a translation quiver without multiple arrows, containing a cyclical path, and satisfying the axioms (C1) to (C6).

4.3 LEMMA. (i) Let $p \in \Gamma_0$ be projective, then Supp $\mathrm{Hom}_{k(\Gamma)}(p, -)$ does not contain all points of an exceptional mesh.

(ii) Let $q \in \Gamma_0$ be injective, then Supp $\mathrm{Hom}_{k(\Gamma)}(-, q)$ does not contain all points of an exceptional mesh.

Proof. This indeed follows from (C3).

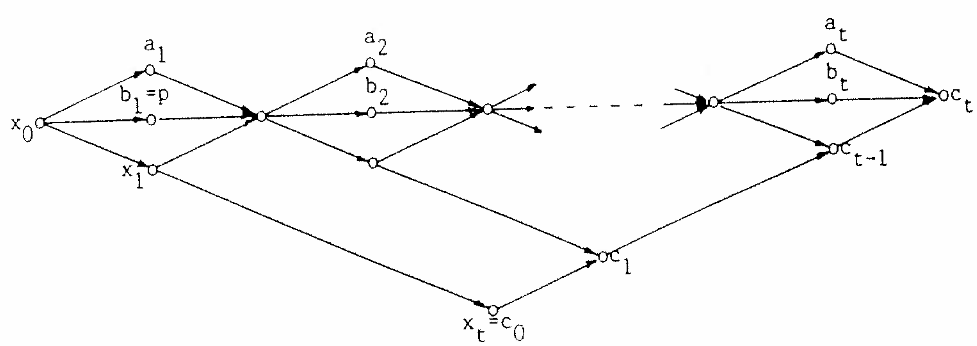
4.4 LEMMA. Consider a mesh-complete translation subquiver of Γ of the form:



Then $\ell(e) - \ell(d) = \ell(f) - \ell(a)$

Proof. We have $\ell(b) = \ell(a) + \ell(c)$. Moreover, by [2] (2.1), $\ell(b) + \ell(e) = \ell(c) + \ell(d) + \ell(f)$. Hence $\ell(a) + \ell(c) + \ell(e) = \ell(c) + \ell(d) + \ell(f)$ and the claim follows.

COROLLARY. Consider a mesh-complete translation subquiver of Γ of the form.



where p is projective. Then at most one of a_t and b_t can be injective. Moreover, if this is the case, and t is odd, then b_t is injective, while, if t is even, then a_t is injective.

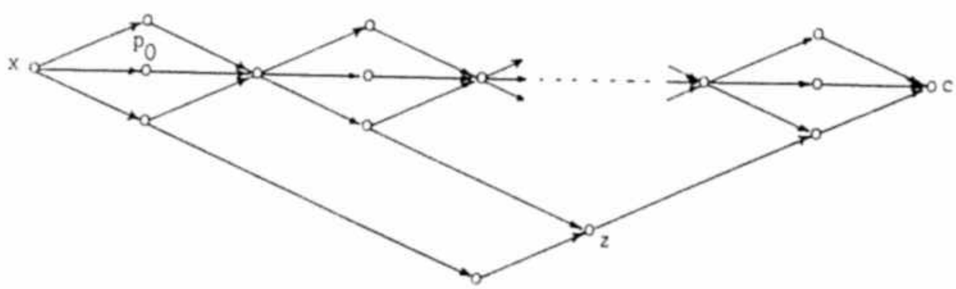
Proof. We shall show that t odd and a_t injective yield a contradiction. Dually, t even and b_t injective yield another contradiction. This would imply the statement.

Assume a_t is injective. Then $\ell(a_t) - \ell(c_t) = 1$. By the lemma, $\ell(b_{t-1}) - \ell(c_{t-1}) = 1$. By induction, and if t is odd, we get $\ell(a_1) - \ell(c_1) = 1$. However, by [2] (2.1), we have $\ell(a_1) + \ell(b_1) + \ell(c_0) = \ell(x_0) + \ell(c_1)$. Since $\ell(b_1) = \ell(x_0) + 1$, we get $\ell(a_1) + \ell(c_0) + 1 = \ell(c_1)$, hence the absurdity $\ell(c_0) = -2$.

4.5. LEMMA. The mesh category k(Γ) contains no oriented cycle of projectives.

Proof. Let $p_0 \to p_1 \to \ldots \to p_t = p_0$ be such a cycle. We claim that there exists another such cycle of projectives with each p_s either exceptional or else such that there exist arrows $q \to x \to p_s$, with q injective.

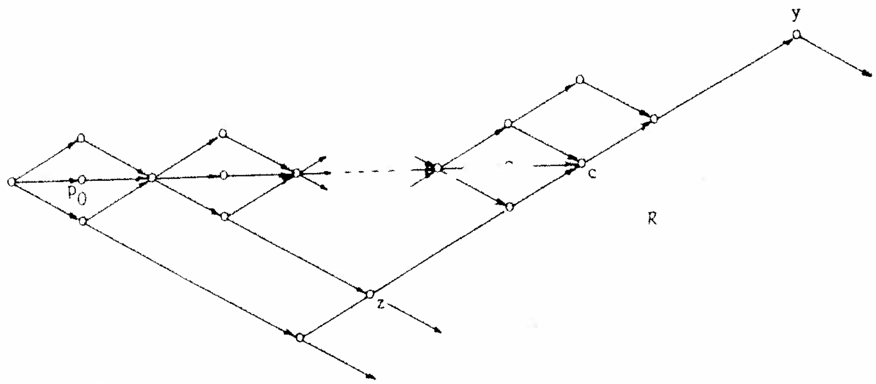
Let us consider p_0 and define a point z as follows. If p_0 is exceptional, let x be its unique direct predecessor, and $c = \tau^{-1}x$ be such that one of the direct predecessors of c is injective (this implies, by (C3), the existence of a coray $]\infty, c]$), then set $z = [p_0, \infty[\cap]\infty, c]$. If p_0 is not exceptional, set $z = p_0$.



