LEFT SECTIONS AND THE LEFT PART OF AN ARTIN ALGEBRA

IBRAHIM ASSEM

INTRODUCTION

Let A be an artin algebra. We are interested in studying the representation theory of A, thus the category $\operatorname{mod} A$ of finitely generated right A-modules. For this purpose, we fix a full subcategory $\operatorname{ind} A$ of $\operatorname{mod} A$ having as objects exactly one representative from each isomorphism class of indecomposable modules. Following Happel, Reiten and Smalø [19], we define the left part $\mathscr{L}_A$ to be the full subcategory of $\operatorname{ind} A$ with objects those modules whose predecessors have projective dimension at most one. The right part is defined dually. These classes, whose definition suggests the interplay between homological properties of an algebra and representation theoretic ones, were heavily investigated and applied (see, for instance, the survey [4]).

The initial motivation for this paper comes from the observations, made in [5, 2, 1], that the left part of an arbitrary artin algebra closely resembles that of a tilted algebra. Tilted algebras, introduced by Happel and Ringel in [20], are among the most important and best understood classes of algebras. Many criteria allow to recognise whether a given algebra is tilted or not. Most of them revolve around the existence of a combinatorial configuration, called "complete slice" or "section" inside the module category, see [20, 15, 22, 13, 28, 29, 7]. Perhaps the most efficient is the Liu-Skowroński criterion: they define (combinatorially) a so-called section in an Auslander-Reiten component and prove that, if there exists a section satisfying reasonable algebraic conditions, then the algebra is tilted (see [26, 30] or [9, (Chapter VIII)]). Surprisingly, however, as is shown in [1], none of the known criteria seems to apply directly to the tilted algebras arising from the study of the left part.

The first aim of this paper is to derive a more suitable version of the Liu-Skowroński criterion, easier to apply in our case. For this purpose, we define a notion of left section in a translation quiver by weakening one of the Liu-Skowroński axioms for section (see (2.1)). Several known results for sections carry over to left sections, sometimes in a restricted form (see, for instance, (2.2) and (3.2)). We thus obtain our first main theorem.

Theorem A. *Let A be an artin algebra, and Σ be a left section in a component Γ of the Auslander-Reiten quiver of A such that $\operatorname{Hom}_A\left(\tau_A^{-1}E', E''\right) = 0$ for all E', E'' in Σ, then $A/\operatorname{Ann}_A\Sigma$ is a tilted algebra having Σ as complete slice.*

If Σ is a section, then the condition that $\operatorname{Hom}_A(\tau_A^{-1}E', E'') = 0$ for all E', E'' in Σ is equivalent to several other conditions, notably that the component Γ which contains it is generalised standard (see [26, 30]). This is not true for left sections. However, if Σ is a left section, then this condition implies (but is not equivalent to say) that the full translation subquiver $\Gamma_{\leqslant\Sigma}$ of Γ consisting of the predecessors of Σ in Γ is generalised standard.

As corollaries of the above theorem, we obtain, not only the Liu-Skowroński criterion, but also the statements necessary for the study of the left part. If, in particular, Σ is a left section which is convex in $\operatorname{ind} A$, then the condition of the theorem is satisfied so

$A/\mathrm{Ann}_A\Sigma$ is tilted (4.3). Also, if A is an algebra over an algebraically closed field, then $A/\mathrm{Ann}_A\Sigma$ coincides with the support algebra of Σ, which is a full convex subcategory of A, see (4.5).

We next apply our criterion to the study of the left part. As shown in [5, 6, 2, 1], the main tool in the proofs of the known results is the description of the Ext-injectives (in the sense of [12]) in the left part $\mathscr{L}_A$. Here, we rather work with a full subcategory $\mathscr{C}$ of $\mathscr{L}_A$ which is closed under predecessors, and we prove that the most useful statements about Ext-injectives in $\mathscr{L}_A$ carry over to this context. This approach allows to work with connected subcategories of $\mathscr{L}_A$ (which is not connected in general). Also, this hypothesis is optimal: easy examples show that the known techniques about $\mathscr{L}_A$ do not carry over to subcategories closed under predecessors which are not contained in $\mathscr{L}_A$ (this more general situation is addressed in a forthcoming work with Coelho and Trepode). This leads to our second main theorem.

Theorem B. *Let A be an artin algebra, and $\mathscr{C} \subseteq \mathscr{L}_A$ be a full subcategory closed under predecessors, having $\mathscr{E}$ as subcategory of Ext-injectives. Let Γ be a component of the Auslander-Reiten quiver of A. Then:*

(a) *If $\Gamma \cap \mathscr{E} = \varnothing$, then either $\Gamma \subseteq \mathscr{C}$ or $\Gamma \cap \mathscr{C} = \varnothing$.*
(b) *If $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$, then Σ is a left section of Γ, convex in $\mathrm{ind}\,A$. Moreover, $A/\mathrm{Ann}_A\Sigma$ is a tilted algebra having Σ as complete slice.*

As corollaries, we obtain the first two main results of [1]. Following this line, we define the support algebra of a subcategory $\mathscr{C}$ as above, thus generalising the notion of left support algebra [5, 32]. As a consequence of the theorem, we describe completely the Auslander-Reiten components which lie entirely inside $\mathscr{C}$ and, for those which intersect $\mathscr{C}$, the part which precedes the left section Σ. This is contained in (7.4)(7.5)(7.6), which generalise the remaining results of [1]. In the last section, we introduce a new class of algebras, called $\mathscr{C}$-supported, modeled after the left supported algebras of [5] and we obtain, in (8.2)(8.8), generalisations of the results of [5, 2].

Clearly, the dual results, for right sections and the right part, hold as well. For the sake of brevity, we refrain from stating them, leaving the primal-dual translation to the reader.

1. Preliminaries

1.1. Notation. Throughout this paper, all our algebras are basic and connected artin algebras. For an algebra A, we denote by $\mathrm{mod}\,A$ its category of finitely generated right modules and by $\mathrm{ind}\,A$ a full subcategory of $\mathrm{mod}\,A$ consisting of one representative from each isomorphism class of indecomposable modules. Whenever we speak about a module (or an indecomposable module), we always mean implicitly that it belongs to $\mathrm{mod}\,A$ (or to $\mathrm{ind}\,A$, respectively). Also, all subcategories of $\mathrm{mod}\,A$ are full and so are identified with their object classes. We sometimes consider an algebra A as a category, in which the object class A_0 is a complete set $\{e_1, e_2, \ldots, e_n\}$ of primitive orthogonal idempotents and the set of morphisms from e_i to e_j is e_iAe_j. An algebra B is a **full subcategory** of A if there is an idempotent $e \in A$, sum of some of the distinguished idempotents e_i, such that $B = eAe$. It is **convex** in A if, for any sequence $e_i = e_{i_0}, e_{i_1}, \ldots, e_{i_t} = e_j$ of objects in A such that $e_{i_\ell}Ae_{i_{\ell+1}} \neq 0$ (with $0 \leq \ell < t$) and e_i, e_j objects in B, then all e_{i_ℓ} lie in B. We denote by P_x (or I_x, or S_x) the indecomposable projective (or injective, or simple, respectively) A-module corresponding to the idempotent e_x.

A subcategory $\mathscr{C}$ of $\mathrm{ind}\,A$ is called **finite** if it has only finitely many objects. We sometimes write $M \in \mathscr{C}$ to express that M is an object in a subcategory $\mathscr{C}$. We denote by

$\operatorname{add}\mathscr{C}$ the subcategory of $\operatorname{mod}A$ with objects the direct sums of summands of modules in $\mathscr{C}$. Given a module M, we let $\operatorname{pd}M$ (or $\operatorname{id}M$) stand for its projective (or injective, respectively) dimension. The global dimension of A is denoted by $\operatorname{gl.dim.}A$ and its Grothendieck group by $K_0(A)$. For a module M, the **support** $\operatorname{Supp}(M,-)$ (or $\operatorname{Supp}(-,M)$) of the functor $\operatorname{Hom}_A(M,-)$ (or $\operatorname{Hom}_A(-,M)$) is the subcategory of $\operatorname{ind}A$ consisting of all X such that $\operatorname{Hom}_A(M,X)\neq 0$ (or $\operatorname{Hom}_A(X,M)\neq 0$, respectively). We denote by $\operatorname{Gen}M$ (or $\operatorname{Cogen}M$) the subcategory of $\operatorname{mod}A$ having as objects all modules generated (or cogenerated, respectively) by M.

For an algebra A, we denote by $\Gamma(\operatorname{mod}A)$ its Auslander-Reiten quiver and by $\tau_A = \operatorname{DTr}$, $\tau_A^{-1} = \operatorname{Tr}\operatorname{D}$ its Auslander-Reiten translations. For further definitions and facts on $\operatorname{mod}A$ or $\Gamma(\operatorname{mod}A)$, we refer to [9, 11]. For tilting theory, we refer to [9].

1.2. Paths. Let A be an algebra. Given $M, N \in \operatorname{ind}A$, a **path** from M to N in $\operatorname{ind}A$ (denoted by $M \rightsquigarrow N$) is a sequence of non-zero morphisms

$$(*) \qquad M = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = N$$

$(t\geq 1)$ where $X_i \in \operatorname{ind}A$ for all i. We then say that M is a **predecessor** of N and N is a **successor** of M. A path from M to M involving at least one non-isomorphism is a **cycle**. A module $M\in\operatorname{ind}A$ which lies on no cycle is **directed**. If each f_i in $(*)$ is irreducible, we say that $(*)$ is a **path of irreducible morphisms**, or path in $\Gamma(\operatorname{mod}A)$. A path $(*)$ of irreducible morphisms is **sectional** if $\tau_A X_{i+1}\neq X_{i-1}$ for all i with $0<i<t$. A **refinement** of $(*)$ is a path in $\operatorname{ind}A$

$$M = X_0' \longrightarrow X_1' \longrightarrow \cdots \longrightarrow X_{s-1}' \longrightarrow X_s' = N$$

such that there exists an order-preserving injection $\sigma : \{1,\ldots,t-1\}\to\{1,\ldots,s-1\}$ satisfying $X_i = X'_{\sigma(i)}$ for all i with $0<i<t$.

A subcategory $\mathscr{C}$ is **closed under predecessors** if, whenever $M\rightsquigarrow N$ is a path in $\operatorname{ind}A$ with $N\in\mathscr{C}$, then $M\in\mathscr{C}$. Equivalently, $\operatorname{add}\mathscr{C}$ is the torsion-free class of a split torsion pair. We define dually subcategories **closed under successors**, which generate torsion classes of split torsion pairs.

Important examples are the left and right parts of $\operatorname{mod}A$, defined in [19]. The **left part** is the full subcategory of $\operatorname{ind}A$ with object class

$$\mathscr{L}_A = \{M\in\operatorname{ind}A \mid \text{for any } L \text{ with } L\rightsquigarrow M, \text{ we have } \operatorname{pd}L\leq 1\}\,.$$

Thus, $\mathscr{L}_A$ is closed under predecessors. The **right part** $\mathscr{R}_A$ is defined dually and is closed under successors. For properties of $\mathscr{L}_A$ and $\mathscr{R}_A$, we refer to [4, 19].

2. Left sections in translation quivers

2.1. In this section, (Γ,τ), or briefly Γ, denotes a translation quiver. Given $x,y\in\Gamma_0$, a **path** from x to y (denoted by $x\rightsquigarrow y$) is a sequence of arrows

$$(*) \qquad x = x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_{t-1} \longrightarrow x_t = y\ .$$

We say that x is a **predecessor** of y, or y is a **successor** of x. If $y=x$ and $t\geq 1$, this path is a **cycle**. A full subquiver Σ of Γ is **acyclic** if it contains no cycle. It is **convex** (in Γ) if, for any path $(*)$ with $x,y\in\Sigma_0$, we have $x_i\in\Sigma_0$ for all i.

