Modules over cluster-tilted algebras that do not lie on local slices

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Abstract. We characterize the indecomposable transjective modules over an arbitrary cluster-tilted algebra that do not lie on a local slice, and we provide a sharp upper bound for the number of (isoclasses of) these modules.

1. Introduction

Cluster-tilted algebras were introduced by Buan, Marsh and Reiten [BMR] and, independently in [CCS] for type $\mathbb{A}$. In [ABS] is given a procedure for constructing cluster-tilted algebras: let C be a triangular algebra of global dimension two over an algebraically closed field k, and consider the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$, where $D = \mathrm{Hom}_k(-, k)$ is the standard duality, with its natural left and right C-actions. The trivial extension of C by this bimodule is called the *relation-extension* of C. It is shown there that, if C is tilted, then its relation-extension is cluster-tilted, and every cluster-tilted algebra occurs in this way.

This relation between tilted and cluster-tilted algebras has been studied further in [ABS2]. Inspired by the complete slices in the module categories of tilted algebras, the authors introduced the concept of *local slices* as a generalization of complete slices, by relaxing a convexity condition. In [ABS2] it is shown that every cluster-tilted algebra B admits a local slice Σ and that, for every such local slice Σ, the quotient algebra $B/\mathrm{Ann}\,\Sigma$ of B by the annihilator of Σ is a tilted algebra with complete slice Σ. Furthermore, there is a unique component in the Auslander-Reiten quiver of B, called the *transjective component*, that contains all local slices. Indecomposable modules in this transjective component are called *transjective*.

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In the module category of a tilted algebra, a complete slice should be thought of as a rather special configuration reproducing the quiver of a hereditary algebra to which our algebra tilts. It is well-known that an algebra is tilted if and only if it admits a complete slice, see, for instance [R]. In contrast to the above situation, the existence of a local slice does *not* characterize cluster-tilted algebras. In [ABS2], it is shown that if the cluster-tilted algebra is of tree type, then *every* indecomposable transjective module lies on a local slice. On the other hand, the authors also gave an example of an indecomposable transjective module over a cluster-tilted algebra of type $\widetilde{\mathbb{A}}_{2,1}$ that does *not* lie on a local slice.

However, the questions 'which indecomposable transjective modules do not lie on local slices", and "how many of these modules do exist", remained open.

It is the purpose of the current paper to answer both questions for arbitrary cluster-tilted algebras. First, we characterize the indecomposable transjective modules that do not lie on a local slice in Theorem 3.6, using the completion of strong sinks defined in [AsScSe]. Then we prove that the number of isoclasses (= isomorphism classes) of indecomposable transjective modules not lying on local slices is finite, and we actually give a sharp bound for this number in Corollary 3.8.

2. Preliminaries

2.1. Notation

Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field k. For an algebra B, we denote by $\operatorname{mod} B$ the category of finitely generated right B-modules. All subcategories are full, and identified with their object classes. Given a category $\mathcal{C}$, we sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$.

For a point x in the ordinary quiver of a given algebra B, we denote by $P(x)$, $I(x)$, $S(x)$ respectively, the indecomposable projective, injective and simple B-modules corresponding to x. We denote by $\Gamma(\operatorname{mod} B)$ the Auslander-Reiten quiver of B and by τ, τ^{-1} the Auslander-Reiten translations. For further definitions and facts, we refer the reader to [ASS].

2.2. Tilting

Let Q be a finite connected and acyclic quiver. A module T over the path algebra kQ of Q is called *tilting* if $\operatorname{Ext}^1_{kQ}(T,T) = 0$ and the number of isoclasses of indecomposable summands of T equals $|Q_0|$, see [ASS]. An algebra C is called *tilted of type* Q if there exists a tilting kQ-module T such that $C = \operatorname{End}_{kQ} T$. An algebra C is tilted if and only if it contains a *complete slice* Σ, see [R], that is, a finite set of indecomposable modules such that

1) $\bigoplus_{U \in \Sigma} U$ is a sincere C-module.
2) If $U_0 \to U_1 \to \ldots \to U_t$ is a sequence of nonzero morphisms between indecomposable modules with $U_0, U_t \in \Sigma$ then $U_i \in \Sigma$ for all i (*convexity*).

