Lectures on split-by-nilpotent extensions

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Dedicated to José Antonio de la Peña for his 60^{th} birthday

Abstract. We survey known properties of split-by-nilpotent extensions of algebras, concentrating on their bound quivers, the change of rings functors and tilting theory.

Introduction

These notes are an updated version of a course given years ago at the Universidad Nacional del Sur in Bahía Blanca (Argentina). Split extensions are fascinating mathematical objects defined as follows. Let A be a finite dimensional algebra over a field $\mathbf{k}$, and E an A-A-bimodule, finite dimensional over $\mathbf{k}$, equipped with an associative product $E \otimes_A E \longrightarrow E$, then the split extension R of A by E is the $\mathbf{k}$-vector space $R = A \oplus E$ with the multiplication

$$(a\, ,\, e)(a'\, ,\, e') = (aa'\, ,\, ea' + ae' + ee')$$

for $a, a' \in A$ and $e, e' \in E$, where ee' stands for the product in E. If E is nilpotent for its product, then R is called a split-by-nilpotent extension. Examples abound in the mathematical literature, the best known being trivial extension algebras. Thus, the study of split-by-nilpotent extensions connects with those of selfinjective algebras and, more recently, cluster tilted algebras.

The general problem of split-by-nilpotent extensions is to predict properties of R knowing properties of A and E, and conversely. In an abstract setting, this is a difficult problem and more information is needed to obtain concrete results.

The objective of these notes is to survey known results about split-by-nilpotent extensions. We tried to keep the notes as selfcontained as possible, providing proofs and examples whenever possible. The first section is devoted to the definition and basic properties of this class. The second section relates the bound quivers of R and A. In the third, we start comparing the module categories of E and A, using the classical change of rings functors of [**21**]. Finally, the fourth section is devoted to the comparison of the tilting theories of R and A, with a particular attention to the induced torsion pairs.

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1. Split-by-nilpotent extensions

1.1. Notation. Throughout, $\mathbf{k}$ denotes an algebraically closed field. By algebra is meant a basic finite dimensional associative $\mathbf{k}$-algebra with an identity.

A ***quiver*** Q is a quadruple $Q = (Q_0\,,\, Q_1, s, t)$, where Q_0, Q_1 are sets whose elements are respectively called ***points*** and ***arrows***, and $s, t\colon Q_1 \longrightarrow Q_0$ are maps which associate to an arrow α its ***source*** $s(\alpha)$ and its ***target*** $t(\alpha)$. Given a connected algebra A, there exists a (unique) connected quiver Q_A and (at least) a surjective algebra morphism $\eta\colon \mathbf{k}Q_A \longrightarrow A$, where $\mathbf{k}Q_A$ is the path algebra of Q_A. The ideal $I = \operatorname{Ker}\eta$ is then ***admissible***, that is, there exists $m \geq 2$ such that $\mathbf{k}Q_A^{+m} \subseteq I \subseteq \mathbf{k}Q_A^{+2}$, where $\mathbf{k}Q_A^{+i}$ is the two-sided ideal of $\mathbf{k}Q_A$ generated by the paths of length at least i. The isomorphism $A \cong \mathbf{k}Q_A/I$ (or the morphism η) is called a ***presentation*** of A, and A is said to be given by the ***bound quiver*** $(Q_A\,,\, I)$, see [**12**]. The ideal I is generated by a finite set of elements called relations: given $x, y \in (Q_A)_0$, a ***relation*** from x to y is a linear combination $\rho = \sum_{i=1}^{m} c_i w_i$ where the c_i are nonzero scalars, and the w_i are paths of length at least two from x to y. The relation ρ is called ***monomial*** if $m = 1$, and ***minimal*** if $m \geq 2$ and, for every nonempty subset $J \subsetneqq \{\, 1, 2, \ldots, m \,\}$, we have $\sum_{j\in J} c_j w_j \notin I$.

Following [**19**], we sometimes consider an algebra $A = \mathbf{k}Q_A/I$ as a $\mathbf{k}$-category, of which the object class A_0 is the set $(Q_A)_0$, while the set of morphisms from x to y is the quotient of the vector space $\mathbf{k}Q_A(x\,,\, y)$ of all $\mathbf{k}$-linear combinations of paths from x to y by the subspace $I(x\,,\, y) = I \cap \mathbf{k}Q_A(x\,,\, y)$. An algebra A is called ***triangular*** if Q_A is acyclic.

We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ a full subcategory containing exactly one representative from each isoclass (= isomorphism class) of indecomposable modules. When we speak about a module, or an indecomposable module, we mean implicitly that it belongs to $\operatorname{mod} A$, or $\operatorname{ind} A$, respectively. If $\mathscr{C}$ is a full subcategory of $\operatorname{mod} A$, we write $M \in \mathscr{C}$ to express that M is an object in $\mathscr{C}$. We denote by $\operatorname{add}\mathscr{C}$ the full subcategory of $\operatorname{mod} A$ having as objects the direct sums of direct summands of objects in $\mathscr{C}$. If there exists a module M such that $\mathscr{C} = \{\, M \,\}$, then we write $\operatorname{add} M$ instead of $\operatorname{add}\mathscr{C}$. Given a module M, we denote by $\operatorname{pd} M$ and $\operatorname{id} M$ its projective and injective dimensions, respectively. The global dimension of A is denoted by $\operatorname{gl.dim.} A$.

For $x \in (Q_A)_0$, we let e_x denote the corresponding primitive idempotent of A, and let S_x, P_x, I_x denote respectively the corresponding simple, indecomposable projective and indecomposable injective modules. The standard duality between right and left modules is denoted by $\mathrm{D} = \operatorname{Hom}_{\mathbf{k}}(-\,,\, \mathbf{k})$ and the Auslander-Reiten translations by $\tau_A = \mathrm{D}\operatorname{Tr}$ and $\tau_A^{-1} = \operatorname{Tr}\mathrm{D}$ (or simply τ, τ^{-1} if there is no ambiguity). For more notions and results about $\operatorname{mod} A$, we refer the reader to [**12, 15**].

1.2. Definition and examples. Let A be an algebra, and E an A-A-bimodule, which is finite dimensional as a $\mathbf{k}$-vector space. We say that E is equipped with an ***associative product*** if there exists a morphism $E \otimes_A E \longrightarrow E$ of A-A-bimodules, denoted as $e \otimes e' \longmapsto ee'$ for $e, e' \in E$ such that

$$e(e'e'') = (ee')e''$$

for all $e, e', e'' \in E$.

Definition 1.2.1. Let A be an algebra and E an A-A-bimodule equipped with an associative product. The $\mathbf{k}$-vector space

$$R = A \oplus E = \{\, (a\,,\, e) \mid a \in A, e \in E \,\}$$

together with the multiplication

$$(a\,,\,e)(a'\,,\,e') = (aa'\,,\,ae' + ea' + ee')$$

for $a, a' \in A$ and $e, e' \in E$, is an algebra, called the ***split extension*** of A by E. If moreover E is nilpotent as a two-sided ideal of R, then R is called a ***split-by-nilpotent extension***.

Clearly, $\dim_{\mathbf{k}} R = \dim_{\mathbf{k}} A + \dim_{\mathbf{k}} E$ and there exists an exact sequence of vector spaces

$$0 \longrightarrow E \xrightarrow{\ \iota\ } R \xrightarrow{\ \pi\ } A \longrightarrow 0$$

where $\iota\colon e \longmapsto (0\,,\,e)$ for $e \in E$, while $\pi\colon (a\,,\,e) \longmapsto a$ for $(a\,,\,e) \in R$. Then π is an algebra morphism and admits as section the algebra morphism $\sigma\colon A \longrightarrow R$, $a \longmapsto (a\,,\,0)$ for $a \in A$. Because ι, π, σ are also A-A-bimodule morphisms, the previous exact sequence may also be considered as a split exact sequence of A-A-bimodules and so, in particular, as a split exact sequence of right, or left, A-modules. Of course, it is also an exact sequence of R-R-bimodules, or of right, or left, R-modules. But then, it is generally not split.

Saying that E is nilpotent amounts to saying that $E \subseteq \operatorname{rad} R$. In the sequel, we always assume that E is nilpotent.

There may be several decompositions of ${}_AR_A$ as a direct sum isomorphic to $A \oplus E$. Therefore the data of an exact sequence as above does not suffice to determine a split extension: one must also fix a direct sum decomposition $R = A \oplus E$, or, equivalently, fix a section σ to π.

EXAMPLES 1.2.2. (a) Because $\mathbf{k}$ is algebraically closed, any algebra can be written as a direct sum $R = (R/\operatorname{rad} R) \oplus \operatorname{rad} R$, so it is a split extension of the semisimple algebra $R/\operatorname{rad} R$ by the nilpotent bimodule $\operatorname{rad} R$.

(b) If $E^2 = 0$, then a split extension of A by E is called a ***trivial extension*** and denoted as $A \ltimes E$. This class plays a very important rôle in the classification results for self-injective algebras. In this case, one takes E to be the minimal injective cogenerator bimodule $E = \mathrm{D}({}_AA_A)$ with its canonical bimodule structure, see [**31, 32**]. Another type of trivial extensions appeared in the theory of cluster algebras: it is indeed proved in [**5**] that an algebra is cluster tilted if and only if it is the trivial extension of a tilted algebra A by the so-called relation bimodule $E = \operatorname{Ext}^2_A(\mathrm{D}A\,,\,A)$ with its canonical bimodule structure.

Perhaps the smallest nontrivial example is the following: let $A = \mathbf{k}$, $E = \mathbf{k}$ with its canonical $\mathbf{k}$-$\mathbf{k}$-bimodule structure. The trivial extension $A \ltimes E$ is the vector space

$$\mathbf{k}^2 = \{\ (a\,,\,b)\ \mid\ a, b \in \mathbf{k}\ \}$$

with the multiplication

$$(a\,,\,b)(a'\,,\,b') = (aa'\,,\,ab' + ba')$$

for $a, a', b, b' \in \mathbf{k}$. Clearly, we have an algebra isomorphism $A \ltimes E \cong \mathbf{k}[t]/(t^2)$.

(c) We now give an example of a split extension which is not a trivial extension. Let $A = \mathbf{k}$, $E = \mathbf{k}^2$ with its canonical bimodule structure and equipped with the (obviously associative) product

$$(b\,,\,c)(b'\,,\,c') = (0\,,\,bb')$$

for $b, b', c, c' \in \mathbf{k}$. The split extension is the three-dimensional vector space

$$R = A \oplus E = \{\ (a\,,\,(b\,,\,c))\ \mid\ a, b, c \in \mathbf{k}\ \}$$

with multiplication

$$(a\,,\,(b\,,\,c))(a'\,,\,(b'\,,\,c')) = (aa'\,,\,(ab'+ba'\,,\,ac'+ca'+bb'))$$

for $a, b, c, a', b', c' \in \mathbf{k}$. It is easy to see that actually $R \cong \mathbf{k}[t]/(t^3)$. One can realise in this way any truncated polynomial algebra $\mathbf{k}[t]/(t^n)$ as split extension of $A = \mathbf{k}$ by $E = \mathbf{k}^{n-1}$.

(d) Let A be given by the quiver

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\beta"'] & \underset{3}{\circ} \arrow[l, "\alpha"'] \end{tikzcd}$$

bound by $\alpha\beta = 0$, and R be given by the quiver

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\beta"'] \arrow[r, bend left, "\eta"] & \underset{3}{\circ} \arrow[l, "\alpha"] \end{tikzcd}$$

bound by $\alpha\beta = 0$, $\eta\alpha\eta\alpha\eta = 0$. Then R is the split extension of A by the bimodule E generated by the arrow η. To find a $\mathbf{k}$-basis of E, we construct those paths (more precisely, classes of paths modulo the binding ideal, but we identify the two) which contain η. This gives the following basis

$$\{\,\eta\,,\,\eta\alpha\,,\,\alpha\eta\,,\,\eta\alpha\eta\,,\,\alpha\eta\alpha\,,\,\eta\alpha\eta\alpha\,,\,\alpha\eta\alpha\eta\,,\,\alpha\eta\alpha\eta\alpha\,\}.$$

The right and left A-module structures of E are computed as follows. We have $A_A = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$ where indecomposable modules are represented by their Loewy series. Similarly, $R_R = 1 \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & & 2 \\ & & 3 \\ & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \end{smallmatrix}$. We next compute R_A: deleting η from the indecomposable R-modules gives their A-module structure. We get $R_A = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)^2 \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)^2$ from where we deduce $E_A = \left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)^4$. Similarly $(\mathrm{D}A)_A = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus 3$ and $(\mathrm{D}R)_R = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 3 \\ 2 \\ 3 \end{smallmatrix}$ yields $(\mathrm{D}E)_A = \left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right)^4$.

1.3. Properties. Our next objective is to describe the quiver Q_R of a split extension R of an algebra A by a nilpotent bimodule E, in terms of the quiver Q_A of A.

Lemma 1.3.1. *Let R be a split extension of A by a nilpotent bimodule E, then* $\operatorname{rad} A = (\operatorname{rad} R)/E$.

Proof. We have $E \subseteq \operatorname{rad} R$ and $(\operatorname{rad} R)/E$ nilpotent as an ideal in $R/E \cong A$. Moreover, $\frac{R/E}{(\operatorname{rad} R)/E} \cong \frac{R}{\operatorname{rad} R}$ is semisimple. Therefore $(\operatorname{rad} R)/E \cong \operatorname{rad}(R/E) \cong \operatorname{rad} A$. $\square$

Theorem 1.3.2 [**10**](1.2)**.** *Let R be a split extension of A by a nilpotent bimodule E. The quiver Q_R of R is constructed as follows:*

(a) $(Q_R)_0 = (Q_A)_0$*;*

(b) for $x, y \in (Q_R)_0$, the set of arrows in Q_R from x to y equals the set of arrows in Q_A from x to y plus

$$\dim_{\mathbf{k}} e_x \left(\frac{E}{E \cdot \operatorname{rad} A + \operatorname{rad} A \cdot E + E^2} \right) e_y$$

additional arrows.

Proof. Because of Lemma 1.3.1, Q_R and Q_A have the same points and moreover, $\operatorname{rad} R = \operatorname{rad} A \oplus E$ as a vector space. Hence

$$\operatorname{rad}^2 R = \operatorname{rad}^2 A \oplus \big(E \cdot \operatorname{rad} A + \operatorname{rad} A \cdot E + E^2\big).$$

The arrows in Q_R from x to y are in bijection with vectors in a basis of the vector space $e_x\big(\frac{\operatorname{rad} R}{\operatorname{rad}^2 R}\big)e_y$. Because $\operatorname{rad}^2 A \subseteq \operatorname{rad} A$ and $E\cdot\operatorname{rad} A+\operatorname{rad} A\cdot E+E^2 \subseteq E$, the statement follows. □

Thus, if A is a connected algebra, then so is R. We now see how split extensions behave upon taking full subcategories.

Lemma 1.3.3 [**14**](1.4). *Let R be a split extension of A by a nilpotent bimodule E and e an idempotent in A. Then eRe is a split extension of eAe by eEe.*

Proof. Clearly, eEe is a two-sided ideal of eRe. Its nilpotency follows from the fact that $eEe \subseteq E$. The map $\pi_e \colon eRe \longrightarrow eAe$, $e(a\,,\,x)e \longmapsto eae$ for $(a\,,\,x) \in R$ is a surjective algebra morphism having as section $\sigma_e \colon eAe \longrightarrow eRe$, $eae \longmapsto e(a\,,\,0)e$ for $a \in A$. Moreover, σ_e is an algebra morphism and $eEe \subseteq \operatorname{Ker} \pi_e$. Because $eRe = eAe \oplus eEe$ as vector spaces, we get the statement by comparing dimensions. □

As we now see, taking split extensions is a transitive procedure.

Lemma 1.3.4 [**10**](1.7). *Let R be a split extension of A by a nilpotent bimodule E and S a split extension of R by a nilpotent bimodule F. Then S is a split extension of A.*

Proof. We have exact sequences of vector spaces

$$0 \longrightarrow E \xrightarrow{\iota} R \underset{\sigma}{\overset{\pi}{\rightleftarrows}} A \longrightarrow 0 \quad , \quad 0 \longrightarrow F \xrightarrow{\iota'} S \underset{\sigma'}{\overset{\pi'}{\rightleftarrows}} R \longrightarrow 0$$

where $\pi, \sigma, \pi', \sigma'$ are algebra morphisms and $\pi\sigma = \mathrm{id}_A$, $\pi'\sigma' = \mathrm{id}_R$. Also, there exist $m, n > 0$ such that $E^m = 0$, $F^n = 0$. We get an exact sequence

$$0 \longrightarrow {\pi'}^{-1}(E) \longrightarrow S \xrightarrow{\pi\pi'} A \longrightarrow 0.$$

Both $\pi\pi', \sigma'\sigma$ are algebra morphisms and $\pi\pi'\sigma'\sigma = \mathrm{id}_A$, so it suffices to prove that ${\pi'}^{-1}(E)$ is nilpotent. We claim that ${\pi'}^{-1}(E)^{mn} = 0$. Let $x_{ij} \in {\pi'}^{-1}(E)$ with $1 \le i \le n$, $1 \le j \le m$. Then $\pi'(x_{ij}) \in E$ for all i, j. Therefore, for each i, we have $\pi'\Big(\prod_{j=1}^m x_{ij}\Big) = \prod_{j=1}^m \pi'(x_{ij}) \in E^m = 0$. Thus, for each i, the product $\prod_{j=1}^m x_{ij}$ lies in $\operatorname{Ker} \pi' = F$. But then $\prod_{i=1}^n \prod_{j=1}^m x_{ij} \in F^n = 0$. □

2. The bound quiver of a split extension

2.1. Presentations. Let R be a split extension of A by a nilpotent bimodule E. Because of 1.3.2, the quiver Q_R of R is obtained from the quiver Q_A of A by adding arrows. It is therefore reasonable to think that E, as an ideal, is generated precisely by the added arrows. Let $\eta_R \colon \mathbf{k}Q_R \longrightarrow R \cong \mathbf{k}Q_R/I_R$ and $\eta_A \colon \mathbf{k}Q_A \longrightarrow A \cong \mathbf{k}Q_A/I_A$ be respectively bound quiver presentations of R and A. For $x, y \in (Q_A)_0$, it follows from the proof of 1.3.2 that there is an inclusion of vector spaces

$$e_x\left(\frac{\operatorname{rad} A}{\operatorname{rad}^2 A}\right)e_y \longrightarrow e_x\left(\frac{\operatorname{rad} R}{\operatorname{rad}^2 R}\right)e_y \ .$$

Therefore, there exists a basis of $e_x\left(\dfrac{\operatorname{rad} R}{\operatorname{rad}^2 R}\right)e_y$ which contains as a subset a basis of $e_x\left(\dfrac{\operatorname{rad} A}{\operatorname{rad}^2 A}\right)e_y$. When the arrows in Q_R are taken in bijection with vectors in such a basis, for any x, y, then we say that the presentation η_R (or (Q_R, I_R)) of R ***respects*** η_A (or (Q_A, I_A), or simply A). The previous comments show that there always exists a presentation of R respecting A.