Definition. *A full subquiver Σ of a translation quiver Γ is called a **left section** if:*

(LS1) *Σ is acyclic;*

(LS2) *for any* $x \in \Gamma_0$ *such that there exist* $y \in \Sigma_0$ *and a path* $x \rightsquigarrow y$*, there exists a unique* $n \geq 0$ *such that* $\tau^{-n}x \in \Sigma_0$*;*

(LS3) Σ *is convex in* Γ.

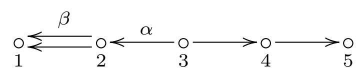
Examples. (a) A full connected subquiver Σ of Γ is a **section** (see [25, 30] or else [9]) if it satisfies (LS1), (LS3) and:

(S2) For any $x \in \Gamma_0$, there exists a unique $n \in \mathbb{Z}$ such that $\tau^n x \in \Sigma_0$.

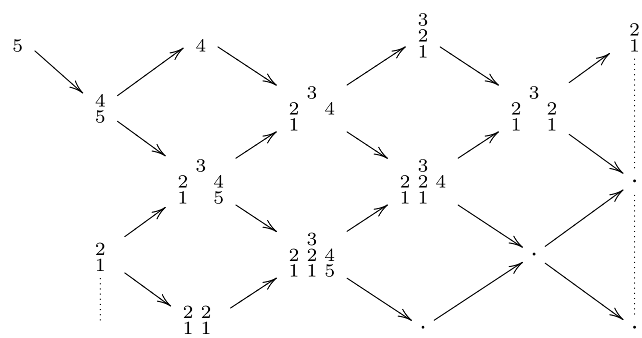
Thus, any section is a left section.

(b) Our second example is the motivating one: let A be an artin algebra, $\mathscr{L}_A$ be the left part of $\operatorname{mod} A$, $\mathscr{E}_A$ be the class of indecomposable Ext-injectives in $\operatorname{add} \mathscr{L}_A$ and Γ be a component of $\Gamma(\operatorname{mod} A)$ such that $\Gamma \cap \mathscr{E}_A \neq \varnothing$. Then $\Gamma \cap \mathscr{E}_A$ is a left section in Γ, but generally not a section [1].

(c) Other examples can be found in the directed part of a semiregular tube (or coil) containing projectives. Let A be given by the quiver



bound by $\alpha\beta = 0$. The projective module P_5 lies in a tube of the form



where modules are represented by their Loewy series and one identifies along the vertical dotted lines. The full subquiver with points $\{ \substack{4\\5}, 4 \}$ is a left section, but not a section.

Lemma. *Let* Σ *be a left section in a translation quiver* Γ*. Then:*

(a) Σ *intersects at most once each* τ*-orbit in* Γ*;*

(b) *every path between two points of* Σ *is sectional;*

(c) *if* $x \in \Gamma_0$ *is injective and precedes* Σ*, then* $x \in \Sigma_0$*;*

(d) *if* $x \to y$*,* $x \in \Sigma_0$ *and* y *is non-projective, then* $y \in \Sigma_0$ *or* $\tau y \in \Sigma_0$*;*

(e) *if* $x \to y$*,* $y \in \Sigma_0$*, then* $x \in \Sigma_0$ *or* $\tau^{-1}x \in \Sigma_0$*;*

(f) *if* $x \in \Sigma_0$ *and* y *precedes* Σ*, then every path from* x *to* y *is sectional and* $y \in \Sigma_0$.

Proof. (a) Follows from the uniqueness in (LS2).

(b) Follows from (a).

(c) There exists $n \geq 0$ such that $\tau^{-n}x \in \Sigma_0$. Since x is injective, $n = 0$.

(d) Since y is non-projective, there is an arrow $\tau y \to x$. By (LS2), there exists a unique $n \geq 0$ such that $\tau^{-n}(\tau y) = \tau^{1-n}y \in \Sigma_0$. If $n > 1$, the path $x \to y \rightsquigarrow \tau^{1-n}y$ and convexity yield $y \in \Sigma_0$. Since $\tau^{1-n}y \in \Sigma_0$, this contradicts (a). Hence $n \in \{0, 1\}$, as required.

(e) This is clear if x is injective. Otherwise, $\tau^{-1}x$ is non-projective and we apply (d) to the arrow $y \to \tau^{-1}x$.

(f) Since y precedes Σ, there exists $n \geq 0$ such that $\tau^{-n}y \in \Sigma_0$. Thus, a path $x \rightsquigarrow y$ induces a path $x \rightsquigarrow y \rightsquigarrow \tau^{-n}y$. By convexity, $y \in \Sigma_0$. Hence $n = 0$ and the sectionality of the path follows from (b).

□

2.2. Lemma. *Let Σ be a left section in Γ. The full subquiver $\Gamma_{\leqslant\Sigma}$ of Γ consisting of all predecessors of Σ in Γ is isomorphic to a full translation subquiver of $\mathbb{Z}\Sigma$ (and, in particular, is acyclic).*

Proof. Repeat the proof of [25, (3.2)] (or [9, (VIII.1.5)]) with the obvious changes. □

2.3. We give necessary and sufficient conditions for a left section to be a section.

Proposition. *Let Σ be a left section in Γ. The following are equivalent:*

(a) *Σ is a section;*
(b) *every projective in Γ precedes Σ;*
(c) *for any projective $p \in \Gamma_0$ such that there exist $x \in \Sigma_0$ and a path $x \rightsquigarrow p$, we have $p \in \Sigma_0$.*

Proof. (a) implies (b). Since Σ is a section, Γ is fully embedded in $\mathbb{Z}\Sigma$, and Σ cuts each τ-orbit of Γ (see [25, (3.2)] or [9, (VII.1.5)]).

(b) implies (c). This follows from convexity.

(c) implies (a). We must show that Σ cuts each τ-orbit of Γ. For this, it suffices to prove that, if $x \in \Sigma_0$ and $z \in \Gamma_0$ are in two neighbouring orbits, then Σ cuts the τ-orbit of z (the statement then follows by induction). Assume that there exist $m \in \mathbb{Z}$ and y in the τ-orbit of x such that we have an arrow $\tau^m x \to y$ or $y \to \tau^m x$. Assume also, without loss of generality, that $|m|$ is minimal. There are three cases:

1) Suppose $m > 0$. If there is an arrow $y \to \tau^m x$, then there is a path $y \to \tau^m x \rightsquigarrow x$, so Σ cuts the τ-orbit of y. If, on the other hand, there is an arrow $\tau^m x \to y$, then there is an arrow $y \to \tau^{m-1}x$, contradicting minimality.
2) Suppose $m < 0$. If there is an arrow $y \to \tau^m x$, then there is an arrow $\tau^{m+1}x \to y$, contradicting minimality. If, on the other hand, there is an arrow $\tau^m x \to y$, then we have two cases. If y is projective, then the path $x \rightsquigarrow \tau^m x \to y$ and the hypothesis imply $y \in \Sigma_0$, hence $m = 0$, a contradiction. If y is non-projective, then there is an arrow $\tau y \to \tau^m x$, hence an arrow $\tau^{m+1}x \to \tau y$, contradicting minimality.
3) Suppose $m = 0$. If there is an arrow $y \to x$, then $y \in \Sigma_0$ or $\tau^{-1}y \in \Sigma_0$. If, on the other hand, there is an arrow $x \to y$, then we have two cases. If y is projective, then $y \in \Sigma_0$ by hypothesis. If y is non-projective, there is an arrow $\tau y \to x$ which, by (2.1)(e), yields $\tau y \in \Sigma_0$ or $y \in \Sigma_0$.

□

3. LEFT SECTIONS AND TILTED ALGEBRAS

3.1. Let A be an artin algebra and Γ be a component of $\Gamma(\mathrm{mod}\, A)$. Recall that the **annihilator** of a full subcategory $\mathscr{C}$ of $\mathrm{ind}\, A$ is defined by $\mathrm{Ann}_A\mathscr{C} = \bigcap_{X\in\mathscr{C}} \mathrm{Ann}_A X$.

Lemma. *If Σ is a finite left section of Γ, then:*

(a) $\mathrm{Ann}_A\Sigma = \mathrm{Ann}_A\Gamma_{\leqslant\Sigma}$*;*
(b) *Σ cogenerates $\Gamma_{\leqslant\Sigma}$.*

Proof. (a) Repeat the proof of [26, (2.1)], [30, (Lemma 3)], with the obvious changes.

(b) Let $X \in \Gamma_{\leqslant\Sigma}$ and $j : X \hookrightarrow I$ be an injective envelope. Since no indecomposable summand of I is a proper predecessor of Σ, then j factors through Σ.

□

3.2. The following is a "left" version of [26, (1.3)], [30, (Theorem 2)].

Proposition. *Let* Σ *be a left section of* Γ. *The following are equivalent:*

(a) $\mathrm{Hom}_A(E', \tau_A E'') = 0$ *for all* $E', E'' \in \Sigma_0$*;*
(b) $|\Sigma_0| \leq \mathrm{rk}\, K_0(A)$ *and* $\mathrm{rad}_A^\infty(E', E'') = 0$ *for all* $E', E'' \in \Sigma_0$*;*
(c) $\Gamma_{\leqslant\Sigma}$ *is generalised standard.*

Proof. (a) implies (b). The first statement follows from Skowroński's lemma [9, (VIII.5.3)], [31, (Lemma 1)], and the second from the fact that, by [9, (VIII.5.4)], any non-zero morphism in $\mathrm{rad}_A^\infty(E', E'')$ factors through $\tau_A\Sigma$.

(b) implies (c). Let $X, Y \in \Gamma_{\leqslant\Sigma}$ be such that $\mathrm{rad}_A^\infty(X, Y) \neq 0$. For each $i \geq 0$, there exists a path in $\mathrm{ind}\, A$

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} X_i \xrightarrow{g_i} Y$$

with all f_j irreducible and $g_i \in \mathrm{rad}_A^\infty(X_i, Y)$ such that $g_i f_1 \cdots f_1 \neq 0$ (see [27]). We claim that there exists i such that $X_i \in \Sigma_0$. Indeed, since Σ is finite, there exists $m \geq 0$ such that $X_m \in \Gamma_{\leqslant\Sigma}$ and $X_{m+1} \notin \Gamma_{\leqslant\Sigma}$. We show that $X_m \in \Sigma$. If X_m is injective, this follows from (2.1)(c). If not, consider the almost split sequence

$$0 \longrightarrow X_m \longrightarrow X_{m+1} \oplus Z \longrightarrow \tau_A^{-1} X_m \longrightarrow 0\ .$$

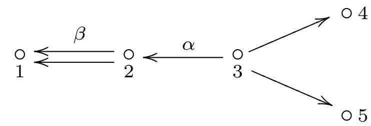
Since $X_{m+1} \notin \Gamma_{\leqslant\Sigma}$, $\tau_A^{-1} X_m \notin \Gamma_{\leqslant\Sigma}$. The conclusion follows from the fact that $X_m \in \Gamma_{\leqslant\Sigma}$, and so Σ cuts the τ_A-orbit of X_m.

We thus have $\mathrm{rad}_A^\infty(X_m, Y) \neq 0$. By (3.1)(b), Y is cogenerated by Σ. Therefore there exists $E' \in \Sigma$ such that $\mathrm{rad}_A^\infty(X_m, E') \neq 0$, a contradiction.

(c) implies (a). Suppose that there exists a non-zero morphism $E' \to \tau_A E''$, with $E', E'' \in \Sigma$. The hypothesis implies the existence of a path $E' \rightsquigarrow \tau_A E''$ in Γ, hence a path $E' \rightsquigarrow \tau_A E'' \to * \to E''$. The convexity of Σ in Γ yields $\tau_A E'' \in \Sigma$, a contradiction.