3) If M is an indecomposable non-projective C-module then at most one of M, τM belongs to Σ.
4) If M, S are indecomposable C-modules, $f\colon M \to S$ an irreducible morphism and $S \in \Sigma$, then either $M \in \Sigma$ or M is non-injective and $\tau^{-1}M \in \Sigma$.

2.3. Cluster-tilted algebras

Let Q be a finite, connected and acyclic quiver. The *cluster category* $\mathcal{C}_Q$ of Q is defined as follows, see [BMRRT]. Let F denote the composition $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}^{-1}$ denotes the inverse Auslander-Reiten translation in the bounded derived category $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} kQ)$, and $[1]$ denotes the shift of $\mathcal{D}$. Then $\mathcal{C}_Q$ is the orbit category $\mathcal{D}/F$: its objects are the F-orbits $\widetilde{X} = (F^iX)_{i\in\mathbb{Z}}$ of the objects $X \in \mathcal{D}$, and the space of morphisms from $\widetilde{X} = (F^iX)_{i\in\mathbb{Z}}$ to $\widetilde{Y} = (F^iY)_{i\in\mathbb{Z}}$ is $\operatorname{Hom}_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}}(X, F^iY)$. Then $\mathcal{C}_Q$ is a triangulated category with almost split triangles and, moreover, for $\widetilde{X}, \widetilde{Y} \in \mathcal{C}_Q$ we have a bifunctorial isomorphism $\operatorname{Ext}^1_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) \cong D\operatorname{Ext}^1_{\mathcal{C}_Q}(\widetilde{Y}, \widetilde{X})$. This is expressed by saying that the category $\mathcal{C}_Q$ is *2-Calabi-Yau*.

An object $\widetilde{T} \in \mathcal{C}_Q$ is called *tilting* if $\operatorname{Ext}^1_{\mathcal{C}_Q}(\widetilde{T}, \widetilde{T}) = 0$ and the number of isoclasses of indecomposable summands of $\widetilde{T}$ equals $|Q_0|$. The endomorphism algebra $B = \operatorname{End}_{\mathcal{C}_Q}\widetilde{T}$ is then called *cluster-tilted* of type Q.

Let now T be a tilting kQ-module, and $C = \operatorname{End}_{kQ}T$ the corresponding tilted algebra. Then it is shown in [ABS] that the trivial extension $\widetilde{C}$ of C by the C-C-bimodule $\operatorname{Ext}^2_C(DC, C)$ with the two natural actions of C, the so-called *relation-extension* of C, is cluster-tilted. Conversely, if B is cluster-tilted, then there exists a tilted algebra C such that $B = \widetilde{C}$.

2.4. Local slices

Let B be a cluster-tilted algebra, then a full connected subquiver Σ of $\Gamma(\operatorname{mod} B)$ is a *local slice*, see [ABS2], if:

1) Σ is a *presection*, that is, if $X \to Y$ is an arrow then:
   (a) $X \in \Sigma$ implies that either $Y \in \Sigma$ or $\tau Y \in \Sigma$
   (b) $Y \in \Sigma$ implies that either $X \in \Sigma$ or $\tau^{-1}X \in \Sigma$.
2) Σ is *sectionally convex*, that is, if $X = X_0 \to X \to \cdots \to X_t = Y$ is a sectional path in $\Gamma(\operatorname{mod} B)$ then $X, Y \in \Sigma$ imply that $X_i \in \Sigma$ for all i.
3) $|\Sigma_0| = \operatorname{rk} K_0(B)$.

Let C be tilted, then, under the standard embedding $\operatorname{mod} C \to \operatorname{mod} \widetilde{C}$, any complete slice in the tilted algebra C embeds as a local slice in $\operatorname{mod} \widetilde{C}$, and any local slice in $\operatorname{mod} \widetilde{C}$ occurs in this way. If B is a cluster-tilted algebra, then a tilted algebra C is such that $B = \widetilde{C}$ if and only if there exists a local slice Σ in $\Gamma(\operatorname{mod} B)$ such that $C = B/\operatorname{Ann}_B\Sigma$, where $\operatorname{Ann}_B\Sigma = \bigcap_{X\in\Sigma} \operatorname{Ann}_B X$, see [ABS2].

