

Laura skew group algebras

Ibrahim Assem, Marcelo Lanzilotta and María Julia Redondo *

1 Introduction

In the representation theory of artin algebras, an important line of research consists in studying those processes which allow to modify, in a predictable way, certain features of the module category of an artin algebra. In this paper, the features we are interested in are the left and the right parts of the module category, introduced by Happel, Reiten and Smalø in [15]. Let A be an artin algebra, $\text{mod } A$ denote the category of finitely generated left A -modules and $\text{ind } A$ denote a full subcategory of $\text{mod } A$ having as objects exactly one representative from each isomorphism class of indecomposable A -modules. The left part \mathcal{L}_A of $\text{mod } A$ is the full subcategory of $\text{ind } A$ having as objects those indecomposable A -modules whose predecessors have projective dimension at most one. The right part \mathcal{R}_A is defined dually. These classes were used successfully in [15] to study the representation theory of quasi-tilted algebras, then, later, to study the many generalisations of this class such as the shod, weakly shod, laura and left (or right) supported algebras (see the survey [5]). However, the definition of \mathcal{L}_A and \mathcal{R}_A is not very practical: it is difficult to find all predecessors (or successors) of a given indecomposable module and thus to say whether it lies in \mathcal{L}_A (or \mathcal{R}_A , respectively) or not. Our first theorem, which generalises [15, II.1.5] and [11, 1.2], simplifies this task: it says that instead of considering all predecessors (or successors) of an indecomposable module, it suffices to look at the “immediate” ones.

THEOREM 1.1 *Let A be an artin algebra, and M be an indecomposable A -module. Then:*

- (a) *M belongs to \mathcal{L}_A if and only if, for every object L in $\text{ind } A$ with projective dimension at least two, we have $\text{Hom}_A(L, M) = 0$.*
- (b) *M belongs to \mathcal{R}_A if and only if, for every object N in $\text{ind } A$ with injective dimension at least two, we have $\text{Hom}_A(M, N) = 0$.*

As a first application of this theorem we consider the indecomposable Ext-injective modules in the additive subcategory of $\text{mod } A$ generated by \mathcal{L}_A , studied and characterised in [4, 3.1]. We give here handier characterisations.

*The third author is a researcher from CONICET, Argentina.

Our main interest in this paper, however, lies in a construction which turns out to behave well when it comes to the left and the right parts: that of the skew group algebra. Let G be a finite group acting on an artin algebra A , the *skew group algebra* $A[G]$ is the free left A -module with basis all the elements in G endowed with the multiplication given by $(a\sigma)(b\zeta) = a\sigma(b)\sigma\zeta$ for all $a, b \in A$ and $\sigma, \zeta \in G$. The study of the representation theory of skew group algebras was started in [19, 17, 18, 14]. We are partly motivated by the fact that finite coverings, and the algebra of invariants, as well as the smash algebras of [10], are particular cases of skew group algebras [17, 7]. The algebra $A[G]$ retains many features from A , such as being representation-finite, being an Auslander algebra, or being a Nakayama algebra (see [7, 19]). However, many properties are not preserved by this construction, like being a basic algebra, or being connected, so we are dealing with essentially different algebras, making it worthwhile to compare their representation theories.

It is shown in [19] that an algebra A and the skew group algebra $A[G]$ share most homological information. Thus, it has been shown that if A is a tilted (or a quasi-tilted) algebra, then so is $A[G]$, see [19] (or [15], respectively). However, for studying generalisations of these classes of algebras (such as, for instance, lura algebras) homological information, by itself, is not sufficient. We also need a nice correspondence between paths in $\text{ind } A$ and in $\text{ind } A[G]$. After establishing this correspondence, we are able to prove our second main theorem.

THEOREM 1.2 *Let A be an artin algebra, G be a finite group acting on A such that $|G|$ is invertible in A , and $R = A[G]^b$ be the basic algebra associated to the skew group algebra. Then:*

- (a) *A is left (or right) supported if and only if so is R .*
- (b) *A is lura if and only if so is R .*
- (c) *A is left (or right) glued if and only if so is R .*
- (d) *A is weakly shod if and only if so is R .*
- (e) *A is shod if and only if so is R .*
- (f) *A is quasi-tilted if and only if so is R .*
- (g) *A is tilted if and only if so is R .*

For the definitions of the above classes, we refer to [5] or to section 5 below. Here, statements (f) and (g) are included for completeness.

Finally, we apply this result to the toupie algebras of [9]. We define a new class, which we call skew toupie algebras, and exhibit a family of lura (actually, weakly shod) skew toupie algebras.

2 A characterisation of the left and right parts.

2.1 Notation.

For an artin algebra A , we denote by $\text{mod } A$ the category of finitely generated left A -modules, and by $\text{ind } A$ a full subcategory of $\text{mod } A$ having as objects a full set of representatives of the isomorphism classes of the indecomposable A -modules. For a subcategory \mathcal{C} of $\text{mod } A$, we write $M \in \mathcal{C}$ to express that M is an object in \mathcal{C} . We denote by $\text{add } \mathcal{C}$ the full subcategory of $\text{mod } A$ having as objects the direct sums of indecomposable summands of objects in \mathcal{C} and, if M is an A -module, we abbreviate $\text{add}\{M\}$ as $\text{add } M$. We say that a full subcategory \mathcal{C} of $\text{ind } A$ is *finite* if it has only finitely many objects. Given two A -modules L and M , we write $L \in M$ to express that L is a direct summand of M . We denote the projective (or injective) dimension of an A -module M by $\text{pd}_A M$ (or $\text{id}_A M$, respectively) and the global dimension of A by $\text{gl.dim. } A$. Finally, we denote by $\Gamma(\text{mod } A)$ the Auslander-Reiten quiver of A , and by τ_A its Auslander-Reiten translation DTr . For further definitions or facts on $\text{mod } A$, $\Gamma(\text{mod } A)$ and τ_A we refer the reader to [7, 20].

Given $M, N \in \text{ind } A$, we write $M \rightsquigarrow N$ in case there exists a path

$$M = M_0 \xrightarrow{f_1} M_1 \rightarrow \cdots \rightarrow M_{t-1} \xrightarrow{f_t} M_t = N \quad (1)$$

($t \geq 0$) from M to N , that is, the f_i are non-zero morphisms, and the M_i are indecomposable A -modules. We then say that M is a *predecessor* of N , and N is a *successor* of M . If each f_i in (1) is an irreducible morphism, we say that (1) is a *path of irreducible morphisms*. A path (1) of irreducible morphisms is *sectional* if $\tau_A M_{j+1} \neq M_{j-1}$ for all j such that $1 \leq j \leq t$.