□

Remark. If Σ is a section then, by [26, (1.3)], [30, (Theorem 2)], the conditions of the proposition are equivalent to saying that $\mathrm{Hom}_A(\tau_A^{-1} E', E'') = 0$ for all $E', E'' \in \Sigma$, or to the condition that Γ be generalised standard. However, there exist left sections lying in non-generalised standard components but satisfying the condition of the proposition. Let A be given by the quiver



bound by $\alpha\beta = 0$. The component containing the projective P_3 is not generalised standard, by [10, (3.2)], but the simple modules $\{S_4, S_5\}$ form a left section in that component satisfying the conditions of the proposition.

3.3. Clearly, if Γ is generalised standard, so is $\Gamma_{\leqslant\Sigma}$, hence Σ satisfies the equivalent conditions of (3.2). We also have the following lemma.

Lemma. *Let Σ be a left section of Γ such that $\mathrm{Hom}_A(\tau_A^{-1}E', E'') = 0$ for all $E', E'' \in \Sigma$, then Σ satisfies the equivalent conditions of* (3.2)*.*

Proof. Indeed, Skowroński's lemma [9, (VIII.5.3)], [31, (Lemma 1)], ensures that $|\Sigma_0| \leq \mathrm{rk}\, K_0(A)$. Let $E', E'' \in \Sigma$. Since, by [9, (VIII.5.4)], any non-zero morphism in $\mathrm{rad}_A^{\infty}(E', E'')$ factors through $\tau_A^{-1}\Sigma$, we infer that $\mathrm{rad}_A^{\infty}(E', E'') = 0$. □

3.4. Lemma. *Let Σ be a left section of Γ satisfying the equivalent conditions of* (3.2) *and let $C = A/I$, where $I \subseteq \mathrm{Ann}_A\Sigma$. Then all indecomposables in Σ lie in the same component Γ' of $\Gamma(\mathrm{mod}\, C)$, Σ is a left section of Γ' and $\Gamma'_{\leqslant\Sigma} = \Gamma_{\leqslant\Sigma}$ is generalised standard in $\mathrm{mod}\, C$.*

Proof. Since $I \subseteq \mathrm{Ann}_A\Sigma$, then all indecomposables in Σ are C-modules. Now, recall that, if C is a quotient of A and X, Y are two indecomposable C-modules such that there is an irreducible morphism $f : X \to Y$ in $\mathrm{mod}\, A$, then f remains irreducible in $\mathrm{mod}\, C$. Therefore, all indecomposables in Σ lie on the same component Γ' of $\Gamma(\mathrm{mod}\, C)$. On the other hand, by (3.1), $\Gamma_{\leqslant\Sigma} \subseteq \mathrm{ind}\, C$ hence $\Gamma_{\leqslant\Sigma} = \Gamma'_{\leqslant\Sigma}$ and consequently Σ is a left section in Γ' as well. The last statement follows from the fact that rad_C^{∞} is contained in rad_A^{∞}. □

3.5. We recall the Liu-Skowroński criterion (see [26, (3.2)], [30, (Theorem 3)] or [9, (VIII.5.6)]. Let A be an artin algebra having a section Σ such that $\mathrm{Hom}_A(E', \tau_A E'') = 0$ for all $E', E'' \in \Sigma$, then $A/\mathrm{Ann}_A\Sigma$ is a tilted algebra having Σ as complete slice. If in particular, Σ is faithful, then A is tilted having Σ as complete slice, and the component in which Σ lies as connecting component.

Theorem. *Let Σ be a left section in a component Γ of $\Gamma(\mathrm{mod}\, A)$ such that $\mathrm{Hom}_A(\tau_A^{-1}E', E'') = 0$ for all E', E'' in Σ. Then $B = A/\mathrm{Ann}_A\Sigma$ is a tilted algebra having Σ as complete slice.*

Proof. By (3.3), Σ satisfies the conditions of (3.2). In particular, Σ is finite so we can set $E = \bigoplus_{U \in \Sigma} U$. By (3.4), all indecomposables in Σ lie in the same component Γ' of $\Gamma(\mathrm{mod}\, B)$ in which Σ is a left section such that $\Gamma'_{\leqslant\Sigma} = \Gamma_{\leqslant\Sigma}$ and moreover, by (3.2), $\mathrm{Hom}_B(E, \tau_B E) = 0$. We also have $\mathrm{Hom}_B(\tau_B^{-1}E, E) = 0$: indeed, assume to the contrary that there exist $E', E'' \in \Sigma$ and a non-zero morphism $\tau_B^{-1}E' \to E''$ then, since by [11, (p. 186–7)], there exists an epimorphism $\tau_A^{-1}E' \to \tau_B^{-1}E'$, we get upon composing a non-zero morphism $\tau_A^{-1}E' \to E''$, a contradiction. In order to complete the proof, it suffices to show that E_B is a tilting B-module with $H = \mathrm{End} E_B$ hereditary. This is done as in [9, (VIII.5.6)], but we include the proof for the benefit of the reader.

Since E_B is faithful, then $\mathrm{pd}\, E_B \leq 1$ and $\mathrm{id}\, E_B \leq 1$, by [9, (VIII.5.1)]. Moreover, $\mathrm{Ext}_B^1(E, E) = 0$, whence E is a partial tilting module. Let $f_1, \ldots, f_d$ be a generating set of the B-module $\mathrm{Hom}_B(B, E)$. Setting $f = [f_1, \ldots, f_d] : B \to E^d$, we have an exact sequence

$$0 \longrightarrow B_B \xrightarrow{f} E^d \longrightarrow X \longrightarrow 0.$$

We claim that $E \oplus X$ is a tilting B-module. Since B_B is projective, we have $\mathrm{pd}\, X \leq 1$. Applying successively $\mathrm{Hom}_B(-, E)$, $\mathrm{Hom}_B(X, -)$ and $\mathrm{Hom}_B(E, -)$ to the preceding exact sequence yields respectively $\mathrm{Ext}_B^1(X, E) = 0$, $\mathrm{Ext}_B^1(X, X) = 0$ and $\mathrm{Ext}_B^1(E, X) = 0$. This establishes our claim.

Assume now that Y is an indecomposable summand of X such that $Y \notin \operatorname{add} E$. The exact sequence above yields a non-zero morphism $E \to Y$. By [9, (VIII.5.4)], $\operatorname{Hom}_B(\tau_B^{-1}E, Y) \neq 0$, hence $\operatorname{Ext}_B^1(Y, T) \neq 0$, a contradiction. This shows that $X \in \operatorname{add} E$ and therefore E_B is a tilting module.

We now prove that H is hereditary. Let P_H be an indecomposable projective H-module and $f : M \to P$ be a monomorphism with M indecomposable. The tilting module E determines a torsion pair $\big(\mathscr{T}(E), \mathscr{F}(E)\big)$ in $\operatorname{mod} B$ and another $\big(\mathscr{X}(E), \mathscr{Y}(E)\big)$ in $\operatorname{mod} H$. Since $P \in \mathscr{Y}(E)$, then $M \in \mathscr{Y}(E)$ hence there exist $g : V \to E'$ with $V \in \mathscr{T}(E)$, $E' \in \Sigma \subseteq \mathscr{T}(E)$ such that $M \cong \operatorname{Hom}_B(E, V)$, $P \cong \operatorname{Hom}_B(E, E')$ and $f \cong \operatorname{Hom}_B(E, g)$. Now, since $M \neq 0$, there exist an indecomposable projective H-module P' and a non-zero morphism $f' : P' \to M$. Again, there exist $E'' \in \Sigma$ and $g' : E'' \to V$ such that $P' \cong \operatorname{Hom}_B(E, E'')$, $f' \cong \operatorname{Hom}_B(E, g')$. Since $ff' \neq 0$, then $gg' \neq 0$. If now $V \notin \Sigma$, then by [9, (VIII.5.4)], gg' factors through $\tau_A E$. But then $\operatorname{Hom}_H(E'', \tau_A E) \neq 0$, a contradiction which shows that $V \in \Sigma$, and thus completes the proof. □

3.6. Clearly, the Liu-Skowroński criterion follows directly from the above theorem. We also have the following easy corollary.

Corollary. *Let Σ be a left section in a generalised standard component Γ of $\Gamma(\operatorname{mod} A)$. Then $A/\operatorname{Ann}_A \Sigma$ is a tilted algebra having Σ as complete slice.*

3.7. Corollary. *An algebra A is tilted if and only if it admits a faithful left section Σ such that $\operatorname{Hom}_A(\tau_A^{-1}E', E'') = 0$ for all $E', E'' \in \Sigma$.*

3.8. Corollary. *Let Σ be a left section in a component Γ of $\Gamma(\operatorname{mod} A)$ such that every projective in Γ precedes Σ and moreover $\operatorname{Hom}_A(E', \tau_A E'') = 0$ for all $E', E'' \in \Sigma$. Then $A/\operatorname{Ann}_A \Sigma$ is a tilted algebra having Σ as complete slice and Γ as connecting component.*

Proof. By (2.3), Σ is a section. We apply the Liu-Skowroński criterion. □

4. LEFT SECTIONS CONVEX IN $\operatorname{ind} A$

4.1. A full subcategory $\mathscr{C}$ of $\operatorname{ind} A$ is called **convex in** $\operatorname{ind} A$ if, for any path

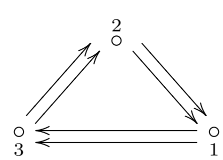
$$X = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_t = Y$$

in $\operatorname{ind} A$, with $X, Y \in \mathscr{C}$, then $X_i \in \mathscr{C}$ for all i.

Lemma. *Let Σ be a left section convex in $\operatorname{ind} A$. Then $\operatorname{Hom}_A(\tau_A^{-1}E', E'') = 0$ for all $E', E'' \in \Sigma$.*

Proof. If $E', E'' \in \Sigma$ are such that $\operatorname{Hom}_A(\tau_A^{-1}E', E'') \neq 0$, then we have a path $E' \to * \to \tau_A^{-1}E' \to E''$ in $\operatorname{ind} A$. Convexity yields $\tau_A^{-1}E' \in \Sigma$, a contradiction to (LS2). □

Remark. It is easy to find examples of left sections (even of sections) which satisfy the conditions of (3.2), but are not convex in $\operatorname{ind} A$. For instance, let A be the radical square zero algebra with quiver



and $E = I_1 \oplus S_2 \oplus P_3$. The set $\{I_1, S_2, P_3\}$ is a section in its component, and satisfies $\operatorname{Hom}_A(\tau_A^{-1}E, E') = 0$, but is not convex in $\operatorname{ind} A$.

4.2. Corollary. *Let A be an algebra having a left section convex in* $\operatorname{ind} A$. *Then* $A/\operatorname{Ann}_A\Sigma$ *is tilted having* Σ *as complete slice.*

4.3. Recall from [21] that an A-module L (not necessarily indecomposable) is called **directed** if there do not exist two indecomposable summands L', L'' of L, an indecomposable non-projective module $N$and a path in $\operatorname{ind} A$ of the form

$$L' \rightsquigarrow \tau_A N \to * \to N \rightsquigarrow L''.$$

Lemma. *Let Σ be a left section convex in* $\operatorname{ind} A$, *and* $E = \bigoplus_{U\in\Sigma} U$. *Then:*

(a) *E is a directed A-module;*
(b) *if E is sincere, then Σ is a section (and, actually, a complete slice);*
(c) *E is sincere if and only if it is faithful.*

Proof. (a) If there exist $E', E'' \in \Sigma$, an indecomposable non-projective module N and a path in $\operatorname{ind} A$ of the form

$$E' \rightsquigarrow \tau_A N \to * \to N \rightsquigarrow E'',$$

then convexity implies $N, \tau_A N \in \Sigma$, a contradiction.