2.5. Completions and reflections

We recall the definition of reflections from [AsScSe]. Let B be a cluster-tilted algebra. Let Σ be a local slice in the transjective component of $\Gamma(\text{mod}\, B)$ having the property that all the sources in Σ are injective B-modules. Then Σ is called a *rightmost* slice of B. Let x be a point in the quiver of B such that $I(x)$ is an injective source of the rightmost slice Σ.

The *completion* H_x *of* x is defined by the following three conditions.

(a) $I(x) \in H_x$.
(b) H_x is closed under predecessors in Σ.
(c) If $L \to M$ is an arrow in Σ with $L \in H_x$ having an injective successor in H_x then $M \in H_x$.

The completion H_x can be constructed inductively in the following way. We let $H_1 = I(x)$, and H'_2 be the closure of H_1 with respect to (c). We then let H_2 be the closure of H'_2 with respect to predecessors in Σ. Then we repeat the procedure; given H_i, we let H'_{i+1} be the closure of H_i with respect to (c) and H_{i+1} be the closure of H'_{i+1} with respect to predecessors. This procedure must stabilize, because the slice Σ is finite. If $H_j = H_k$ with $k > j$, we let $H_x = H_j$.

We can decompose H_x as the disjoint union of three sets as follows. Let $\mathcal{J}$ denote the set of injectives in H_x, let $\mathcal{J}^-$ be the set of non-injectives in H_x which have an injective successor in H_x, and let $\mathcal{E} = H_x \setminus (\mathcal{J} \cup \mathcal{J}^-)$ denote the complement of $(\mathcal{J} \cup \mathcal{J}^-)$ in H_x. Thus $H_x = \mathcal{J} \sqcup \mathcal{J}^- \sqcup \mathcal{E}$ is a disjoint union. The *reflection of the slice* Σ *in* x is defined as

$$\sigma_x^+ \Sigma = \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1}\mathcal{E} \cup (\Sigma \setminus H_x),$$

where $\tau^{-2}\mathcal{J}$ stands for the set of all indecomposable projectives $P(y)$ such that the corresponding injective $I(y)$ is in the set $\mathcal{J}$.

Theorem 2.1. [AsScSe, Theorem 4.4] *Let* Σ *be a rightmost local slice in* $\text{mod}\, B$ *with injective source* $I(x)$*. Then the reflection* $\sigma_x^+ \Sigma$ *is a local slice as well.*

3. Main results

In this section, we prove our main results. We start with two preparatory lemmas.

Definition 3.1. Let B be a representation-infinite cluster-tilted algebra and let Σ, Σ' be two local slices in $\text{mod}\, B$ and $\widetilde{\Sigma}, \widetilde{\Sigma'}$ be their lifts in the cluster category $\mathcal{C}$. Then for every indecomposable module X in Σ, we define $d_X(\Sigma, \Sigma')$ to be the unique integer k such that $\tau_{\mathcal{C}}^{-k}\widetilde{X}$ lies in $\widetilde{\Sigma'}$, where $\widetilde{X}$ is the lift of X in $\mathcal{C}$.

Remark 3.2. In the above definition, the condition that B is representation-infinite is necessary for the uniqueness of the integer k.

Lemma 3.3. *Let B be a representation-infinite cluster-tilted algebra. Let Σ be a rightmost local slice in* $\operatorname{mod} B$ *with source $I(x)$, and H_x the completion in Σ. Suppose that Σ' is another local slice such that $d_{I(x)}(\Sigma, \Sigma') \geq 2$. Then for every indecomposable module Y in H_x we have*

$$d_Y(\Sigma, \Sigma') \geq 1.$$

In particular, for every injective indecomposable $I(y)$ in H_x we have

$$d_{I(y)}(\Sigma, \Sigma') \geq 2.$$

Proof. Let $\{I(x)\} = H_1 \subset H_2 \subset \cdots \subset H_r = H_x$ be the recursive construction of H_x as in section 2.5 above. Recall that given H_{i-1}, the set H_i' is the closure of H_{i-1} with respect to condition (c) of the definition of H_x, and H_i is the closure of H_i' under predecessors. Let $Y \in H_i \setminus H_{i-1}$. We will prove the result by induction on i.