Let A be a basic and connected artin algebra. Following [15], we define the left part \mathcal{L}_A of $\text{mod } A$ to be the full subcategory of $\text{ind } A$ consisting of all the modules M such that, if $L \rightsquigarrow M$, then $\text{pd}_A L \leq 1$. Dually, the right part \mathcal{R}_A is the full subcategory of $\text{ind } A$ consisting of all the modules M such that, if $M \rightsquigarrow N$, then $\text{id}_A N \leq 1$. Clearly, \mathcal{L}_A is closed under predecessors, while \mathcal{R}_A is closed under successors.

For the sake of brevity, we refrain from now on from stating the dual of each statement and leave the primal-dual translation to the reader.

The following result generalises [15, II.1.5] and [11, 1.2] and its proof is heavily inspired from the proofs of these statements.

LEMMA 2.1 *Let A be an artin algebra, and M be an indecomposable A -module such that there exists a path $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M$ in $\text{ind } A$ with $\text{pd}_A M_0 \geq 2$. Then there exists an indecomposable A -module L with $\text{pd}_A L \geq 2$ and $\text{Hom}_A(L, M) \neq 0$.*

Proof. Assume that this is not the case, that is, there exist $M \in \text{ind } A$ and a path $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M$ with $\text{pd}_A M_0 \geq 2$ and moreover $\text{Hom}_A(L, M) = 0$ for all $L \in \text{ind } A$ with

$\text{pd}_A L \geq 2$. We may clearly choose the path $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M$ with $\text{pd}_A M_0 \geq 2$ so that the length $\ell(M_1)$ of M_1 is minimal.

It follows from our hypothesis that $f_1 f_0 = 0$ (in particular, f_0 is not an epimorphism) and also that $\text{pd}_A M_1 \leq 1$.

We claim that $C = \text{Coker } f_0$ is indecomposable. Since $f_1 f_0 = 0$, we have $\text{Hom}_A(C, M) \neq 0$. So, if C were decomposable, there would exist an indecomposable $C' \in C$ such that $\text{Hom}_A(C', M) \neq 0$. Let $C \xrightarrow{p} C' \xrightarrow{i} C$ denote respectively the canonical projection and injection. We have a fiber product diagram

$$\begin{array}{ccccccccc} (\delta i) & 0 & \rightarrow & \text{Im } f_0 & \xrightarrow{f'} & M'_1 & \xrightarrow{g'} & C' & \rightarrow & 0 \\ & & & & \parallel & \downarrow h & & \downarrow i & & \\ (\delta) & 0 & \rightarrow & \text{Im } f_0 & \xrightarrow{f} & M_1 & \xrightarrow{g} & C & \rightarrow & 0 \end{array}$$

where f is the canonical inclusion. We claim that the upper sequence is not zero. Indeed, assume that it is, and let $g'' : C' \rightarrow M'_1$ be such that $g' g'' = 1_{C'}$. We have $(pg)(hg'') = p(gh)g'' = p(i g')g'' = 1_{C'}$, hence $C' \in M_1$. Since M_1 is indecomposable, $C' \simeq M_1$ and then $\text{Im } f_0 = 0$, a contradiction. This shows that the upper sequence is not split.

Since, by hypothesis, $\text{Hom}_A(C', M) \neq 0$ and g' is an epimorphism, $\text{Hom}_A(M'_1, M) \neq 0$. Thus, there exists an indecomposable summand M''_1 of M'_1 such that $\text{Hom}_A(M''_1, M) \neq 0$. On the other hand, $\text{Im } f_0$ maps non-trivially on every indecomposable summand of M'_1 (because the upper sequence is not split). In particular, $\text{Hom}_A(\text{Im } f_0, M''_1) \neq 0$. Composing with the canonical projection $f'_0 : M_0 \rightarrow \text{Im } f_0$ yields a non-zero morphism $M_0 \rightarrow M''_1$. Then we have a path $M_0 \rightarrow M''_1 \rightarrow M$ in $\text{ind } A$. Our minimality assumption yields $\ell(M_1) \leq \ell(M''_1) \leq \ell(M'_1) \leq \ell(M_1)$ so that $M_1 \simeq M''_1 = M'_1$. Therefore, h is an isomorphism. Hence so is i . This establishes our claim that C is indecomposable.

The indecomposability of C implies that $\text{pd}_A C \leq 1$, because $\text{Hom}_A(C, M) \neq 0$. This, and the short exact sequence

$$0 \rightarrow \text{Im } f_0 \xrightarrow{f} M_1 \xrightarrow{g} C \rightarrow 0$$

imply that $\text{pd}_A \text{Im } f_0 \leq 1$. Therefore, f_0 is not a monomorphism.

On the other hand, $\text{pd}_A C \leq 1$ implies $\text{Ext}_A^2(C, \text{Ker } f_0) = 0$. In particular, the class in $\text{Ext}_A^2(C, \text{Ker } f_0)$ of the exact sequence

$$(\epsilon) \quad 0 \rightarrow \text{Ker } f_0 \rightarrow M_0 \xrightarrow{f_0} M_1 \xrightarrow{g} C \rightarrow 0$$

vanishes. Letting ϵ_1 and ϵ_2 denote respectively the short exact sequences

$$(\epsilon_1) \quad 0 \rightarrow \text{Ker } f_0 \rightarrow M_0 \xrightarrow{f'_0} \text{Im } f_0 \rightarrow 0$$

$$(\epsilon_2) \quad 0 \rightarrow \text{Im } f_0 \xrightarrow{f} M_1 \xrightarrow{g} C \rightarrow 0$$

we have $0 = \epsilon = \epsilon_1 \epsilon_2$. Applying $\text{Hom}_A(-, \text{Ker } f_0)$ to ϵ_2 yields an exact sequence

$$\cdots \rightarrow \text{Ext}_A^1(M_1, \text{Ker } f_0) \rightarrow \text{Ext}_A^1(\text{Im } f_0, \text{Ker } f_0) \rightarrow \text{Ext}_A^2(C, \text{Ker } f_0) = 0.$$

Then, there exists $\zeta \in \text{Ext}_A^1(M_1, \text{Ker } f_0)$ such that $\zeta f = \epsilon_1$. That is, there exists an A -module N and a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Ker } f_0 & \rightarrow & M_0 & \rightarrow & \text{Im } f_0 & \rightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow f & & \\
0 & \rightarrow & \text{Ker } f_0 & \rightarrow & N & \rightarrow & M_1 & \rightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & C & = & C & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}$$

from which we deduce a short exact sequence

$$0 \rightarrow M_0 \rightarrow \text{Im } f_0 \oplus N \rightarrow M_1 \rightarrow 0.$$