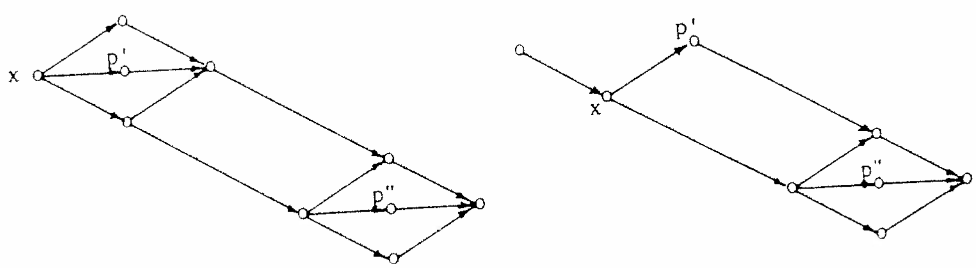
Let us now consider the set of points y on the sectional path from z to the mouth such that there exists a ray $[y, \infty[$. This set is non-empty (it contains z). Let y be a maximal element in this set (that is, closer to the mouth) and let [z, y] denote the sectional path from z to y. By (C1), for each v on [z, y], there exists a ray $[v, \infty[$. Let $\mathcal{R}$ denote the mesh-complete translation subquiver consisting of all points lying on these rays. Then $\mathcal{R} \subseteq \mathrm{Supp\ Hom}_{k(\Gamma)}(p_0, -)$: indeed, this is clear if $z = p_0$, otherwise it follows from the fact that c and exactly two of its direct predecessors (namely, the one lying on [z, y], and the injective direct predecessor) belong to $\mathrm{Supp\ Hom}_{k(\Gamma)}(p_0, -)$. By (4.3), $\mathcal{R}$ contains no exceptional mesh.

We claim that $[y, \infty[$ contains either an arrow $q \to x$, with q injective and x a direct predecessor of a projective p', or else a point x which is the direct predecessor of an exceptional projective p'. Indeed, assume that this is not the case. Since

$\mathrm{Hom}_{k(\Gamma)}(p_0, p_1) \neq 0$ then, by (C1) and the existence of the rays $[v, \infty[$ with v on $[z, y]$, we must have that p_1 belongs to $[c, y]$. Since $\mathrm{Hom}_{k(\Gamma)}(p_i, p_{i+1}) \neq 0$ for all $1 \leq i \leq t-2$ and $\mathcal{R}$ consists of ordinary meshes, we deduce that $p_2, \dots p_{t-1}$ belong all to $[c, y]$. This implies, by the definition of y and our hypothesis that $[y, \infty[$ contains no point as required, that $\mathrm{Hom}_{k(\Gamma)}(p_{t-1}, p_0) = 0$, a contradiction.



Note that p', as defined in the previous claim, is the unique projective having a direct predecessor on $[y, \infty[$. For, if this is not the case, the existence of the ray $[p', \infty[$ implies that we have one of the following two cases.



in particular, the projective p'' is necessarily exceptional, and hence projective-injective; we thus get a contradiction by (4.3), since the lower mesh lies in $\mathrm{Supp}\,\mathrm{Hom}_{k(\Gamma)}(p_0, -)$.

Let i be such that the projectives $p_1, \dots p_i$ belong to $\mathcal{R}$ while $p_{i+1} \notin \mathcal{R}$. Then either $p_{i+1} = p'$, or the (only) morphism $p_i \to p_{i+1}$ factors through p'. If $p_{i+1} = p'$,we replace $p_1, \dots p_{i+1}$ by p', while if $p_i \to p_{i+1}$ factors through p', we replace $p_1, \dots p_i$ by p'. In this way, we replace inductively the given cycle by a cycle of projectives satisfying the required property. Actually, we have even defined a cycle of the form

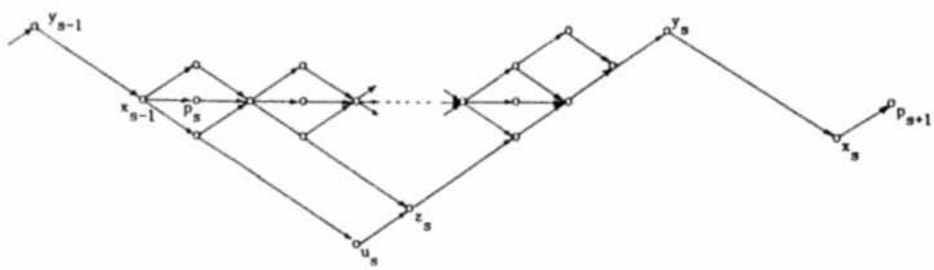
$$\dots \to p_s \to \dots \to z_s \to \dots \to y_s \to \dots \to p_{s+1} \to \dots$$

where we have sectional paths $[p_s, z_s]$ pointing to infinity, $[z_s, y_s]$ pointing to the mouth, and $[y_s, x_s]$ (where x_s is a direct predecessor of p_{s+1} on this cycle) pointing to infinity.

For each s, let $u_s = [y_{s-1}, \infty[\cap]\infty, y_s]$. Then u_s is clearly a direct predecessor of z_s and we have a cycle

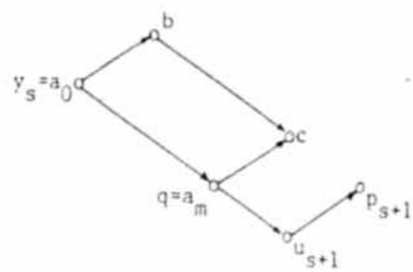
$$\dots \to u_s \to \dots \to y_s \to \dots \to u_{s+1} \to \dots$$

where paths correspond to sectional paths, $[u_s, y_s]$ pointing to the mouth, $[y_s, u_{s+1}]$ pointing to infinity. If follows from the above discussion that for each v on $[u_s, y_s]$, there exists a ray $[v, \infty[$ and a coray $]\infty, v]$.



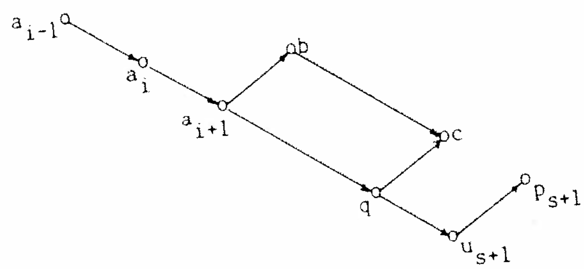
We shall now show that $\ell(u_{s+1}) < \ell(y_s)$ for all s. There are two cases according as p_{s+1} is an ordinary or an exceptional projective.

(a) Assume p_{s+1} is ordinary. Then there exist arrows $q \to u_{s+1} = x_s \to p_{s+1}$ with q injective. Consider the sectional path $y_s = a_0 \to a_1 \to \dots \to a_m = q \to u_{s+1} \to p_{s+1}$. Assume first that no a_i is injective.