If $i = 1$ then $Y = I(x)$ and we have $d_Y(\Sigma, \Sigma') \geq 2$ by assumption. Now assume that $i > 1$. Then there are two possibilities

a) Suppose first that $Y \in H_i'$. Then there exists an arrow $L \to Y$ in Σ with $L \in H_{i-1}$ having an injective successor I in H_{i-1}. So there is a path

$$\ell : L = L_0 \to L_1 \to \cdots \to L_{s-1} \to L_s = I$$

in H_{i-1} and our induction hypothesis yields $d_{L_s}(\Sigma, \Sigma') = k \geq 2$. In the cluster category $\mathcal{C}$, denote by $\widetilde{I}$, $\widetilde{L}_i$ and $\widetilde{\Sigma'}$ the lifts of I, L_i and Σ', respectively. Then $\tau_{\mathcal{C}}^{-k}\widetilde{I} \in \widetilde{\Sigma'}$. Moreover, since $\widetilde{L}_{s-1} \to \widetilde{L}_s$ is an arrow in $\widetilde{\Sigma}$, there is an arrow $\tau_{\mathcal{C}}^{-k}\widetilde{L}_{s-1} \to \tau_{\mathcal{C}}^{-k}\widetilde{L}_s$ in the Auslander-Reiten quiver of $\mathcal{C}$, and because $\widetilde{\Sigma'}$ is a local slice, this implies that either $\tau_{\mathcal{C}}^{-k}\widetilde{L}_{s-1}$ or $\tau_{\mathcal{C}}^{-(k+1)}\widetilde{L}_{s-1}$ is in $\widetilde{\Sigma'}$. In particular $d_{L_{s-1}}(\Sigma, \Sigma') \geq d_{L_s}(\Sigma, \Sigma') \geq 2$. Repeating this argument for every arrow in the path ℓ we see that $d_{L_i}(\Sigma, \Sigma') \geq 2$, for all i, and thus $d_L(\Sigma, \Sigma') \geq 2$. This implies that $d_Y(\Sigma, \Sigma') \geq 1$, since there is an arrow $L \to Y$.

b) Now suppose that $Y \in H_i \setminus H_i'$. Thus Y is obtained by closing under predecessors. Hence there is a path $\ell' : Y = L_0' \to L_1' \to \cdots \to L_t'$ with $L_t' \in H_i'$. In particular, $d_{L_t'}(\Sigma, \Sigma') \geq 1$, by part a). By the same argument as in case a), going back along the path ℓ' will not decrease the values of the function d, so we see that $d_Y(\Sigma, \Sigma') \geq d_{L_t'}(\Sigma, \Sigma') \geq 1$. This shows the first claim. Now, if Y is injective then $d_Y(\Sigma, \Sigma')$ cannot be equal to 1, because $\tau^{-1}Y = 0$ is not in Σ'. This shows the second claim. □

For the proof of the next lemma, we need the following construction. Let (Γ, τ) be a translation quiver, and X be a point in Γ. Then we define

$$\Sigma(\to X) = \left\{ Y \in \Gamma \;\middle|\; \begin{array}{l} \text{there exists a sectional path from } Y \text{ to } X \text{ in } \Gamma \\ \text{and every path from } Y \text{ to } X \text{ in } \Gamma \text{ is sectional.} \end{array} \right\},$$

$$\Sigma(X \to) = \left\{ Y \in \Gamma \;\middle|\; \begin{array}{l} \text{there exists a sectional path from } X \text{ to } Y \text{ in } \Gamma \\ \text{and every path from } X \text{ to } Y \text{ in } \Gamma \text{ is sectional.} \end{array} \right\}.$$

Proposition 3.4. [R, 4.2 (6), p. 185] *Let Y be an indecomposable sincere module in a postprojective or preinjective component. Then both $\Sigma(\to Y)$ and $\Sigma(Y \to)$ are complete slices.*

Lemma 3.5. *Let M be an indecomposable transjective B-module which does not lie on a local slice. Then there exist an indecomposable injective B-module $I(j)$ and a local slice Σ containing a sectional path $v : \tau M \to \cdots \to I(j)$.*

Proof. Let A be a hereditary algebra and $T \in \mathcal{C}_A$ a cluster-tilting object such that $B = \mathrm{End}_{\mathcal{C}_A}(T)$. Let M be an indecomposable B-module in the transjective component $\mathcal{T}$ of $\Gamma(\mathrm{mod}\, B)$, and let $\widetilde{M} \in \mathcal{C}_A$ be an indecomposable object such that $\mathrm{Hom}_{\mathcal{C}_A}(T, \widetilde{M}) = M$. Finally, let $\widetilde{\Sigma} = \Sigma(\widetilde{M} \to)$ in the cluster category $\mathcal{C}_A$.