Lemma 2.1.1 [**10**](1.5)**.** *Let R be a split extension of A by a nilpotent bimodule E. Then there exists a presentation of R respecting A such that E is generated by the classes of arrows in Q_R which are not in Q_A.*

Proof. Let (Q_R, I_R) be a bound quiver presentation of R which respects A and $\{\rho_1, \dots, \rho_s\}$ be the preimage modulo I_R of any linearly independent set of generators for E. We may assume that each ρ_i is a linear combination of paths having the same source and the same target: for, if this is not the case, then we multiply each ρ_i on the left and on the right by stationary paths and we obtain such a set. Because Q_A, Q_R have the same points, all paths involved in the ρ_i have length at least one. Moreover, as seen in 1.3.2, the top of E is contained in $\operatorname{rad} R/\operatorname{rad}^2 R$, that is $\rho_i + \operatorname{rad}^2 R \in \operatorname{rad} R/\operatorname{rad}^2 R$ for all i with $1 \leq i \leq s$. So we have

$$\rho_i = \alpha_i + \sum_j \lambda_j w_j$$

where α_i is an arrow in Q_R and $\sum_j \lambda_j w_j$ a linear combination of paths of length at least one. Because the ρ_i are linearly independent modulo I_R, we define a new presentation by replacing α_i by

$$\alpha_i' = \alpha_i + \sum_j \lambda_j w_j.$$

In this presentation, E is indeed generated by $\alpha_1', \dots, \alpha_s'$. □

Corollary 2.1.2 [**10**](2.1)**.** *Let R be a split extension of A by a nilpotent bimodule E. Given a presentation $\eta_A\colon \mathbf{k}Q_A \longrightarrow A \cong \mathbf{k}Q_A/I_A$, there exists a presentation $\eta_R\colon \mathbf{k}Q_R \longrightarrow R \cong \mathbf{k}Q_R/I_R$ respecting A such that:*

(a) E is an ideal of R generated by classes of arrows,

(b) there exist algebra morphisms $\tilde\pi\colon \mathbf{k}Q_R \longrightarrow \mathbf{k}Q_A$, $\tilde\sigma\colon \mathbf{k}Q_A \longrightarrow \mathbf{k}Q_R$ such that $\tilde\pi\tilde\sigma = \mathrm{id}_{\mathbf{k}Q_A}$, $\eta_R\tilde\sigma = \sigma\eta_A$ and $\tilde\sigma(I_A) \subseteq I_R$,

(c) there exists a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde E \cap I_R \arrow[r] \arrow[d] & I_R \arrow[r] \arrow[d] & I_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde E \arrow[r] \arrow[d] & \mathbf{k}Q_R \arrow[r, shift left, "\tilde\pi"] \arrow[d, "\eta_R"'] & \mathbf{k}Q_A \arrow[l, shift left, "\tilde\sigma"] \arrow[r] \arrow[d, "\eta_A"] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & R \arrow[r, shift left, "\pi"] \arrow[d] & A \arrow[l, shift left, "\sigma"] \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Proof. Because of 1.3.2, we may (and shall) identify Q_A to a nonfull subquiver of Q_R. Applying [**12**](II.1.8), the inclusion $Q_A \hookrightarrow Q_R$ extends to an algebra morphism $\tilde{\sigma}\colon \mathbf{k}Q_A \longrightarrow \mathbf{k}Q_R$ preserving stationary paths and arrows. Letting $\eta_R\colon \mathbf{k}Q_R \longrightarrow \mathbf{k}Q_R/I_R \cong R$ be a presentation constructed as in 2.1.1, we then have $\eta_R\tilde{\sigma} = \sigma\eta_A$. Moreover, there exists a set S of arrows in Q_R such that E is the ideal generated by the classes of the elements of S. Let $\tilde{E}$ be the lifted ideal in $\mathbf{k}Q_R$, that is, the one generated by the elements of S. Applying again [**12**](II.1.8), there exists a surjective algebra morphism $\tilde{\pi}\colon \mathbf{k}Q_R \longrightarrow \mathbf{k}Q_A$ preserving stationary paths and such that, for an arrow β,

$$\tilde{\pi}(\beta) = \begin{cases} \beta & \text{if } \beta \in (Q_R)_1 \setminus S \\ 0 & \text{if } \beta \in S. \end{cases}$$

We deduce an exact sequence of vector spaces

$$0 \longrightarrow \tilde{E} \longrightarrow \mathbf{k}Q_R \longrightarrow \mathbf{k}Q_A \longrightarrow 0$$

and also $\eta_A\tilde{\pi} = \pi\eta_R$. A direct calculation shows that $\tilde{\pi}\tilde{\sigma} = \mathrm{id}_{\mathbf{k}Q_A}$ and (c) follows by passing to kernels. □

2.2. The relations. We have seen that, if R is a split extension of A by E, then E may be assumed to be generated by arrows in Q_R. But what is not clear is whether, if we choose an arbitrary set of arrows in Q_R, and call E the ideal they generate, then R is a split extension of R/E by E or not. Actually, this is not always the case, as the following example shows.

Example 2.2.1. Let R be given by the quiver

$$\begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 1 & & 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & 3 \arrow[ul, "\delta"] & \end{tikzcd}$$

bound by $\alpha\beta = \gamma\delta$. Let E be the ideal generated by α. Then $A = R/E$ is not a subalgebra of R: indeed, the product of (the classes of) γ and δ is zero in A, but not in R. Thus, R is not a split extension of A by E.

If, on the other hand, we let E' be generated by α and γ, then it is easily seen that R is a split extension of R/E' by E'.

This example shows that, when passing from R to A, any deletion of arrows must take into account the relations.

Lemma 2.2.2 [10](2.1)(2.3). *Let $\eta_R\colon \mathbf{k}Q_R \longrightarrow \mathbf{k}Q_R/I_R \cong R$ be a presentation, S a set of arrows in Q_R and E the ideal in R generated by S.*

(a) *Setting $A = R/E$, there exists a presentation $\eta_A\colon \mathbf{k}Q_A \longrightarrow \mathbf{k}Q_A/I_A \cong A$ such that we have a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \tilde{E} \cap I_R \arrow[r] \arrow[d] & I_R \arrow[r] \arrow[d] & I_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{E} \arrow[r] \arrow[d] & \mathbf{k}Q_R \arrow[r, shift left, "\tilde{\pi}"] \arrow[d, "\eta_R"'] & \mathbf{k}Q_A \arrow[l, shift left, "\tilde{\sigma}"] \arrow[r] \arrow[d, "\eta_A"] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[d] & R \arrow[r, "\pi"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $\tilde{\pi}$, $\tilde{\sigma}$ are algebra morphisms such that $\tilde{\pi}\tilde{\sigma} = \mathrm{id}_{\mathbf{k}Q_A}$.

(b) *If moreover $\tilde{\sigma}(I_A) \subseteq I_R$, then the lower sequence realises R as a split extension of A by E.*

Proof. (a) Let Q be the quiver having the same points as Q_R and arrows all arrows of Q_R except those in S. Using [**12**](II.1.8) there exists a surjective algebra morphism $\tilde{\pi}\colon \mathbf{k}Q_R \longrightarrow \mathbf{k}Q$, preserving stationary paths and such that

$$\tilde{\pi}(\beta) = \begin{cases} \beta & \text{if } \beta \in (Q_R)_1 \setminus S \\ 0 & \text{if } \beta \in S. \end{cases}$$

Let $\pi\colon R \longrightarrow A$ be the projection and $\tilde{E} = \operatorname{Ker} \tilde{\pi}$. Then $\tilde{E}$ is the ideal of $\mathbf{k}Q_R$ generated by the arrows in S. Clearly, $\pi\eta_R(\tilde{E}) = 0$ hence there exists a unique algebra morphism $\eta_A\colon \mathbf{k}Q \longrightarrow A$ such that $\eta_A\tilde{\pi} = \pi\eta_R$. Moreover, η_A is surjective, because so are π and η_R.

We claim that $I_A = \operatorname{Ker} \eta_A$ is an admissible ideal of $\mathbf{k}Q$. We first prove that $I_A \subseteq \mathbf{k}Q^{+2}$. If this is not the case, let $\gamma \in I_A \setminus \mathbf{k}Q^{+2}$. There exist $\alpha_1, \ldots, \alpha_t \in Q_1$, nonzero scalars $c_1, \ldots, c_t$ and $\gamma' \in \mathbf{k}Q^{+2}$ such that

$$\gamma = \sum_{i=1}^{t} c_i\alpha_i + \gamma'.$$

Considering γ as an element of $\mathbf{k}Q_R$, we have

$$\pi\eta_R(\gamma) = \eta_A\tilde{\pi}(\gamma) = \eta_A(\gamma) = 0.$$

Hence $\eta_R(\gamma) \in \operatorname{Ker} \pi = E$. Therefore there exist nonzero scalars $d_1, \ldots, d_s$ and arrows $\beta_1, \ldots, \beta_s \in S$ such that

$$\sum_{i=1}^{t} c_i\alpha_i + \gamma' + I_R = \gamma + I_R = \sum_{j=1}^{s} d_j\beta_j + I_R.$$

Because I_R is admissible and $\gamma' \in \mathbf{k}Q^{+2}$, this equality yields, because of the grading,

$$\sum_{i=1}^{t} c_i\alpha_i = \sum_{j=1}^{s} d_j\beta_j.$$

Now the β_j lie in S, while the α_i do not. This absurdity yields $I_A \subseteq \mathbf{k}Q^{+2}$. On the other hand, there exists $m \geq 2$ such that $\mathbf{k}Q_R^{+m} \subseteq I_R$. Because Q is a subquiver of Q_R, we have $\mathbf{k}Q^{+m} \subseteq \mathbf{k}Q_R^{+m}$ so that $\mathbf{k}Q^{+m} \subseteq I_R$. Because of the definition of η_R, the last inclusion reads as $\mathbf{k}Q^{+m} \subseteq I_A$. This establishes our claim.

Therefore $\eta_A\colon \mathbf{k}Q \longrightarrow \mathbf{k}Q/I_A \cong A$ is a presentation of A. Because the quiver of an algebra is uniquely determined, we have $Q = Q_A$. Moreover that $\eta_A\tilde{\pi} = \pi\eta_R$ and $E, \tilde{E}$ are the respective kernels of $\pi, \tilde{\pi}$ imply that the shown diagram is commutative with exact rows and columns.

Finally, the (non full) quiver inclusion $Q_A \hookrightarrow Q_R$ yields, because of [**12**](II.1.8), an algebra morphism $\tilde{\sigma}\colon \mathbf{k}Q_A \longrightarrow \mathbf{k}Q_R$ such that $\tilde{\pi}\tilde{\sigma} = \mathrm{id}_{\mathbf{k}Q_A}$.

(b) The hypothesis yields a morphism of abelian groups $\sigma\colon A \longrightarrow R$ such that $\sigma\eta_A = \eta_R\tilde{\sigma}$. Because $E \subseteq \operatorname{rad} R$, it suffices to prove that σ is an algebra morphism and a section to π. Let w, w' be paths in Q_A, then

$$\begin{aligned}\sigma\big((w + I_A)(w' + I_A)\big) &= \sigma(ww' + I_A) = \sigma\eta_A(ww') = \eta_R\tilde{\sigma}(ww')\\ &= \eta_R\tilde{\sigma}(w)\eta_R\tilde{\sigma}(w') = \sigma\eta_A(w)\sigma\eta_A(w')\\ &= \sigma(w + I_A)\sigma(w' + I_A).\end{aligned}$$

Thus, σ is an algebra morphism. Also, $\tilde{\pi}\tilde{\sigma} = \mathrm{id}_{\mathbf{k}Q_A}$ implies that $\pi\sigma\eta_A = \pi\eta_R\tilde{\sigma} = \eta_A\tilde{\pi}\tilde{\sigma} = \eta_A$. The surjectivity of η_A yields $\pi\sigma = \mathrm{id}_A$, as required. □

Let w be a path in a quiver and α an arrow on w, that is, such that there exist subpaths w_1, w_2 of w satisfying $w = w_1\alpha w_2$, then we write $\alpha \mid w$. Let now S be a set of arrows and $\rho = \sum_{i=1}^{t} \lambda_i w_i$ a relation, with the c_i nonzero scalars and the w_i paths. We say that ρ is ***consistently cut*** if, for any i, if there exists an arrow $\alpha_i \in S$ such that $\alpha_i \mid w_i$ then for every $j \neq i$, there exists $\alpha_j \in S$ such that $\alpha_j \mid w_j$. That is, if S cuts one branch of ρ, then it cuts all its branches.

In 2.2.1, the relation $\alpha\beta = \gamma\delta$ is not consistently cut by the set $\{\, \alpha \,\}$, but it is consistently cut by $\{\, \alpha, \gamma \,\}$.

Because relations in a bound quiver may be assumed monomial or minimal, and because monomial relations are trivially consistently cut, the definition above applies only to minimal relations.

Theorem 2.2.3 [**10**](2.5). *Let $\eta_R\colon \mathbf{k}Q_R \longrightarrow R \cong \mathbf{k}Q_R/I_R$ be a presentation, S a set of arrows in Q_R, E the ideal they generate and $\pi\colon R \longrightarrow R/E = A$ the projection. Then:*

(a) If every minimal relation in I_R is consistently cut, then the exact sequence

$$0 \longrightarrow E \longrightarrow R \xrightarrow{\ \pi\ } A \longrightarrow 0$$

realises R as a split extension of A by E.

(b) Conversely, if the sequence in (a) is a split extension and η_R respects A, then every minimal relation is consistently cut.

Proof. (a) Assume that every minimal relation in I_R is consistently cut. As seen in 2.2.2(a), the projection π lifts to an algebra morphism $\tilde{\pi}\colon \mathbf{k}Q_R \longrightarrow \mathbf{k}Q_A$. Let $\rho \in I_R$ be a relation, then $\rho = \sum c_i w_i$ where the c_i are nonzero scalars and the w_i paths. Because ρ is consistently cut, if there exists i such that $\tilde{\pi}(w_i) = 0$ then, for each $j \neq i$, we have $\tilde{\pi}(w_j) = 0$. This proves that, for any relation ρ, we have either $\tilde{\pi}(\rho) = \rho$ or $\tilde{\pi}(\rho) = 0$.

In order to prove our statement, it suffices, because of 2.2.2(b), to prove that the algebra morphism $\tilde{\sigma}\colon \mathbf{k}Q_A \longrightarrow \mathbf{k}Q_R$ induced by the inclusion $Q_A \hookrightarrow Q_R$ satisfies $\tilde{\sigma}(I_A) \subseteq I_R$.

Let $\rho \in I_A$ be nonzero. We may assume, without loss of generality, that ρ is a relation. Because the restriction $\tilde{\pi}|_{I_R} : I_R \longrightarrow I_A$ is surjective, there exists $\rho' \in I_R$ such that $\tilde{\pi}(\rho') = \rho$. Then ρ' can be written as $\rho' = \sigma + \eta$, where $\sigma = \sum_i \sigma_i$ is a sum of monomial relations and $\eta = \sum_j \eta_j$ is a sum of minimal relations. We distinguish two cases:

1) Assume ρ is monomial. Because each of the $\tilde{\pi}(\sigma_i)$, $\tilde{\pi}(\eta_j)$ is a summand of $\rho = \tilde{\pi}(\rho')$, then $\tilde{\pi}(\eta_j) = 0$ for all j and there exists a unique i such that $\rho = \tilde{\pi}(\sigma_i) = \sigma_i$. We thus have $\tilde{\pi}(\rho) = \rho$ and so $\tilde{\sigma}(\rho) = \rho \in I_R$.
2) If ρ is minimal, then, for each i, we have $\tilde{\pi}(\sigma_i) = 0$ because otherwise, σ_i would be a summand of the minimal relation ρ, a contradiction. Similarly, if j, k are distinct indices such that $\tilde{\pi}(\eta_j) = \eta_j$ and $\tilde{\pi}(\eta_k) = \eta_k$, then $\eta_j + \eta_k$ would be a summand of ρ, another contradiction to minimality. Hence there exists a unique j such that $\rho = \tilde{\pi}(\eta_j) = \eta_j$. Again, we have $\rho = \tilde{\pi}(\rho)$ and $\tilde{\sigma}(\rho) = \rho \in I_R$.

This completes the proof of (a).