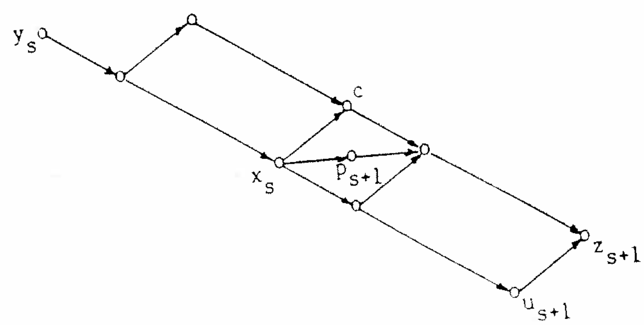
Then $\ell(q) + \ell(b) = \ell(y_s) + \ell(c)$ and $\ell(q) = \ell(c) + \ell(u_{s+1}) + 1$. Hence $\ell(q) + \ell(b) = \ell(c) + \ell(b) + \ell(u_{s+1}) + 1$ and $\ell(y_s) = \ell(b) + \ell(u_{s+1}) + 1 > \ell(u_{s+1})$.

On the other hand, if i is the largest index $\leq m$ such that a_{i-1} is injective, we have as before

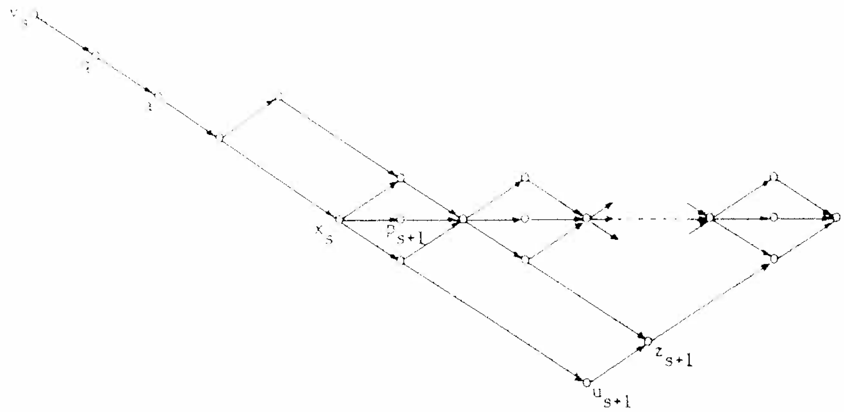


$\ell(a_{i+1}) = \ell(u_{s+1}) + \ell(b) + 1$. However, $\ell(a_{i+1}) = \ell(a_i) + \ell(b)$ yields $\ell(a_i) = \ell(u_{s+1}) + 1 > \ell(u_{s+1})$. Repeating this procedure over all injectives among the a_i, we deduce our statement.

(b) Assume p_{s+1} is exceptional. If no point on the sectional path from y_s to x_s is injective,

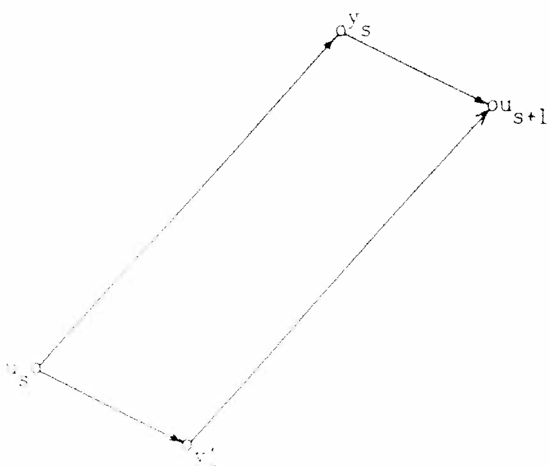


we have, by [2] (2.1) $\ell(p_{s+1}) + \ell(u_{s+1}) = \ell(y_s) + \ell(z_{s+1})$ and, by (4.4), $\ell(p_{s+1}) = \ell(z_{s+1}) + 1$. Thus $\ell(y_s) = \ell(u_{s+1}) + 1 > \ell(u_{s+1})$. On the other hand, if there exists an injective on the sectional path from y_s to x_s, we let q be the injective closer to p_{s+1} and a be the direct successor of q on this path.



We again have $\ell(a) + \ell(z_{s+1}) = \ell(p_{s+1}) + \ell(u_{s+1})$ and $\ell(p_{s+1} = \ell(z_{s+1}) + 1$. Hence $\ell(a) = \ell(u_{s+1}) + 1$. Continuing the argument as in (a) above, we deduce our statement.

Now we shall define another cycle lying lower in Γ. For each s, let $y'_s = [u_s, \infty[\cap]\infty, u_{s+1}]$.



We thus obtain a cycle

$$\dots \to u_s \to \dots \to y'_s \to \dots \to u_{s+1} \to \dots \to y'_{s+1} \to \dots$$

with all paths sectional, $[u_s, y'_s]$ pointing to infinity and $[y'_s, u_{s+1}]$ pointing to the mouth. We have seen that the shown rectangle contains no exceptional mesh. Consequently, we have $\ell(y'_s) < \ell(u_s)$ for all s.

By construction of our new cycle, there are no projectives on rays starting from points on the cycle. On the other hand, $\sum_s \{\ell(u_s)+\ell(y'_s)\} < \sum_s \{\ell(u_s)+\ell(y_s)\}$

Now, for each s, letting $u'_{s+1} = [y'_s, \infty] \cap]\infty, y'_{s+1}]$, we obtain a third cycle

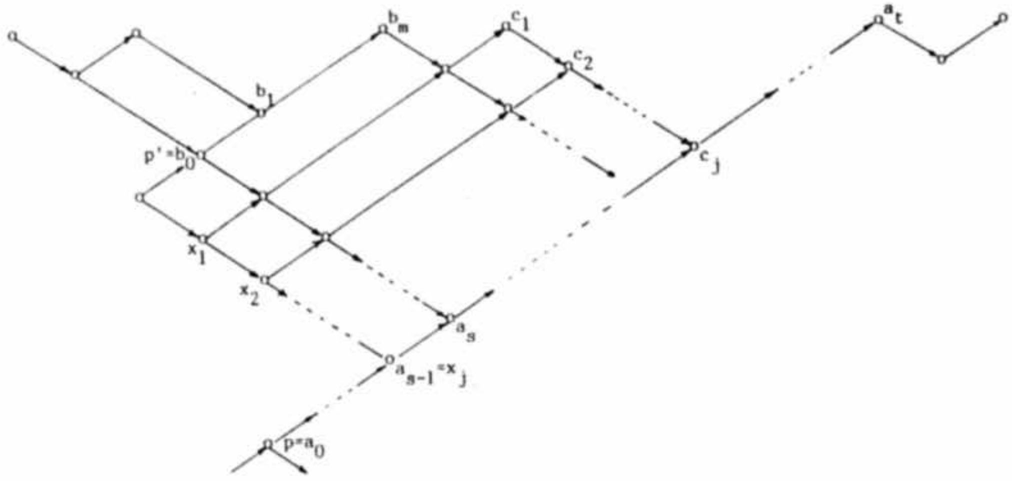
$$\dots \to y'_s \to \dots \to u'_{s+1} \to \dots \to y'_{s+1} \to \dots$$

lying lower in Γ, such that $\sum\{\ell(u'_s)+\ell(y'_s)\} < \sum\{\ell(u_s)+\ell(y'_s)\}$. Continuing in this way, we eventually obtain negative values, a contradiction.