Since $\text{pd}_A M_0 \geq 2$ while $\text{pd}_A \text{Im } f_0 \leq 1$ and $\text{pd}_A M_1 \leq 1$, then N has an indecomposable summand N' with $\text{pd}_A N' \geq 2$. On the other hand, the middle column of the above diagram is not split (otherwise, the right column would split too, a contradiction). Hence, every indecomposable summand of N maps non-trivially to C . We consider the resulting path $N' \rightarrow C \rightarrow M$ in $\text{ind } A$. Since $\text{pd}_A N' \geq 2$ while $\ell(C) < \ell(M_1)$, we get a contradiction to our minimality assumption. \square

Proof of Theorem 1.1: It suffices to prove (a), since (b) is dual. Since the necessity of the condition is obvious, we prove the sufficiency. Assume that M is such that, for every $L \in \text{ind } A$ with $\text{pd}_A L \geq 2$, we have $\text{Hom}_A(L, M) = 0$. We must show that every predecessor L' of M has projective dimension at most one. Assume to the contrary that $\text{pd}_A L' \geq 2$, and use induction on the length t of a shortest path from L' to M in $\text{ind } A$:

$$L' = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_t} M_t = M.$$

If $t \in \{0, 1\}$, we have a contradiction to the hypothesis. If $t = 2$, the contradiction follows from Lemma 2.1. Assume that $t \geq 3$. By Lemma 2.1, there exists an indecomposable N and a non-zero morphism $f : N \rightarrow M_2$ such that $\text{pd}_A N \geq 2$. But then the path

$$N \xrightarrow{f} M_2 \xrightarrow{f_3} M_3 \cdots \xrightarrow{f_t} M_t = M$$

of length $t - 1$ yields a contradiction to the induction hypothesis. \square

3 Ext-injectives in $\text{add } \mathcal{L}_A$.

Let A be a basic and connected artin algebra. We recall from [8] that an indecomposable module $M \in \mathcal{L}_A$ is called *Ext-injective* in $\text{add } \mathcal{L}_A$ whenever $\text{Ext}_A^1(L, M) = 0$ for all $L \in \mathcal{L}_A$. It is shown in [8, 3.4] that $M \in \mathcal{L}_A$ is Ext-injective in $\text{add } \mathcal{L}_A$ if and only if $\tau_A^{-1}M \notin \mathcal{L}_A$. Further, it is shown in [4, 3.1] that an indecomposable A -module M is Ext-injective in $\text{add } \mathcal{L}_A$ if and only if it belongs to one of the following subsets of $\text{ind } A$:

- (a) $\mathcal{E}_1 = \mathcal{E}_1(A) = \{L \in \mathcal{L}_A : \text{there exists an injective } I \in \text{ind } A, \text{ and a path } I \rightsquigarrow L \text{ in } \text{ind } A\}$, and
- (b) $\mathcal{E}_2 = \mathcal{E}_2(A) = \{L \in \mathcal{L}_A \setminus \mathcal{E}_1 : \text{there exists a projective } P \in \text{ind } A \setminus \mathcal{L}_A, \text{ and a sectional path } P \rightsquigarrow \tau_A^{-1}L\}$.

Clearly, \mathcal{E}_2 is contained in the (apparently) larger set:

$$\mathcal{E}'_2 = \mathcal{E}'_2(A) = \{L \in \mathcal{L}_A \setminus \mathcal{E}_1 : \text{there exists a projective } P \in \text{ind } A \setminus \mathcal{L}_A, \text{ and a path } P \rightsquigarrow \tau_A^{-1}L\}.$$

Our objective in this section is to prove that $\mathcal{E}_2 = \mathcal{E}'_2$. This yields an easier characterisation of the Ext-injectives in $\text{add } \mathcal{L}_A$ which, apart from its theoretical interest, will be used essentially in section 5.

PROPOSITION 3.1 *Let A be an artin algebra. An indecomposable A -module M is Ext-injective in $\text{add } \mathcal{L}_A$ if and only if $M \in \mathcal{E}_1 \cup \mathcal{E}'_2$.*

Proof. Assume M to be indecomposable and Ext-injective in $\text{add } \mathcal{L}_A$. Then $\tau_A^{-1}M \notin \mathcal{L}_A$. By Theorem 1.1, there exists an indecomposable A -module L such that $\text{pd}_A L \geq 2$ and $\text{Hom}_A(L, \tau_A^{-1}M) \neq 0$. Hence, there exists an indecomposable injective I such that $\text{Hom}_A(I, \tau_A L) \neq 0$. So, either $\text{Hom}_A(\tau_A L, M) \neq 0$ and the path $I \rightarrow \tau_A L \rightarrow M$ yields $M \in \mathcal{E}_1$, or else $\text{Hom}_A(\tau_A L, M) = 0$ in which case the Auslander-Reiten formula gives $\underline{\text{Hom}}_A(L, \tau_A^{-1}M) \simeq \overline{\text{Hom}}_A(\tau_A L, M) = 0$. Since $\text{Hom}_A(L, \tau_A^{-1}M) \neq 0$, there exists an indecomposable projective P and a path $L \rightarrow P \rightarrow \tau_A^{-1}M$. Since $\text{pd}_A L \geq 2$, we have $L \notin \mathcal{L}_A$, so $P \notin \mathcal{L}_A$ and consequently, $M \in \mathcal{E}'_2$.

Conversely, let $M \in \mathcal{E}_1 \cup \mathcal{E}'_2$. Clearly, if $M \in \mathcal{E}'_2$, then $M \in \mathcal{L}_A$ but $\tau_A^{-1}M \notin \mathcal{L}_A$ (because $\tau_A^{-1}M$ succeeds a projective not in \mathcal{L}_A) so that M is Ext-injective in $\text{add } \mathcal{L}_A$. If $M \in \mathcal{E}_1$, then either M is injective, or else there exists an indecomposable injective I and a path $I \rightsquigarrow M \rightarrow * \rightarrow \tau_A^{-1}M$. By [3, 1.6], we infer that $\tau_A^{-1}M \notin \mathcal{L}_A$. Thus, M is Ext-injective in $\text{add } \mathcal{L}_A$.

This establishes the first assertion. The second follows immediately. □

The following corollary gives equivalent characterisations of the sets \mathcal{E}_1 and \mathcal{E}_2 .

COROLLARY 3.2 *Let $M \in \mathcal{L}_A$. Then:*

- (a) *The following conditions are equivalent:*

- i) There exist an injective $I \in \text{ind } A$ and a path $I \rightsquigarrow M$;
 - ii) There exist an injective $I \in \text{ind } A$ and a path of irreducible morphisms $I \rightsquigarrow M$;
 - iii) There exist an injective $I \in \text{ind } A$ and a sectional path $I \rightsquigarrow M$;
 - iv) There exists an injective $I \in \text{ind } A$ such that $\text{Hom}_A(I, M) \neq 0$.
- (b) The following conditions are equivalent:
- i) There exist a projective $P \in \text{ind } A \setminus \mathcal{L}_A$ and a path $P \rightsquigarrow \tau_A^{-1}M$;
 - ii) There exist a projective $P \in \text{ind } A \setminus \mathcal{L}_A$ and a path of irreducible morphisms $P \rightsquigarrow \tau_A^{-1}M$;
 - iii) There exist a projective $P \in \text{ind } A \setminus \mathcal{L}_A$ and a sectional path $P \rightsquigarrow \tau_A^{-1}M$;
 - iv) There exists a projective $P \in \text{ind } A \setminus \mathcal{L}_A$ such that $\text{Hom}_A(P, \tau_A^{-1}M) \neq 0$.