(b) Conversely, assume that the given sequence is a split extension and that η_R respects A. Let $\rho = \sum_i c_i w_i$ be a minimal relation in I_R with the c_i nonzero scalars and the w_i paths. Assume there exist i and $\alpha_i \in S$ such that $\alpha_i \mid w_i$. Let J be the proper subset of $\{ 1, \dots, t \}$ consisting of those j such that there is no arrow $\alpha_j \in S$ such that $\alpha_j \mid w_j$. We must prove that $J = \varnothing$. If not, then we can write

$$\rho = \sum_{i \notin J} c_i w_i + \sum_{j \in J} c_j w_j .$$

The commutative diagram of 2.2.2 yields $\tilde{\pi}(\rho) = \sum_{j\in J} c_j w_j$ in $\mathbf{k}Q_A$. Because $\rho \in I_R$, we have $\eta_A \tilde{\pi}(\rho) = \pi \eta_R(\rho) = 0$. Hence $\tilde{\pi}(\rho) \in \operatorname{Ker} \eta_A = I_A$. Because the given exact sequence is a split extension, it follows from 2.1.2 that $\tilde{\sigma}(I_A) \subseteq I_R$. But then we get $\sum_{j\in J} c_j w_j \in I_R$, which contradicts the minimality of ρ. □

We recall that an algebra is ***monomial*** if it admits a presentation such that the binding ideal is generated by monomials. ***String algebras*** are special types of monomial algebras for which we refer to [**20**]. ***Gentle algebra*** are special types of string algebras, see[**12**] Chapter X. For ***special biserial algebras***, we refer to [**30**].

Corollary 2.2.4. *Let R be a split extension of A by a nilpotent bimodule E, with a presentation respecting A. If R is monomial, string, gentle or special biserial, then so is A.*

Proof. In each case, the defining conditions on the bound quiver of R remain satisfied if one cuts arrows so that the conditions of 2.2.3 are satisfied. □

As an interesting particular case, if R is a trivial extension of A (by either the minimal injective cogenerator $\mathrm{D}({}_A A_A)$ or the relation bimodule $\operatorname{Ext}^2_A(\mathrm{D}A\,,\,A)$) and R is monomial, string, gentle or special biserial, then so is A.

The reader will connect the notion of consistent cut of relation with that of admissible cut of an algebra, introduced in [**24**] in the case of selfinjective trivial extensions and in [**16**] in the case of cluster tilted algebras, see 2.2.5(b) below, and also [**2**].

EXAMPLES 2.2.5. (a) We show that one-point extensions may be viewed as split extensions. Let B be an algebra, and M a B-module, then

$$R = B[M] = \begin{pmatrix} B & 0 \\ M & \mathbf{k} \end{pmatrix} = \left\{ \begin{pmatrix} b & 0 \\ m & \lambda \end{pmatrix} \;\middle|\; b \in B, m \in M, \lambda \in \mathbf{k} \right\}$$

becomes an algebra when equipped with the ordinary matrix addition and the multiplication induced from the B-module structure of M. It is called the ***one-point extension*** of R by M, see [**27**]. The quiver Q_R equals Q_B plus an extra point x, called the extension point, which is a source in Q_R.

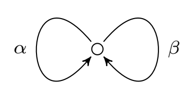
Cutting all arrows having x as a source is certainly a consistent cut. Therefore R is a split extension of $A = B \times \mathbf{k}$ by the bimodule E such that $E_A = M$ while $\mathrm{D}({}_AE) = S_x{}^{\dim_{\mathbf{k}} M}$.

(b) Let Q be a quiver with neither loops nor cycles of length two. A full subquiver of Q is a ***chordless cycle*** if it is induced by a set of points $\{ x_1, \dots, x_p \}$ such that the only edges on it are precisely the x_i —— x_{i+1} , where we set $x_{p+1} = x_1$, see [**17**]. A quiver Q is called ***cyclically oriented*** if each chordless cycle is an oriented cycle, see [**18**].

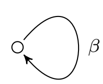
Let R be a cluster tilted algebra with a cyclically oriented quiver, for instance a representation-finite cluster tilted algebra, then Q_R is consistently cut by exactly one arrow from each branch of a relation if and only if the resulting algebra A is an admissible cut of R, that is, R is the trivial extension of A by its relation bimodule. This indeed follows easily from [**18**](4.2)(3.4)(4.7).

(c) The following example, due to M. I. Platzeck (private communication) shows that in 2.2.3(b), the condition that η_R respects A is necessary.

Assume $\mathrm{char}\,\mathbf{k} \neq 2$ and let R be given by the quiver

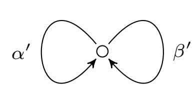


bound by $\alpha^2 = 0$, $\alpha\beta = \beta\alpha$, $\beta^2 = 0$. Let E be the ideal generated by α, then R is the split extension of $A = R/E$ given by the quiver



bound by $\beta^2 = 0$. Here, the given presentation of R respects A.

Let now $\alpha' = \alpha$, $\beta' = \alpha + \beta$. Then R is given by the quiver



bound by $\alpha'^2 = 0$, $\beta'^2 = 2\beta'\alpha'$, $\alpha'\beta' = \beta'\alpha'$. Taking E' as the ideal generated by α', we get $R/E' \cong A$, as before. However, the second relation is not consistently cut. This presentation of R does not respect A.

3. Modules over split-by-nilpotent extensions

3.1. The change of rings functors. Let R be a split extension of A by a nilpotent bimodule E. There is an obvious embedding of $\operatorname{mod} A$ into $\operatorname{mod} R$, but the latter is in general much larger than the former. For instance, the path algebra of the Kronecker quiver

$$1 \circ \overset{\alpha}{\underset{\beta}{\leftleftarrows}} \circ 2$$

is a split extension of the path algebra of the quiver

$$1 \circ \xleftarrow{\alpha} \circ 2.$$

The first is representation-infinite while the second has only 3 isoclasses of indecomposable modules.

Because A is a quotient of R, we have the classical change of rings functors, see [**21**]:

(a) The ***induction*** functor $- \otimes_A R_R \colon \operatorname{mod} A \longrightarrow \operatorname{mod} R$. Modules in its image are called ***induced***.
(b) The ***restriction*** functor $- \otimes_R A_A \colon \operatorname{mod} R \longrightarrow \operatorname{mod} A$. Modules in its image are called ***restricted***.
(c) The ***coinduction*** functor $\operatorname{Hom}_A({}_R R_A\,, -) \colon \operatorname{mod} A \longrightarrow \operatorname{mod} R$. Modules in its image are called ***coinduced***.
(d) The ***corestriction*** functor $\operatorname{Hom}_R({}_A A_R\,, -) \colon \operatorname{mod} R \longrightarrow \operatorname{mod} A$. Modules in its image are called ***corestricted***.

We also have obvious functors:

(e) the ***forgetful*** functor $- \otimes_R R_A$ or $\operatorname{Hom}_R({}_A R_R\,, -) \colon \operatorname{mod} R \longrightarrow \operatorname{mod} A$
(f) the ***embedding*** functor $- \otimes_A A_R$ or $\operatorname{Hom}_A({}_R A_A\,, -) \colon \operatorname{mod} A \longrightarrow \operatorname{mod} R$.

Besides the usual adjunction relations, we have the following lemma.

Lemma 3.1.1. *We have isomorphisms of functors*
(a) $- \otimes_A R \otimes_R A_A \cong \mathrm{id}_{\operatorname{mod} A}$,
(b) $\operatorname{Hom}_R\big({}_A A_R\,, \operatorname{Hom}_A({}_R R_A\,, -)\big) \cong \mathrm{id}_{\operatorname{mod} A}$.

Proof. (a) is obvious and (b) follows from the isomorphisms of functors

$$\operatorname{Hom}_R\big({}_A A_R\,, \operatorname{Hom}_A({}_R R_A\,, -)\big) \cong \operatorname{Hom}_A({}_A A \otimes_R R_A\,, -) \cong \operatorname{Hom}_A(A\,, -). \quad \square$$

In the next corollary, we use for the first time a notation that we follow until the end of these notes. Because we deal with modules over two algebras, in order to avoid confusion, we denote A-modules by $L, M, N, \ldots$ and R-modules by $X, Y, Z, \ldots$

Corollary 3.1.2. *The following conditions are equivalent for two A-modules L and M:*

(a) $L \cong M$ *(b)* $L \otimes_A R \cong M \otimes_A R$ *(c)* $\operatorname{Hom}_A(R\,, L) \cong \operatorname{Hom}_A(R\,, M)$. $\square$

Lemma 3.1.3. *(a) An R-module X is projective if and only if:*
i) $X \otimes_R A$ *is projective in* $\operatorname{mod} A$*, and*
ii) $X \otimes_R A \otimes_A R \cong X$ *in* $\operatorname{mod} R$.

Moreover, in this case, X is indecomposable if and only if so is $X \otimes_R A$.

(b) An R-module Y is injective if and only if:
   i) $\mathrm{Hom}_R(A\,,\,Y)$ is injective in $\mathrm{mod}\,A$, and
   ii) $\mathrm{Hom}_A\big({}_RR_A\,,\,\mathrm{Hom}_A(A\,,\,Y)\big) \cong Y$ in $\mathrm{mod}\,R$.

Moreover, in this case, Y is indecomposable if and only if so is $\mathrm{Hom}_R(A\,,\,Y)$.

Proof. (a) Let $e \in R$ be an idempotent such that $X = eR$. Then $X \otimes_R A = eR \otimes_R A = eA$ is projective in $\mathrm{mod}\,A$. Also, $X \otimes_R A \otimes_A R \cong eA \otimes_A R \cong eR = X$. So X satisfies i) and ii). Conversely, if X satisfies i) and ii), there exists an idempotent e such that $X \otimes_R A = eA$. But then ii) gives $X \cong X \otimes_R A \otimes_A R \cong eA \otimes_A R \cong eR$. This establishes the first statement.

Assume that X is decomposable, say $X = X_1 \oplus X_2$ with X_1, X_2 nonzero, but that $X \otimes_R A = (X_1 \otimes_R A) \oplus (X_2 \otimes_R A)$ is indecomposable. Then one of the summands is zero, say $X_1 \otimes_R A = 0$. But then $X_1 \cong X_1 \otimes_R A \otimes_A R = 0$, a contradiction. Therefore X is indecomposable. Similarly, X indecomposable implies $X \otimes_R A$ indecomposable. $\square$

Thus, there exists a bijection between isoclasses of indecomposable projective A- and R-modules given by $eA \mapsto eR$, where e is a primitive idempotent. Also, there exists a similar bijection for the injectives.

Because $A = R/E$, the category $\mathrm{mod}\,A$ may be identified with the full subcategory of $\mathrm{mod}\,R$ of the modules X such that $XE = 0$. Given any R-module X, there exists a largest R-submodule of X which is annihilated by E, that is, which is an A-module. This is $K_X = \{\, x \in X \mid xE = 0 \,\}$.

Lemma 3.1.4. *Let X be an R-module. We have functorial isomorphisms:*
(a) $X \otimes_R A \cong X/XE$,
(b) $\mathrm{Hom}_R(A\,,\,X) \cong K_X$.

Proof. (a) Applying $X \otimes_R -$ to the exact sequence of R-R-bimodules

$$0 \longrightarrow E \xrightarrow{\iota} R \xrightarrow{\pi} A \longrightarrow 0 \qquad (*)$$

yields a commutative diagram with exact rows in $\mathrm{mod}\,R$

$$\begin{tikzcd} & X \otimes_R E \arrow[r] \arrow[d, "\mu'"] & X \otimes_R R \arrow[r] \arrow[d, "\mu"] & X \otimes_R A \arrow[r] \arrow[d, "\mu''"] & 0 \\ 0 \arrow[r] & XE \arrow[r, "i"] & X \arrow[r, "p"] & X/XE \arrow[r] & 0 \end{tikzcd}$$

where i, p are respectively the inclusion and projection, μ, μ' are the multiplication maps $x \otimes r \mapsto xr$ and μ'' is induced by passing to cokernels.

Clearly, μ' is surjective, because of the definition of XE. It is well-known that μ is an isomorphism. Therefore the snake lemma implies that μ'' is injective and also surjective. So it is an isomorphism.

(b) Let $f\colon \mathrm{Hom}_R(R\,,\,X) \longrightarrow X\,,\; u \mapsto u(1)$ be the well-known functorial isomorphism. For every $u \in \mathrm{Hom}_R(A\,,\,X)$, we have

$$f\,\mathrm{Hom}_A(\pi\,,\,X)(u) = f(u\pi) = u\pi(1).$$

So, for $x \in E$, we have $u\pi(1)x = u\pi(x) = 0$. Therefore, the image of the composition $f\,\mathrm{Hom}_A(\pi\,,\,X)$ lies in K_X. That is, there exists $f'\colon \mathrm{Hom}_R(A\,,\,X) \longrightarrow K_X$ making the following square commutative

$$\begin{tikzcd} \mathrm{Hom}_R(A\,,\,X) \arrow[r, "\mathrm{Hom}_A(\pi\,,\,X)"] \arrow[d, "f'"'] & \mathrm{Hom}_R(R\,,\,X) \arrow[d, "f"] \\ K_X \arrow[r, hook, "j"] & X \end{tikzcd}$$

where j is the inclusion. Applying $\mathrm{Hom}_R(-\,,\,X)$ to the exact sequence $(*)$ above shows that $\mathrm{Hom}_R(\pi\,,\,X)$ is injective. Therefore so is $f\,\mathrm{Hom}_R(\pi\,,\,X)$ and so is f'.

We prove that f' is surjective. Let $x \in K_X$. Because $x \in X$, there exists $u_x \in \mathrm{Hom}_R(R\,,\,X)$ such that $x = u_x(1)$. But then $u_x(E) = u_x(1)E = xE = 0$ hence there exists $v_x \colon A \longrightarrow X$ such that $u_x = v_x\pi$. Then $x = u_x(1) = v_x\pi(1) = f'(v_x)$ and so f' is surjective. Therefore it is an isomorphism. $\square$

3.2. Projective covers and injective envelopes. For the notions of superfluous epimorphisms and essential monomorphisms, we refer the reader, for example, to [**3**].

Lemma 3.2.1 [**11**](1.1). *Let X be an R-module.*

(a) The canonical epimorphism $p_X \colon X \longrightarrow X/XE$ is superfluous.
(b) The canonical monomorphism $j_X \colon K_X \hookrightarrow X$ is essential.

Proof. (a) Because of Nakayama's lemma, the canonical epimorphism $f \colon X \longrightarrow X/X \cdot \mathrm{rad}\,R$ is superfluous. Because $E \subseteq \mathrm{rad}\,R$, there exists an epimorphism $g \colon X/XE \longrightarrow X/X \cdot \mathrm{rad}\,R$ such that $f = gp_X$. Assume h is such that $p_X h$ is an epimorphism. Then so is $fh = gp_X h$. Because f is superfluous, h is an epimorphism.
(b) Let Y be a nonzero submodule of X. Because E is nilpotent, there exists $s \geq 1$ such that $YE^{s-1} \neq 0$ but $YE^s = 0$. Let $y \in YE^{s-1}$ be nonzero. Then $yE = 0$ so that $y \in K_X$. Therefore $K_X \cap Y \neq 0$ and we are done. $\square$

Corollary 3.2.2. [**11**](1.2) *Let M be an A-module.*

(a) There is a bijection between the isoclasses of indecomposable summands of M in $\mathrm{mod}\,A$ and $M \otimes_A R$ in $\mathrm{mod}\,R$, given by $N \longmapsto N \otimes_A R$.
(b) There is a bijection between the isoclasses of indecomposable summands of M in $\mathrm{mod}\,A$ and $\mathrm{Hom}_A(R\,,\,M)$ in $\mathrm{mod}\,R$, given by $N \longmapsto \mathrm{Hom}_A(R\,,\,N)$.

Proof. (a) Suppose N is indecomposable in $\mathrm{mod}\,A$ but $N \otimes_A R = X_1 \oplus X_2$ in $\mathrm{mod}\,R$. Then $N \cong N \otimes_A R \otimes_R A \cong (X_1 \otimes_R A) \oplus (X_2 \otimes_R A)$. Because N is indecomposable, $X_1 \otimes_R A$, say, is zero. So $X_1/X_1E = 0$. But p_{X_1} is superfluous so $X_1 = 0$. Thus $N \otimes_A R$ is indecomposable. The rest of the proof is an application of 3.1.2. $\square$

Lemma 3.2.3. [**11**](1.3) *Let M be an A-module.*

(a) If $f \colon P \longrightarrow M$ is a projective cover in $\mathrm{mod}\,A$, then $f \otimes_A R \colon P \otimes_A R \longrightarrow M \otimes_A R$ is a projective cover in $\mathrm{mod}\,R$.
(b) If $g \colon M \longrightarrow I$ is an injective envelope in $\mathrm{mod}\,A$, then $\mathrm{Hom}_A(R\,,\,g) \colon \mathrm{Hom}_A(R\,,\,M) \longrightarrow \mathrm{Hom}_A(R\,,\,I)$ is an injective envelope in $\mathrm{mod}\,R$.

Proof. (a) Clearly, $P \otimes_A R$ is projective in $\operatorname{mod} R$ and $f \otimes_A R$ is an epimorphism. Consider the commutative square:

$$\begin{tikzcd} P \otimes_A R \arrow[r, "f \otimes_A R"] \arrow[d, "p_{P\otimes_A R}"'] & M \otimes_A R \arrow[d, "p_{M\otimes_A R}"] \\ P \arrow[r, "f"] & M \end{tikzcd}$$

where we have used that $(M \otimes_A R)/(M \otimes_A R)E \cong M \otimes_A R \otimes_R A \cong M$ and similarly for P. It suffices to prove that $f \otimes_A R$ is superfluous. Let h be such that $(f \otimes_A R)h$ is an epimorphism. Then $p_{M\otimes_A R}(f \otimes_A R)h = f p_{P\otimes_A R} h$ is an epimorphism. Because both f and $p_{P\otimes_A R}$ are superfluous, h is an epimorphism. □

We have a similar result when passing from $\operatorname{mod} R$ to $\operatorname{mod} A$.

Lemma 3.2.4. [**14**](3.1) *Let X be an R-module.*

(a) If $f : \tilde{P} \longrightarrow X$ is a projective cover in $\operatorname{mod} R$, then $f \otimes_R A \colon \tilde{P} \otimes_R A \longrightarrow X \otimes_R A$ is a projective cover in $\operatorname{mod} A$.