REMARK. It follows from the proof that all projectives in Γ lie above some cyclical path in Γ, and consequently Γ has only finitely many projectives.

4.6. Proof of the theorem. We shall show the sufficiency of the axioms by induction on the number of projectives in Γ. Indeed, if Γ has neither projectives nor injectives, then Γ is a stable tube. By duality, we may assume that Γ contains at least one projective.

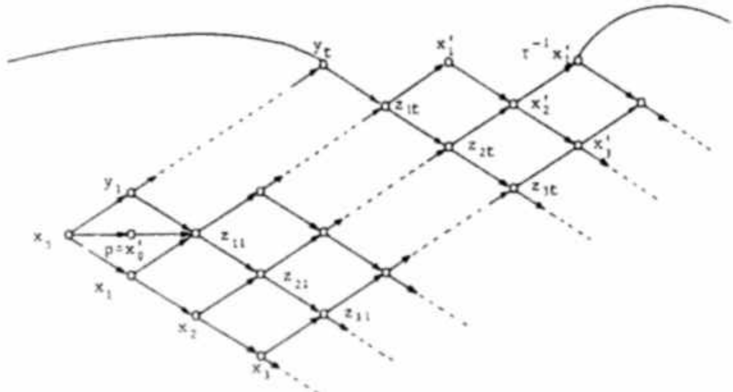
By (4.5), there exists a projective $p \in \Gamma_0$ such that Supp $\mathrm{Hom}_{k(\Gamma)}(p, -)$ contains no projective. Assume first that p is an ordinary projective and consider the sectional path from p pointing to the mouth $p = a_0 \to a_1 \to \dots \to a_t$, with a_t lying on the mouth. Let s be the largest index such that there exists a projective $p' \in \Gamma_0$ and a sectional path $[p', a_s]$ pointing to infinity. We can clearly choose p' so that its successors on $[p', a_s]$ are not projective. Observe that Supp $\mathrm{Hom}_{k(\Gamma)}(p', -)$ contains no projective: indeed, by definition of s, no projective lies on a sectional path pointing to infinity and passing through a_r, $s < r \leq t$; moreover, by our assumption on p, the sectional path $[a_t, \infty[$ contains no direct predecessor of a projective. Also, p' is necessarily ordinary: for, if $p \neq p'$, then $\mathrm{Hom}_{k(\Gamma)}(p, p') = 0$, p' lies above the sectional path $[p, a_t]$ and hence it belongs to no cyclical path (by (C1) and the fact that at most one of the direct predecessors of p lies on a cyclical path and then it lies on the coray $]\infty, a_t]$). Therefore Γ contains a mesh-complete translation subquiver of the form:



where possibly $p = p'$, $b_m = a_t$ or $b_0 = b_m$. Denote by $\mathcal{R}$ the set of the points in Γ lying on sectional paths from the mouth to infinity passing through $b_0, \ldots b_m$, and by Γ_1 the translation quiver obtained from Γ by deleting $\mathcal{R}$ and replacing the sectional paths $x_i \to \ldots \to c_i$ (if they exist) by arrows $x_i \to c_i$, $i \geq 1$. Clearly Γ_1 still satisfies the axioms and has at least one projective less than Γ.

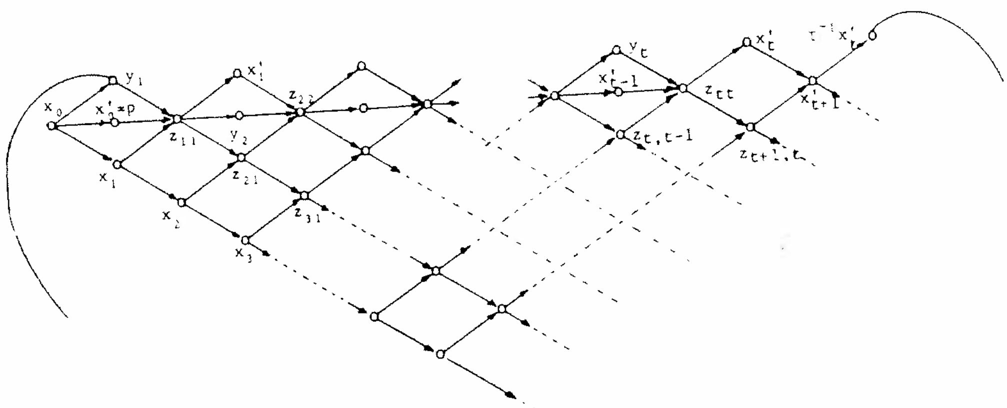
We may thus suppose that no sink in the full subcategory of $k(\Gamma)$ consisting of projectives is ordinary. Let thus p be a sink in this category. Then p is exceptional and we have two cases to consider.

Assume that p is injective. Then Supp $\mathrm{Hom}_{k(\Gamma)}(p, -)$ is formed by the vertices x'_j and z_{ij} of a mesh-complete translation subquiver of Γ of the form.