(b) To show that Σ is a section, it suffices, by (2.3), to prove that, if P is a projective module such that there exist $E' \in \Sigma$ and a path of irreducible morphisms $E' \rightsquigarrow P$, then $P \in \Sigma$. Since E is sincere, there exists $E'' \in \Sigma$ such that $\operatorname{Hom}_A(P, E'') \neq 0$. Thus we have a path $E' \rightsquigarrow P \to E''$ in $\operatorname{ind} A$ and convexity forces $P \in \Sigma$. Hence Σ is a section. The second statement follows from the observation that any sincere section which is convex in $\operatorname{ind} A$ is a complete slice (see [28, 29]).

(c) This follows from (b).

□

4.4. Let Σ be a finite left section and $E = \bigoplus_{U\in\Sigma} U$. The **support** of Σ is the full subcategory $\operatorname{Supp}\Sigma = eAe$ where e is the sum of those primitive idempotents e_x of A such that $Ee_x \neq 0$.

Lemma. *Assume A is a finite dimensional algebra over an algebraically closed field, and Σ is a left section convex in* $\operatorname{ind} A$. *Then* $\operatorname{Supp}\Sigma$ *is a full convex subcategory of A.*

Proof. We slightly modify Bongartz' convexity argument [16]. If $\operatorname{Supp}\Sigma$ is not convex, there exists a path $x_0 \xrightarrow{\alpha_1} x_1 \longrightarrow \cdots \longrightarrow x_{m-1} \xrightarrow{\alpha_m} x_m$ in the quiver of A such that $m \geq 2$, $x_0, x_m \in \operatorname{Supp}\Sigma$ and $x_i \notin \operatorname{Supp}\Sigma$ for $1 \leq i < m$. Let $\alpha_1 = \beta_1, \ldots, \beta_s$ be all the arrows from x_0 to x_1 and $\alpha_m = \gamma_1, \ldots, \gamma_t$ be all those from x_{m-1} to x_m. Let J be the two-sided ideal of $\operatorname{Supp}\Sigma$ generated by all paths of the forms $\beta_i\delta$ or $\delta\gamma_j$, then consider $A' = \operatorname{Supp}\Sigma/J$. Since $EJ = 0$, then E is an A'-module. Denoting by P'_{x_0}, I'_{x_m}, respectively, the indecomposable projective A'-module at x_0 and injective A'-module at x_m, we have $\operatorname{Hom}_{A'}(P'_{x_0}, E) \neq 0$ and $\operatorname{Hom}_{A'}(E, I'_{x_m}) \neq 0$. Let $\begin{pmatrix} S_y \\ S_z \end{pmatrix}$ be the uniserial module of length two having the simple S_y as top and S_z as socle. We get $E', E'' \in \Sigma$ and a path in $\operatorname{ind} A'$ (hence in $\operatorname{ind} A$) of the form

$$E' \longrightarrow I'_{x_m} \longrightarrow S_{x_{m-1}} \longrightarrow \begin{pmatrix} S_{x_{m-2}} \\ S_{x_{m-1}} \end{pmatrix} \longrightarrow \cdots \longrightarrow \begin{pmatrix} S_{x_1} \\ S_{x_2} \end{pmatrix} \longrightarrow S_{x_1} \longrightarrow P'_{x_0} \longrightarrow E''.$$

Since Σ is convex in $\operatorname{ind} A$, we get $S_{x_i} \in \Sigma$ for all i, a contradiction. □

4.5. Theorem. *Let A be a finite dimensional algebra over an algebraically closed field, and Σ be a left section convex in $\operatorname{ind} A$. Then $\operatorname{Supp} \Sigma \cong A/\operatorname{Ann}_A \Sigma$ is a tilted algebra having Σ as complete slice and is a full convex subcategory of A.*

Proof. By (4.2), $A/\operatorname{Ann}_A\Sigma$ is tilted and has Σ as complete slice. Also, by (4.4), $\operatorname{Supp}\Sigma$ is a full convex subcategory of A. In particular, $\operatorname{Supp}\Sigma \cong A/I(\Sigma)$, where $I(\Sigma)$ is the two-sided ideal of A generated by those primitive idempotents e_x such that $Ee_x = 0$. Since $I(\Sigma) \subseteq \operatorname{Ann}_A\Sigma$, it follows from (3.4) that all points of Σ lie in the same component Γ' of $\Gamma\big(\operatorname{mod} A/I(\Sigma)\big)$ and Σ is a left section in Γ'. Further, Σ is convex in $\operatorname{ind} A/I(\Sigma)$, because it is so in $\operatorname{ind} A$. By (4.2), Σ is a complete slice in $A/I(\Sigma)$. We have established that each of $A/\operatorname{Ann}_A\Sigma$ and $\operatorname{Supp}\Sigma \cong A/I(\Sigma)$ is tilted and that these algebras have Σ as common complete slice. Therefore they are isomorphic (see, for instance, [9, (VIII.5.6)]). □

5. Subcategories closed under predecessors

5.1. Throughout this section, $\mathscr{C}$ is a full subcategory of $\operatorname{ind} A$, closed under predecessors. We first characterise the relative projectives and injectives in $\operatorname{add}\mathscr{C}$.

Lemma. *Let $P_0 \in \mathscr{C}$. The following are equivalent:*

(a) *P_0 is Ext-projective in $\operatorname{add}\mathscr{C}$ (that is, $\operatorname{Ext}^1_A(P_0, -)|_{\mathscr{C}} = 0$).*
(b) *If $0 \to L \to M \to N \to 0$ is a short exact sequence lying in $\operatorname{add}\mathscr{C}$, then the induced sequence $0 \to \operatorname{Hom}_A(P_0, L) \to \operatorname{Hom}_A(P_0, M) \to \operatorname{Hom}_A(P_0, N) \to 0$ is exact.*
(c) *Every short exact sequence of the form $0 \to L \to M \to P_0 \to 0$ splits.*
(d) *P_0 is a projective A-module.*

Proof. We prove that (c) implies (d) (the other implications are trivial). Let $f : P \to P_0$ be a projective cover in $\operatorname{mod} A$. By hypothesis, the sequence $0 \longrightarrow \operatorname{Ker} f \longrightarrow P \xrightarrow{f} P_0 \longrightarrow 0$ splits, so P_0 is projective. □

5.2. Lemma. *Let $E_0 \in \mathscr{C}$. The following are equivalent:*

(a) *E_0 is Ext-injective in $\operatorname{add}\mathscr{C}$ (that is $\operatorname{Ext}^1_A(-, E_0)|_{\mathscr{C}} = 0$).*
(b) *If $0 \to L \to M \to N \to 0$ is a short exact sequence lying in $\operatorname{add}\mathscr{C}$, then the induced sequence $0 \to \operatorname{Hom}_A(N, E_0) \to \operatorname{Hom}_A(M, E_0) \to \operatorname{Hom}_A(L, E_0) \to 0$ is exact.*
(c) *Every short exact sequence of the form $0 \to E_0 \to M \to N \to 0$ with $N \in \operatorname{add}\mathscr{C}$ splits.*
(d) $\tau_A^{-1}E_0 \notin \mathscr{C}$.

Proof. The equivalence of (a), (b), (c) is trivial and that of (a), (d) follows from [12, (3.4)]. □

5.3. While the Ext-projectives in $\operatorname{add}\mathscr{C}$ are perfectly characterised above, the same is not true of the Ext-injectives. Let $\mathscr{E}$ denote the full subcategory of $\mathscr{C}$ (hence of $\operatorname{ind} A$) with objects the Ext-injectives in $\operatorname{add}\mathscr{C}$. Clearly, any injective A-module lying in $\mathscr{C}$ belongs to $\mathscr{E}$, but the converse is generally not true. Note also that we may have $\mathscr{E} = \varnothing$ (take, for instance, A hereditary and representation-infinite, and $\mathscr{C}$ consisting of all postprojective modules).

Lemma. (a) *For every $E', E'' \in \mathscr{E}$, we have $\operatorname{Hom}_A(\tau_A^{-1}E', E'') = 0$.*
(b) $|\mathscr{E}| \leq \operatorname{rk} K_0(A)$.
(c) *$\mathscr{E}$ cuts each τ_A-orbit in $\Gamma(\operatorname{mod} A)$ at most once.*

(d) *Every path of irreducible morphisms contained in $\mathscr{E}$ is sectional.*
(e) *$\mathscr{E}$ contains no cycle of irreducible morphisms.*

Proof. (a) Indeed, $E'' \in \mathscr{C}$, while $\tau_A^{-1}E' \notin \mathscr{C}$.
(b) Follows from (a) and Skowroński's lemma [9, (VIII.5.3)], [31, (Lemma 1)].
(c) Assume, $E', \tau_A^{-t}E' \in \mathscr{E}$ (with $t > 0$). We have a path of irreducible morphisms $E' \to * \to \tau_A^{-1}E' \rightsquigarrow \tau_A^{-t}E'$. Since $\tau_A^{-t}E' \in \mathscr{C}$, we have $\tau_A^{-1}E' \in \mathscr{C}$, a contradiction.
(d) Follows from (c).
(e) Let $E_0 \to E_1 \to \cdots \to E_t = E_0$ be a cycle of irreducible morphisms in $\mathscr{E}$. By the Bautista-Smalø theorem [14, 17], it is not sectional. Hence $E_1 = \tau_A^{-1}E_{t-1}$ or there exists i with $1 \leq i < t$ such that $E_{i+1} = \tau_A^{-1}E_{i-1}$, thus contradicting (a). □

5.4. Lemma.

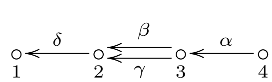
(a) *Let $E' = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t$ be a path in $\operatorname{ind} A$, with $E' \in \mathscr{E}$ and $M_t \in \mathscr{C}$. If no f_i factors through an injective module, then $M_i \in \mathscr{E}$ for all i.*
(b) *Let Γ be a component of $\Gamma(\operatorname{mod} A)$ not containing injectives and such that $\Gamma \cap \mathscr{E} \neq \varnothing$. Then, for every $M \in \mathscr{C}$, there exists a unique $m \geq 0$ such that $\tau_A^{-m}M \in \mathscr{E}$.*

Proof. (a) Since $M_t \in \mathscr{C}$, then $M_i \in \mathscr{C}$ for all i. Also, no M_i is injective and the Auslander-Reiten isomorphism $\underline{\operatorname{Hom}}_A(\tau_A^{-1}M_{i-1}, \tau_A^{-1}M_i) \cong \overline{\operatorname{Hom}}_A(M_{i-1}, M_i)$ yields a non-zero morphism $\tau_A^{-1}M_{i-1} \to \tau_A^{-1}M_i$ (for each i). This yields a path $\tau_A^{-1}E' = \tau_A^{-1}M_0 \to \tau_A^{-1}M_1 \to \cdots \to \tau_A^{-1}M_t$. Since $\tau_A^{-1}E' \notin \mathscr{C}$, then $\tau_A^{-1}M_i \notin \mathscr{C}$ for all i. Consequently, $M_i \in \mathscr{E}$ for all i.
(b) Since Γ contains no injectives (hence is right stable) but contains at least one Ext-injective in $\operatorname{add}\mathscr{C}$ (the set of which is finite, by (5.3)(b)), we may take $\ell > 0$ so that $\tau_A^{-\ell}M$ succedes $\tau_A^{-1}E' \notin \mathscr{C}$ (for every $E' \in \Gamma \cap \mathscr{E}$), then $\tau_A^{-\ell}M \notin \mathscr{C}$. Hence there exists $m \geq 0$ such that $\tau_A^{-m}M \in \mathscr{C}$ but $\tau_A^{-m-1}M \notin \mathscr{C}$. Thus $\tau_A^{-m}M \in \mathscr{E}$. Uniqueness of m follows from (5.3)(c). □

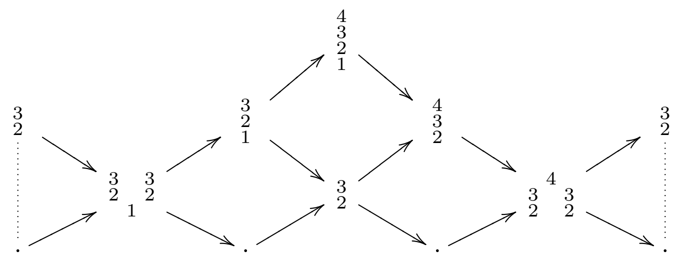
5.5. Proposition. *Let Γ be a component of $\Gamma(\operatorname{mod} A)$ not containing injectives and such that $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$. Then Σ is a left section in Γ and $A/\operatorname{Ann}_A\Sigma$ is a tilted algebra having Σ as complete slice.*

Proof. We check the axioms for a left section: (LS1) follows from (5.3)(e) and (LS3) from (5.4)(a), then let M be a predecessor of Σ in Γ, we have $M \in \Gamma \cap \mathscr{C}$ and so (LS2) follows from (5.4)(b). The statement then follows from (5.3)(a) and (3.5). □

Remark. The statement is false if Γ contains injectives. Let A be given by the quiver



bound by $\alpha\beta = 0$, $\beta\delta = 0$ and $\mathscr{C}$ be the full subcategory of $\operatorname{ind} A$ having as objects all non-preinjective modules. Then the only Ext-injective in $\operatorname{add}\mathscr{C}$ is the projective-injective module $P_4 = I_1$ which lies in a tube



Clearly, $\Gamma \cap \mathscr{E}$ satisfies neither (LS2) nor (LS3).