(b) If $g \colon X \longrightarrow \tilde{I}$ is an injective envelope in $\operatorname{mod} R$, then $\operatorname{Hom}_R(A, g) \colon \operatorname{Hom}_R(A, X) \longrightarrow \operatorname{Hom}_R\left(A, \tilde{I}\right)$ is an injective envelope in $\operatorname{mod} A$.

Proof. (a) First $\tilde{P} \otimes_R A$ is projective in $\operatorname{mod} A$, see 3.1.3, and $f \otimes_R A$ is an epimorphism. Next,

$$\begin{aligned} \operatorname{top}(\tilde{P} \otimes_R A) = \operatorname{top}(\tilde{P}/\tilde{P}E) &\cong \frac{\tilde{P}/\tilde{P}E}{\tilde{P}/\tilde{P}E \cdot \operatorname{rad} A} \cong \frac{\tilde{P}/\tilde{P}E}{\tilde{P}/\tilde{P}E \cdot \operatorname{rad}(R/E)} \\ &\cong \frac{\tilde{P}/\tilde{P}E}{(\tilde{P} \cdot \operatorname{rad} R)/\tilde{P}E} \cong \frac{\tilde{P}}{\tilde{P} \cdot \operatorname{rad} A} \cong \frac{X}{X \cdot \operatorname{rad} R} \cong \frac{X/XE}{X/XE \cdot (\operatorname{rad} R)/E} \\ &\cong \operatorname{top}(X \otimes_R A) \end{aligned}$$

where the last isomorphism comes from 3.1.4. □

3.3. Presentations. We now compute minimal projective presentations and injective copresentations of R-modules

Corollary 3.3.1 [**11**](1.3). *Let M be an A-module.*

(a) If $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ is a projective presentation in $\operatorname{mod} A$, then $P_1 \otimes_A R \xrightarrow{f_1 \otimes_A R} P_0 \otimes_A R \xrightarrow{f_0 \otimes_A R} M \otimes_A R \longrightarrow 0$ is a projective presentation in $\operatorname{mod} R$. Further, if the first is minimal, then so is the second.

(b) If $0 \longrightarrow M \xrightarrow{g^0} I^0 \xrightarrow{g^1} I^1$ is an injective copresentation in $\operatorname{mod} A$, then $0 \longrightarrow \operatorname{Hom}_A(R, M) \xrightarrow{\operatorname{Hom}_A(R, g^0)} \operatorname{Hom}_A\left(R, I^0\right) \xrightarrow{\operatorname{Hom}_A(R, g^1)} \operatorname{Hom}_A\left(R, I^1\right)$ is an injective copresentation in $\operatorname{mod} R$. Further, if the first is minimal, then so is the second.

Proof. (a) The first statement is clear. If the given projective presentation of M is minimal, then, because of 3.2.3, $f_0 \otimes_A R \colon P_0 \otimes_A R \longrightarrow M \otimes_A R$ is a projective cover in $\operatorname{mod} R$. Because $f_1 \colon P_1 \longrightarrow f_1(P_1)$ is a projective cover in $\operatorname{mod} A$, so is $f_1 \otimes_A R \colon P_1 \otimes_A R \longrightarrow f_1(P_1) \otimes_A R \cong (f_1 \otimes_A R)(P_1 \otimes_A R) \cong \operatorname{Ker}(f_0 \otimes_A R)$ in $\operatorname{mod} R$. □

Clearly, if $\tilde{P}_1 \longrightarrow \tilde{P}_0 \longrightarrow X \longrightarrow 0$ is a projective presentation of X in $\operatorname{mod} R$, then $\tilde{P}_1 \otimes_R A \longrightarrow \tilde{P}_0 \otimes_R A \longrightarrow X \otimes_R A \longrightarrow 0$ is a projective presentation in $\operatorname{mod} A$.

But here, the minimality of the first presentation does not imply that of the second.

Examples 3.3.2. Let A be given by the quiver

$$1 \circ \xleftarrow{\alpha} \circ\ 2$$

and R by the quiver

$$\begin{tikzcd} 1 \circ \arrow[r, shift left, "\beta"] & \circ\ 2 \arrow[l, shift left, "\alpha"] \end{tikzcd}$$

bound by $\alpha\beta\alpha = 0$, $\beta\alpha\beta = 0$. The simple R-module S_1 has a minimal projective presentation

$$e_2 R \longrightarrow e_1 R \longrightarrow S_1 \longrightarrow 0.$$

Applying $- \otimes_R A$ yields a projective presentation

$$e_2 A \longrightarrow e_1 A \longrightarrow S_1 \otimes_R A \longrightarrow 0.$$

But $\operatorname{Hom}_A(e_2 A\,,\, e_1 A) = 0$, hence $S_1 \otimes_R A \cong e_1 A$ and the previous presentation is not minimal.

We need, for later purposes, to compute the minimal projective presentation of an A-module, considered as an R-module under the embedding $\operatorname{mod} A \hookrightarrow \operatorname{mod} R$.

Lemma 3.3.3. *Let M be an A-module.*

(a) If $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ is a minimal projective presentation of M in $\operatorname{mod} A$ and P the projective cover of $P_0 \otimes_A E_A$ in $\operatorname{mod} A$, then there exists a direct summand P' of P such that

$$(P_1 \oplus P') \otimes_A R \longrightarrow P_0 \otimes_A R \longrightarrow M \longrightarrow 0$$

is a minimal projective presentation in $\operatorname{mod} R$.

(b) If $0 \longrightarrow M \xrightarrow{g^0} I^0 \xrightarrow{g^1} I^1$ is a minimal injective copresentation of M in $\operatorname{mod} A$, and I the injective envelope of $\operatorname{Hom}_A\big(E\,,\, I^0\big)$ in $\operatorname{mod} A$, then there exists a direct summand I' of I such that

$$0 \longrightarrow M \longrightarrow \operatorname{Hom}_A\big(R\,,\, I^0\big) \longrightarrow \operatorname{Hom}_A\big(R\,,\, I^1 \oplus I'\big)$$

is a minimal injective copresentation in $\operatorname{mod} R$.

Proof. (a) Let $p_{M\otimes_A R}\colon M \otimes_A R \longrightarrow M$ be the canonical surjection. Because of 3.2.1, it is a superfluous epimorphism. Because of 3.2.3, so is $f_0 \otimes_A R\colon P_0 \otimes_A R \longrightarrow M \otimes_A R$. Then their composition $p_{M\otimes_A R}(f_0 \otimes_A R)$ is a superfluous epimorphism, hence it is a projective cover in $\operatorname{mod} R$.

As A-modules, we have $P_0 \otimes_A R \cong P_0 \otimes_A (A \oplus E) \cong P_0 \oplus (P_0 \otimes_A E)$ and similarly $(M \otimes_A R)_A \cong M \oplus (M \otimes_A E)$. The morphism $f_0 \otimes_A R$ then takes the form $\begin{pmatrix} f_0 & 0 \\ 0 & f_0 \otimes_A E \end{pmatrix}$. Because $p_{M\otimes_A R}\colon x \otimes (a, e) \longmapsto xa$, for $x \in M$ and $(a\,,\, e) \in R$, we get $p_{M\otimes_A R}(f_0 \otimes_A R) = (f_0\,,\, 0)$.

Let $\tilde{P}_1$ be the projective cover of $\operatorname{Ker}\big(p_{M\otimes_A R}(f_0 \otimes_A R)\big) = \Omega^1_R M$. Because $p_{M\otimes_A R}(f_0 \otimes_A R) = (f_0, 0)$, then $P_0 \otimes_A E$ is actually a direct summand of $\Omega^1_R M$, when the latter is viewed as A-module. In fact, $\Omega^1_R M \cong \Omega^1_A M \oplus (P_0 \otimes_A E)$ in

$\operatorname{mod} A$. The projective cover of $\Omega^1_A M$ in $\operatorname{mod} A$ is P_1, while that of $P_0 \otimes_A E$ is P. Then, we have a commutative diagram in $\operatorname{mod} R$ with exact rows

$$\begin{tikzcd}
P_1 \otimes_A R \arrow[r, "f_1 \otimes_A R"] \arrow[d, "\binom{1}{0}"'] & P_0 \otimes_A R \arrow[r, "f_0 \otimes_A R"] \arrow[d, equal] & M \otimes_A R \arrow[r] \arrow[d, "p_{M \otimes_A R}"] & 0 \\
(P_1 \oplus P) \otimes_A R \arrow[r, "(f_1 \otimes_A R\, ,\, \tilde{f})"] & P_0 \otimes_A R \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where $\tilde{f}\colon P \otimes_A R \longrightarrow P_0 \otimes_A R$ is the composition of the embedding $P_0 \otimes_A E_R \longrightarrow P_0 \otimes_A R_R$, induced from the embedding ${}_A E_R \subseteq {}_A R_R$ because of the projectivity of P_0, with the projective cover $P_A \longrightarrow P_0 \otimes_A E$ in $\operatorname{mod} A$.

The lower row in the preceding diagram is a projective presentation in $\operatorname{mod} R$, but is not necessarily minimal. Assume P'' is a direct summand of $P_1 \oplus P$ such that we have a minimal projective presentation in $\operatorname{mod} R$

$$P'' \otimes_A R \longrightarrow P_0 \otimes_A R \longrightarrow M \longrightarrow 0.$$

Because M is an A-module, it is annihilated by E when viewed as R-module. Hence $M \cong M \otimes_R A$ because of 3.1.4. Applying $- \otimes_R A$ to the previous presentation yields a commutative diagram with exact rows in $\operatorname{mod} A$

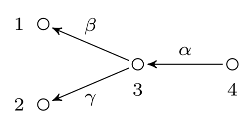
$$\begin{tikzcd}
P'' \arrow[r] & P_0 \arrow[r, "f_0"] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\
P_1 \arrow[r, "f_1"] & P_0 \arrow[r, "f_0"] & M \arrow[r] & 0.
\end{tikzcd}$$

Because P_1 is the projective cover of $\Omega^1_A M$, there exists an epimorphism $P'' \longrightarrow P_1$ making the diagram commute. Therefore $P'' = P_1 \oplus P'$ and we have a minimal projective presentation in $\operatorname{mod} R$

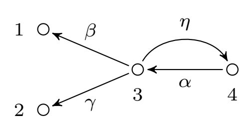
$$(P_1 \oplus P') \otimes_A R \xrightarrow{(f_1 \otimes_A R\, ,\, \tilde{f}')} P_0 \otimes_A R \xrightarrow{p_{M \otimes_A R}(f_0 \otimes_A R)} M \longrightarrow 0$$

where $\tilde{f}'$ is the restriction of $\tilde{f}$ to $P' \otimes_A R$. $\square$

Example 3.3.4. Let A be given by the quiver



bound by $\alpha\beta = 0$, $\alpha\gamma = 0$, and R be given by the quiver



bound by $\alpha\beta = 0$, $\alpha\gamma = 0$, $\eta\alpha\eta\alpha = 0$. Then R is the split extension of A by the nilpotent bimodule E generated by η. The indecomposable (injective) module $M_A = \begin{smallmatrix}3\\2\end{smallmatrix}$ has the minimal projective presentation in $\operatorname{mod} A$

$$0 \longrightarrow e_1 A = {\scriptstyle 1} \longrightarrow e_3 A = \begin{smallmatrix} & 3 & \\ 2 & & 1 \end{smallmatrix} \longrightarrow M \longrightarrow 0.$$

Because of 3.3.1, a minimal projective presentation for $M \otimes_A R$ is given by

$$e_1 R = {\scriptstyle 1} \xrightarrow{\ f\ } e_3 R = \begin{smallmatrix} & & 3 & \\ 1 & 2 & & 4 \\ & & & 3 \\ & & & 4 \end{smallmatrix} \longrightarrow M \otimes_A R \longrightarrow 0.$$

Then, $M \otimes_A R \cong \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ & & 3 \\ & & 4 \end{smallmatrix}$ and also f is a monomorphism, so that $\operatorname{pd}(M \otimes_A R) \leq 1$. Considering $M \otimes_A R$ as an A-module, we get $(M \otimes_A R)_A \cong \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus {\scriptstyle 4}$. In particular $M \otimes_A E_A = \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \oplus {\scriptstyle 4}$ has as projective cover $P = \left(\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right)^2$. Therefore there exists a projective presentation in $\operatorname{mod} R$

$$(e_4 R)^2 \oplus e_1 R = \begin{pmatrix} \begin{smallmatrix} 4 \\ 3 \\ 4 \\ 3 \\ 4 \end{smallmatrix} \end{pmatrix}^2 \oplus {\scriptstyle 1} \longrightarrow e_3 R = \begin{smallmatrix} & & 3 & \\ 1 & 2 & & 4 \\ & & & 3 \\ & & & 4 \end{smallmatrix} \longrightarrow M = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \longrightarrow 0.$$

It is not minimal, but letting $P' = e_4 A = \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$, we get a minimal projective presentation in $\operatorname{mod} R$

$$e_4 R \oplus e_1 R \longrightarrow e_3 R \longrightarrow M \longrightarrow 0.$$

3.4. Homological dimension one. Working with homological dimension one is easier than with other dimensions, due to its connection with the Auslander-Reiten translation, see [**12**](IV.2.7).

Lemma 3.4.1 [**11**](2.1)**.** *For any A-module M, we have*

(a) $\tau_R(M \otimes_A R) \cong \operatorname{Hom}_A({}_R R_A\,,\, \tau_A M)$

(b) $\tau_R^{-1} \operatorname{Hom}_A({}_R R_A\,,\, M) \cong (\tau_A^{-1} M) \otimes_A R.$

Proof. (a) A minimal projective presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in $\operatorname{mod} A$ induces, because of 3.3.1, a minimal projective presentation

$$P_1 \otimes_A R \longrightarrow P_0 \otimes_A R \longrightarrow M \otimes_A R \longrightarrow 0.$$

We deduce a commutative diagram with exact rows in $\operatorname{mod} R^{\mathrm{op}}$

$$\begin{array}{ccccccc}
\operatorname{Hom}_R(P_0 \otimes_A R\,,\, R) & \longrightarrow & \operatorname{Hom}_R(P_1 \otimes_A R\,,\, R) & \longrightarrow & \operatorname{Tr}(M \otimes_A R) & \longrightarrow & 0 \\
\cong \big\downarrow f & & \cong \big\downarrow g & & \big\downarrow h & & \\
R \otimes_A \operatorname{Hom}_A(P_0\,,\, A) & \longrightarrow & R \otimes_A \operatorname{Hom}_A(P_1\,,\, A) & \longrightarrow & R \otimes_A \operatorname{Tr} M & \longrightarrow & 0
\end{array}$$

where the functorial isomorphisms f, g are defined as follows: if e is an idempotent, then $\operatorname{Hom}_R(eA \otimes_A R\,,\, R) \cong \operatorname{Hom}_R(eR\,,\, R) \cong Re \cong R \otimes_A Ae \cong R \otimes_A \operatorname{Hom}_A(eA\,,\, A)$. Then h is deduced by passing to the cokernels and so is an isomorphism. We thus have

$$\begin{aligned} \tau_R(M \otimes_A R) &\cong \mathrm{D}\operatorname{Tr}(M \otimes_A R) \cong \mathrm{D}(R \otimes_A \operatorname{Tr} M) \\ &\cong \operatorname{Hom}_A(R\,,\, \mathrm{D}\operatorname{Tr} M) \cong \operatorname{Hom}_A(R\,,\, \tau_A M). \end{aligned}$$ □

Corollary 3.4.2 [**11**]2.2**.** *For any A-module M, we have*

(a) $\operatorname{pd}(M \otimes_A R) \leq 1$ *if and only if* $\operatorname{pd} M_A \leq 1$ *and* $\operatorname{Hom}_A(\mathrm{D}E\,,\, \tau_A M) = 0$

(b) $\operatorname{id} \operatorname{Hom}_A(R\,,\, M) \leq 1$ *if and only if* $\operatorname{id} M_A \leq 1$ *and* $\operatorname{Hom}_A\left(\tau_A^{-1} M\,,\, E\right) = 0.$

Proof. (a) Because of [**12**](IV.2.7), $\operatorname{pd}(M \otimes_A R) \leq 1$ if and only if $\operatorname{Hom}_R\big(\mathrm{D}R\,,\, \tau_R(M \otimes_A R)\big) = 0$. Now, we have

$$\begin{aligned}
\operatorname{Hom}_R\big(\mathrm{D}R\,,\, \tau_R(M \otimes_A R)\big) &\cong \operatorname{Hom}_R\big(\mathrm{D}R\,,\, \operatorname{Hom}_A(R\,,\, \tau_A M)\big)\\
&\cong \operatorname{Hom}_A(\mathrm{D}R \otimes_R R_A\,,\, \tau_A M)\\
&\cong \operatorname{Hom}_A(\mathrm{D}R_A\,,\, \tau_A M)\\
&\cong \operatorname{Hom}_A(\mathrm{D}A\,,\, \tau_A M) \oplus \operatorname{Hom}_A(\mathrm{D}E\,,\, \tau_A M).
\end{aligned}$$

The result follows from another application of [**12**](IV.2.7). $\square$

Example 3.4.3. We give an example showing that both conditions are necessary. Let A be the path algebra of the quiver

$$\begin{tikzcd}
\underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\beta"'] & \underset{3}{\circ} \arrow[l, "\alpha"']
\end{tikzcd}$$

and R be given by the quiver

$$\begin{tikzcd}
\underset{1}{\circ} \arrow[rr, bend left, "\eta"] & \underset{2}{\circ} \arrow[l, "\beta"] & \underset{3}{\circ} \arrow[l, "\alpha"]
\end{tikzcd}$$

bound by $\alpha\beta\eta = 0$. In this case, one easily sees that $E_A = \left(\begin{smallmatrix}3\\2\\1\end{smallmatrix}\right)^2$, $(\mathrm{D}E)_A = \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)^3$. Let $M = \begin{smallmatrix}3\\2\end{smallmatrix}$. We have a minimal projective presentation of M in $\operatorname{mod} A$

$$0 \longrightarrow e_1A \longrightarrow e_3A \longrightarrow M \longrightarrow 0.$$

Applying $- \otimes_A R$ yields a minimal projective presentation in $\operatorname{mod} R$

$$e_1R \xrightarrow{f} e_3R \longrightarrow M \otimes_A R \longrightarrow 0.$$

Therefore, $M \otimes_A R \cong \begin{smallmatrix}3\\2\end{smallmatrix}$. The projective dimension of $\begin{smallmatrix}3\\2\end{smallmatrix}$ in $\operatorname{mod} R$ equals 2. Indeed $\operatorname{Ker} f = e_3R$, so that $\operatorname{pd} M_A \leq 1$ but $\operatorname{pd}(M \otimes_A R) > 1$. This shows that the second condition of the corollary is necessary. Actually, $\tau_A\left(\begin{smallmatrix}3\\2\end{smallmatrix}\right) = \begin{smallmatrix}2\\1\end{smallmatrix}$ so $\operatorname{Hom}_A(\mathrm{D}E\,,\, \tau_A M) = 0$.