Proof.

- (a) i) implies ii) and ii) implies iii) follow from [3, 1.6], the other implications are trivial.
- (b) i) implies iii) by the above proposition. The other implications are trivial. \square

4 Preliminaries on skew group algebras.

We recall the relevant definitions, see [7, 19]. Let A be an artin k -algebra, and G be a finite group with identity 1. We say that G acts on A if there is a function $G \times A \rightarrow A$, $(\sigma, a) \mapsto \sigma(a)$ such that:

- (a) For each $\sigma \in G$, the map $a \mapsto \sigma(a)$ is a k -linear automorphism of A .
- (b) $(\sigma_1\sigma_2)(a) = \sigma_1(\sigma_2(a))$ for all $\sigma_1, \sigma_2 \in G$ and $a \in A$.
- (c) $1(a) = a$ for all $a \in A$.

Such an action induces an action of G on $\text{mod } A$ as follows. Let M be an A -module, and $\sigma \in G$. We define ${}^\sigma M$ to be the A -module with the additive structure of M but where the multiplication is given by $a.x = \sigma^{-1}(a)x$, for $a \in A$ and $x \in M$.

LEMMA 4.1 *Let $\sigma \in G$. The mapping $M \mapsto {}^\sigma M$ (where M is an A -module) induces an homomorphism of G into the group of automorphisms of the category $\text{mod } A$.*

Proof. We define a functor $\sigma(-) : \text{mod } A \rightarrow \text{mod } A$ on objects by $M \mapsto {}^\sigma M$. Let now $f : L \rightarrow M$ be a morphism of A -modules and define ${}^\sigma f : {}^\sigma L \rightarrow {}^\sigma M$ by $x \mapsto f(x)$ for $x \in L$. This is an A -linear map because

$${}^\sigma f(a.x) = {}^\sigma f(\sigma^{-1}(a)x) = f(\sigma^{-1}(a)x) = \sigma^{-1}(a)f(x) = a. {}^\sigma f(x)$$

for all $a \in A$ and $x \in L$. This clearly defines an endofunctor of $\text{mod } A$. The lemma then follows from the observation that, for any $\sigma_1, \sigma_2 \in G$, we have $\sigma_1(\sigma_2(-)) = \sigma_1\sigma_2(-)$ and, in particular, $\sigma(-) \cdot \sigma^{-1}(-) = \text{id}_{\text{mod } A} = \sigma^{-1}(-) \cdot \sigma(-)$ for any $\sigma \in G$. \square

Assume that G acts on A . The *skew group algebra* $A[G]$ has as underlying A -module the free left A -module having as basis all the elements in G , with the multiplication defined by

$$(a\sigma)(b\zeta) = a\sigma(b)\sigma\zeta$$

for all $a, b \in A$ and $\sigma, \zeta \in G$.

Throughout this paper, we assume that the order $|G|$ of G is invertible in A . Since the skew group algebra $A[G]$ is not basic in general, even if A is so (see [7]), we agree to always consider the basic form $R = A[G]^b$ of $A[G]$. Also, in order to avoid confusion, we denote the A -modules by the letters L, M, N, \dots and the R -modules by the letters X, Y, Z, \dots

The natural ring inclusion $A \hookrightarrow R$ given by $a \mapsto a.1$ for $a \in A$, induces the change of rings functors ${}_R R_A \otimes - : \text{mod } A \rightarrow \text{mod } R$ and $\text{Hom}_R({}_R R_A, -) : \text{mod } R \rightarrow \text{mod } A$. The following proposition summarises the properties of these functors, as in [19, 1.1 and 1.8 (a)(b)(c)].

PROPOSITION 4.2 *Let G be a finite group acting on an artin algebra A , and $R = A[G]^b$. Assume that $|G|$ is invertible in A . Then:*

- (a) $(R \otimes_A -, \text{Hom}_R(R, -))$ and $(\text{Hom}_R(R, -), R \otimes_A -)$ are two adjoint pairs of functors.
- (b) (1) The natural morphism $\text{id}_{\text{mod } A} \rightarrow \text{Hom}_R(R, R \otimes_A -)$ is a section of functors.
(2) The natural morphism $R \otimes_A \text{Hom}_R(R, -) \rightarrow \text{id}_{\text{mod } R}$ is a retraction of functors.
- (c) If $M, N \in \text{ind } A$, then
 - (1) $\text{Hom}_R(R, R \otimes_A M) \simeq \bigoplus_{\sigma \in G} {}^\sigma M$.
 - (2) $R \otimes_A M \simeq R \otimes_A N$ if and only if there exists $\sigma \in G$ such that $M \simeq {}^\sigma N$.
 - (3) If $R \otimes_A M \simeq \bigoplus_{i=1}^n X_i$ is an indecomposable decomposition, then, for each i , the A -module $\text{Hom}_R(R, X_i)$ has an indecomposable summand from each isomorphism class of the ${}^\sigma M$, with $\sigma \in G$.

It follows from (a) above that both functors $R \otimes_A -$ and $\text{Hom}_R(R, -)$ are exact and preserve projectives and injectives. On the other hand, as seen in (c), they do not preserve indecomposability.