This raises the question of finding the right conditions so that the statement of (5.5) stays valid in a component containing injectives. As is easy to see, the condition is that every object in $\mathscr{C}$ should have projective dimension at most one (or, equivalently, $\mathscr{C} \subseteq \mathscr{L}_A$).

6. Full subcategories of $\mathscr{L}_A$

6.1. In this section, we assume that $\mathscr{C}$ is a full subcategory of $\mathscr{L}_A$, closed under predecessors. The first result generalises [3, (1.5)].

Lemma. *Let I be an indecomposable injective A-module.*

(a) *There exist at most $\operatorname{rk} K_0(A)$ indecomposable modules $N \in \mathscr{C}$ such that there exists a path $I \rightsquigarrow N$ in $\operatorname{ind} A$.*
(b) *Every such path is refinable to a path of irreducible morphisms.*
(c) *Every such path of irreducible morphisms is sectional.*
(d) *Every such module $N \in \mathscr{C}$ is Ext-injective in $\operatorname{add} \mathscr{C}$.*

Proof. If $I \notin \mathscr{C}$, then no successor of I lies in $\mathscr{C}$ and the statement holds trivially. Assume that $I \in \mathscr{C}$ has infinitely many successors in $\mathscr{C}$. Then, for each $s \geq 0$, there exists a path in $\operatorname{ind} A$

$$I = L_0 \longrightarrow L_1 \longrightarrow \cdots \longrightarrow L_{s-1} \longrightarrow L_s$$

with $L_i \in \mathscr{C}$ for all i. An easy induction (as in [3, (1.5)]) shows that this path induces another one

$$(*) \qquad I = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_i} M_i \longrightarrow L_j$$

with $j \leq i$, $M_\ell \in \mathscr{C}$ and f_ℓ irreducible for all ℓ, and $f'_i \neq 0$.

We prove that $(*)$ is sectional. If this is not the case, there exists a least i such that the subpath $I = M_0 \rightsquigarrow M_i$ is sectional and $M_{i+1} = \tau_A^{-1} M_{i-1}$. In particular, by [17], $\operatorname{Hom}_A(I, M_{i-1}) \neq 0$, hence $\operatorname{pd} \tau_A^{-1} M_{i-1} > 1$. But $M_{i+1} \in \mathscr{C}$ implies $\operatorname{pd} M_{i+1} \leq 1$, a contradiction.

Since I is injective, it is Ext-injective in $\operatorname{add} \mathscr{C}$. Moreover, $\operatorname{Hom}_A(I, M_i) \neq 0$ implies $\tau_A^{-1} M_i \notin \mathscr{C}$ (because either M_i is injective or $\operatorname{pd} \tau_A^{-1} M_i > 1$). Since $M_i \in \mathscr{C}$, we infer that M_i is Ext-injective for each i. Invoking (5.3)(b) finishes the proof. □

6.2. A module $M \in \mathscr{C}$ is called Ext-injective **of the first kind** if there exist an injective module I and a path $I \rightsquigarrow M$ in $\operatorname{ind} A$. We denote by $\mathscr{E}_1$ the class of Ext-injectives of the first kind. An Ext-injective which is not of the first kind is **of the second kind**, and the class of Ext-injectives of the second kind is denoted by $\mathscr{E}_2$ $(= \mathscr{E} \setminus \mathscr{E}_1)$. The following result generalises part of [5, (3.1)], [6, (3.1)].

Corollary. *Let $M \in \mathscr{C}$. The following are equivalent:*

(a) *M is Ext-injective of the first kind;*
(b) *there exists an injective module I and a path of irreducible morphisms $I \rightsquigarrow M$;*
(c) *there exists an injective module I and a sectional path of irreducible morphisms $I \rightsquigarrow M$;*
(d) *there exists an injective module I such that* $\operatorname{Hom}_A(I, M) \neq 0$.

Proof. That (a) implies (b) and that (b) implies (c) follow from (6.1), that (c) implies (d) follows from [17] and finally (d) implies (a) is trivial. □

Remark. If $\mathscr{C} = \mathscr{L}_A$, then Ext-injectives of the second kind are characterised as being for instance, those $M \in \mathscr{L}_A \setminus \mathscr{E}_1$ such that there exists a projective module $P \notin \mathscr{L}_A$ satisfying $\operatorname{Hom}_A(P, \tau_A^{-1} M) \neq 0$ (see [5, (3.1)], [6, (3.1)]). No such characterisation is known in general.

6.3. The next result generalises [5, (3.4)].

Proposition. *Assume that $E' \in \mathscr{E}$ and $M \in \mathscr{C}$ are such that there exists a path $E' \rightsquigarrow M$ in* $\operatorname{ind} A$. *Then this path can be refined to a sectional path of irreducible morphisms and $M \in \mathscr{E}$. In particular, $\mathscr{E}$ is convex in* $\operatorname{ind} A$.

Proof. Let $E' = X_0 \xrightarrow{f_1} X_1 \longrightarrow \cdots \xrightarrow{f_t} X_t = M$ be the given path. We first show that $X_i \in \mathscr{E}$ for each i. If no f_i factors through an injective, this follows from (5.4)(a). Otherwise, let i be minimal such that $f_i : X_{i-1} \to X_i$ factors through an injective I. We thus have a subpath $I \to X_i \rightsquigarrow X_t = M$. By (6.1), $X_j \in \mathscr{E}$ for $j \geq i$. But, on the other hand, we have another subpath $E' = X_0 \rightsquigarrow X_{i-1}$ where, because of the minimality of i, none of the morphisms factors through an injective. Since, for every $j < i$, we have $X_j \in \mathscr{C}$, we deduce from (5.4)(a) that $X_j \in \mathscr{E}$.

This establishes the convexity of $\mathscr{E}$ in $\operatorname{ind} A$. There remains to show that each f_i lies in a finite power of the radical of $\operatorname{mod} A$. Indeed, if this is not the case for some f_i, then, for every $s \geq 1$, the given path has a refinement

$$E' = X_0 \rightsquigarrow X_{i-1} = Y_0 \to Y_1 \to \cdots \to Y_s = X_i \rightsquigarrow X_t = M.$$

By convexity in $\operatorname{ind} A$, we have $Y_\ell \in \mathscr{E}$ for each ℓ. This contradicts (5.3)(b). □

6.4. Corollary. *The modules in $\mathscr{E}$ are directed in* $\operatorname{ind} A$.

Proof. Let $E' = M_0 \to M_1 \to \cdots \to M_t = E'$ be a cycle with $E' \in \mathscr{E}$. By (6.3), it can be refined to a cycle of irreducible morphisms contained in $\mathscr{E}$, contradicting (5.3)(e). □

6.5. Corollary. *Let Γ be a component of* $\Gamma(\operatorname{mod} A)$ *containing an injective. Then every module of $\Gamma \cap \mathscr{C}$ is directed.*

Proof. Let $M_0 \to M_1 \to \cdots \to M_t = M_0$ be a cycle with $M_0 \in \Gamma \cap \mathscr{C}$. By [3, (1.4)], there exist an injective module $I \in \Gamma$ and a path $I \rightsquigarrow M_0$. We compose this path with two copies of the cycle to get a longer path from I to M_0 which, by (6.3), is refinable to a sectional path of irreducible morphisms, yielding a contradiction to [14]. □

6.6. Proposition. *Let Γ be a component of* $\Gamma(\operatorname{mod} A)$ *such that $\Gamma \cap \mathscr{E} \neq \varnothing$. Then:*

(a) *for every $M \in \Gamma \cap \mathscr{C}$, there exists a unique $m \geq 0$ such that $\tau_A^{-m} M \in \mathscr{E}$;*
(b) *the number of τ_A-orbit of $\Gamma \cap \mathscr{C}$ equals $|\Gamma \cap \mathscr{E}|$ (hence is finite);*
(c) *$\Gamma \cap \mathscr{C}$ contains no modules on a cycle between modules in Γ.*

Proof. (a) By (5.4), we may assume that Γ contains an injective. Suppose that, for any $\ell \geq 0$, we have $\tau_A^{-\ell} M \in \mathscr{C}$. Then M is right stable (and not periodic, by (6.5)). Since Γ contains an injective, there is a walk from this injective to the τ_A-orbit of M. Among all such injectives, choose one, say I, such that there is a walk of least length from I to the τ_A-orbit of M. Minimality implies that all modules on this walk, except I, are right stable. Hence there exist $s \geq 0$ and a path of irreducible morphisms $I \rightsquigarrow \tau_A^{-s} M$. Since $I \in \mathscr{E}$, we have $\tau_A^{-s} M \in \mathscr{E}$ by (6.3), hence $\tau_A^{-s-1} M \notin \mathscr{C}$, a contradiction. Thus there exists $m \geq 0$ such that $\tau_A^{-m} M \in \mathscr{C}$ but $\tau_A^{-m-1} M \notin \mathscr{C}$, so $\tau_A^{-m} M \in \mathscr{E}$. Uniqueness of m follows from (5.3).

(b) Let n be the number of τ_A-orbits of $\Gamma \cap \mathscr{C}$. By (a), $n \leq |\Gamma \cap \mathscr{E}| \leq \operatorname{rk} K_0(A)$. The statement follows because each element of $\Gamma \cap \mathscr{E}$ lies in exactly one τ_A-orbit of $\Gamma \cap \mathscr{C}$.

(c) By (6.5), we may assume that Γ contains no injectives. Let

$$M_0 \to M_1 \to \cdots \to M_t = M_0$$

be a cycle with $M_0 \in \Gamma \cap \mathscr{C}$ and all $M_i \in \Gamma$. Clearly, $M_i \in \mathscr{C}$ for all i. By (6.4), $M_i \notin \mathscr{E}$ for all i, and also no f_i factors through an injective. Thus this cycle induces a new one

$$\tau_A^{-1} M_0 \to \tau_A^{-1} M_1 \to \cdots \to \tau_A^{-1} M_t = \tau_A^{-1} M_0$$

with $\tau_A^{-1} M_i \in \Gamma \cap \mathscr{C}$ for all i. Repeating this procedure indefinitely, we get that, for all $m \geq 0$, the module $\tau_A^{-m} M_0$ lies in $\Gamma \cap \mathscr{C}$ and in a cycle. This contradicts (a).