Lemma 3.4.4 [**14**](2.3)**.** *Let M be an A-module.*

(a) If $\operatorname{pd} M_R \leq 1$, then $\operatorname{pd} M_A \leq 1$.
(b) If $\operatorname{id} M_R \leq 1$, then $\operatorname{id} M_A \leq 1$.

Proof. (a) Because of 3.4.2, it suffices to prove that $\operatorname{pd}(M \otimes_A R) \leq 1$. Let $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ be a minimal projective presentation. Because of 3.3.1, and with its notation, there exists a commutative diagram with exact rows

$$\begin{tikzcd}
P_1 \otimes_A R \arrow[r, "f_1 \otimes_A R"] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & P_0 \otimes_A R \arrow[r, "f_0 \otimes_A R"] \arrow[d, equal] & M \otimes_A R \arrow[r] \arrow[d] & 0\\
(P_1 \oplus P') \otimes_A R \arrow[r, "(f_1 \otimes_A R, \tilde{f}')"] & P_0 \otimes_A R \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Because $\operatorname{pd} M_R \leq 1$, the morphism $(f_1 \otimes_A R, \tilde{f}')$ is injective. Because so is $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, the morphism $f_1 \otimes_A R$ is injective. Therefore $\operatorname{pd}(M \otimes_A R) \leq 1$, thus establishing our claim. $\square$

An easy application of this lemma is the following: if R is hereditary, then so is A. But we have a much stronger result, due to Suarez.

Theorem 3.4.5 [**33**](3.2)(3.5)**.** *Let R be a split extension of A by a nilpotent bimodule E. Then* $\mathrm{gl.\,dim.}\,A \leq \mathrm{gl.\,dim.}\,R \leq \mathrm{gl.\,dim.}\,A + \mathrm{pd}\,A_R$. □

We can also apply 3.4.4 to the study of the left and right parts of an algebra. Recall from [**25**] that, if C is an algebra, the ***left part*** $\mathscr{L}_C$ of $\mathrm{mod}\,C$ is the full subcategory of $\mathrm{ind}\,C$ consisting of those indecomposable modules U such that, if there exists V indecomposable and a sequence of nonzero morphisms between indecomposable C-modules

$$V = V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_m = U$$

then $\mathrm{pd}\,V_C \leq 1$. One defines dually the ***right part*** $\mathscr{R}_C$ of $\mathrm{mod}\,C$.

Lemma 3.4.6 [**14**](2.4)**.** *Let M be an indecomposable A-module.*

(a) If $M \otimes_A R$ belongs to $\mathscr{L}_R$, then M belongs to $\mathscr{L}_A$.
(b) If $\mathrm{Hom}_A(R\,,\,M)$ belongs to $\mathscr{R}_R$, then M belongs to $\mathscr{R}_A$.
(c) If $M \otimes_A R$ belongs to $\mathscr{R}_R$, then M belongs to $\mathscr{R}_A$.
(d) If $\mathrm{Hom}_A(R\,,\,M)$ belongs to $\mathscr{L}_R$, then M belongs to $\mathscr{L}_A$.

Proof. (a) Let $L = L_0 \xrightarrow{f_1} L_1 \longrightarrow \cdots \xrightarrow{f_m} L_m = M$ be a sequence of nonzero morphisms between indecomposable A-modules. For each i, $L_i \otimes_A R$ is indecomposable and $f_i \otimes_A R$ is nonzero. So we have a sequence of nonzero morphisms between indecomposable R-modules

$$L \otimes_A R = L_0 \otimes_A R \xrightarrow{f_1 \otimes_A R} L_1 \otimes_A R \longrightarrow \cdots \xrightarrow{f_m \otimes_A R} L_m \otimes_A R = M \otimes_A R.$$

Because $M \otimes_A R \in \mathscr{L}_R$, then $\mathrm{pd}(L \otimes_A R) \leq 1$. Because of 3.4.2, we get $\mathrm{pd}\,L_A \leq 1$.

(c) We have isomorphisms of k-vector spaces

$$\begin{aligned}\mathrm{Hom}_R\big(M \otimes_A R\,,\, \mathrm{Hom}_A({}_R R_A\,,\,M)\big) &\cong \mathrm{Hom}_A(M \otimes_A R \otimes_R R_A\,,\,M)\\ &\cong \mathrm{Hom}_A(M \otimes_A R_A\,,\,M)\\ &\cong \mathrm{Hom}_A\big(M \otimes_A (A \oplus E)\,,\,M\big)\\ &\cong \mathrm{Hom}_A(M\,,\,M) \oplus \mathrm{Hom}_A(M \otimes_A E\,,\,M).\end{aligned}$$

Because $\mathrm{Hom}_A(M\,,\,M) \neq 0$, there exists a nonzero morphism $M \otimes_A R \longrightarrow \mathrm{Hom}_A(R\,,\,M)$. Now $M \otimes_A R \in \mathscr{R}_R$, which is closed under successors. Hence $\mathrm{Hom}_A(R\,,\,M) \in \mathscr{R}_R$. Applying (b), which is proved exactly as (a), we get $M \in \mathscr{R}_A$, as required. □

We now consider different classes of algebras. An algebra C is called ***laura*** if $\mathscr{L}_C \cup \mathscr{R}_C$ is cofinite in $\mathrm{ind}\,C$, see [**9**] or [**29**]. It is ***left glued*** if $\mathscr{L}_C$ is cofinite in $\mathrm{ind}\,C$, see [**8**]. ***Right glued*** algebras are defined similarly. An algebra C is ***weakly shod*** if the length of any path from an indecomposable not in $\mathscr{L}_C$ to one not in $\mathscr{R}_C$ is bounded, see [**23**]. It is ***shod*** if every indecomposable has projective dimension or injective dimension at most one, see [**22**]. It is ***quasi-tilted*** if $C_C \in \mathscr{L}_C$, see [**25**]. For ***tilted*** algebras, we refer to [**12**], Chapter VIII. The algebra C is ***right ada*** if $C_C \in \mathrm{add}(\mathscr{L}_C \cup \mathscr{R}_C)$ and ***left ada*** if $\mathrm{D}C_C \in \mathrm{add}(\mathscr{L}_C \cup \mathscr{R}_C)$, see [**1**]. Finally, C is ***ada*** if it is both right and left ada, see [**7**].

Theorem 3.4.7 [**14**](2.5) [**35**](1.10) [**1**](3.6) [**7**](2.9)**.** *Let R be a split extension of A by the nilpotent bimodule E.*

(a) If R is laura, then so is A.
(b) If R is right or left glued, then so is A.

(c) If R is weakly shod, then so is A.
(d) If R is shod, then so is A.
(e) If R is quasi-tilted, then so is A.
(f) If R is tilted, then so is A.
(g) If R is right or left ada, then so is A.
(h) If R is ada, then so is A.

Proof. (a) Because of 3.4.6, if an indecomposable A-module M does not lie in $\mathscr{L}_A \cup \mathscr{R}_A$, then $M \otimes_A R \notin \mathscr{L}_R \cup \mathscr{R}_R$. Because R is laura, $\mathscr{L}_R \cup \mathscr{R}_R$ is cofinite in $\operatorname{ind} R$. Therefore, $\mathscr{L}_A \cup \mathscr{R}_A$ is cofinite in $\operatorname{ind} A$.
(b) is proved in the same way.
(c) Let $M_0 \notin \mathscr{L}_A$, $M_t \notin \mathscr{R}_A$ be indecomposable A-modules. As seen in 3.4.6, a sequence of nonzero morphisms between indecomposable R-modules $M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t$ induces a sequence of nonzero morphisms between indecomposable R-modules $M_0 \otimes_A R \longrightarrow M_1 \otimes_A R \longrightarrow \cdots \longrightarrow M_t \otimes_A R$. Because R is weakly shod, t is bounded.
(d) Let M be an indecomposable A-module. Because R is shod, $\operatorname{pd} M_R \leq 1$ or $\operatorname{id} M_R \leq 1$. Then 3.4.4 gives $\operatorname{pd} M_A \leq 1$ or $\operatorname{id} M_A \leq 1$.
(e) Let P be an indecomposable projective A-module. Then $P \otimes_A R$ is an indecomposable projective R-module. Because R is quasi-tilted, $P \otimes_A R \in \mathscr{L}_R$. Because of 3.4.6, $P \in \mathscr{L}_A$.
(f) We refer the reader to [**35**].
(g) Let P be an indecomposable projective A-module. If R is right ada, then $P \otimes_A R \in \mathscr{L}_R \cup \mathscr{R}_R$. Because of 3.4.6, $P \in \mathscr{L}_A$.
(h) Follows from (g). □

3.5. Almost split sequences. We now look for a criterion allowing to verify when an almost split sequence in $\operatorname{mod} A$ embeds as an almost split sequence in $\operatorname{mod} R$.

Lemma 3.5.1 [**13**](1.1). *Let M be an indecomposable A-module.*
(a) Let P_0 be a projective cover of M and P a projective cover of $P_0 \otimes_A E$ in $\operatorname{mod} A$. Then there exist a direct summand P' of P and an exact sequence in $\operatorname{mod} A$

$$0 \longrightarrow \tau_A M \oplus \operatorname{Hom}_A(E, \tau_A M) \longrightarrow \tau_R M \longrightarrow P' \otimes_A \mathrm{D}R \longrightarrow \operatorname{Ker}(p_{M\otimes_A R} \otimes_A \mathrm{D}R) \longrightarrow 0.$$

(b) Let I_0 be an injective envelope of M and I an injective envelope of $\operatorname{Hom}_A(E\,,\, I^0)$ in $\operatorname{mod} A$. Then there exist a direct summand I' of I and an exact sequence in $\operatorname{mod} A$

$$0 \to \operatorname{Coker} \operatorname{Hom}_A(\mathrm{D}R, j_{\operatorname{Hom}_A(R,M)}) \to \operatorname{Hom}_A\big(\mathrm{D}R, I'\big) \to \tau_R^{-1} M \to \tau_A^{-1} M \oplus (\tau_A^{-1} M \otimes_A E) \to 0.$$

Proof. (a) Let $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ be a minimal projective presentation in $\operatorname{mod} A$. Because of 3.3.1, we have a minimal projective presentation

$$P_1 \otimes_A R \xrightarrow{f_1 \otimes_A R} P_0 \otimes_A R \xrightarrow{f_0 \otimes_A R} M \otimes_A R \longrightarrow 0$$

in $\operatorname{mod} R$. Because of 3.3.3, there exists a direct summand P' of P such that we have a commutative diagram with exact rows in $\operatorname{mod} R$

$$\begin{tikzcd}
P_1 \otimes_A R \arrow[r, "f_1 \otimes_A R"] \arrow[d, "\binom{1}{0}"'] & P_0 \otimes_A R \arrow[r, "f_0 \otimes_A R"] \arrow[d, equal] & M \otimes_A R \arrow[r] \arrow[d, "p_{M \otimes_A R}"] & 0 \\
(P_1 \oplus P') \otimes_A R \arrow[r, "\left(f_1 \otimes_A R\,,\, \tilde{f}'\right)"] & P_0 \otimes_A R \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is a minimal projective presentation.

Applying the Nakayama functor $-\otimes_R \mathrm{D}R$ yields another commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \tau_R(M\otimes_A R) \arrow[r, "j"] \arrow[d, "u"] & P_1\otimes_A \mathrm{D}R \arrow[r, "f_1\otimes_A \mathrm{D}R"] \arrow[d, "\binom{1}{0}"] & P_0\otimes_A \mathrm{D}R \arrow[r, "f_0\otimes_A \mathrm{D}R"] \arrow[d, equal] & M\otimes_A \mathrm{D}R \arrow[r] \arrow[d, "p_{M\otimes R}\otimes_A \mathrm{D}R"] & 0 \\
0 \arrow[r] & \tau_R M \arrow[r] & (P_1\oplus P')\otimes_A \mathrm{D}R \arrow[r] & P_0\otimes_A \mathrm{D}R \arrow[r] & M\otimes_R \mathrm{D}R \arrow[r] & 0
\end{tikzcd}$$

where j is the inclusion and u is induced by passing to kernels. Because $\binom{1}{0}j$ is injective, so is u.

This diagram induces the two commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \tau_R(M\otimes_A R) \arrow[r] \arrow[d, "u"] & P_1\otimes_A \mathrm{D}R \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & \tau_R M \arrow[r] & (P_1\oplus P')\otimes_A \mathrm{D}R \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "v"] & P_0\otimes_A \mathrm{D}R \arrow[r] \arrow[d, equal] & M\otimes_A \mathrm{D}R \arrow[r] \arrow[d, "p_{M\otimes R}\otimes_A \mathrm{D}R"] & 0 \\
0 \arrow[r] & Y \arrow[r] & P_0\otimes_A \mathrm{D}R \arrow[r] & M\otimes_R \mathrm{D}R \arrow[r] & 0
\end{tikzcd}$$

where v is the induced morphism. The snake lemma applied to the second diagram yields v injective and $\operatorname{Coker} v \cong \operatorname{Ker}(p_{M\otimes_A R}\otimes_A \mathrm{D}R)$, and to the first diagram an exact sequence

$$0 \longrightarrow \operatorname{Coker} u \longrightarrow P'\otimes_A \mathrm{D}R \longrightarrow \operatorname{Coker} v \longrightarrow 0.$$

This latter sequence splices with the exact sequence

$$0 \longrightarrow \tau_R(M\otimes_A R) \xrightarrow{u} \tau_R M \longrightarrow \operatorname{Coker} u \longrightarrow 0$$

to give an exact sequence

$$0 \to \tau_R(M\otimes_A R) \to \tau_R M \to P'\otimes_A \mathrm{D}R \to \operatorname{Ker}(p_{M\otimes_A R}\otimes_A \mathrm{D}R) \to 0.$$

Finally, as A-modules, we have

$$\tau_R(M\otimes_A R)\cong \operatorname{Hom}_A(R\,,\,\tau_A M)\cong \tau_A M\oplus \operatorname{Hom}_A(E\,,\,\tau_A M). \qquad \square$$

Theorem 3.5.2 **[13]**(2.1)**.** *Let M be an indecomposable A-module.*

(a) If M is nonprojective, then the following conditions are equivalent:
- *i) the almost split sequences ending with M in $\operatorname{mod} A$ and $\operatorname{mod} R$ coincide*
- *ii) $\tau_A M\cong \tau_R M$*
- *iii) $\operatorname{Hom}_A(E\,,\,\tau_A M)=0$ and $M\otimes_A E=0$.*

(b) If M is noninjective, then the following conditions are equivalent:
- *i) the almost split sequences starting with M in $\operatorname{mod} A$ and $\operatorname{mod} R$ coincide*
- *ii) $\tau_A^{-1}M\cong \tau_R^{-1}M$*
- *iii) $\operatorname{Hom}_A(E\,,\,M)=0$ and $\tau_A^{-1}M\otimes_A E=0$.*

Proof. (a) i) implies ii). This is trivial.

ii) implies i). Let $0 \longrightarrow \tau_R M \longrightarrow X \longrightarrow M \longrightarrow 0$ be almost split in $\operatorname{mod} R$. Because $\tau_R M \cong \tau_A M$, the whole sequence lies in $\operatorname{mod} A$. It does not split in $\operatorname{mod} A$, because otherwise it would split in $\operatorname{mod} R$. Let $h \colon L \longrightarrow M$ be a nonretraction in $\operatorname{mod} A$. Then h is a nonretraction in $\operatorname{mod} R$. Therefore, there exists $h' \colon L \longrightarrow X$ such that $h = gh'$. Because L, X are A-modules, h' is a morphism in $\operatorname{mod} A$.

ii) implies iii). Because of 3.5.1, there exists a monomorphism $u \colon \tau_A M \oplus \operatorname{Hom}_A(E\,, \tau_A M) \longrightarrow \tau_R M$ in $\operatorname{mod} A$. Then $\tau_R M \cong \tau_A M$ forces $\operatorname{Hom}_A(E\,, \tau_A M) = 0$. Moreover, u is an isomorphism $\tau_R(M \otimes_A R) \cong \tau_R M$. But then $M \otimes_A R \cong M$ and so $M \otimes_A E = 0$.

iii) implies ii). Because $M \otimes_A E = 0$, we have $M \otimes_A R \cong M$, therefore

$$\Omega^1_R(M \otimes_A R) \cong \Omega^1_R M \cong \Omega^1_A M \oplus (P_0 \otimes_A E)$$

using the notation and the proof of 3.3.3. Let $\tilde{P}$ be a projective cover of $\Omega^1_R(M \otimes_A R)$ in $\operatorname{mod} A$. Then $\tilde{P} \otimes_A R$ is a projective cover of $\Omega^1_R(M \otimes_A R)$ in $\operatorname{mod} R$, because of 3.3.3. Now $\Omega^1_R(M \otimes_A R) \cong \Omega^1_A M$ yields $\tilde{P} \otimes_A R \cong P_1 \otimes_A R$ hence $\tilde{P} = P_1$. Therefore the tops of $\Omega^1_R(M \otimes_A R)$ and $\Omega^1_A M$ are equal in $\operatorname{mod} A$, and hence $P_0 \otimes_A E = 0$. But then its projective cover P is zero and so $P' = 0$. Then we have $\tau_R M \cong \tau_A M \oplus \operatorname{Hom}_A(E\,, \tau_A M)$. Thus, $\operatorname{Hom}_A(E\,, \tau_A M) = 0$ implies $\tau_R M \cong \tau_A M$, as desired. □

The following corollary, due to Hoshino [**26**], played an important rôle in the classification of the representation-finite selfinjective algebras.