COROLLARY 4.3 *Let X be an indecomposable R -module. Then there exists $M \in \text{ind } A$ such that $M \in \text{Hom}_R(R, X)$ and $X \in R \otimes_A M$.*

Proof. By 4.2(b)(2), there is a retraction $R \otimes_A \text{Hom}_R(R, X) \rightarrow X$. The statement follows from the indecomposability of X . \square

LEMMA 4.4 *Let M, N be indecomposable A -modules such that $\text{Hom}_A(M, N) \neq 0$.*

(a) *For any indecomposable $X \in R \otimes_A M$, we have $\text{Hom}_R(X, R \otimes_A N) \neq 0$.*

(b) *For any indecomposable $Y \in R \otimes_A N$, we have $\text{Hom}_R(R \otimes_A M, Y) \neq 0$.*

Proof. We only prove (a) since (b) is similar. By 4.2(c)(3), we have an indecomposable decomposition in $\text{mod } R$

$$R \otimes_A M = \bigoplus_{i=1}^m X_i$$

such that $\text{Hom}_R(R, X_i) = \bigoplus_{\sigma \in H_i} {}^\sigma M$ for some $H_i \subseteq G$. Moreover, for any i , and any $\zeta \in G$, there exists $\sigma \in H_i$ such that

$${}^\sigma M \simeq {}^\zeta M.$$

We need to show that, for any i , $\text{Hom}_R(X_i, R \otimes_A N) \simeq \text{Hom}_A(\text{Hom}_R(R, X_i), N) \neq 0$. It follows from $\text{Hom}_A(M, N) \neq 0$ and 4.2(b)(1) that

$$\text{Hom}_A(\text{Hom}_R(R, R \otimes_A M), \text{Hom}_R(R, R \otimes_A N)) \neq 0.$$

Therefore there exist $\sigma, \zeta \in G$ such that $\text{Hom}_A({}^\sigma M, {}^\zeta N) \neq 0$. By 4.1, $\text{Hom}_A({}^{\zeta^{-1}\sigma} M, N) \neq 0$. Thus $\text{Hom}_A(\text{Hom}_R(R, X_i), N) \neq 0$ for any i . \square

COROLLARY 4.5 *Let $M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_t$ be a path in $\text{ind } A$. Then:*

(a) *For any indecomposable $X_1 \in R \otimes_A M_1$, there exists a path $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_t$ in $\text{ind } R$ such that, for any i , $X_i \in R \otimes_A M_i$ and $M_i \in \text{Hom}_R(R, X_i)$.*

(b) *For any indecomposable $Y_t \in R \otimes_A M_t$, there exists a path $Y_1 \rightarrow Y_2 \rightarrow \cdots \rightarrow Y_t$ in $\text{ind } R$ such that, for any i , $Y_i \in R \otimes_A M_i$ and $M_i \in \text{Hom}_R(R, Y_i)$.*

Proof. We only prove (a) since (b) is similar. By 4.2(c)(3), $X_i \in R \otimes_A M_i$ implies $M_i \in \text{Hom}_R(R, X_i)$ for any i . Since $X_1 \in R \otimes_A M_1$ and $\text{Hom}_A(M_1, M_2) \neq 0$, we have $\text{Hom}_A(X_1, R \otimes_A M_2) \neq 0$ by 4.4(a). Hence there exists an indecomposable R -module $X_2 \in R \otimes_A M_2$ such that $\text{Hom}_R(X_1, X_2) \neq 0$. The statement follows from an obvious induction. \square

LEMMA 4.6 *Let X, Y be indecomposable R -modules such that $\text{Hom}_R(X, Y) \neq 0$. Let M be an indecomposable summand of $\text{Hom}_R(R, X)$ such that $X \in R \otimes_A M$. Then there exist $\sigma \in G$ and an indecomposable summand $N \in \text{Hom}_R(R, Y)$ such that $Y \in R \otimes_A {}^\sigma N$ and $\text{Hom}_A(M, {}^\sigma N) \neq 0$.*

Proof. By 4.2(c)(2), $R \otimes_A N \simeq R \otimes_A {}^\sigma N$ for all $\sigma \in G$ and all $N \in \text{ind } A$. Since $X \in R \otimes_A M$ and $\text{Hom}_R(X, Y) \neq 0$ then, for each $N \in \text{ind } A$ such that $N \in \text{Hom}_R(R, Y)$ and $Y \in R \otimes_A N$, we have $\text{Hom}_R(R \otimes_A M, R \otimes_A N) \neq 0$. Adjunction gives $\text{Hom}_A(M, \text{Hom}_R(R, R \otimes_A N)) \neq 0$. By 4.2(c)(1), $\text{Hom}_R(R, R \otimes_A N) \simeq \bigoplus_{\sigma \in G} {}^\sigma N$. Hence there exists $\sigma \in G$ such that $\text{Hom}_A(M, {}^\sigma N) \neq 0$ and $Y \in R \otimes_A N \simeq R \otimes_A {}^\sigma N$. \square

COROLLARY 4.7 *Let $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_t$ be a path in $\text{ind } R$. Then:*

- (a) *For any indecomposable $M_1 \in \text{Hom}_R(R, X_1)$ such that $X_1 \in R \otimes_A M_1$, there exist $\sigma_2, \dots, \sigma_t \in G$ and a path*

$$M_1 \rightarrow {}^{\sigma_2}M_2 \rightarrow \cdots \rightarrow {}^{\sigma_{t-1}}M_{t-1} \rightarrow {}^{\sigma_t}M_t$$

in $\text{ind } A$, with $M_i \in \text{Hom}_R(R, X_i)$ and $X_i \in R \otimes_A {}^{\sigma_i}M_i$ for any i .

- (b) *For any indecomposable $N_t \in \text{Hom}_R(R, X_t)$ such that $X_t \in R \otimes_A N_t$, there exist $\zeta_1, \dots, \zeta_{t-1} \in G$ and a path*

$${}^{\zeta_1}N_1 \rightarrow {}^{\zeta_2}N_2 \rightarrow \cdots \rightarrow {}^{\zeta_{t-1}}N_{t-1} \rightarrow N_t$$

in $\text{ind } A$, with $N_i \in \text{Hom}_R(R, X_i)$ and $X_i \in R \otimes_A {}^{\zeta_i}N_i$ for any i .

Proof. The proof is easy and left to the reader. □

5 Laura skew group algebras.

Throughout this section, we assume that A is a basic and connected artin algebra, that G is a group acting on A , with $|G|$ invertible in A , and we let $R = A[G]^b$.

LEMMA 5.1 *For any $\sigma \in G$, we have ${}^\sigma\mathcal{L}_A = \mathcal{L}_A$.*

Proof. We first show that ${}^\sigma\mathcal{L}_A \subseteq \mathcal{L}_A$. Let $L \in {}^\sigma\mathcal{L}_A$, then $\sigma^{-1}L \in \mathcal{L}_A$. Let M be an indecomposable A -module such that $\text{Hom}_A(M, L) \neq 0$. Then $\text{Hom}_A(\sigma^{-1}M, \sigma^{-1}L) \neq 0$. Since $\sigma^{-1}L \in \mathcal{L}_A$, we have $\text{pd}_A \sigma^{-1}M \leq 1$. Hence $\text{pd}_A M \leq 1$. By Theorem 1.1, $L \in \mathcal{L}_A$, as required. Conversely, if $N \in \mathcal{L}_A$, then $\sigma^{-1}N \in {}^{\sigma^{-1}}\mathcal{L}_A$ so that $\sigma^{-1}N \in \mathcal{L}_A$ and $N \in {}^\sigma\mathcal{L}_A$. □

LEMMA 5.2

- (a) $\text{add}(R \otimes_A \mathcal{L}_A) = \text{add } \mathcal{L}_R$.
(b) $\text{add } \text{Hom}_R(R, \mathcal{L}_R) = \text{add } \mathcal{L}_A$.