□

6.7. Theorem. *Let A be an artin algebra, and $\mathscr{C} \subseteq \mathscr{L}_A$ be a full subcategory closed under predecessors, having $\mathscr{E}$ as subcategory of Ext-injectives. Let Γ be a component of $\Gamma(\operatorname{mod} A)$. Then:*

(a) *If $\Gamma \cap \mathscr{E} = \varnothing$, then either $\Gamma \subseteq \mathscr{C}$ or $\Gamma \cap \mathscr{C} = \varnothing$.*

(b) *If $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$, then Σ is a left section of Γ, convex in $\operatorname{ind} A$. Moreover $A/\operatorname{Ann}_A \Sigma$ is tilted having Σ as complete section.*

Proof. (a) Assume $\Gamma \cap \mathscr{E} = \varnothing$. If Γ contains a module in $\mathscr{C}$ and one not in $\mathscr{C}$, then there exists an irreducible morphism $X \to Y$ with $X \in \Gamma \cap \mathscr{C}$ and $Y \in \Gamma \setminus \mathscr{C}$. Since $\Gamma \cap \mathscr{E} = \varnothing$, then X is not injective, so we have an irreducible morphism $Y \to \tau_A^{-1} X$. Since $Y \notin \mathscr{C}$, then $\tau_A^{-1} X \notin \mathscr{C}$. Thus $X \in \mathscr{E}$, a contradiction.

(b) If Γ contains no injective, then Σ is a left section by (5.5) and is convex in $\operatorname{ind} A$ by (6.3). If Γ contains an injective, then (LS1) follows from (5.3)(e), convexity in $\operatorname{ind} A$ (hence (LS3)) follows from (6.3) and finally (LS2) follows easily from (6.6)(a) (for, if M precedes Σ in Γ, then $M \in \Gamma \cap \mathscr{C}$).

The last statement follows from (4.2).

□

6.8. Corollary. *Let A be a finite dimensional algebra over an algebraically closed field, $\mathscr{C} \subseteq \mathscr{L}_A$ be a full subcategory closed under predecessors, having $\mathscr{E}$ as subcategory of Ext-injectives and Γ be a component of $\Gamma(\operatorname{mod} A)$ such that $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$. Then $\operatorname{Supp} \Sigma \cong A/\operatorname{Ann}_A \Sigma$ is a tilted algebra having Σ as complete slice and is a full convex subcategory of A closed under successors.*

Proof. By (4.5) and (6.7), it suffices to prove that $\operatorname{Supp} \Sigma$ is closed under successors. Let $x \to y$ be an arrow with $x \in \operatorname{Supp} \Sigma$. Then we have a non-zero morphism $P_y \to P_x$. Also, there exists an embedding $P_x \hookrightarrow \bar{E}$, with $\bar{E} \in \operatorname{add}\Sigma$. This yields a non-zero morphism $P_y \to E'$ for some $E' \in \Sigma$. Thus $y \in \operatorname{Supp}\Sigma$. □

6.9. The next result generalises [1, (Theorem A)].

Corollary. *Let A be an artin algebra, and Γ be a component of $\Gamma(\operatorname{mod} A)$ which intersects the class $\mathscr{E}_A$ of Ext-injective indecomposables in $\operatorname{add}\mathscr{L}_A$. Then:*

(a) *each τ_A-orbit of $\Gamma \cap \mathscr{L}_A$ intersects $\mathscr{E}_A$ exactly once;*
(b) *the number of τ_A-orbits of $\Gamma \cap \mathscr{L}_A$ equals $|\Gamma \cap \mathscr{E}_A|$;*
(c) *$\Gamma \cap \mathscr{L}_A$ contains no modules lying on a cycle between modules in Γ;*
(d) *$A/\operatorname{Ann}_A(\Gamma \cap \mathscr{E}_A)$ is a tilted algebra having $\Gamma \cap \mathscr{E}_A$ as complete slice.*

If, on the other hand, $\Gamma \cap \mathscr{E}_A = \varnothing$ then either $\Gamma \subseteq \mathscr{L}_A$ or $\Gamma \cap \mathscr{L}_A = \varnothing$.

Proof. This follows from (6.6) and (6.7). □

6.10. The following corollary generalises [1, (Theorem B)].

Corollary. *With the notation of* (6.7)*, if $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$ and all projectives in Γ belong to $\mathscr{C}$, then:*

(a) *Σ is a section in Γ;*
(b) *Γ is generalised standard;*
(c) *$A/\operatorname{Ann}_A\Sigma$ is a tilted algebra having Γ as connecting component and Σ as complete slice.*

Proof. By (6.7) and (2.3), Σ is a section in Γ. The rest follows. □

7. THE SUPPORT ALGEBRA

7.1. Definition. *Let $\mathscr{C}$ be a full subcategory of $\mathscr{L}_A$, closed under predecessors. Its **support algebra** $A(\mathscr{C})$ is the endomorphism algebra of the direct sum of all indecomposable projectives lying in $\mathscr{C}$ (that is, $A(\mathscr{C}) = \operatorname{End}\left(\bigoplus_{P_x \in \mathscr{C}} P_x\right)$).*

Clearly, this generalises the left support of an artin algebra [5, (2.2)], [32, (3.1)]. Note that $A(\mathscr{C})$ is a full subcategory of A, closed under successors (and hence convex).

Lemma. *The support algebra $A(\mathscr{C})$ is a direct product of connected quasi-tilted algebras.*

Proof. By the above remark, A may be written in matrix form

$$\begin{bmatrix} A(\mathscr{C}) & 0 \\ M & B \end{bmatrix}$$

where M is a B-$A(\mathscr{C})$-bimodule. Moreover, $\mathscr{C} \subseteq \operatorname{ind} A(\mathscr{C})$ for, if $L \in \mathscr{C}$ and P_x is an indecomposable projective such that $\operatorname{Hom}_A(P_x, L) \neq 0$, then $P_x \in \mathscr{C}$. By [5, (2.1)], $\mathscr{L}_A \subseteq \mathscr{L}_{A(\mathscr{C})}$. Hence $\mathscr{C} \subseteq \mathscr{L}_{A(\mathscr{C})}$. The statement follows because any indecomposable projective $A(\mathscr{C})$-module (= projective A-module lying in $\mathscr{C}$) belongs to $\mathscr{L}_{A(\mathscr{C})}$. □

7.2. Let $\mathscr{E}$ denote the full subcategory of $\mathscr{C}$ consisting of the Ext-injectives in $\operatorname{add}\mathscr{C}$ and $E = \bigoplus_{U \in \mathscr{E}} U$. Denote by F the direct sum of all indecomposable projectives which are not (!) in $\mathscr{C}$, and set $T = E \oplus F$. We call a partial tilting A-module **convex** if the class of its indecomposable summands is convex in $\operatorname{ind} A$, see [7].

Lemma. (a) *E is a convex partial tilting A-module.*
(b) *E is a convex partial tilting $A(\mathscr{C})$-module. In particular, $|\mathscr{E}| \leq \operatorname{rk} K_0(A(\mathscr{C}))$.*
(c) *T is a partial tilting module.*

(d) *T is a tilting A-module if and only if $|\mathscr{E}|$ equals the number of projectives in $\mathscr{C}$ or if and only if E is a tilting $A(\mathscr{C})$-module.*

(e) *If T is a tilting A-module, then the associated torsion pair $\big(\mathscr{T}(T), \mathscr{F}(T)\big)$ is given by $\mathscr{F}(T) = \operatorname{add}(\mathscr{C} \setminus \mathscr{E})$ and $\mathscr{T}(T) = \operatorname{add}(\operatorname{ind} A \setminus \mathscr{F}(T))$.*

Proof. (a) Since $\mathscr{C} \subseteq \mathscr{L}_A$, then $\operatorname{pd} E \leq 1$. Clearly, $\operatorname{Ext}^1_A(E, E) = 0$ so E is partial tilting. Its convexity follows from (6.3).

(b) Since $A(\mathscr{C})$ is a full convex subcategory of A, and $\mathscr{C} \subseteq \mathscr{L}_A$, then E is a partial tilting $A(\mathscr{C})$-module. It is convex because any path in $\operatorname{ind} A(\mathscr{C})$ induces one in $\operatorname{ind} A$.

(c) Since $\operatorname{pd} T \leq 1$, it suffices to observe that $\operatorname{Ext}^1_A(E, F) \cong \operatorname{DHom}_A(F, \tau_A E) = 0$, because $\tau_A E \in \operatorname{add} \mathscr{C}$ while no indecomposable summand of F lies in $\mathscr{C}$.

(d) This is clear.

(e) Assume $M \in \mathscr{C} \setminus \mathscr{E}$. If $M \notin \mathscr{F}(T)$, then $\operatorname{Hom}_A(T, M) \neq 0$. Since no summand of F lies in $\mathscr{C}$, $\operatorname{Hom}_A(F, M) = 0$. Hence there exists $E_0 \in \mathscr{E}$ such that $\operatorname{Hom}_A(E_0, M) \neq 0$. By (6.3), $M \in \mathscr{E}$, a contradiction. Thus $\mathscr{C} \setminus \mathscr{E} \subseteq \mathscr{F}(T)$. Conversely, let $M \in \mathscr{F}(T) = \operatorname{Cogen} \tau_A T$. There exist $E' \in \mathscr{E}$ and a path $M \to \tau_A E' \to * \to E'$. In particular, $M \in \mathscr{E}$. On the other hand, $M \notin \mathscr{E}$ since $\mathscr{E} \subseteq \operatorname{add} T$. This shows the first statement. The second follows.

□

7.3. Lemma. *If an indecomposable injective $A(\mathscr{C})$-module I precedes $\mathscr{E}$, then $I \in \mathscr{E}$.*

Proof. This is clear if I is injective in $\operatorname{mod} A$. Assume it is not. Since I precedes $\mathscr{E}$, then $I \in \mathscr{C}$. But then $\tau_A^{-1} I \notin \mathscr{C}$ (for, otherwise, $\tau_A^{-1} I = \tau_{A(\mathscr{C})}^{-1} I$, a contradiction to the injectivity of I in $\operatorname{mod} A(\mathscr{C})$). Therefore $I \in \mathscr{E}$. □

7.4. The following result generalises [1, (Theorem C)].

Theorem. *Let A be an artin algebra and $\mathscr{C} \subseteq \mathscr{L}_A$ be a full convex subcategory closed under predecessors, having $\mathscr{E}$ as subcategory of Ext-injectives. Let Γ be a component of $\Gamma(\operatorname{mod} A(\mathscr{C}))$ such that $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$. Then:*

(a) *Σ is a section in Γ, convex in $\operatorname{ind} A(\mathscr{C})$;*

(b) *Γ is directed, and generalised standard;*

(c) *$A(\mathscr{C})/\operatorname{Ann}_A \Sigma$ is a tilted algebra having Γ as connecting component and Σ as complete slice.*

Proof. (a) First, we show that there exists a unique component Γ' of $\Gamma(\operatorname{mod} A)$ which contains Σ. Let indeed $E_1 \to E_2$ be an irreducible morphism in $\operatorname{mod} A(\mathscr{C})$ with $E_1, E_2 \in \Sigma$. By (6.3), it induces a path of irreducible morphisms $E_1 \rightsquigarrow E_2$ in $\operatorname{mod} A$. In particular, E_1 and E_2 lie in the same component.

We next show that $\Gamma'_{\leqslant \Sigma} = \Gamma_{\leqslant \Sigma}$. Indeed, if $X \to Y$ is irreducible in $\Gamma'_{\leqslant \Sigma}$, it stays so in $\Gamma_{\leqslant \Sigma}$. Conversely, let $X \to Y$ be irreducible in $\Gamma_{\leqslant \Sigma}$. If $X \notin \mathscr{E}$, then X is not injective and $\tau_A^{-1} X = \tau_{A(\mathscr{C})}^{-1} X$. Therefore the almost split sequence starting with X is the same in both categories, and $X \to Y$ is irreducible in $\Gamma'_{\leqslant \Sigma}$. If $X \in \mathscr{E}$ then, by (6.3), $Y \in \mathscr{E}$. If Y is not projective, the almost split sequence ending with Y is the same in both categories. If Y is projective, then $\operatorname{rad} Y$ is the same in both categories. In any case, $X \to Y$ remains irreducible in $\Gamma'_{\leqslant \Sigma}$. This establishes the claim.