Since $B \cong \mathrm{End}_{\mathcal{C}_A}(\tau^l_{\mathcal{C}_A} T)$ for all $l \in \mathbb{Z}$, we may assume without loss of generality that $\widetilde{\Sigma}$ lies in the postprojective component of $\mathrm{mod}\, A$. Furthermore, we may assume that every postprojective successor of $\widetilde{\Sigma}$ in $\mathrm{mod}\, A$ in sincere. Indeed this follows from the fact that there are only finitely many isoclasses of indecomposable postprojective A-modules that are not sincere. For tame algebras this holds, because non-sincere modules are supported on a Dynkin quiver, and for wild algebras see [Ke, Corollary 2.3].

Now since $\widetilde{M}$ is a sincere A-module, Proposition 3.4 implies that $\widetilde{\Sigma}$ is a slice in $\mathrm{mod}\, A$, and therefore a local slice in $\mathcal{C}_A$. Let $\Sigma_1 = \mathrm{Hom}_{\mathcal{C}_A}(T, \Sigma(\widetilde{M} \to))$. Then $M \in \Sigma_1$, and thus by assumption Σ_1 is not a local slice in $\mathrm{mod}\, B$. Therefore, there exists an indecomposable direct summand T_j of T such that $\tau T_j \in \Sigma(\widetilde{M} \to)$. Moreover, by definition of $\Sigma(\widetilde{M} \to)$ there is a sectional path $\widetilde{M} \to \cdots \to \tau T_j$ and every path from $\widetilde{M}$ to τT_j is sectional. Applying τ we see that there exists a sectional path $\tilde{v} : \tau\widetilde{M} \to \cdots \to \tau^2 T_j$ and every path from $\tau\widetilde{M}$ to $\tau^2 T_j$ is sectional. Thus the local slice $\Sigma(\to \tau^2 T_j)$ in $\mathcal{C}_A$ contains the path $\tilde{v}$. If there exists a summand T_i of T such that $\tau T_i \in \Sigma(\to \tau^2 T_j)$ then $0 \neq \mathrm{Hom}_{\mathcal{C}_A}(\tau T_i, \tau^2 T_j) \cong D\mathrm{Ext}^1_{\mathcal{C}_A}(T_j, T_i)$ which is impossible. Thus, the local slice $\Sigma(\to \tau^2 T_j)$ does not contain summands of τT. Therefore, $\Sigma = \mathrm{Hom}_{\mathcal{C}_A}(T, \Sigma(\to \tau^2 T_j))$ is a local slice in $\mathrm{mod}\, B$ containing τM and containing a sectional path $v = \mathrm{Hom}_{\mathcal{C}_A}(T, \tilde{v}) : \tau M \to \cdots \to I(j)$. ☐

We are now ready for our main result.

Theorem 3.6. *Let B be a cluster-tilted algebra and M an indecomposable transjective B-module. Then the following are equivalent.*

(a) *M does not lie on a local slice.*

(b) *There exist a rightmost slice Σ with source $I(x)$ such that the completion H_x contains a sectional path*

$$\omega : I(x) \to \cdots \to \tau M \to \cdots \to I(j)$$

with $I(j)$ injective. In particular $\tau M \in \mathcal{J}^-(H_x)$.

Proof. (a) ⇒ (b). By Lemma 3.5, there is an indecomposable injective $I(j)$ and a local slice Σ_1 containing a sectional path $v : \tau M \to \cdots \to I(j)$. Without loss of generality we may assume that there is no other injective on the path v and that Σ_1 is a rightmost local slice. Let $u_1 : I(x_1) \to \cdots \to \tau M$ be a maximal path in Σ_1 ending in τM. Thus $I(x_1)$ is a source in the rightmost local slice Σ_1, hence $I(x_1)$ is injective. Moreover, since τM is not an injective module, $I(x_1) \neq \tau M$. We distinguish two cases.