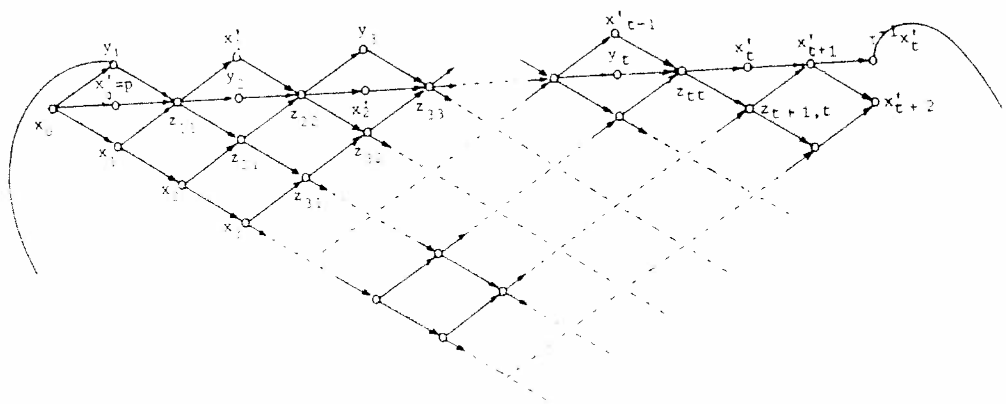


Denote by $\mathcal{R}$ the set of the points in Γ of the forms x'_i, $i \geq 0$, z_{ij}, $i \geq 1$, $1 \leq j \leq t$ and by Γ_1 the translation quiver obtained from Γ by deleting $\mathcal{R}$ and replacing the sectional paths $x_i \to \dots \to \tau^{-1}x'_{i-1}$ (if they exist) by arrows $x_i \to \tau^{-1}x'_{i-1}$. Clearly, Γ_1 still satisfies the axioms and has one projective less than Γ.

Finally, assume that p is not injective. Then, by (C1), and the fact that p is a sink in the full subcategory of $k(\Gamma)$ consisting of projectives, we infer that Supp $\mathrm{Hom}_{k(\Gamma)}(p, -)$ is formed by the points x'_i and z_{ij} of a mesh-complete translation subquiver of Γ of the form



if t is odd (see (4.4), Corollary), or



if t is even.

Denote

j ≤ t, and by Γ

the sectional

sectional path

paths $x_i \to$ ..

Hence Γ_1 is c

Clear

finishes the

5. Full

5.1.

multicoil al

characterisa

structure of

LEM

algebra A.

Proo

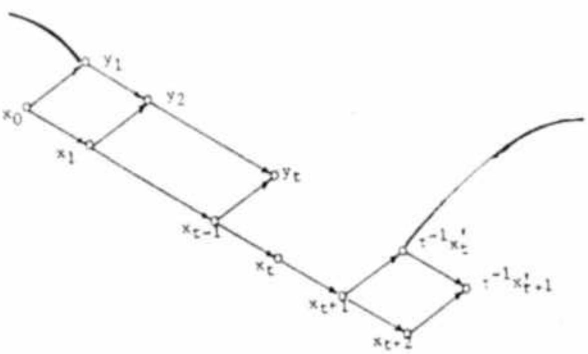
Supp Γ = S

consisting

C = Supp I

if t is even.

Denote by $\mathcal{R}$ the set of the points in Γ of the forms x'_i, $i \geq 0$, z_{ij}, $i \geq 1$, $1 \leq j \leq t$, and by Γ_1 the translation quiver obtained from Γ by deleting $\mathcal{R}$, and replacing the sectional paths $x_i \to \cdots \to y_{i+1}$ by arrows $x_i \to y_{i+1}$ $(0 \leq i \leq t-1)$, the sectional paths $y_i \to z_{ii} \to y_{i+1}$ by arrows $y_i \to y_{i+1}$ $(1 \leq i \leq t\text{-}1)$ and the sectional paths $x_i \to \ldots \to x'_i \to \tau^{-1}x'_{i-1}$ (if they exist) by arrows $x'_i \to \tau^{-1}x'_{i-1}$, $i \geq t+1$. Hence Γ_1 is of the form



Clearly, Γ_1 satisfies the axioms and has one projective less than Γ. This finishes the proof of the theorem.

5. Full convex subcategories of multicoil algebras.

5.1. In this section, we shall prove that a full convex subcategory of a multicoil algebra is also a multicoil algebra. We shall deduce from this result the characterisation of the minimal representation-infinite multicoil algebras, and the structure of the non-stable coils in a multicoil algebra.

LEMMA. Let Γ be a coil in the Auslander-Reiten quiver Γ_A of a multicoil algebra A. Then $C = \text{Supp}\,\Gamma$ is a full convex subcategory of A.

Proof. It follows easily from the inductive construction of coils that $\text{Supp}\,\Gamma = \text{Supp}\,\Gamma_\gamma$, where Γ_γ denotes, as in section (3), the full subquiver of Γ consisting of all points lying on a cyclical path. If suffices thus to show that $C = \text{Supp}\,\Gamma_\gamma$ is convex in A. This is done by the well-known convexity argument of

Bongartz (see, for instance, [2](3.1)) using the fact that, since Γ_γ is a coil in the multicoil algebra A, any cycle of non-zero non-isomorphisms with a module in Γ_γ must lie completely in Γ_γ.

5.2. REMARK. Let A be a multicoil algebra, and X be an indecomposable A-module lying in a stable tube of Γ_A. Then $C = \operatorname{Supp}(\bigoplus_{t \geq 0} \tau^t X)$ is a full convex subcategory of A.

5.3. LEMMA. Let $A = B[X]$ be a multicoil algebra, with extension point a, such that P(a) belongs to a proper coil Γ in Γ_A. Then the full subcategory $\mathfrak{C}$ of ind Γ of the B-modules lying on a cycle in mod B is standard and its quiver is a coil.

Proof. By [8] (3.5), if $0 \to L \to M \to N \to 0$ is the almost split sequence in mod A starting with L, where L is a B-module which is not injective in mod B, then the almost split sequence in mod B starting with L is the lower row of the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M' & \to & N' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & L & \to & M'' & \to & N'' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where M' and N' are the largest quotients of M and N, respectively, to be B-modules, and the morphisms $K \to M'$, $K \to N'$ are sections. The lemma follows by applying this statement to compute the quiver of $\mathfrak{C}$ by calculating the almost split sequences starting at the indecomposable objects of $\mathfrak{C}$ (and, in particular, at the X_i, in the notation of (2.1)). The standardness of the new coil follows from the standardness of Γ.

5.4. COROLLARY. Let A be a multicoil algebra and Γ be a coil in Γ_A. Then Supp Γ contains a full convex subcategory C such of A that Γ is obtained from a sincere stable tube Γ_1 of Γ_C by a sequence of admissible operations and Supp Γ is obtained from Supp Γ_1 = C by the corresponding sequence of one-point extensions and coextensions.