By (6.7), $\Sigma = \Gamma \cap \mathscr{E} = \Gamma' \cap \mathscr{E}$ is a left section and, by (7.2)(b), is convex in $\operatorname{ind} A(\mathscr{C})$. In order to show that Σ is a section, consider, according to (2.3), a projective P such that there exist $E' \in \Sigma$ and a path $E' \rightsquigarrow P$ in Γ. Since P is a projective $A(\mathscr{C})$-module, it lies in $\mathscr{C}$. But then (6.3) yields $P \in \mathscr{E}$.

(b) Since Γ contains a section, it is directed by [25, (3.2)], [9, (VIII.1.5)]. It is generalised standard because so are the directed Auslander-Reiten components of a quasi-tilted algebra [18].

(c) Follows from (a) and the Liu-Skowroński criterion.

□

7.5. The following result generalises [1, (4.6)].

Theorem. *Let A be an artin algebra and $\mathscr{C} \subseteq \mathscr{L}_A$ be a full convex subcategory closed under predecessors, having $\mathscr{E}$ as subcategory of Ext-injectives. Let B be a connected component of $A(\mathscr{C})$ such that $\operatorname{mod} B \cap \mathscr{E} \neq \varnothing$ and Γ be a component of $\Gamma(\operatorname{mod} B)$ such that $\Sigma = \Gamma \cap \mathscr{E} \neq \varnothing$. Then B is tilted having Γ as connecting component and Σ as complete slice.*

If A is a finite dimensional algebra over an algebraically closed field, then $B \cong \operatorname{Supp} \Sigma$.

Proof. In order to show that Γ is a connecting component, we start by assuming that Γ is postprojective non-connecting. We claim that there exists an indecomposable projective B-module not in Γ. Otherwise, indeed, the number of τ_B-orbits in Γ equals $\operatorname{rk} K_0(B)$, so Γ is connecting, a contradiction which establishes the claim.

Suppose $Q \notin \Gamma$ is indecomposable projective. There exists a walk of indecomposable projective B-modules $P_0 — P_1 — \cdots — P_s \cong Q$ with $P_0 \in \Gamma$ and therefore there exist $P_i \in \Gamma$ and $P_{i+1} \notin \Gamma$ such that $\operatorname{Hom}_B(P_i, P_{i+1}) \neq 0$. By [27], there exists, for each $t > 0$, a path

$$P_i = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t \xrightarrow{f} P_{i+1}$$

with all M_i indecomposable, all f_i irreducible and $f f_t \cdots f_1 \neq 0$. Since t is arbitrary, we may assume M_t successor of $\tau_B^{-1}\Sigma$. But then $P_{i+1} \notin \mathscr{C}$, a contradiction.

Assume now that Γ is preinjective non-connecting. The same argument yields an $M \in \Gamma$, proper predecessor of Σ, and an indecomposable injective $I \notin \Gamma$ such that $\operatorname{Hom}_B(I, M) \neq 0$. Since I precedes Σ, then $I \in \mathscr{E}$. By (6.3), $M \in \mathscr{E}$, a contradiction.

Since, by (7.4), Γ is directed, it is connecting by [18]. Moreover, again by (7.4), Σ is a section in Γ, convex in $\operatorname{ind} B$. By the Liu-Skowroński criterion, B is tilted having Γ as connecting component and Σ as complete slice. The final statement is clear. □

7.6. The following corollary generalises [5, (5.4)], [1, (2.6)].

Corollary. *Let A be representation-infinite and $\mathscr{C} \subseteq \mathscr{L}_A$ be a full subcategory closed under predecessors. The following are equivalent:*

(a) *there exists a component $\Gamma \subseteq \mathscr{C}$;*
(b) *$\mathscr{C}$ is infinite;*
(c) *$\mathscr{C}$ contains a postprojective component without injectives.*

If, moreover, A is not hereditary, then such a component Γ is postprojective or regular, or obtained from a stable tube or a component of type $\mathbb{Z}\mathbb{A}_\infty$ by ray extensions.

Proof. (a) implies (b). Indeed, otherwise A would be representation-finite, a contradiction.

(b) implies (c). Since $\mathscr{C}$ is infinite, there exists a connected component B of $A(\mathscr{C})$ such that $\operatorname{mod} B \cap \mathscr{C}$ is infinite. Also, B is quasi-tilted, by (7.1). Let Γ be a postprojective component of $\Gamma(\operatorname{mod} B)$. Suppose Γ contains an injective. Then Γ is connecting, it is the only postprojective component and $\Gamma \cap \mathscr{C}$ is finite. Since $\mathscr{C}$ is infinite, there exists $X \in \mathscr{C} \setminus \Gamma$ and then one can easily find a morphism from a module in $\Gamma \setminus \mathscr{C}$ to X, a contradiction. Therefore Γ has no injectives. Now if $\Gamma \nsubseteq \mathscr{C}$, the existence of an $M \in \Gamma \cap \mathscr{C}$ and an $N \in \Gamma \setminus \mathscr{C}$ implies the existence of an Ext-injective in Γ. Again, we get that $\mathscr{C}$ is finite, a contradiction. Hence $\Gamma \subseteq \mathscr{C}$.

Since (c) implies (a) trivially, and the last statement follows from the description of the components of quasi-tilted algebras, [23, 24, 18], the proof is complete. □

8. ALGEBRAS SUPPORTED BY SUBCATEGORIES

8.1. Definition. *Let $\mathscr{C}$ be a full subcategory of* $\operatorname{ind} A$*, closed under predecessors. The algebra A is called* ***$\mathscr{C}$-supported*** *if* $\operatorname{add}\mathscr{C}$ *has enough Ext-injectives (that is, if* $\operatorname{add}\mathscr{C} = \operatorname{Cogen} E$*, where $E = \bigoplus_{U \in \mathscr{E}} U$).*

This generalises the left supported algebras of [5] which, in this terminology, are $\mathscr{L}_A$-supported.

Lemma. *An algebra A is $\mathscr{C}$-supported if and only if* $\operatorname{add}\mathscr{C}$ *is contravariantly finite in* $\operatorname{mod} A$.

Proof. Assume A is $\mathscr{C}$-supported. Since $\operatorname{add}\mathscr{C}$ is a torsion-free class, then by [33], it is contravariantly finite. Conversely, if $\operatorname{add}\mathscr{C}$ is contravariantly finite, then by [33], there exists an Ext-injective $N \in \operatorname{add}\mathscr{C}$ such that $\operatorname{add}\mathscr{C} = \operatorname{Cogen} N$. Since $N \in \operatorname{add} E$, we have

$$\operatorname{add}\mathscr{C} = \operatorname{Cogen} N \subseteq \operatorname{Cogen} E \subseteq \operatorname{add}\mathscr{C}$$

and equality follows. □

Remarks. (a) If A is representation-finite, then A is $\mathscr{C}$-supported for any full subcategory $\mathscr{C}$ of $\operatorname{ind} A$.

(b) In general, this property depends on the chosen subcategory: let A be tame hereditary, and $\mathscr{C}$ consist of all postprojective modules, then A is not $\mathscr{C}$-supported. If, on the other hand, $\mathscr{C}'$ is a finite subcategory of $\operatorname{ind} A$ consisting of postprojective modules, then A is $\mathscr{C}'$-supported.

8.2. From now on, we assume again $\mathscr{C} \subseteq \mathscr{L}_A$. We denote by $\operatorname{Pred} E$ the full subcategory of $\operatorname{ind} A$ consisting of those X such that there exist $E' \in \mathscr{E}$ and a path $X \rightsquigarrow E'$ in $\operatorname{ind} A$. Also, an A-module L is called **almost directed** [2, (2.2)] if there do not exist two indecomposable summands L', L'' of L and a path $L' \rightsquigarrow \tau_A L''$ in $\operatorname{ind} A$. The dual notion is that of **almost codirected** module.

Theorem. *Let A be an artin algebra, and $\mathscr{C}$ be a full subcategory of $\mathscr{L}_A$ closed under predecessors. The following are equivalent:*

(a) *A is $\mathscr{C}$-supported;*
(b) $\operatorname{add}\mathscr{C}$ *is contravariantly finite;*
(c) *$T = E \oplus F$ is a tilting A-module;*
(d) *$|\mathscr{E}|$ equals the number of projectives in $\mathscr{C}$;*
(e) *E is a tilting $A(\mathscr{C})$-module;*
(f) *E is a cotilting $A(\mathscr{C})$-module;*
(g) $\mathscr{C} = \operatorname{Supp}(-, E)$*;*

(h) *there exists an almost codirected module* L_A *such that* $\mathscr{C} = \operatorname{Supp}(-, L)$*;*
(i) *there exists a module* L_A *such that* $\operatorname{Hom}_A(\tau_A^{-1}L, L) = 0$ *and* $\mathscr{C} = \operatorname{Supp}(-, L)$*;*
(j) $\mathscr{C} = \operatorname{Pred} E$*;*
(k) E *is a sincere* $A(\mathscr{C})$*-module;*
(l) *for every connected component* B *of* $A(\mathscr{C})$*, we have* $\operatorname{mod} B \cap \mathscr{E} \neq \varnothing$*;*
(m) *every connected component* B *of* $A(\mathscr{C})$ *is tilted and has* $\operatorname{mod} B \cap \mathscr{E}$ *as complete slice;*
(n) *every projective* A*-module in* $\mathscr{C}$ *precedes* $\mathscr{E}$.

Proof. The proofs of [2, (Theorem A)], [1, (2.1)], [5, (Theorems A,B)] carry over with the obvious changes. □

8.3. Corollary. *If* A *is* $\mathscr{C}$*-supported, then:*
(a) *the* $A(\mathscr{C})$*-modules not in* $\mathscr{F}(T)$ *are those of* $\operatorname{Gen} E$*;*
(b) F *is the Bongartz complement of* E.

Proof. The proof of [5, (5.3)] applies with the obvious changes. □

8.4. The following is a new characterisation of supported algebras.

Corollary. *An algebra* A *is* $\mathscr{C}$*-supported if and only if every morphism* $f : L \to M$ *with* $L \in \mathscr{C}$ *and* $M \notin \mathscr{C}$ *factors through* $\operatorname{add} E$.

Proof. Necessity. If A is $\mathscr{C}$-supported then, by (8.3), $T = E \oplus F$ is a tilting module. Since $M \notin \mathscr{C}$, then $M \in \mathscr{T}(T)$, by (7.2). Let $\{g_1, \ldots, g_d\}$ be a generating set of the $\operatorname{End}T$-module $\operatorname{Hom}_A(T, M)$, then $g = [g_1, \ldots, g_d] : T^d \to M$ is surjective and $K = \operatorname{Ker} g$ belongs to $\mathscr{T}(T)$ (see [9, (VI.2.5)]). Applying $\operatorname{Hom}_A(L, -)$ to the exact sequence

$$0 \longrightarrow K \longrightarrow T^d \xrightarrow{g} M \longrightarrow 0$$

yields an exact sequence.

$$\operatorname{Hom}_A(L, T^d) \longrightarrow \operatorname{Hom}_A(L, M) \longrightarrow \operatorname{Ext}^1_A(L, K).$$

Now $\operatorname{Ext}^1_A(L, K) \cong \mathrm{D}\overline{\operatorname{Hom}}_A(K, \tau_A L) = 0$ because $K \in \mathscr{T}(T)$ while $\tau_A L \in \mathscr{C} \subseteq \mathscr{F}(T)$. Hence $\operatorname{Hom}_A(L, g)$ is surjective, so f factors through $\operatorname{add} T$. Since $\operatorname{Hom}_A(L, F) = 0$, we have that, in fact, f factors through $\operatorname{add} E$.