(1) If $I(j) \in H_{x_1}$, then the composition $\omega = u_1 v$ lies entirely inside H_{x_1}, because H_{x_1} is closed under predecessors and we are done.

(2) Now suppose that $I(j) \notin H_{x_1}$.

(2.1) If $\tau M \in H_{x_1}$ then τM must lie in $\mathcal{J}^-$ of H_{x_1}, because otherwise the reflection $\sigma^+_{x_1}\Sigma_1$ would be a local slice containing M which is impossible by (a). But $\tau M \in \mathcal{J}^-$ implies the existence of a path $v' : \tau M \to \cdots \to I'$ in $H_{x_1} \subset \Sigma_1$ with I' injective, and then the path

$$\omega = u_1 v' : I(x_1) \to \cdots \to \tau M \to \cdots \to I'$$

lies entirely inside H_{x_1}, and we are done. Note that ω is sectional since it is a path in a local slice.

(2.2) If $\tau M \notin H_{x_1}$, then the path v lies entirely in $\Sigma_1 \setminus H_{x_1}$ and thus v lies entirely in the local slice $\Sigma_2 = \sigma^+_{x_1}\Sigma_1$. Repeating the argument, we either obtain a local slice with source $I(x)$ such that $I(j) \in H_x$ and we conclude by the argument of case (1), or we obtain a local slice $\Sigma_k = \sigma^+_{x_{k-1}} \cdots \sigma^+_{x_2}\sigma^+_{x_1}\Sigma_1$ containing v and a path $u_k : I(x_k) \to \cdots \to \tau M$ with $I(x_k)$ an injective source and $\tau M \in H_{x_k}$, and we conclude by the argument of case (2.1).

(b) $\Rightarrow$ (a). We want to show that M does not lie on a local slice. Suppose to the contrary that there exists a local slice Σ_M containing M. Let ω, Σ and H_x be as in the statement of the theorem. By the argument of the first part of the proof, we may assume without loss of generality that $I(j) \in H_x$. We use the following notation for the path ω

$$I(x) \to \cdots \to I(i) \to L_{-s} \to \cdots \to L_{-2} \to L_{-1} \to \tau M \to L_1 \to L_2 \to \cdots \to L_r \to I(j),$$

and we assume without loss of generality that none of the L_i is injective. Let γ be the path obtained by applying τ^{-1} to a part of ω, such that

$$\gamma : \tau^{-1}L_{-s} \to \cdots \to \tau^{-1}L_{-2} \to \tau^{-1}L_{-1} \to M \to \tau^{-1}L_1 \to \tau^{-1}L_2 \to \cdots \to \tau^{-1}L_r.$$

Since M lies in the local slice Σ_M and $M \to \tau^{-1}L_1$ is an arrow in the Auslander-Reiten quiver, we have that either $\tau^{-1}L_1$ or L_1 is in Σ_M. If $\tau^{-1}L_1 \in \Sigma_M$ then by the same argument, we have that either $\tau^{-1}L_2$ or L_2 is in Σ_M. Repeating this reasoning, we see that either there is an $L_i \in \Sigma_M$ or Σ_M contains all the $\tau^{-1}L_i$ for $i = 1, 2, \cdots, r$. In the latter case, we have an arrow $I(j) \to \tau^{-1}L_r$ with $\tau^{-1}L_r \in \Sigma_M$ and thus $I(j)$ must be in Σ_M, since $\tau^{-1}I(j) = 0$. Thus in both cases $\Sigma_M \cap \omega \neq \emptyset$ and

$$d_{I(j)}(\Sigma, \Sigma_M) \leq 0. \tag{3.1}$$

A similar argument along the part of the path γ from $\tau^{-1}L_{-s}$ to M, we see that $\Sigma_M \cap \tau^{-1}\gamma \neq \emptyset$ and $d_{I(i)}(\Sigma, \Sigma_M) \geq 2$. Going back along the initial segment of the path $\omega : I(x) \to \cdots \to I(i)$ the values of the function d cannot decrease, thus $d_{I(x)}(\Sigma, \Sigma_M) \geq 2$ as well. Now using Lemma 3.3, we see that $d_{I(j)}(\Sigma, \Sigma_M) \geq 2$, which is a contradiction to the inequality (3.1). □

Remark 3.7. For cluster-tilted algebras of tree type, in particular for representation-finite cluster-tilted algebras, we know from [ABS2] that every indecomposable module lies on a local slice. Thus condition (b) cannot hold in a cluster-tilted algebra of tree type.