Proof. We first show that $\text{add}(R \otimes_A \mathcal{L}_A) \subseteq \text{add } \mathcal{L}_R$. Let $M \in \mathcal{L}_A$, and $R \otimes_A M = \bigoplus_{i=1}^m X_i$ be a decomposition into indecomposable modules in $\text{mod } R$. We claim that $X_i \in \mathcal{L}_R$ for any i . By Theorem 1.1, we must prove that, for each $Y \in \text{ind } R$ such that $\text{Hom}_R(Y, X_i) \neq 0$, we have $\text{pd}_R Y \leq 1$. By 4.6, there exist $L, N \in \text{ind } A$ such that $L \in \text{Hom}_R(R, Y)$, $Y \in R \otimes_A L$, $N \in \text{Hom}_R(R, X_i)$, $X_i \in R \otimes_A {}^\sigma N$ for some $\sigma \in G$, and $\text{Hom}_A(L, {}^\sigma N) \neq 0$. On the other hand, by 4.2(c), we have an indecomposable decomposition in $\text{mod } A$

$$\text{Hom}_R(R, X_i) = \bigoplus_{\zeta \in H_i} {}^\zeta M$$

for some $H_i \subseteq G$. Hence, there exists $\zeta \in H_i$ such that $\text{Hom}_A(L, \sigma^\zeta M) \neq 0$. By 5.1, $\sigma^\zeta M \in \mathcal{L}_A$ hence $\text{pd}_A L \leq 1$. Therefore $\text{pd}_R R \otimes_A L \leq 1$. Since $Y \in R \otimes_A L$, we infer that $\text{pd}_R Y \leq 1$, as required.

We next prove that $\text{add Hom}_R(R, \mathcal{L}_R) \subseteq \text{add } \mathcal{L}_A$. Let $X \in \mathcal{L}_R$ and $\text{Hom}_R(R, X) = \bigoplus_{i=1}^m M_i$ be a decomposition into indecomposable modules in $\text{mod } A$. We claim that $M_i \in \mathcal{L}_A$ for any i . By Theorem 1.1, we must prove that for each $L \in \text{ind } A$ such that $\text{Hom}_A(L, M_i) \neq 0$, we have $\text{pd}_A L \leq 1$. Let $R \otimes_A L = \bigoplus_{j=1}^n Y_j$ be an indecomposable decomposition in $\text{mod } R$. Since $\text{Hom}_A(L, M_i) \neq 0$, we have $\text{Hom}_R(R \otimes_A L, X) \simeq \text{Hom}_A(L, \text{Hom}_R(R, X)) \neq 0$, so there exists $j_0 \in \{1, \dots, n\}$ such that $\text{Hom}_R(Y_{j_0}, X) \neq 0$. Then $\text{pd}_R Y_{j_0} \leq 1$. On the other hand, $\text{Hom}_R(R, Y_{j_0}) \simeq \bigoplus_{\sigma \in H_{j_0}} \sigma L$ for some $H_{j_0} \subseteq G$. Hence $\text{pd}_R Y_{j_0} = \text{pd}_A \text{Hom}_R(R, Y_{j_0}) = \text{pd}_A \sigma L = \text{pd}_A L$ for each $\sigma \in H_{j_0}$. Therefore, $\text{pd}_A L \leq 1$.

There remains to prove that equality holds in each case. Assume $X \in \mathcal{L}_R$. Then $\text{Hom}_R(R, X) \in \text{add } \mathcal{L}_A$. By 4.2(b), $X \in R \otimes_A \text{Hom}_R(R, X)$. Hence $X \in \text{add}(R \otimes_A \mathcal{L}_A)$. Similarly, if $M \in \mathcal{L}_A$, then $M \in \text{Hom}_R(R, R \otimes_A M)$ by 4.2(b) so $M \in \text{add Hom}_R(R, \mathcal{L}_R)$. \square

COROLLARY 5.3

- (a) $\text{add}(R \otimes_A \text{ind } A \setminus \mathcal{L}_A) = \text{add}(\text{ind } R \setminus \mathcal{L}_R)$.
- (b) $\text{add Hom}_R(R, \text{ind } R \setminus \mathcal{L}_R) = \text{add}(\text{ind } A \setminus \mathcal{L}_A)$.

Proof. We only prove (a) since (b) is similar. Let $M \in \text{ind } A \setminus \mathcal{L}_A$ and $R \otimes_A M \simeq \bigoplus_{i=1}^m X_i$ be an indecomposable decomposition in $\text{mod } R$. By 4.2(c), for any i , we have $\text{Hom}_R(R, X_i) = \bigoplus_{\sigma \in H_i} \sigma M$ for some $H_i \subseteq G$. By 5.1, $\sigma M \notin \mathcal{L}_A$ for any σ . Hence, by 5.2, $X_i \notin \mathcal{L}_R$ for any i . Conversely, assume $X \in \text{ind } R \setminus \mathcal{L}_R$. By 4.3, there exists $M \in \text{ind } A$ such that $X \in R \otimes_A M$. By 5.2, $M \notin \mathcal{L}_A$. \square

We now show that the Ext-injectives in $\text{add } \mathcal{L}_A$ correspond to those in $\text{add } \mathcal{L}_R$. For this purpose, we denote by $\mathcal{E}_1(A)$, $\mathcal{E}'_2(A)$ and by $\mathcal{E}_1(R)$, $\mathcal{E}'_2(R)$, respectively, the sets described in section 3 for the algebras A and R .

LEMMA 5.4

- (a) $\text{add}(R \otimes_A \mathcal{E}_1(A)) = \text{add } \mathcal{E}_1(R)$, $\text{add}(R \otimes_A \mathcal{E}'_2(A)) = \text{add } \mathcal{E}'_2(R)$.
- (b) $\text{add Hom}_R(R, \mathcal{E}_1(R)) = \text{add } \mathcal{E}_1(A)$, $\text{add Hom}_R(R, \mathcal{E}'_2(R)) = \text{add } \mathcal{E}'_2(A)$.