Corollary 3.5.3 [**13**](2.3)**.** *Assume $E = {}_A\mathrm{D}A_A$ and let M be an indecomposable A-module.*

(a) If M is nonprojective, then $\tau_A M \cong \tau_R M$ if and only if $\operatorname{pd} M_A \leq 1$, $\operatorname{id} \tau_A M \leq 1$.
(b) If M is noninjective, then $\tau_A^{-1} M \cong \tau_R^{-1} M$ if and only if $\operatorname{pd} \tau_A^{-1} M \leq 1$, $\operatorname{id} M_A \leq 1$.

Proof. (a) We have $\operatorname{pd} M_A \leq 1$ if and only if $\operatorname{Hom}_A(\mathrm{D}A\,, \tau_A M) = 0$, see [**12**](IV.2.7), and $\operatorname{id} \tau_A M \leq 1$ if and only if $M \otimes_A \mathrm{D}A \cong \mathrm{D}\operatorname{Hom}_A(M\,, A) = 0$. □.

Assume now that A is a tilted algebra and $E = \operatorname{Ext}^2_A(\mathrm{D}A\,, A)$ so that $R = A \ltimes E$ is cluster tilted. It is shown in [**4**] that any complete slice in $\operatorname{mod} A$ embeds in $\operatorname{mod} R$ as what is called a local slice, a result extended in [**6**] to algebras B such that there exist surjective morphisms of algebras $R \longrightarrow B \longrightarrow A$. The decisive step was the proof that, if M is an indecomposable lying on a complete slice in $\operatorname{mod} A$, then, if M is nonprojective in $\operatorname{mod} A$, we have $\tau_A M \cong \tau_R M$ and if it is noninjective, then $\tau_A^{-1} M \cong \tau_R^{-1} M$, see [**6**](3.2.1).

In [**34**](5.9), Treffinger obtained necessary and sufficient conditions for a τ-slice in $\operatorname{mod} A$ to embed as a τ-slice in $\operatorname{mod} R$.

Example 3.5.4. Let A be given by the quiver

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\beta"'] & \underset{3}{\circ} \arrow[l, "\alpha"'] \end{tikzcd}$$

bound by $\alpha\beta = 0$, and R be given by the quiver

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\beta"'] \arrow[r, bend left, "\eta"] & \underset{3}{\circ} \arrow[l, "\alpha"'] \end{tikzcd}$$

bound by $\alpha\beta = 0$, $\eta\alpha = 0$. Here we find $E_A = ({\scriptstyle 3})^2$, while $(\mathrm{D}E)_A = {\substack{3\\2}}$. Consider first the simple module $S_2 = {\scriptstyle 2}$. We have $\tau_A S_2 = {\scriptstyle 1}$ hence $\mathrm{Hom}_A(E\,,\,\tau_A S_2) = 0$. On the other hand, $S_2 \otimes_A E \cong \mathrm{DHom}_A(S_2\,,\,\mathrm{D}E) = \mathrm{DHom}_A\big({\scriptstyle 2}\,,\,{\substack{3\\2}}\big) \neq 0$. Therefore the almost split sequences ending with S_2 in $\mathrm{mod}\,A$ and $\mathrm{mod}\,R$ do not coincide. In fact, a quick calculation shows that the first is $0 \longrightarrow {\scriptstyle 1} \longrightarrow {\substack{2\\1}} \longrightarrow {\scriptstyle 2} \longrightarrow 0$ while the second is $0 \longrightarrow {\substack{2\\1\;3}} \longrightarrow {\substack{2\\3}} \oplus {\substack{2\\1}} \longrightarrow {\scriptstyle 2} \longrightarrow 0$.

On the other hand, looking at $S_3 = {\scriptstyle 3}$ we have $\tau_A S_3 = {\scriptstyle 2}$, so that $\mathrm{Hom}_A(E\,,\,\tau_A S_3) = 0$. Also $S_3 \otimes_A E \cong \mathrm{DHom}_A(S_3\,,\,\mathrm{D}E) = \mathrm{DHom}_A\big({\scriptstyle 3}\,,\,{\substack{3\\2}}\big) = 0$. Therefore, the almost split sequences ending in S_3 in $\mathrm{mod}\,A$ and $\mathrm{mod}\,R$ coincide.

4. Tilting modules

4.1. Extendable tilting modules. For tilting theory, we refer the reader to [**12**] Chapter IV. Let, as usual, R be a split extension of A by a nilpotent bimodule E.

Theorem 4.1.1 [11](2.3). *Let T be an A-module, then*

(a) $T \otimes_A R$ is a partial tilting (or tilting) R-module if and only if T is a partial tilting (or tilting, respectively) A-module, $\mathrm{Hom}_A(T \otimes_A E\,,\,\tau_A T) = 0$ and $\mathrm{Hom}_A(\mathrm{D}E\,,\,\tau_A T) = 0$;

(b) $\mathrm{Hom}_A(R\,,\,T)$ is a partial cotilting (or cotilting) R-module if and only if T is a partial cotilting (or cotilting, respectively) A-module, $\mathrm{Hom}_A\Big(\tau_A^{-1}T\,,\,\mathrm{Hom}_A(E\,,\,T)\Big) = 0$ and $\mathrm{Hom}_A\Big(\tau_A^{-1}T\,,\,E\Big) = 0$.

Proof. (a) Because of 3.2.2, the number of isoclasses of indecomposable summands of T equals that of $T \otimes_A R$. Also, because of 2.1.2, the ranks of the Grothendieck groups of A and R are equal. Therefore, it suffices to prove the statement about partial tilting modules.

We have isomorphisms of vector spaces

$$\begin{aligned}\mathrm{Hom}_R\big(T \otimes_A R\,,\,\tau_R(T \otimes_A R)\big) &\cong \mathrm{Hom}_R\big(T \otimes_A R\,,\,\mathrm{Hom}_A(R\,,\,\tau_A T)\big)\\ &\cong \mathrm{Hom}_A(T \otimes_A R \otimes_R R\,,\,\tau_A T)\\ &\cong \mathrm{Hom}_A(T \otimes_A R\,,\,\tau_A T)\\ &\cong \mathrm{Hom}_A(T\,,\,\tau_A T) \oplus \mathrm{Hom}_A(T \otimes_A E\,,\,\tau_A T).\end{aligned}$$

If T is a partial tilting module then $\mathrm{pd}\,T_A \leq 1$ implies $\mathrm{Hom}_A(T\,,\,\tau_A T) \cong \mathrm{DExt}^1_A(T\,,\,T) = 0$. Further, $\mathrm{Hom}_A(\mathrm{D}E\,,\,\tau_A T) = 0$ implies $\mathrm{pd}(T \otimes_A R) \leq 1$ because of 3.4.2. Therefore $\mathrm{Hom}_A(T \otimes_A E\,,\,\tau_A T) = 0$ implies

$$\mathrm{Ext}^1_R(T \otimes_A R\,,\,T \otimes_A R) \cong \mathrm{DHom}_R\big(T \otimes_A R\,,\,\tau_R(T \otimes_A R)\big) = 0$$

and so $T \otimes_A R$ is a partial tilting R-module.

Conversely, if $T \otimes_A R$ is a partial tilting R-module, 3.4.2 gives $\mathrm{pd}\,T_A \leq 1$ and $\mathrm{Hom}_A(\mathrm{D}E\,,\,\tau_A T) = 0$. Moreover $\mathrm{Hom}_R\big(T \otimes_A R\,,\,\tau_R(T \otimes_A R)\big) = 0$ yields $\mathrm{Hom}_A(T \otimes_A E\,,\,\tau_A T) = 0$ and $\mathrm{Ext}^1_A(T\,,\,T) \cong \mathrm{DHom}_A(T\,,\,\tau_A T) = 0$, so T_A is a partial tilting module. □

Definition 4.1.2. (a) A partial tilting (or tilting) A-module is called ***extendable*** if $T \otimes_A R$ is a partial tilting (or tilting, respectively) R-module.

(b) A partial cotilting (or cotilting) A-module is called ***coextendable*** if $\mathrm{Hom}_A(R\,,\,T)$ is a partial cotilting (or cotilting, respectively) R-module.

One reason for looking at this class of (co)tilting modules is that they preserve the splitting character of the algebra.

Proposition 4.1.3 [**11**](2.5). *(a) If T is an extendable tilting A-module, then $S = \mathrm{End}(T \otimes_A R)$ is the split extension of $B = \mathrm{End}\, T_A$ by the nilpotent bimodule ${}_BW_B = \mathrm{Hom}_A({}_BT_A\,,\,{}_BT \otimes_A E)$.*
(b) If T is a coextendable cotilting A-module, then $S = \mathrm{End}\,\mathrm{Hom}_A(R\,,\,T)$ is the split extension of $B = \mathrm{End}\, T_A$ by the nilpotent bimodule ${}_BW_B = \mathrm{Hom}_A\big(\mathrm{Hom}_A(E\,,\,T)\,,\,T\big)$.

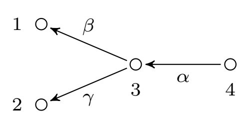
Proof. (a) We have vector space isomorphisms

$$\begin{aligned} S = \mathrm{Hom}_R(T \otimes_A R\,,\,T \otimes_A R) &\cong \mathrm{Hom}_A\big(T\,,\,\mathrm{Hom}_R({}_AR_A\,,\,T \otimes_A R)\big) \\ &\cong \mathrm{Hom}_A(T\,,\,T \otimes_A R_A) \cong \mathrm{Hom}_A(T\,,\,T) \oplus \mathrm{Hom}_A(T\,,\,T \otimes_A E). \end{aligned}$$

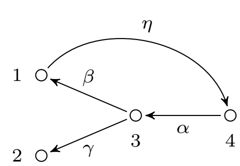
We thus have an exact sequence $0 \longrightarrow W \longrightarrow S \xrightarrow{\varphi} B \longrightarrow 0$ where φ is an algebra morphism, and the ideal structure of W is induced from its B-B-bimodule structure. There remains to prove that W is nilpotent. The multiplication in W is that of S and, for any $w \in W$, its image is contained in $T \otimes_A E$. Because E is nilpotent, there exists $s \geq 0$ such that, for any sequence $w_1, \ldots, w_s$ of elements of W, the image of $w_1 \cdots w_s$ lies in $T \otimes_A E^s = 0$. Therefore $W^s = 0$. $\square$

The proof shows that the nilpotency index of W in S does not exceed that of E in R. Thus, if R is a trivial extension of A by E, then S is a trivial extension of B by W.

Example 4.1.4. Let A be the path algebra of the quiver



and R be given by the quiver



bound by $\beta\eta = 0$, $\eta\alpha\beta = 0$, $\eta\alpha\gamma = 0$. It is easily seen that the A-module

$$T = e_1A \oplus e_4A \oplus \mathrm{D}(Ae_1) \oplus \mathrm{D}(Ae_4) = 1 \oplus \begin{smallmatrix} & 4 & \\ & 3 & \\ 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \\ 1 \end{smallmatrix} \oplus 4$$

is tilting. We claim it is extendable. We first observe that $E_A = \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$, $(\mathrm{D}E)_A = (\,1\,)^2$. In particular $\mathrm{D}E$ is generated by T so that $\mathrm{Hom}_A(\mathrm{D}E\,,\,\tau_AT) \cong \mathrm{D}\mathrm{Ext}^1_A(T\,,\,\mathrm{D}E) = 0$. We now compute $T \otimes_A R$. We have $e_1A \otimes_A R \cong e_1R = \begin{smallmatrix} 1 \\ 4 \\ 3 \end{smallmatrix}$ and $e_4A \otimes_A R \cong e_4R = \begin{smallmatrix} & 4 & \\ & 3 & \\ 1 & & 2 \end{smallmatrix}$. Also the minimal projective presentations

$$0 \longrightarrow e_2A \longrightarrow e_4A \longrightarrow \begin{smallmatrix} 4 \\ 3 \\ 1 \end{smallmatrix} \longrightarrow 0$$

$$0 \longrightarrow e_3A \longrightarrow e_4A \longrightarrow 4 \longrightarrow 0$$

induce respectively the minimal projective presentations

$$e_2R \longrightarrow e_4R \longrightarrow \begin{smallmatrix}4\\3\\1\end{smallmatrix} \otimes_A R \longrightarrow 0$$

$$e_3R \longrightarrow e_4R \longrightarrow {\scriptstyle 4} \otimes_A R \longrightarrow 0\ .$$

Therefore $\begin{smallmatrix}4\\3\\1\end{smallmatrix} \otimes_A R \cong \begin{smallmatrix}4\\3\\1\end{smallmatrix}$ and ${\scriptstyle 4} \otimes_A R \cong {\scriptstyle 4}$ and $T \otimes_A R = \begin{smallmatrix}1\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}&4&\\&3&\\2&&1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix} \oplus {\scriptstyle 4}$ so that $T \otimes_A E = \begin{smallmatrix}4\\3\end{smallmatrix}$, which is generated by T. Therefore $\operatorname{Hom}_A(T \otimes_A E\,,\, \tau_A T) = 0$ and T is extendable.

The algebra $\operatorname{End} T$ is given by the quiver

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\nu"'] & \underset{3}{\circ} \arrow[l, "\mu"'] & \underset{4}{\circ} \arrow[l, "\lambda"'] \end{tikzcd}$$

bound by $\lambda\mu\nu = 0$, while $\operatorname{End}(T \otimes_A R)$ is given by the quiver

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l, "\nu"'] \arrow[rr, bend left, "\sigma"] & \underset{3}{\circ} \arrow[l, "\mu"'] & \underset{4}{\circ} \arrow[l, "\lambda"'] \end{tikzcd}$$

bound by $\lambda\mu\nu = 0$, $\lambda\mu\sigma = 0$, $\sigma\lambda = 0$. It is the split extension of $\operatorname{End} T$ by the bimodule generated by σ.

On the other hand, the tilting A-module $T' = \begin{smallmatrix}&4&\\&3&\\2&&1\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\2\end{smallmatrix} \oplus \begin{smallmatrix}4\\3\\1\end{smallmatrix} \oplus {\scriptstyle 4}$ is not extendable, because $\mathrm{D}E$ is not generated by T' and then $\operatorname{Ext}^1_A\big(T'\,,\,\mathrm{D}E\big) \neq 0$.

Another example of extendable partial tilting module can be found in [**28**] where the authors study cluster tilted algebras from the point of view of induced and coinduced modules. They prove in [**28**](4.9) that, if A is a tilted algebra, $E = \operatorname{Ext}^2_A(\mathrm{D}A\,,\,A)$ and $R = A \ltimes E$, then $(\mathrm{D}E)_A$ is an extendable partial tilting module, and E_A is a coextendable partial cotilting module.

4.2. Induced torsion pairs. For a module M, the notations $\operatorname{Gen} M$ and $\operatorname{Cogen} M$ stand respectively for the class of modules generated and cogenerated by M.

Associated with a tilting A-module T is a torsion pair $\big(\mathscr{T}(T_A), \mathscr{F}(T_A)\big)$ in $\operatorname{mod} A$ defined by

$$\mathscr{T}(T_A) = \Big\{\, M_A \;\Big|\; \operatorname{Ext}^1_A(T\,,\,M) = 0 \,\Big\} = \operatorname{Gen} T$$
$$\mathscr{F}(T_A) = \{\, M_A \mid \operatorname{Hom}_A(T\,,\,M) = 0 \,\} = \operatorname{Cogen}(\tau_A T).$$

Similarly, associated with a cotilting A-module T is a torsion pair $\big(\mathscr{T}'(T_A), \mathscr{F}'(T_A)\big)$ given by

$$\mathscr{T}'(T_A) = \{\, M_A \mid \operatorname{Hom}_A(T\,,\,M) = 0 \,\} = \operatorname{Gen}(\tau_A^{-1} T)$$
$$\mathscr{F}'(T_A) = \Big\{\, M_A \;\Big|\; \operatorname{Ext}^1_A(T\,,\,M) = 0 \,\Big\} = \operatorname{Cogen} T.$$

Proposition 4.2.1. *(a) If T is an extendable tilting A-module then*
$X_R \in \mathscr{T}(T \otimes_A R)$ if and only if $X_A \in \mathscr{T}(T)$
$X_R \in \mathscr{F}(T \otimes_A R)$ if and only if $X_A \in \mathscr{F}(T)$.
(b) If T is a coextendable cotilting A-module then
$X_R \in \mathscr{T}'\big(\operatorname{Hom}_A(R\,,\,T)\big)$ if and only if $X_A \in \mathscr{T}'(T)$
$X_R \in \mathscr{F}'\big(\operatorname{Hom}_A(R\,,\,T)\big)$ if and only if $X_A \in \mathscr{F}'(T)$.