Proof. By induction on the number of projectives and injectives in Γ, using (5.1) and (5.3). Assume that Γ is obtained from Γ' by an admissible operation which, by duality, may be taken to be the insertion of a projective (or projectives) and the corresponding rectangle in Γ'. By (5.1) D = Supp Γ is a full convex subcategory of A. It is obtained from D' = Supp Γ' by a one-point extension and, by (5.3), Γ'_γ is a standard coil in $\Gamma_{D'}$. Moreover, Supp Γ'_γ = Supp Γ' = D' is a full convex subcategory of D, and D is obtained from D' by an admissible operation. The proof is completed by induction.

5.5.DEFINITION. Let Γ be a coil, and M be a mouth module in Γ_γ. The mitre $\widehat{M}$ determined by M is defined to be the full translation subquiver of Γ consisting of all modules N such that there exist sectional paths $X \to \ldots \to N$ for some X on the (unique) coray $]\infty, M]$ and $N \to \ldots \to Y$ for some Y on the (unique) ray $[M, \infty[$.

LEMMA. Let A = B[X] be a multicoil algebra. If Γ is a stable tube containing at least one A-module which is not a B-module, there exists a mouth module $M \in \Gamma_0$ such that $\widehat{M}$ contains no B-module. Consequently, the full subcategory of mod B consisting of the indecomposables in Γ has finitely many non-isomorphic indecomposables and contains no cycles.

Proof. Let P(a) be the unique indecomposable projective A-module which is not a B-module. We shall prove the existence of a mouth module $M \in \Gamma_0$ such that $\mathrm{Hom}_A(P(a), M) \neq 0$. Let $L \in \mathrm{Supp}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma}$ and Ω denote the maximal coray passing through L. If the mouth module on Ω belongs to $\mathrm{Supp}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma}$, we are done. If not, we may assume that L is chosen so that, if $L \to N$ is the arrow on Ω, then $\mathrm{Hom}_A(P(a), N) = 0$. Let Σ be the maximal ray passing through L. We claim that $\mathrm{Hom}_A(P(a), U) \neq 0$ for all $U \in \Sigma_0$. Let f: $P(a) \to L$ be non-zero. Since Γ is a stable tube, the irreducible morphisms on Σ are injective. Hence the compositions of f with the morphisms determined by the subpaths of Σ starting with L are non-zero, and consequently $\mathrm{Hom}_A(P(a), U) \neq 0$ for all $U \in \Sigma_0$ lying between L and infinity. Consider the almost split sequence

$$0 \to \tau N \to L \oplus K \to N \to 0$$

Since $\mathrm{Hom}_A(P(a), L \oplus K) \neq 0$, $\mathrm{Hom}_A(P(a), N) = 0$, then $\mathrm{Hom}_A(P(a), \tau N) \neq 0$. Moreover, since the irreducible morphism $K \to N$ is injective, we have

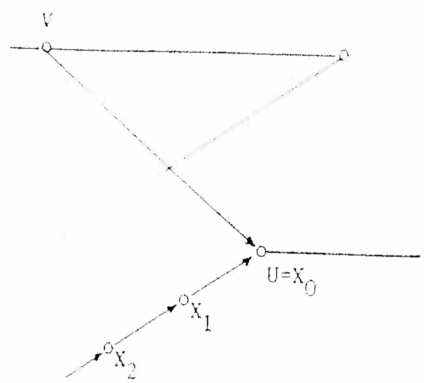
$\mathrm{Hom}_A(P(a), K) = 0$. Using induction on the number of modules on Σ from the mouth to L, we deduce that $\mathrm{Hom}_A(P(a), U) \neq 0$ for all $U \in \Sigma_0$ between the mouth and L. This proves the existence of the wanted mouth module M, and also shows that the ray Σ lies entirely in Supp $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma}$. Finally, since morphisms induced by sectional subpaths of corays ending at a point on Σ are surjective, and P(a) is projective, then $\widehat{M}$ lies entirely in Supp $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma}$. Consequently, Ker $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma}$ contains only finitely many non-isomorphic indecomposables. Moreover, it contains no cycle because the mouth module of Γ lying on Σ belongs to Supp $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma}$.

5.6. THEOREM. Let A be a multicoil algebra, and B be a full convex subcategory of A. Then B is a multicoil algebra.

__Proof__. Since, by (3.5), A is triangular, it suffices to prove the statement in case A is a one-point extension of B. Any cycle in mod B is also a cycle in mod A and hence belongs to a standard coil in Γ_A. Hence, it suffices to show that, if Γ is a standard coil in Γ_A and $\mathcal{C}$ the full subcatory of ind Γ consisting of the B-modules in Γ which lie on a cycle in mod B, and if $\mathcal{C} \neq \emptyset$, then the quiver of $\mathcal{C}$ is a standard coil.

Let P(a) be the unique projective A-module which is not a B-module. Clearly, the B-modules in Γ belonging to a cycle in mod B must belong to a cycle in Γ_γ. First, if $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_\gamma} = 0$, then Γ_γ consists entirely of B-modules. Hence the quiver of $\mathcal{C}$ is equal to Γ_γ and thus is a coil (by (3.3)); further, it is standard because it is so in mod A. We may thus assume that $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_\gamma} \neq 0$. If $P(a) \in (\Gamma_\gamma)_0$, the statement follows from (5.3). If $P(a) \notin (\Gamma_\gamma)_0$ and Γ is a stable tube, the statement follows from (5.5) ($\mathcal{C}$ is empty in this case). There remains to consider the case where $P(a) \notin (\Gamma_\gamma)_0$ and Γ is not a stable tube. There exists a sequence $\Gamma_1, \Gamma_2 \ldots \Gamma_m = \Gamma$ where Γ_1 is a stable tube, and Γ_{i+1} is obtained from Γ_i by an admissible operation. By (5.4), we may assume that Supp Γ_1 is a full convex subcategory of Supp Γ_γ, and the latter is obtained from the former by a sequence of admissible operations. We shall show that, if $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_1} \neq 0$, then Ker $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_\gamma}$ contains only finitely many non-isomorphic indecomposable objects and no cycle in mod B (so that $\mathcal{C} = \emptyset$).