Proof. Let $E_1 \in \mathcal{E}_1(A)$ and $X \in R \otimes_A E_1$ be indecomposable. By 5.2, $X \in \mathcal{L}_R$. Now, there exists a path $I \rightsquigarrow E_1$ in $\text{ind } A$ with I injective. By 4.5, this path induces a path $I' \rightsquigarrow X$ in $\text{ind } R$, with $I' \in R \otimes_A I$ so that I' is an injective R -module. Thus $X \in \mathcal{E}_1(R)$. Let now $E_2 \in \mathcal{E}'_2(A)$ and $X \in R \otimes_A E_2$ be indecomposable. By 5.2, $X \in \mathcal{L}_R$. Moreover, there exists a path $P \rightsquigarrow \tau^{-1} E_2$ in $\text{ind } A$, with $P \notin \mathcal{L}_A$ projective. By [19, 3.8], we have $\tau_R^{-1} X \in R \otimes_A (\tau_A^{-1} E_2)$. Applying 4.5 yields a path $P' \rightsquigarrow \tau_R^{-1} X$ in $\text{ind } R$, with $P' \in R \otimes_A P$, that is, P' is a projective R -module. By 5.3, $P' \notin \mathcal{L}_R$. Then, $X \in \mathcal{E}_1(R) \cup \mathcal{E}'_2(R)$.

One proves in exactly the same way (using 4.7 instead of 4.5) that $\text{add Hom}_R(R, \mathcal{E}_1(R)) \subseteq \text{add } \mathcal{E}_1(A)$ and $\text{add Hom}_R(R, \mathcal{E}'_2(R)) \subseteq \text{add}(\mathcal{E}_1(A) \cup \mathcal{E}'_2(A))$.

Since $\text{add}(R \otimes_A \mathcal{E}_1(A)) \subseteq \text{add } \mathcal{E}_1(R)$ and $\text{add Hom}_R(R, \mathcal{E}_1(R)) \subseteq \text{add } \mathcal{E}_1(A)$, a simple application of 4.2(b) implies that equality holds in each of these cases.

Now, let $E_2 \in \mathcal{E}'_2(A)$ and assume that an indecomposable summand $X \in R \otimes_A E_2$ lies in $\mathcal{E}_1(R)$. Then $\text{Hom}_R(R, X) \in \text{add } \mathcal{E}_1(A)$. However, $\text{Hom}_R(R, X) = \bigoplus_{\sigma \in H} {}^\sigma E_2$ for some $H \subseteq G$ so ${}^\sigma E_2 \in \mathcal{E}_1(A)$ for any σ , hence $E_2 \in \mathcal{E}_1(A)$ a contradiction. This shows that $\text{add}(R \otimes_A \text{add } \mathcal{E}'_2(A)) \subseteq \text{add } \mathcal{E}'_2(R)$. Similarly, $\text{add Hom}_R(R, \mathcal{E}'_2(R)) \subseteq \text{add } \mathcal{E}'_2(A)$. Finally, applying 4.2(b) yields equality in each of these cases. \square

We recall some definitions. An artin algebra A is *left supported* if $\text{add } \mathcal{L}_A$ is contravariantly finite [4]. We know that an algebra A is left supported if and only if $\text{add } \mathcal{L}_A$ is cogenerated by the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-injectives in $\text{add } \mathcal{L}_A$ (see [4]). Right supported algebras are defined dually. An artin algebra A is a *laura algebra* if $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\text{ind } A$, that is, if $\text{ind } A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is finite [3]. It is called *right glued* if the class of all $M \in \text{ind } A$ such that $\text{pd}_A M \leq 1$ is cofinite in $\text{ind } A$, see [1], or, equivalently, if \mathcal{L}_A is cofinite in $\text{ind } A$, see [3, 2.2]. Left glued algebras are defined dually. Clearly, left and right glued algebras are laura.

An artin algebra is called *weakly shod* if the length of any path in $\text{ind } A$ from an injective to a projective is bounded [12] or, equivalently, if there exists $l \geq 0$ such that any path in $\text{ind } A$ from an indecomposable not in \mathcal{L}_A to one not in \mathcal{R}_A has length at most l , see [2, 1.4]. The algebra A is called *shod* if, for each $M \in \text{ind } A$, we have $\text{pd}_A M \leq 1$ or $\text{id}_A M \leq 1$, or, equivalently, if $\mathcal{L}_A \cup \mathcal{R}_A = \text{ind } A$, see [11]. Shod algebras are weakly shod algebras, and weakly shod algebras are laura algebras. Finally, A is *quasi-tilted* if it is shod and $\text{gl.dim. } A \leq 2$, see [15]. Those laura algebras which are not quasi-tilted are left and right supported, see [4, 4.4].

Proof of Theorem 1.2: The proof of (f) is in [15, II.1.6]. Actually the same proof establishes (e). The proof of (g) in the representation-finite case is in [19, 4.6]. It carries over to the general case using, for instance, the Liu-Skowroński criterion. We just show statements (a) to (d).

- (a) Assume A to be left supported and denote by E (or Q) the direct sum of a complete set of representatives of the isomorphism classes of indecomposable Ext-injectives in $\text{add } \mathcal{L}_A$ (or $\text{add } \mathcal{L}_R$, respectively). We know that $\text{add } \mathcal{L}_A$ is the class of A -modules cogenerated by E . Let $X \in \mathcal{L}_R$. Then $\text{Hom}_R(R, X) \in \text{add } \mathcal{L}_A$ by 5.2, hence there exist $m > 0$ and a monomorphism $\text{Hom}_R(R, X) \hookrightarrow E^{(m)}$. Since $X \in R \otimes_A \text{Hom}_R(R, X)$ by 4.2 (b), and $R \otimes_A -$ is exact, we deduce a monomorphism

$$X \hookrightarrow R \otimes_A \text{Hom}_R(R, X) \hookrightarrow (R \otimes_A E)^{(m)}.$$

Since $R \otimes_A E \in \text{add } Q$ by 5.4, X is cogenerated by Q . So R is left supported. The converse is proven in the same way.

- (b) Assume A to be laura. Then $\text{ind } A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$ is finite. Hence the set \mathcal{X}_R of all indecomposable R -modules in $\text{add}(R \otimes_A \text{ind } A \setminus (\mathcal{L}_A \cup \mathcal{R}_A))$ is finite. Let $X \notin \mathcal{L}_R \cup \mathcal{R}_R$ be an indecomposable R -module. By 4.3, there exists $M \in \text{ind } A$ such that $M \in \text{Hom}_R(R, X)$ and $X \in R \otimes_A M$. By 5.2, $M \notin \mathcal{L}_A \cup \mathcal{R}_A$. Hence $X \in \mathcal{X}_R$ and $\text{ind } R \setminus (\mathcal{L}_R \cup \mathcal{R}_R) \subseteq \mathcal{X}_R$. In particular, it is finite so R is laura.