Proof. (a) The statement follows from the vector space isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_R(T\otimes_A R, X) &\cong \mathrm{DHom}_R\big(X\,,\,\tau_R(T\otimes_A R)\big)\cong \mathrm{DHom}_R\big(X\,,\,\mathrm{Hom}_A(R\,,\,\tau_A T)\big)\\ &\cong \mathrm{DHom}_A(X\otimes_R R_A\,,\,\tau_A T)\cong \mathrm{DHom}_A(X_A\,,\,\tau_A T)\cong \mathrm{Ext}^1_A(T,X)\end{aligned}$$

and $\mathrm{Hom}_R(T\otimes_A R\,,\,X)\cong \mathrm{Hom}_A\big(T\,,\,\mathrm{Hom}_R({}_AR_R\,,\,X)\big)\cong \mathrm{Hom}_A(T\,,\,X_A)$. □

Corollary 4.2.2. *(a) If T is an extendable tilting module, then*

i) $\mathcal{T}(T)\otimes_A R\subseteq \mathcal{T}(T\otimes_A R)$ always, and
$\mathcal{T}(T)\otimes_A R\supseteq \mathcal{T}(T\otimes_A R)$ if and only if $\mathrm{Im}(-\otimes_A R)\supseteq \mathcal{T}(T\otimes_A R)$.

ii) $\mathcal{F}(T)\otimes_A R\subseteq \mathcal{F}(T\otimes_A R)$ if and only if $\mathcal{F}(T)\otimes_A E\subseteq \mathcal{F}(T)$,
$\mathcal{F}(T)\otimes_A R\supseteq \mathcal{F}(T\otimes_A R)$ if and only if $\mathrm{Im}(-\otimes_A R)\supseteq \mathcal{F}(T\otimes_A R)$.

(b) If T is a coextendable cotilting module, then

i) $\mathrm{Hom}_A\big(R\,,\,\mathcal{F}'(T)\big)\subseteq \mathcal{F}'\big(\mathrm{Hom}_A(R\,,\,T)\big)$ always, and
$\mathrm{Hom}_A\big(R\,,\,\mathcal{F}'(T)\big)\supseteq \mathcal{F}'\big(\mathrm{Hom}_A(R\,,\,T)\big)$ if and only if $\mathrm{Im}\,\mathrm{Hom}_A(R\,,\,-)\supseteq \mathcal{F}'\Big(\mathrm{Hom}_A(R\,,\,T)\Big)$.

ii) $\mathrm{Hom}_A\big(R\,,\,\mathcal{T}'(T)\big)\subseteq \mathcal{T}'\big(\mathrm{Hom}_A(R\,,\,T)\big)$ if and only if $\mathrm{Hom}_A\big(E\,,\,\mathcal{T}'(T)\big)\subseteq \mathcal{T}'(T)$,
$\mathrm{Hom}_A\big(R\,,\,\mathcal{T}'(T)\big)\supseteq \mathcal{T}'\big(\mathrm{Hom}_A(R\,,\,T)\big))$ if and only if $\mathrm{Im}\,\mathrm{Hom}_A(R\,,\,-)\supseteq \mathcal{T}'\Big(\mathrm{Hom}_A(R\,,\,T)\Big)$.

Proof. (a) i) Let $M\in\mathcal{T}(T)$. In order to show that $M\otimes_A R\in\mathcal{T}(T\otimes_A R)$, we need, because of 4.2.1, to prove that $M\otimes_A R_A = M\oplus(M\otimes_A E)$ lies in $\mathcal{T}(T)$. We know that $M\in\mathcal{T}(T)$. But M generated by T implies $M\otimes_A E$ generated by $T\otimes_A E$ and the latter is generated by T, because of 4.1.1. This establishes the first statement.

The necessity part of the second statement is clear, so we prove sufficiency. Let $X\in\mathcal{T}(T\otimes_A R)$. The hypothesis says that there exists M_A such that $X\cong M\otimes_A R$. It suffices to prove that $M\in\mathcal{T}(T)$. But this follows from the facts that $X_A\in\mathcal{T}(T)$ and $X_A\cong M\oplus(M\otimes_A E)$.

ii) Let $N\in\mathcal{F}(T)$. We have $N\otimes_A R\in\mathcal{F}(T\otimes_A R)$ if and only if $N\otimes_A R_A = N\oplus(N\otimes_A E)\in\mathcal{F}(T)$ if and only if $N\otimes_A E\in\mathcal{F}(T)$. This implies the first statement. The second one is proved as the corresponding one for $\mathcal{T}(T)$. □

Corollary 4.2.3. *(a) Let T be an extendable tilting module, then*

i) If $\big(\mathcal{T}(T\otimes_A R),\mathcal{F}(T\otimes_A R)\big)$ splits in $\mathrm{mod}\,R$, then $\big(\mathcal{T}(T),\mathcal{F}(T)\big)$ splits in $\mathrm{mod}\,A$.

ii) If $\big(\mathcal{T}(T),\mathcal{F}(T)\big)$ splits in $\mathrm{mod}\,A$ and $\mathcal{F}(T\otimes_A R)\subseteq \mathrm{Im}(-\otimes_A R)$, then $\big(\mathcal{T}(T\otimes_A R),\mathcal{F}(T\otimes_A R)\big)$ splits in $\mathrm{mod}\,R$.

(b) Let T be a coextendable cotilting module, then

i) If $\Big(\mathcal{T}'\big(\mathrm{Hom}_A(R\,,\,T)\big),\mathcal{F}'\big(\mathrm{Hom}_A(R\,,\,T)\big)\Big)$ splits in $\mathrm{mod}\,R$, then $\big(\mathcal{T}'(T),\mathcal{F}'(T)\big)$ splits in $\mathrm{mod}\,A$.

ii) If $\big(\mathcal{T}'(T),\mathcal{F}'(T)\big)$ splits in $\mathrm{mod}\,A$ and $\mathcal{T}'(\mathrm{Hom}_A(R\,,\,T))\subseteq \mathrm{Im}\,\mathrm{Hom}_A(R\,,\,-)$, then $\Big(\mathcal{T}'\big(\mathrm{Hom}_A(R\,,\,T)\big),\mathcal{F}'\big(\mathrm{Hom}_A(R\,,\,T)\big)\Big)$ splits in $\mathrm{mod}\,R$.

Proof. (a) i) Let $M\in\mathcal{T}(T)$, $N\in\mathcal{F}(T)$. We claim that $\mathrm{Ext}^1_A(N\,,\,M)=0$. Because of 4.2.1, we have $M_R\in\mathcal{T}(T\otimes_A R)$, $N_R\in\mathcal{F}(T\otimes_A R)$. But then $\mathrm{Ext}^1_R(N\,,\,M)=0$. This implies $\mathrm{Ext}^1_A(N\,,\,M)=0$.

ii) Let $X \in \mathscr{T}(T\otimes_A R), Y \in \mathscr{F}(T\otimes_A R)$. We claim that $\mathrm{Ext}^1_R(X\,,\,Y) = 0$. Because of the hypothesis and 4.2.2, there exists $N \in \mathscr{F}(T)$ such that $Y \cong N\otimes_A R$. Also, $X_R \in \mathscr{T}(T\otimes_A R)$ implies that $X_A \in \mathscr{T}(T)$, that is, $X \in \mathrm{Gen}(T)$. This implies that $X\otimes_A E \in \mathrm{Gen}(T\otimes_A E)$. Because $T\otimes_A E \in \mathrm{Gen}\,T$, see 4.1.1, we get $X\otimes_A E \in \mathscr{T}(T)$. Therefore, $X\otimes_A R_A \cong X \oplus (X\otimes_A E) \in \mathscr{T}(T)$. Hence

$$
\begin{aligned}
\mathrm{Ext}^1_R(Y,X) &\cong \mathrm{D}\overline{\mathrm{Hom}}_R(X,\tau_R Y) \subseteq \mathrm{DHom}_R(X\,,\,\tau_R Y) \cong \mathrm{DHom}_R\big(X\,,\,\tau_R(N\otimes_A R)\big)\\
&\cong \mathrm{DHom}_R\big(X\,,\,\mathrm{Hom}_A(R\,,\,\tau_A N)\big) \cong \mathrm{DHom}_A(X\otimes_R R_A\,,\,\tau_A N)\\
&\cong \mathrm{DHom}_A(X\,,\,\tau_A N) \oplus \mathrm{DHom}_A(X\otimes_A E\,,\,\tau_A N) = 0
\end{aligned}
$$

because $N \in \mathscr{F}(T)$ and $\big(\mathscr{T}(T), \mathscr{F}(T)\big)$ split imply $\tau_A N \in \mathscr{F}(T)$. $\square$

Example 4.2.4. Let $e \in A$ be an idempotent such that eA is simple projective noninjective, and E is a nilpotent bimodule such that $eE = Ee = 0$. Then the APR-tilting module $T = \tau_A^{-1}(eA) \oplus (1-e)A$ is extendable.

Indeed, we must show that $\mathrm{D}E$ and $T\otimes_A E$ are generated by T. Now $\mathrm{Ext}^1_A(T\,,\,\mathrm{D}E) \cong \mathrm{DHom}_A(\mathrm{D}E\,,\,\tau_A T) = \mathrm{DHom}_A(\mathrm{D}E\,,\,eA)$ is nonzero if and only if eA is a direct summand of $\mathrm{D}E$. But $\mathrm{Hom}_A(eA\,,\,\mathrm{D}E) \cong (\mathrm{D}E)e \cong \mathrm{D}(eE) = 0$. Hence $\mathrm{Ext}^1_A(T\,,\,\mathrm{D}E) = 0$ and so $\mathrm{D}E \in \mathrm{Gen}\,T$.

Moreover, there exists an idempotent $e' \in A$ such that we have an almost split sequence

$$
0 \longrightarrow eA \longrightarrow e'A \longrightarrow \tau_A^{-1}(eA) \longrightarrow 0. \qquad (*)
$$

Applying $-\otimes_A E$ yields $e'E \cong \tau_A^{-1}(eA)\otimes_A E$. Now $e'E \in \mathrm{Gen}\,T$ because $e'Ee = 0$, hence so is $\tau_A^{-1}(eA)\otimes_A E$. Therefore $T\otimes_A E \in \mathrm{Gen}\,T$.

Furthermore, $T\otimes_A R$ is also an APR-tilting module.

Indeed, we first prove that eR is simple projective noninjective in $\mathrm{mod}\,R$. If this is not the case, there exists $\alpha \in (Q_R)_1$ starting at the point corresponding to e. There is no such arrow in Q_A, hence α belongs to E and $\alpha = e\alpha = 0$ gives a contradiction. Next, applying $-\otimes_A R$ to $(*)$ yields an exact sequence

$$
0 \longrightarrow \mathrm{Tor}^A_1\Big(\tau_A^{-1}(eA)\,,\,R\Big) \longrightarrow eR \xrightarrow{\ j\ } e'R \longrightarrow \tau_A^{-1}(eA)\otimes_A R \longrightarrow 0\ .
$$

Because $\mathrm{Tor}^A_1\Big(\tau_A^{-1}(eA)\,,\,R\Big) \cong \mathrm{DExt}^1_A\Big(\tau_A^{-1}(eA)\,,\,\mathrm{D}R\Big) = 0$ (for, $\mathrm{D}E \in \mathrm{Gen}\,T$), we deduce that j is a monomorphism. Hence $\tau_A^{-1}(eA)\otimes_A R \cong \tau_R^{-1}(eR)$ and so $T\otimes_A R$ is indeed an APR-tilting module.

Finally, $\mathscr{F}(T\otimes_A R) = \mathrm{add}(eR) = \mathrm{add}(eA\otimes_A R) \subseteq \mathrm{Im}(-\otimes_A R)$ so that the conditions of 4.2.3 are satisfied in this case.

4.3. Restrictions of tilting modules. We consider the reverse problem: given a tilting R-module U, under which conditions is the restricted module $U\otimes_R A$ a tilting A-module (and similarly for cotilting modules)?

Lemma 4.3.1. *(a) Let U_R be such that $\mathrm{Tor}^R_1(U,A) = 0$ and $0 \to \tilde{P}_1 \xrightarrow{f_1} \tilde{P}_0 \xrightarrow{f_0} U \to 0$ a minimal projective resolution for U, then*

$$
0 \longrightarrow \tilde{P}_1\otimes_R A \xrightarrow{f_1\otimes_R A} \tilde{P}_0\otimes_R A \xrightarrow{f_0\otimes_R A} U\otimes_R A \longrightarrow 0
$$

is a minimal projective resolution for $U\otimes_R A$. In particular, $\mathrm{pd}(U\otimes_R A) \leq 1$.

(b) Let U_R be such that $\mathrm{Ext}^1_R(A\,,\,U)=0$ *and* $0\longrightarrow U\xrightarrow{g^0}\tilde{I}^0\xrightarrow{g^1}\tilde{I}^1\longrightarrow 0$ *a minimal injective coresolution for U, then*

$$0\longrightarrow \mathrm{Hom}_R(A\,,\,U)\xrightarrow{\mathrm{Hom}_R(A\,,\,g^0)}\mathrm{Hom}_R\Big(A\,,\,\tilde{I}^0\Big)\xrightarrow{\mathrm{Hom}_R(A\,,\,g^1)}\mathrm{Hom}_R\Big(A\,,\,\tilde{I}^1\Big)\longrightarrow 0$$

is a minimal injective coresolution for $\mathrm{Hom}_R(A\,,\,U)$*. In particular,* $\mathrm{id}\,\mathrm{Hom}_R(A\,,\,U)\le 1$.

Proof. (a) Applying $-\otimes_R A$ to the given minimal projective resolution of U_R and using that $\mathrm{Tor}^R_1(U\,,\,A)=0$ yields an exact sequence

$$0\longrightarrow \tilde{P}_1\otimes_R A\xrightarrow{f_1\otimes_R A}\tilde{P}_0\otimes_R A\xrightarrow{f_0\otimes_R A}U\otimes_R A\longrightarrow 0\ .$$

Because $\tilde{P}_0\otimes_R A$, $\tilde{P}_1\otimes_R A$ are projective A-modules, this is a projective resolution. In particular, $\mathrm{pd}(U\otimes_R A)\le 1$. Minimality follows from the fact that, because of 3.2.4, $\tilde{P}_0\otimes_R A$ is a projective cover of $U\otimes_R A$. □

Lemma 4.3.2. *(a) Let U_R be such that* $\mathrm{pd}\,U\le 1$ *and* $\mathrm{Tor}^R_1(U\,,\,A)=0$*, then*

$$\tau_A(U\otimes_R A)\cong \mathrm{Hom}_R(A\,,\,\tau_R U).$$

(b) Let U_R be such that $\mathrm{id}\,U\le 1$ *and* $\mathrm{Ext}^1_R(A\,,\,U)=0$*, then*

$$\tau_A^{-1}\,\mathrm{Hom}_R(A\,,\,U)\cong(\tau_R^{-1}U)\otimes_R A.$$

Proof. (a) Because of 4.3.1, a minimal projective resolution $0\longrightarrow\tilde{P}_1\longrightarrow\tilde{P}_0\longrightarrow U\longrightarrow 0$ induces a minimal projective resolution in $\mathrm{mod}\,R$

$$0\longrightarrow \tilde{P}_1\otimes_R A\longrightarrow\tilde{P}_0\otimes_R A\longrightarrow U\otimes_R A\longrightarrow 0\ .$$

Applying $\mathrm{Hom}_A(-\,,\,A)$ yields a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_A\Big(\tilde{P}_0\otimes_R A, A\Big) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A\Big(\tilde{P}_1\otimes_R A\,,\,A\Big) \arrow[r] \arrow[d, "\cong"] & \mathrm{Tr}(U\otimes_R A) \arrow[r] & 0 \\
\mathrm{Hom}_A\Big(\tilde{P}_0\,,\mathrm{Hom}_A({}_RA, A)\Big) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A\Big(\tilde{P}_1,\mathrm{Hom}_A({}_RA, A)\Big) \arrow[d, "\cong"] & & \\
\mathrm{Hom}_R\Big(\tilde{P}_0\,,\,A\Big) \arrow[r] & \mathrm{Hom}_R\Big(\tilde{P}_1\,,\,A_R\Big) \arrow[r] & \mathrm{Ext}^1_R(U, A_R) \arrow[r] & 0
\end{tikzcd}$$

where the lower row is obtained by applying $\mathrm{Hom}_R(-\,,\,A_R)$ to the original minimal projective resolution of U_R. Thus $\mathrm{Tr}(U\otimes_R A)\cong\mathrm{Ext}^1_R(U\,,\,A)$ and therefore $\tau_A(U\otimes_R A)\cong \mathrm{DExt}^1_R(U\,,\,A)\cong\mathrm{Hom}_R(A\,,\,\tau_R U)$ because $\mathrm{pd}\,U_R\le 1$. □

Theorem 4.3.3 [14](3.3)**.** *(a) Let U_R be a partial tilting (or tilting) R-module such that* $\mathrm{Tor}^R_1(U\,,\,A)=0$*, then $U\otimes_R A$ is a partial tilting (or tilting, respectively) A-module.*

(b) Let U_R be a partial cotilting (or cotilting) R-module such that $\mathrm{Ext}^1_R(A\,,\,U)=0$*, then* $\mathrm{Hom}_R(A\,,\,U)$ *is a partial cotilting (or cotilting, respectively) A-module.*

Proof. (a) Assume first that U_R is partial tilting and such that $\operatorname{Tor}_1^R(U\,,\,A) = 0$. Because of 4.3.1, we have $\operatorname{pd}(U \otimes_R A) \leq 1$. Also we have vector space isomorphisms

$$\begin{aligned}
\operatorname{DExt}^1_A(U \otimes_R A\,,\,U \otimes_R A) &\cong \operatorname{Hom}_A\big(U \otimes_R A\,,\,\tau_A(U \otimes_R A)\big) \\
&\cong \operatorname{Hom}_R\Big(U\,,\,\operatorname{Hom}_A\big({}_RA\,,\,\tau_A(U \otimes_R A)\big)\Big) \\
&\cong \operatorname{Hom}_R\Big(U\,,\,\operatorname{Hom}_A\big({}_RA\,,\,\operatorname{Hom}_R(A\,,\,\tau_R U)\big)\Big) \\
&\cong \operatorname{Hom}_R\big(U\,,\,\operatorname{Hom}_R({}_RA \otimes_A A_R\,,\,\tau_R U)\big) \\
&\cong \operatorname{Hom}_R\big(U\,,\,\operatorname{Hom}_R({}_RA_R\,,\,\tau_R U)\big).
\end{aligned}$$

Applying $\operatorname{Hom}_R(-\,,\,\tau_R U)$ to the exact sequence $0 \longrightarrow {}_RE_R \longrightarrow {}_RR_R \longrightarrow {}_RA_R \longrightarrow 0$ yields a monomorphism

$$0 \longrightarrow \operatorname{Hom}_R({}_RA_R\,,\,\tau_R U) \longrightarrow \operatorname{Hom}_R({}_RR_R\,,\,\tau_R U) \cong \tau_R U.$$

Applying next $\operatorname{Hom}_R(U\,,\,-)$ yields another monomorphism

$$0 \longrightarrow \operatorname{Hom}_R\big(U\,,\,\operatorname{Hom}_R({}_RA_R\,,\,\tau_R U)\big) \longrightarrow \operatorname{Hom}_R(U\,,\,\tau_R U) \cong \operatorname{DExt}^1_R(U\,,\,U) = 0.$$

Thus $\operatorname{Ext}^1_A(U \otimes_R A\,,\,U \otimes_R A) = 0$ and $U \otimes_R A$ is a partial tilting A-module.