Sufficiency. The inclusion of any $L \in \mathscr{C}$ into its injective envelope factors through $\operatorname{add} E$. Therefore $\operatorname{add} \mathscr{C} = \operatorname{Cogen} E$. □

8.5. A particular case of $\mathscr{L}_A$-supported algebras was studied in [8]. Recall that a full subcategory $\mathscr{C}$ of $\operatorname{ind} A$, closed under predecessors, is **abelian exact** if $\operatorname{add} \mathscr{C}$ is abelian and the inclusion $\operatorname{add} \mathscr{C} \hookrightarrow \operatorname{mod} A$ is an exact functor.

Corollary. *Let* A *and* $\mathscr{C}$ *be such that* $\mathscr{C} \subseteq \mathscr{L}_A$ *is closed under predecessors and abelian exact.*
(a) $A \cong \begin{bmatrix} A(\mathscr{C}) & 0 \\ X & B \end{bmatrix}$, *with* $A(\mathscr{C})$ *hereditary and* $X_{A(\mathscr{C})}$ *injective.*
(b) A *is* $\mathscr{C}$*-supported.*
(c) *If* A *is triangular, then* $A = A(\mathscr{C})$ *(in particular, is hereditary).*

Proof. (a) By [8, (2.5)], $A \cong \begin{bmatrix} C & 0 \\ X & B \end{bmatrix}$ with C hereditary, X_C injective and $\operatorname{add} \mathscr{C} \cong \operatorname{mod} C$. Therefore $C = A(\mathscr{C})$.

(b) Since $\operatorname{mod} A(\mathscr{C})$ is cogenerated by its minimal injective cogenerator, it follows from [33] that $\operatorname{add}\mathscr{C}$ is contravariantly finite.
(c) This follows from [8, (3.2)].

□

8.6. Denote by $\mathscr{C}^c = \operatorname{ind} A \setminus \mathscr{C}$ the complement of $\mathscr{C}$ in $\operatorname{ind} A$ and by $\mathscr{E}_1$ the subcategory of Ext-injectives of the first kind (see (6.2)). We define the **almost complement** of $\mathscr{C}$ to be the full subcategory $\mathscr{C}^* = \mathscr{C}^c \cup \mathscr{E}_1$ of $\operatorname{ind} A$. If, for instance, $\mathscr{C} = \mathscr{L}_A$, then $\mathscr{C}^*$ is the class $\mathscr{R}_0$ consisting of all $M \in \operatorname{ind} A$ such that there exist an injective module I and a path $I \rightsquigarrow M$ in $\operatorname{ind} A$ (see [2, (5.1)]).

Lemma. *$\mathscr{C}^*$ is closed under successors.*

Proof. Assume $X \in \mathscr{C}^*$ and we have a path $X \rightsquigarrow Y$ in $\operatorname{ind} A$. If $X \in \mathscr{C}^c$, then $Y \in \operatorname{ind} A$. If $X \in \mathscr{E}_1$, then there is an injective I and a path $I \rightsquigarrow X$. If $Y \in \mathscr{C}^c$ there is nothing to show while, if $Y \in \mathscr{C}$, the composed path $I \rightsquigarrow X \rightsquigarrow Y$ yields $Y \in \mathscr{E}_1$, by (6.1). □

8.7. We set $E_1 = \bigoplus_{X \in \mathscr{E}_1} X$, $E_2 = \bigoplus_{Y \in \mathscr{E}_2} Y$ and $U = E_1 \oplus \tau_A^{-1} E_2 \oplus F$ (note that, by definition, no summand of E_2 is injective).

Lemma. (a) *Let $M \in \operatorname{ind} A$, then:*
(i) *M is Ext-projective in $\operatorname{add}\mathscr{C}^*$ if and only if $M \in \operatorname{add} U$;*
(ii) *M is Ext-injective in $\operatorname{add}\mathscr{C}^*$ if and only if M is injective.*
(b) *U is a partial tilting module.*
(c) *U is a tilting module if and only if $T = E_1 \oplus E_2 \oplus F$ is a tilting module.*
(d) *If U is a tilting module, then the resulting torsion pair $\big(\mathscr{T}(U), \mathscr{F}(U)\big)$ is given by $\mathscr{T}(U) = \operatorname{add}\mathscr{C}^*$ and $\mathscr{F}(U) = \operatorname{add}(\operatorname{ind} A \setminus \mathscr{C}^*)$.*

Proof. The proofs of [2, (5.3)(5.4)(5.5)] apply with the obvious changes. □

8.8. We now generalise [2, (Theorem B)]. For a functor F on $\operatorname{mod} A$, we denote by $\operatorname{Ker} F$ the full subcategory of $\operatorname{mod} A$ consisting of the modules M such that $FM = 0$.

Theorem. *Let A be an artin algebra and $\mathscr{C} \subseteq \mathscr{L}_A$ be a full subcategory closed under predecessors. The following are equivalent:*

(a) *A is $\mathscr{C}$-supported;*
(b) *$\operatorname{add}\mathscr{C}^*$ is covariantly finite;*
(c) *$\operatorname{add}\mathscr{C}^* = \operatorname{Gen} U$;*
(d) *U is a tilting A-module;*
(e) *$\mathscr{C}^* = \operatorname{Supp}(U, -)$;*
(f) *there exists an almost directed module R_A such that $\mathscr{C}^* = \operatorname{Supp}(R, -)$;*
(g) *there exists a module R_A such that $\operatorname{Hom}_A(R, \tau_A R) = 0$ and $\mathscr{C}^* = \operatorname{Supp}(R, -)$;*
(h) *$\operatorname{add}\mathscr{C}^* = \operatorname{Ker}\operatorname{Ext}^1_A(U, -)$;*
(i) *$\operatorname{Ker}\operatorname{Hom}_A(U, -) = \operatorname{add}(\mathscr{C} \setminus \mathscr{E}_1)$.*

Proof. The proof of [2, (5.6)] applies with the obvious changes. □

8.9. Corollary. *Let $\mathscr{C} \subseteq \mathscr{L}_A$ be a full subcategory closed under predecessors, then $\operatorname{add}\mathscr{C}$ is contravariantly finite if and only if $\operatorname{add}\mathscr{C}^c$ is covariantly finite.*

Proof. The proof of [2, (5.8)] applies with the obvious changes. □

8.10. Corollary. *Let A be $\mathscr{C}$-supported, then F is the Bongartz complement of $E_1 \oplus \tau_A^{-1} E_2$.*

Proof. The proof of [2, (5.10)] applies with the obvious changes. □

Acknowledgment. The author gratefully acknowledges partial support from the NSERC of Canada and the Université de Sherbrooke. He also thanks Shiping Liu, David Smith and Sonia Trepode for useful discussions.

References

[1] I. Assem, J. A. Cappa, M. I. Platzeck, and S. Trepode. The left part and the Auslander-Reiten components of an artin algebra. *Revista de la Unión Mat. Arg.*, 46(2):19–33, 2005.
[2] I. Assem, J. A. Cappa, M. I. Platzeck, and S. Trepode. Some characterisations of supported algebras. *J. Pure Appl. Algebra*, 208(3):1121–1135, 2007.
[3] I. Assem and F. U. Coelho. Two-sided gluings of tilted algebras. *J. Algebra*, 269(2):456–479, 2003.
[4] I. Assem, F. U. Coelho, M. Lanzilotta, D. Smith, and S. Trepode. Algebras determined by their left and right parts. In *Algebraic structures and their representations*, volume 376 of *Contemp. Math.*, pages 13–47. Amer. Math. Soc., Providence, RI, 2005.
[5] I. Assem, F. U. Coelho, and S. Trepode. The left and the right parts of a module category. *J. Algebra*, 281(2):518–534, 2004.
[6] I. Assem, M. Lanzilotta, and M. J. Redondo. Laura skew group algebras. *Comm. Algebra*, to appear.
[7] I. Assem, M. I. Platzeck, and S. Trepode. On the representation dimension of tilted and laura algebras. *J. Algebra*, 296(2):426–439, 2006.
[8] I. Assem and M. Saorín. Abelian exact subcategories closed under predecessors. *Comm. Algebra*, 33(4):1205–1216, 2005.
[9] I. Assem, D. Simson, and A. Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.
[10] I. Assem and A. Skowroński. Algebras with cycle-finite derived categories. *Math. Ann.*, 280(3):441–463, 1988.
[11] M. Auslander, I. Reiten, and S. O. *Representation Theory of Artin Algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
[12] M. Auslander and S. O. Smalø. Almost split sequences in subcategories. *J. Algebra*, 69(2):426–454, 1981. Addendum: *J. Algebra*, 71(2):592–594, 1981.
[13] Ø. Bakke. Some characterizations of tilted algebras. *Math. Scand.*, 63(1):43–50, 1988.
[14] R. Bautista and S. O. Nonexistent cycles. *Comm. Algebra*, 11(16):1755–1767, 1983.
[15] K. Bongartz. Tilted algebras. In *Representations of algebras (Puebla, 1980)*, volume 903 of *Lecture Notes in Math.*, pages 26–38. Springer, Berlin, 1981.
[16] K. Bongartz. Algebras and quadratic forms. *J. London Math. Soc. (2)*, 28(3):461–469, 1983.
[17] K. Bongartz. On a result of Bautista and Smalø on cycles. *Comm. Algebra*, 11(18):2123–2124, 1983.
[18] F. U. Coelho. Directing components for quasitilted algebras. *Colloq. Math.*, 82(2):271–275, 1999.
[19] D. Happel, I. Reiten, and S. O. Smalø. Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.*, 120(575):viii+ 88, 1996.
[20] D. Happel and C. M. Ringel. Tilted algebras. *Trans. Amer. Math. Soc.*, 274(2):399–443, 1982.
[21] D. Happel and C. M. Ringel. Directing projective modules. *Arch. Math. (Basel)*, 60(3):237–246, 1993.
[22] M. Hoshino. Happel-Ringel's theorem on tilted algebras. *Tsukuba J. Math.*, 6(2):289–292, 1982.
[23] H. Lenzing and A. Skowroński. Quasi-tilted algebras of canonical type. *Colloq. Math.*, 71(2):161–181, 1996.
[24] S. Liu. The connected components of the Auslander-Reiten quiver of a tilted algebra. *J. Algebra*, 161(2):505–523, 1993.
[25] S. Liu. Semi-stable components of an Auslander-Reiten quiver. *J. London Math. Soc. (2)*, 47(3):405–416, 1993.
[26] S. Liu. Tilted algebras and generalized standard Auslander-Reiten components. *Arch. Math. (Basel)*, 61(1):12–19, 1993.
[27] C. M. Ringel. On algorithms for solving vector space problems. I. Report on the Brauer-Thrall conjectures: Rojter's theorem and the theorem of Nazarova and Rojter. In *Representation theory, I (Proc. Workshop,*

Carleton Univ., Ottawa, Ont., 1979), volume 831 of *Lecture Notes in Math.*, pages 104–136. Springer, Berlin, 1980.

[28] C. M. Ringel. *Tame algebras and integral quadratic forms*, volume 1099 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[29] C. M. Ringel. The regular components of the Auslander-Reiten quiver of a tilted algebra. *Chinese Ann. Math. Ser. B*, 9(1):1–18, 1988.

[30] A. Skowroński. Generalized standard Auslander-Reiten components without oriented cycles. *Osaka J. Math.*, 30(3):515–527, 1993.

[31] A. Skowroński. Regular Auslander-Reiten components containing directing modules. *Proc. Amer. Math. Soc.*, 120(1):19–26, 1994.

[32] A. Skowroński. On Artin algebras with almost all indecomposable modules of projective or injective dimension at most one. *Cent. Eur. J. Math.*, 1(1):108–122 (electronic), 2003.

[33] S. O. Smalø. Torsion theories and tilting modules. *Bull. London Math. Soc.*, 16(5):518–522, 1984.

DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE, QUÉBEC, J1K 2R1, CANADA

E-mail address: Ibrahim.Assem@USherbrooke.ca