By (5.5), there exists a mouth module $U \in (\Gamma_1)_0$ such that $\widehat{U}$ lies entirely outside Ker $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_1}$. We claim that such a mitre exists for any Γ_i and is determined by a mouth module which is not projective, lies on a cycle in Γ_i, and is such that the almost split sequence ending at this module has indecomposable middle term. Inductively, assume such a mitre exists for Γ_{i-1}. Since P(a) maps non-trivially to modules in $\widehat{U}$, and a is a source, we cannot use as radical of a new projective any module from $\widehat{U}$. If we use ad1), ad2) or ad3), with pivot $X = X_0$, then $\widehat{U} \cap \mathrm{Supp}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_i}$ contains a ray $[X_m, \infty[$ for some m, which may be taken minimal with this property. Considering U as a module in Γ_i, we clearly have $U \not\twoheadrightarrow X_m$ (for, if $U \twoheadrightarrow X_m$, the almost split sequence ending in U has more than one indecomposable middle term). Consequently, if we look at U as a module in Γ_i, then $\widehat{U}$ lies entirely outside $\mathrm{Ker}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_i}$: indeed, in the notation of (2.1), we have $\mathrm{Hom}_A(P(a), X_p) \neq 0$ for $p \geq m$, and since we have monomorphisms $X_p \to Z_{pj}$, $X_p \to X'_p$ then $\mathrm{Hom}_A(P(a), Z_{pj}) \neq 0$ and $\mathrm{Hom}_A(P(a), X'_p) \neq 0$ for $p \geq m$. The same argument carries over to the cases ad2*), ad3*) and even ad1*) except if $U = X_0$ is the pivot.



In this latter case, let V be the mouth module in Γ_i such that the ray starting with V in Γ_i passes through U. Then V is not projective, and the almost split sequence ending with V has indecomposable middle term. Using again the notation of (2.1), we have, for any m, j, epimorphisms $Z_{mj} \to X_m$, $X'_m \to X_m$. Thus the mitre $\widehat{V}$ lies entirely outside $\mathrm{Ker}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_i}$ and we have proved our claim. By induction, such a mitre exists for Γ_γ and hence, if $\mathrm{Supp}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_1} \neq 0$, then $\mathrm{Ker}\,\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_\gamma}$ has finitely many non-isomorphic indecomposable objects and contains no cycle in mod B.

There remains the case where $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_1} = 0$ but $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_\gamma} \neq 0$. We claim that any non-zero morphism from P(a) to Γ_γ factors through modules not lying on a cycle, and belonging to a multicoil containing Γ. If this is not the case, and $\mathrm{Hom}_A(P(a), M) \neq 0$ for $M \in (\Gamma_\gamma)_0$, there exists a sequence of irreducible morphisms

$$\cdots \to M_t \xrightarrow{f_t} M_{t-1} \to \cdots \to M_1 \xrightarrow{f_1} M_0 = M$$

and for each t, g_t: $P(a) \to M_t$ is such that $f_1 \dots f_t g_t \neq 0$ and all $M_t \in (\Gamma_\gamma)_0$. We shall show that this implies $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_1} \neq 0$, a contradiction. In any sequence as before, we have infinitely many non-isomorphic M_t's: indeed, if an indecomposable N occurs infinitely many times among the M_t's, there exist $h_i \in$ rad End N such that $h_i \dots h_1 \neq 0$ for all $i \geq 1$, a contradiction to the nilpotency of rad End N. This implies that infinitely many M_t's do not belong to rectangles determined by projectives in Γ. Since, by hypothesis, $\mathrm{Hom}_A(P(a), -)|_{\mathrm{ind}\Gamma_1} = 0$, we have that infinitely many M_t's belong to a rectangle determined by an injective I(b) (through one of the operations ad1*) ad2*) or ad3*)). Actually, there exists $t_0 \in \mathbb{N}$ such that, for $t \geq t_0$, all M_t belong to the same coray inside this rectangle and thus, in the notation of (2.1), are of the form Z_{mj} for some j. Since a is a source, $a \neq b$ and hence $\mathrm{Hom}_A(P(a), S(b)) = 0$. But this implies that we may replace the Z_{mj} by the corresponding X_m. If these X_m belong to a rectangle determined by another injective, we repeat this reasoning. Since the coil contains at most finitely many injectives, we thus obtain a factorisation through modules in Γ_1. This yields the required contradiction and the proof of our claim. Let again $M \in (\Gamma_\gamma)_0$ be such that $\mathrm{Hom}_A(P(a), M) \neq 0$. A non-zero morphism $P(a) \to M$ must factor through modules not lying on a cycle and belonging to the multicoil containing Γ. In particular, M belongs to a rectangle determined by a projective P (c) in Γ. This implies that the set $J = \{j \mid \mathrm{Hom}_A(P(a), Y_j) \neq 0\}$ is not empty, where the Y_j are as in (2.1), corresponding to the projective P (c). Each such Y_j determines a ray $[Y_j, \infty[$ consisting of A-modules which are not B-modules. Thus the B-modules inside Γ_γ form a coil obtained from Γ_γ by deleting the rays $[Y_j, \infty[$ for all $j \in J$. The proof is now complete.

REMARK. The above reasoning yields an alternative proof of [2] (2.3).

5.7. We deduce the following characterisation of the minimal representation-infinite multicoil algebras (compare with [2] (2.3)).

THEOREM. Let A be a basic and connected finite dimensional algebra over an algebraically closed field. The following conditions are equivalent:

(i) A is a tame concealed algebra.

(ii) A is a representation-infinite multicoil algebra, and, for every $0 \neq e^2 = e \in A$, A/AeA is representation-finite.

(iii) A is a representation-infinite multicoil algebra, and every proper full convex subcategory of A is representation-finite.

Proof. As in [2], it suffices to prove that (iii) implies (i). Since A is a multicoil algebra, it is tame (3.4). By [11], there exists a vector $\underline{d}$ in the Grothendieck group $K_0(A)$ of A such that infinitely many non-isomorphic indecomposable modules lying in stable tubes of rank 1 have $\underline{d}$ as a dimension-vector. Then C = Supp $\underline{d}$ is, by (5.1), a full convex subcategory of A. Since C is representation-infinite, the minimality of A implies that A = C. Therefore A has stable tubes containing sincere indecomposables. By [3] (2.9), all coils in Γ_A are standard tubes which contain either projectives or injectives but not both. Thus A is a coil algebra in the sense of [2] and our statement follows from the main result (4.1) there.

5.8. COROLLARY. Let A be a representation-infinite multicoil algebra, then A contains a tame concealed full convex subcategory.

Proof. Repeat the proof of [2] (4.2).

5.9. The following characterisation of non-stable coils in a multicoil algebra uses the main result (4.1) of [3], which is, in turn, a consequence of the above results.