We now prove that the number of transjective modules over a cluster-tilted algebra which do not lie on a local slice is finite.

Corollary 3.8. *Let B be a cluster-tilted algebra. Denote by n the number of isoclasses of indecomposable projective B-modules, and define t as the maximum of the number 1 and the number of isoclasses of indecomposable transjective projective B-modules. Then the number of isoclasses of indecomposable transjective B-modules that do not lie on a local slice is at most*

$$(2^{t-1} - 1)(n - 2).$$

Proof. First observe that if B is representation-finite, then the result trivially holds by Remark 3.7. Assume therefore that B is representation-infinite. By Theorem 3.6 we have that the number of indecomposable transjective B-modules that do not lie on a local slice is bounded above by the cardinality of the set $\cup_\Sigma \cup_x \mathcal{J}^-(H_x)$, where Σ runs over all rightmost local slices and x runs over all points such that $I(x)$ is a source in Σ. Since $\mathcal{J}^-(H_x) \subset \Sigma$, we have

$$\cup_x \mathcal{J}^-(H_x) \subset \{L \in \Sigma \mid L \text{ is a noninjective indecomposable } B\text{-module}\}$$

and thus

$$\left|\cup_x \mathcal{J}^-(H_x)\right| \leq n - 2$$

because we need at least two injectives in Σ for $\mathcal{J}^-(H_x) \neq \emptyset$.

Let $B = \operatorname{End}_{\mathcal{C}}(T)$ where T is a cluster-tilting object over a cluster category $\mathcal{C}$. Given a local slice Σ in $\operatorname{mod} B$ let $\widetilde{\Sigma}$ be the lift of Σ to the cluster category, that is $\Sigma = \operatorname{Hom}_{\mathcal{C}}(T, \widetilde{\Sigma})$. We claim that the number of rightmost local slices Σ in $\operatorname{mod} B$ is at most $2^{t-1} - 1$.

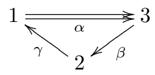
Observe that for every indecomposable transjective summand T_i of T we have that τT_i is a predecessor or a successor of the local slice $\widetilde{\Sigma}$ in $\mathcal{C}$. Moreover, since the slice Σ is rightmost it is determined by the predecessors and successors in τT of the corresponding $\widetilde{\Sigma}$. We have to subtract 1 because if $\widetilde{\Sigma}$ has no transjective successors in τT then Σ is not rightmost. This shows that the number of local slices is at most $2^t - 1$.

Finally, for $\mathcal{J}^-(H_x) \neq \emptyset$ there must be at least two summands of τT which cannot be separated by a local slice, because $\mathcal{J}^-(H_x) \neq \emptyset$ implies that there is a sectional path $\omega : I(i) \to \cdots \to \tau M \to \cdots \to I(j)$ and in the cluster category this yields a sectional path $\tau^{-1}\tilde{\omega} : \tau T_i \to \cdots \widetilde{M} \to \cdots \to \tau T_j$ and $M = \operatorname{Hom}_{\mathcal{C}}(T, \widetilde{M})$ does not lies on a local slice. This shows that the number of local slices is at most $2^{t-1} - 1$. □

4. Two examples

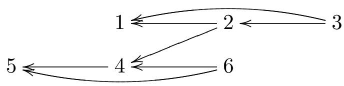
We conclude with two examples. The first example shows that the bound in Corollary 3.8 is sharp, and the second example illustrates the statement of the theorem.

Example 4.1. Let B be the cluster-tilted algebra of type $\widetilde{\mathbb{A}}_{2,1}$ given by the following quiver with relations $\alpha\beta = \beta\gamma = \gamma\alpha = 0$.