If U_R is tilting, then there exists an exact sequence $0 \longrightarrow R_R \longrightarrow U_0 \longrightarrow U_1 \longrightarrow 0$ with $U_0, U_1 \in \operatorname{add} U$. Because $\operatorname{Tor}_1^R(U\,,\,A) = 0$, applying $- \otimes_R A$ yields an exact sequence

$$0 \longrightarrow A_A \longrightarrow U_0 \otimes_R A \longrightarrow U_1 \otimes_R A \longrightarrow 0.$$

Because $U_0 \otimes_R A, U_1 \otimes_R A \in \operatorname{add}(U \otimes_R A)$, this finishes the proof. $\square$

Definition 4.3.4. (a) A partial tilting, or tilting, R-module U is called ***restrictable*** provided $\operatorname{Tor}_1^R(U\,,\,A) = 0$ and then $U \otimes_R A$ is called its ***restriction***.

(b) A partial cotilting, or cotilting, R-module U is called ***corestrictable*** provided $\operatorname{Ext}^1_R(A\,,\,U) = 0$ and then $\operatorname{Hom}_R(A\,,\,U)$ is called its ***corestriction***.

Lemma 4.3.5. *(a) Let T be an extendable partial tilting (or tilting) A-module, then $T \otimes_A R$ is a restrictable partial tilting (or tilting, respectively) R-module, with restriction T.*

(b) Let T be a coextendable partial cotilting (or cotilting) A-module, then $\operatorname{Hom}_A(R\,,\,T)$ is a corestrictable partial cotilting (or cotilting, respectively) R-module, with corestriction T.

Proof. (a) Assume that T is an extendable partial tilting, or tilting, A-module. We claim that $T \otimes_A R$ is restrictable, that is $\operatorname{Tor}_1^R(T \otimes_A R\,,\,A) = 0$. Let $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow T \longrightarrow 0$ be a minimal projective resolution in $\operatorname{mod} A$. Because T is extendable, $\operatorname{pd}(T \otimes_A R) \leq 1$, hence 3.3.1 gives a minimal projective resolution

$$0 \longrightarrow P_1 \otimes_A R \longrightarrow P_0 \otimes_A R \longrightarrow T \otimes_A R \longrightarrow 0\ .$$

Applying $-\otimes_R$ yields a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Tor}_1^R(T \otimes_A R, A) \arrow[r] & P_1 \otimes_A R \otimes_R A \arrow[r] \arrow[d, equal, "\wr"] & P_0 \otimes_A R \otimes_R A \arrow[r] \arrow[d, equal, "\wr"] & T \otimes_A R \otimes_R A \arrow[r] \arrow[d, equal, "\wr"] & 0 \\
& 0 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & T \arrow[r] & 0.
\end{tikzcd}$$

Thus $\mathrm{Tor}_1^R(T \otimes_A R\,,\, A) = 0$ and so $T \otimes_A R$ is restrictable. On the other hand, $T \otimes_A R \otimes_R A \cong T$: the restriction of $T \otimes_A R$ is T. $\square$

THEOREM 4.3.6 [14](3.4). *(a) The functors $- \otimes_A R$ and $- \otimes_R A$ induce mutually inverse bijections*

$$\{\textit{extendable tilting } A\textit{-modules}\} \underset{-\otimes_R A}{\overset{-\otimes_A R}{\rightleftarrows}} \{\textit{restrictable induced tilting } R\textit{-modules}\}.$$

(b) The functors $\mathrm{Hom}_A(R\,,\,-)$ and $\mathrm{Hom}_R(A\,,\,-)$ induce mutually inverse bijections

$$\{\textit{coextendable cotilting } A\textit{-modules}\} \underset{\mathrm{Hom}_R(A\,,\,-)}{\overset{\mathrm{Hom}_A(R\,,\,-)}{\rightleftarrows}} \{\textit{corestrictable coinduced } R\textit{-modules}\}.$$

PROOF. (a) If T is an extendable tilting A-module, then $T \otimes_A R$ is induced by definition and it is restrictable because of 4.3.5.

Conversely, if U_R is an induced restrictable tilting module, then $\mathrm{Tor}_1^R(U\,,\,A) = 0$. Because of 4.3.5, $U \otimes_R A$ is a tilting A-module. On the other hand, there exists M_A such that $U \cong M \otimes_A R$. But then $U \otimes_R A \cong M \otimes_A R \otimes_R A \cong M \otimes_A A \cong M$ so that $(U \otimes_R A) \otimes_A R \cong M \otimes_A R \cong U$. Thus $U \otimes_R A$ is extendable. $\square$.

The modern guise of tilting theory is τ-tilting theory. In [33], Suarez obtained a similar result for (support) τ-tilting modules.

EXAMPLE 4.3.7. There exist restrictable tilting R-modules which are not induced. Let A be the path algebra of the quiver

$$1 \circ \xleftarrow{\alpha} \circ\ 2$$

and R the path algebra of the Kronecker quiver

$$1 \circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ\ 2\ .$$

We claim that the APR-tilting module $U_R = \tau_R^{-1}(e_1R) \oplus e_2R$ is restrictable but not induced.

To prove that U is not induced, it suffices to prove that $\tau_R^{-1}(e_1R) = {}_{1}{}^{2}{}_{1}{}^{2}{}_{1}$ is not induced. Because A has only 3 isoclasses of indecomposable modules of which two are projective, it suffices to compute the R-module induced by the remaining indecomposable $S_2 = {}_2$. The minimal projective resolution

$$0 \longrightarrow e_1A \longrightarrow e_2A \longrightarrow S_2 \longrightarrow 0$$

in $\mathrm{mod}\, A$ induces one in $\mathrm{mod}\, R$

$$0 \longrightarrow e_1A \longrightarrow e_2A \longrightarrow S_2 \otimes_A R \longrightarrow 0\ .$$

Therefore $S_2 \otimes_A R \cong \substack{2\\1} \ncong \tau_R^{-1}(e_1R)$.

To prove that U is restrictable, we must show that $\mathrm{Tor}_1^R(U\,,\,A) \cong \mathrm{DExt}_R^1(U\,,\,\mathrm{D}A) = 0$. But this amounts to showing that $(\mathrm{D}A)_R$ is generated by U in $\mathrm{mod}\, R$. Now $(\mathrm{D}A)_R = \substack{2\\1} \oplus {}_2$ and both of its summands are generated by U.

Finally, we compute the restriction of U. We have a minimal projective resolution

$$0 \longrightarrow e_1R \longrightarrow (e_2R)^2 \longrightarrow \tau_R^{-1}(e_1R) \longrightarrow 0\ .$$

Applying $-\otimes_A R$ yields an exact sequence

$$0 \longrightarrow e_1A \longrightarrow (e_2A)^2 \longrightarrow \tau_R^{-1}(e_1R)\otimes_R A \longrightarrow 0\ .$$

Therefore $\tau_R^{-1}(e_1R)\otimes_R A \cong {}^{2}_{1} \oplus 2$. Because $e_2R\otimes_R A \cong e_2A = {}^{2}_{1}$, we deduce that $U\otimes_R A = \left({}^{2}_{1}\right)^2 \oplus 2$.

Proposition 4.3.8 [**14**](3.5). *(a) Let U be a restrictable tilting R-module, then*
$M_A \in \mathcal{T}(U\otimes_R A)$ if and only if $M_R \in \mathcal{T}(U)$,
$M_A \in \mathcal{F}(U\otimes_R A)$ if and only if $M_R \in \mathcal{F}(U)$.
(b) Let U be a corestrictable cotilting R-module, then
$M_A \in \mathcal{T}'(\mathrm{Hom}_R(A\,,\,U))$ if and only if $M_R \in \mathcal{T}'(U)$,
$M_A \in \mathcal{F}'(\mathrm{Hom}_R(A\,,\,U))$ if and only if $M_R \in \mathcal{F}'(U)$.

Proof. (a) This follows from the vector space isomorphisms

$$\begin{aligned}
\mathrm{Ext}^1_A(U\otimes_R A\,,\,M) &\cong \mathrm{DHom}_A\big(M\,,\,\tau_A(U\otimes_R A)\big)\\
&\cong \mathrm{DHom}_A\big(M\,,\,\mathrm{Hom}_R(A\,,\,\tau_A U)\big)\\
&\cong \mathrm{DHom}_R(M\otimes_A A_R\,,\,\tau_A U)\\
&\cong \mathrm{DHom}_R(M_R\,,\,\tau_R U)\\
&\cong \mathrm{Ext}^1_R(U\,,\,M_R),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hom}_A(U\otimes_R A\,,\,M) &\cong \mathrm{Hom}_R\big(U\,,\,\mathrm{Hom}_A(A\,,\,M)\big)\\
&\cong \mathrm{Hom}_R(U\,,\,M_R).
\end{aligned}$$ □

Corollary 4.3.9 [**14**](3.5). *(a) If U is a restrictable tilting R-module such that $\big(\mathcal{T}(U),\mathcal{F}(U)\big)$ splits in $\mathrm{mod}\,R$, then $\big(\mathcal{T}(U\otimes_R A),\mathcal{F}(U\otimes_R A)\big)$ splits in $\mathrm{mod}\,A$.*
(b) If U is a corestrictable cotilting R-module such that $\big(\mathcal{T}'(U),\mathcal{F}'(U)\big)$ splits in $\mathrm{mod}\,R$, then $\big(\mathcal{T}'(\mathrm{Hom}_R(A\,,\,U)),\mathcal{F}'(\mathrm{Hom}_R(A\,,\,U))\big)$ splits in $\mathrm{mod}\,A$.

Proof. (a) Let $M\in\mathcal{T}(U\otimes_R A)$, $N\in\mathcal{F}(U\otimes_R A)$. Then $M_R\in\mathcal{T}(U)$, $N\in\mathcal{F}(U)$. Because of the hypothesis, $\mathrm{Ext}^1_R(N\,,\,M)=0$. Therefore $\mathrm{Ext}^1_A(N\,,\,M)=0$. □

Remark 4.3.10 ([**14**](3.7)). It is useful to observe that an R-module U verifies $\mathrm{Tor}_1^R(U\,,\,A)=0$ if and only if the multiplication $U\otimes_R E \longrightarrow UE\,,\ x\otimes e\mapsto xe$ (for $x\in U$, $e\in E$) is an isomorphism of A-modules. Indeed, applying $U\otimes_R -$ to the exact sequence $0\longrightarrow {}_RE_A \longrightarrow {}_RR_A \longrightarrow {}_RA_A \longrightarrow 0$ yields a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}_1^R(U\otimes_A R, A) \arrow[r] & U\otimes_R E \arrow[r] \arrow[d, "\mu'"] & U\otimes_R R \arrow[r] \arrow[d, "\mu"] & U\otimes_R A \arrow[r] \arrow[d, "\mu''"] & 0\\
& 0 \arrow[r] & UE \arrow[r, "i"] & U \arrow[r, "p"] & U/UE \arrow[r] & 0
\end{tikzcd}$$

where μ,μ' are the multiplication maps and μ'' is induced by passing to cokernels. As seen in 3.1.4, μ'' is an isomorphism and μ is well-known to be so. Moreover, μ' is surjective so that we have an exact sequence in $\mathrm{mod}\,A$

$$0 \longrightarrow \mathrm{Tor}_1^R(U\otimes_A R\,,\,A) \longrightarrow U\otimes_R E \xrightarrow{\mu'} UR \longrightarrow 0\ .$$

The statement follows.

References

1. E. R. Alvares, I. Assem, D. Castonguay, and R. R. S. Vargas, *Right ADA algebras*, J. Algebra Appl. **16** (2017), no. 11, 175–210.
2. A. Alvarez, D. Bravo, E. Fernández, V. Guazzelli, M. Muller, N. Rojas, and S. Trepode, *Trivial extensions, Admissible cuts and HW-reflections*, in preparation.
3. F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, Grad. Texts in Math., Springer-Verlag, 1974.
4. I. Assem, T. Brüstle, and R. Schiffler, *Cluster-tilted algebras and slices*, J. Algebra **319** (2008), no. 8, 3464–3479.
5. ———, *Cluster-tilted algebras as trivial extensions*, Bull. Lond. Math. Soc **40** (2008), no. 1, 151–162.
6. I. Assem, J. C. Bustamante, J. Dionne, P. Le Meur, and D. Smith, *Representation theory of partial relation extensions*, Colloq. Math. **155** (2019), 157–186.
7. I. Assem, D. Castonguay, M. Lanzilotta, and R. R. S. Vargas, *Algebras determined by their supports*, J. Pure Appl. Algebra **216** (2012), no. 5, 1134–1145.
8. I. Assem and F. U. Coelho, *Glueings of tilted algebras*, J. Pure Appl. Algebra **96** (1994), no. 3, 225–243.
9. ———, *Two-sided gluings of tilted algebras*, J. Algebra **269** (2003), no. 2, 456–479.
10. I. Assem, F. U. Coelho, and S. Trepode, *The bound quiver of a split extension*, J. Algebra Appl. **7** (2008), no. 4, 405–423.
11. I. Assem and N. Marmaridis, *Tilting modules over split-by-nilpotent extensions*, Colloq. Math. **26** (1998), no. 5, 1547–1555.
12. I. Assem, D. Simson, and A. Skowroński, *Elements of the Representation Theory of Associative Algebras: Volume 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, 2006.
13. I. Assem and D. Zacharia, *Full embeddings of almost split sequences over split-by-nilpotent extensions*, Colloq. Math. **81** (1999), 21–31.
14. ———, *On split-by-nilpotent extensions*, Colloq. Math. **98** (2003), no. 2, 259–275.
15. M. Auslander, I. Reiten, and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math., vol. 36, Cambridge Univ. Press, 1997.
16. M. Barot, E. Fernández, M. I. Platzeck, N. I. Pratti, and S. Trepode, *From iterated tilted algebras to cluster-tilted algebras*, Adv. Math. **223** (2010), no. 4, 1468–1494.
17. M. Barot, C. Geiss, and A. Zelevinsky, *Cluster algebras of finite type and positive symmetrizable matrices*, J. Lond. Math. Soc. **73** (2006), no. 3, 545–564.
18. M. Barot and S. Trepode, *Cluster tilted algebras with a cyclically oriented quiver*, Comm. Algebra **41** (2013), no. 10, 3613–3628.
19. K. Bongartz and P. Gabriel, *Covering spaces in representation-theory*, Invent. Math. **65** (1982), no. 3, 331–378.
20. M. C. R. Butler and C. M. Ringel, *Auslander-Reiten sequences with few middle terms and applications to string algebras*, Comm. Algebra **15** (1987), no. 1-2, 145–179.
21. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Math. Ser, vol. 19, Princeton Univ. Press, 1956.
22. F. U. Coelho and M. A. Lanzilotta, *Algebras with small homological dimensions*, Manuscripta Math. **100** (1999), no. 1, 1–11.
23. ———, *Weakly shod algebras*, J. Algebra **265** (2003), no. 1, 379–403.
24. E. Fernández, *Extensiones triviales y álgebras inclinadas iteradas*, Ph.D. thesis, Universidad Nacional del Sur, Argentina, 1999.
25. D. Happel, I. Reiten, and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc., vol. 575, Amer. Math. Soc., 1996.
26. M. Hoshino, *Trivial extensions of tilted algebras*, Comm. Algebra **10** (1982), no. 18, 1965–1999.
27. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Math., vol. 1099, Springer, 2006.
28. R. Schiffler and K. Serhiyenko, *Induced and coinduced modules over cluster-tilted algebras*, J. Algebra **472** (2017), 226–258.
29. A. Skowroński, *On Artin algebras with almost all indecomposable modules of projective or injective dimension at most one*, Cent. Eur. J. Math. **1** (2003), no. 1, 108–122.
30. A. Skowroński and J. Waschbüsch, *Representation-finite biserial algebras*, J. Reine Angew. Math. **1983** (1983), no. 345, 172–181.
31. A. Skowroński and K. Yamagata, *Frobenius Algebras I: Basic Representation Theory*, EMS Textbk. Math., vol. 12, Eur. Math. Soc., 2011.
32. ———, *Frobenius Algebras II: Tilted and Hochschild Extension Algebras*, EMS Textbk. Math., vol. 19, Eur. Math. Soc., 2017.
33. P. Suarez, *Split-by-nilpotent extensions and support τ-tilting modules*, to appear.
34. H. Treffinger, *τ-Tilting theory and τ-slices*, J. Algebra **481** (2017), 362–392.

35. S. Zito, *Short proof of a conjecture concerning split-by-nilpotent extensions*, Arch. Math. **111** (2018), no. 5, 479–483.

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