THEOREM. Let A be a multicoil algebra and Γ be a non-stable coil of Γ_A. Then there exists a tame concealed full convex subcategory C of A and a stable tube Γ_1 of Γ_C such that Γ is obtained from Γ_1 by a sequence of admissible operations and Supp Γ is obtained from C = Supp Γ_1 by the corresponding sequence of one-point extensions and coextensions.

Proof. By (5.4), A contains a full convex subcategory C such that Γ is obtained from a sincere stable tube Γ_1 of Γ_C by a sequence of admissible operations and Supp Γ is obtained from Supp Γ_1 = C by the corresponding sequence of one-point extensions and coextensions. Then, by (5.6), C is a multicoil algebra and Γ_C has a sincere stable tube. Therefore, by [3] (4.1), C is either tame concealed or tubular. We claim that C is tame concealed. Suppose that C is tubular. Then, in the notation of [21] (5.2), we have that ind C is of the form $\mathcal{P}_0 \vee \mathcal{T}_0 \vee (\vee_q \mathcal{T}_q) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$, where $\mathcal{P}_0$ is a postprojective component, $\mathcal{Q}_\infty$ a preinjective component and $\mathcal{T}_q, q \in \mathbb{Q}^+ \cup \{0, \infty\}$, are $\mathbb{P}_1$ (k)-families of tubes. Since Γ is a non-stable coil, A contains a one-point extension or coextension of C by a module M whose restriction $M|_C$ to C lies in Γ_1. By duality, we may assume that it is a one-point extension. Since A is tame (as a multicoil algebra), this one-point extension is also tame, and consequently, by [3] (3.2), Γ_1 is a stable tube of the family $\mathcal{T}_\infty$. But then Γ_1 is not a sincere tube of Γ_C, because $\mathcal{T}_\infty$ admits a tube containing an injective (see [21](5.2)) and different tubes of $\mathcal{T}_\infty$ are pairwise orthogonal, a contradiction. Thus C is tame concealed and this finishes our proof.

REFERENCES

1. ASSEM, I. and SKOWROŃSKI, A. : Algebras with cycle-finite derived categories, Math. Ann. 280 (1988) 441-463.

2. ASSEM, I. and SKOWROŃSKI, A. : Minimal representation-infinite coil algebras, Manuscripta Math. 67 (1990) 305-331.

3. ASSEM, I. and SKOWROŃSKI, A. : Indecomposable modules over multicoil algebras, Math. Scand. 71(1992),31-61.

4. ASSEM, I. and SKOWROŃSKI, A. : Sincere indecomposable modules lying in quasi-tubes, in preparation.

5. ASSEM, I. and SKOWROŃSKI, A. : Indecomposable modules lying in coils, in preparation.

6. ASSEM, I., NEHRING, J. and SKOWROŃSKI, A. : Domestic trivial extensions of simply connected algebras, Tsukuba J. Math., Vol. 13 No. 1 (1989), 31-72.

7. AUSLANDER, M. and REITEN, I. : Representation theory of artin algebras III and IV, Comm. Algebra 3 (1975), 239-294 and 5 (1977), 443-518.

8. AUSLANDER, M. and SMALØ, S.O. : Almost split sequences in subcategories, J. Algebra 69 (1981), No. 2, 426-454.

9. BONGARTZ, K. : On a result of Bautista and Smalø on cycles, Comm. Algebra 11 (18) (1983), 2123-2124.

10. BONGA theory, I

11. CRAWL Math. Sc

12. DROZD 1979), L

13. D'ESTE, 201.

14. HAPPEI represen (1983) 2

15. KERNEI No. 21, 2

16. NEHRIN extension

17. NEHRIN Thesis, N

18. DE LA P Algebra

19. RIEDTM Zurück,

20. RINGEL Lecture N

21. RINGEL Mathema

10. BONGARTZ, K. and GABRIEL, P. : Covering spaces in representation theory, Invent. Math. 65 (1981/82) No.3, 331-378.

11. CRAWLEY - BOEVEY, W. : On tame algebras and BOCS's, Proc. London Math. Soc., Vol. 56, No. 3 (1988) 451-483.

12. DROZD, Ju. : Tame and wild matrix problems, Proc. ICRA II (Ottawa, 1979), Lecture Notes in Mathematics 832, Springer, Berlin (1980), 240-258.

13. D'ESTE, G. and RINGEL, C. M. : Coherent tubes, J. Algebra 87 (1984) 150-201.

14. HAPPEL, D. and VOSSIECK, D. : Minimal algebras of infinite representation type with preprojective component, Manuscripta Math. 42 (1983) 221-243.

15. KERNER, O. : Tilting wild algebras, J. London Math. Soc. (2) 39 (1989) No. 21, 29-47.

16. NEHRING, J. and SKOWROŃSKI, A. : Polynomial growth trivial extensions of simply connected algebras, Fund. Math. 132 (1989) 117-134.

17. NEHRING, J. : Trywialne rozszerzenia wielomianowego wzrostu, Ph.D. Thesis, Nicholas Copernicus University (1989).

18. DE LA PEÑA, J.-A. and TOMÉ, B. : Iterated tubular algebras, J. Pure Appl. Algebra 64 (1990) 303-314.

19. RIEDTMANN, C. : Algebren, Darstellungsköcher, Überlagerungen und Zurück, Comment. Math. Helv. 55 (1980), No. 2, 199-224.

20. RINGEL, C.M. : Tame algebras, Proc. Worshop ICRA II (Ottawa, 1979), Lecture Notes in Mathematics 831, Springer, Berlin (1980) 137-287.

21. RINGEL, C.M. : Tame algebras and integral quadratic forms, Lecture Notes in Mathematics 1099, Springer, Berlin (1984).

22. SKOWROŃSKI, A. : Group algebras of polynomial growth, Manuscripta Math. 59 (1987), 499-516.

23. SKOWROŃSKI, A. : Self-injective algebras of polynomial growth, Math. Ann. 285 (1989), 177-199.

24. SKOWROŃSKI, A. : Algebras of polynomial growth, Topics in Algebra, Banach Center Publications, Vol. 26, Part 1, PWN, Warsaw (1990), 535-568.

25. SKOWROŃSKI, A. : Standard algebras of polynomial growth, in preparation.

26. SKOWROŃSKI, A. and WENDERLICH, M. : Artin algebras with directing indecomposable projective modules, Preprint, Nicholas Copernicus University (1992).

Mathématiques et Informatique
Université de Sherbrooke
Sherbrooke, Québec
Canada, J1K 2R1

Institute of Mathematics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń
Poland