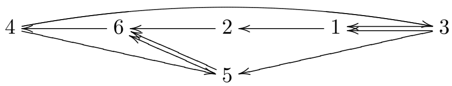


The projective B-modules $P(1)$ and $P(3)$ lie in the transjective component of $\Gamma(\operatorname{mod} B)$ while the projective $P(2)$ lies in a tube. The only transjective B-module not lying on a local slice is $S(2)$. On the other hand the formula in Corollary 3.8 gives $(2^{t-1} - 1)(n-2) = (2^{2-1} - 1)(3-2) = 1$.

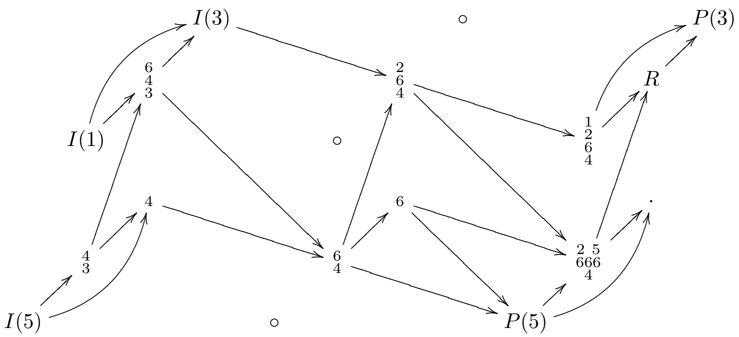
Example 4.2. We give an example to illustrate Theorem 3.6. Let A be the path algebra of the quiver



Mutating at the vertices 2,4 and 6 yields the cluster-tilted algebra B with quiver



In the Auslander-Reiten quiver of $\operatorname{mod} B$ we have the following local configuration.



$$I(1) = \begin{smallmatrix} & 2 & \\ 6 & & 6 \\ 4 & & 4 \\ 3 & & 3 \\ & 1 & \end{smallmatrix} \quad I(3) = \begin{smallmatrix} 2 \\ 6 \\ 4 \\ 3 \end{smallmatrix} \quad I(5) = \begin{smallmatrix} 4 & \\ 3 & 4 \\ 5 & \end{smallmatrix} \quad P(5) = \begin{smallmatrix} & 5 & \\ 6 & & 6 \\ & 4 & \end{smallmatrix} \quad R = \begin{smallmatrix} 1 & \\ 2 & 5 \\ 666 & \\ 4 & \end{smallmatrix} \quad P(3) = \begin{smallmatrix} & 3 & \\ 1 & & 1 \\ 2 & & 2\;5 \\ 6 & & 666 \\ 4 & & 4 \end{smallmatrix}$$

The 6 modules on the left form a rightmost local slice

$$\Sigma = \{I(1), \begin{smallmatrix} 6 \\ 4 \\ 3 \end{smallmatrix}, I(3), I(5), \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}, 4\}$$

in which both $I(1)$ and $I(5)$ are sources. Their completions are $H_1 = \Sigma$ and $H_5 = \{I(5), \begin{smallmatrix}4\\3\end{smallmatrix}, \begin{smallmatrix}4\end{smallmatrix}\}$.

The module $\begin{smallmatrix}6\\4\\3\end{smallmatrix}$ satisfies condition (b) of the theorem with respect to H_1. Therefore the module $\tau^{-1}\begin{smallmatrix}6\\4\\3\end{smallmatrix} = \begin{smallmatrix}2\\6\\4\end{smallmatrix}$ does not lie on a local slice.

The module $\begin{smallmatrix}4\\3\end{smallmatrix}$ does not satisfy condition (b) of the theorem. Indeed, in H_5 it does not have an injective successor, and in H_1 it is not a successor of $I(1)$. The theorem implies that the module $\tau^{-1}\begin{smallmatrix}4\\3\end{smallmatrix} = \begin{smallmatrix}6\\4\end{smallmatrix}$ does lie on a local slice. This local slice is the reflection $\sigma_5^+\Sigma = \{I(1), \begin{smallmatrix}6\\4\\3\end{smallmatrix}, I(3), \begin{smallmatrix}6\\4\end{smallmatrix}, \begin{smallmatrix}6\end{smallmatrix}, P(5)\}$.

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