Conversely, assume R to be laura. Hence the set \mathcal{M}_A of all indecomposable A -modules in $\text{add } \text{Hom}_R(R, \text{ind } R \setminus (\mathcal{L}_R \cup \mathcal{R}_R))$ is finite. Let $M \notin \mathcal{L}_A \cup \mathcal{R}_A$ be an indecomposable A -module. If $R \otimes_A M = \bigoplus_{i=1}^m X_i$ is an indecomposable decomposition in $\text{mod } R$, then, by 5.3, $X_i \notin \mathcal{L}_R \cup \mathcal{R}_R$ for any i . By 4.2(b), $M \in \text{Hom}_R(R, R \otimes_A M) \simeq \bigoplus_{i=1}^m \text{Hom}_R(R, X_i)$. Consequently, $M \in \mathcal{M}_A$ and $\text{ind } A \setminus (\mathcal{L}_A \cup \mathcal{R}_A) \subseteq \mathcal{M}_A$. In particular, it is finite, so A is laura.

- (c) Assume A to be right glued. Then $\text{ind } A \setminus \mathcal{L}_A$ is finite. By 5.3 and 4.3, $\text{ind } R \setminus \mathcal{L}_R$ is finite. Hence R is right glued. The converse is proven similarly.
- (d) Assume A to be weakly shod, and let $l \geq 0$ be such that any path from an indecomposable not in \mathcal{L}_A to one not in \mathcal{R}_A has length at most l . Let

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_s = Y \quad (2)$$

be a path in $\text{ind } R$ with $X \notin \mathcal{L}_R$ and $Y \notin \mathcal{R}_R$. By 4.7, it induces a path in $\text{ind } A$ of the form

$$M_0 \rightarrow \sigma^1 M_1 \rightarrow \sigma^2 M_2 \rightarrow \cdots \rightarrow \sigma^s M_s$$

with $M_j \in \text{Hom}_R(R, X_j)$ and $X_j \in R \otimes_A \sigma^j M_j \simeq R \otimes_A M_j$. By 5.3 and 5.1, $M_0 \notin \mathcal{L}_A$ and $\sigma^s M_s \notin \mathcal{R}_A$. Then $s \leq l$ and R is weakly shod.

The converse is proven in the same way, using 4.5 instead of 4.7.

□

We now assume that the group G acts freely on the idempotents. In this case, $R = A[G]^b$ and the algebra of invariants A^G are Morita equivalent. Indeed, recall, for instance from [13], that this is the case if and only if there exist an element $x \in A$ such that $\sum_{g \in G} g(x) = 1$ and two families of elements $\{x_1, x_2, \dots, x_r\}$, $\{y_1, y_2, \dots, y_r\}$ of A such that $\sum_{i=1}^r x_i \cdot y_i = 1$ and $\sum_{i=1}^r x_i \cdot g(y_i) = 0$ for all $g \neq 1$. Now, since $|G|$ is invertible in A , the element $x = |G|^{-1}$ verifies the first condition. The second follows upon taking $x_i = y_i = e_i$, where $\{e_1, e_2, \dots, e_r\}$ is a complete set of primitive orthogonal idempotents. This shows our assertion. Observe also, that, in this case, A is a finite Galois covering of $A[G]$ (or equivalently A^G) with group G (see [6] or [13]). We may now state:

COROLLARY 5.5 *Let A be an artin algebra, G be a finite group acting on A such that $|G|$ is invertible in A and G acts freely on the idempotents of A . Then:*

- (a) A is left (or right) supported if and only if so is A^G ;

- (b) A is laura if and only if so is A^G ;
- (c) A is left (or right) glued if and only if so is A^G ;
- (d) A is weakly shod if and only if so is A^G ;
- (e) A is shod if and only if so is A^G ;
- (f) A is quasitilted if and only if so is A^G ;
- (e) A is tilted if and only if so is A^G .

The Auslander-Reiten components of a laura or a supported algebra have been described in [3, 4]. We notice that, if A, R are as above, and Γ is a component of the Auslander-Reiten quiver $\Gamma(\text{mod } A)$ of A (or $\Gamma(\text{mod } R)$ of R), then, in general, the images of the indecomposables of Γ lie in several components of $\Gamma(\text{mod } R)$ (or $\Gamma(\text{mod } A)$, respectively).

LEMMA 5.6 *Let A be an artin algebra, Γ be a component of $\Gamma(\text{mod } A)$ and Γ' be the unique component of $\Gamma(\text{mod } R)$ containing an indecomposable $X \in R \otimes_A M$, with $M \in \Gamma$. Then:*

- (a) Γ is postprojective if and only if so is Γ' ;
- (b) Γ is preinjective if and only if so is Γ' ;
- (c) Γ is regular if and only if so is Γ' ;
- (d) Γ is non-semiregular if and only if so is Γ' ;
- (e) Γ is semiregular if and only if so is Γ' .

Proof. For the proofs of (a), (b) and (c), we refer to [19, 4.3]. We now prove (d). Suppose that Γ is non-semiregular, then there exist an indecomposable injective A -module $I \in \Gamma$, an indecomposable projective A -module $P \in \Gamma$, and a walk of irreducible morphisms between indecomposables

$$I = L_0 - L_1 - \cdots - L_s = M = N_0 - N_1 - \cdots - N_t = P.$$

Applying [19, 4.1] and induction yields a walk of irreducible morphisms between indecomposable R -modules in Γ'

$$I' = Y_0 - Y_1 - \cdots - Y_s = X = Z_0 - Z_1 - \cdots - Z_t = P'$$

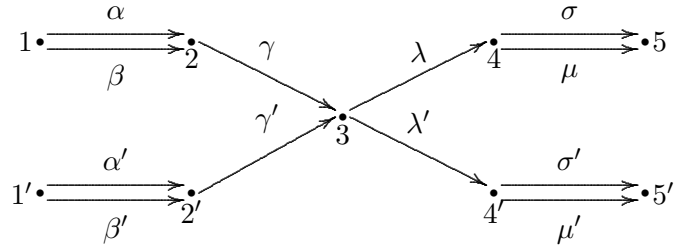
with $Y_i \in R \otimes_A L_i$ for any i (thus, I' is injective) and $Z_j \in R \otimes_A N_j$ for any j (thus, P' is projective). This shows the sufficiency. The necessity is shown in the same way taking into account that, under the stated hypothesis, $M \in \text{Hom}_R(R, X)$. Finally, the proof of (e) is also similar. \square

Here, we are interested mainly in the case where A is a strict (that is, not quasi-tilted) laura algebra.

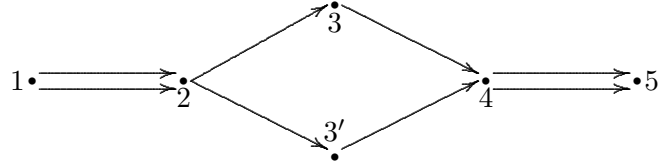
COROLLARY 5.7 *Let A be a strict lura algebra and Γ, Γ' be as in the hypothesis of the lemma. Then Γ is the unique faithful non-semiregular component of $\Gamma(\text{mod } A)$ if and only if Γ' is the unique faithful non-semiregular component of $\Gamma(\text{mod } R)$.*

EXAMPLE 5.8

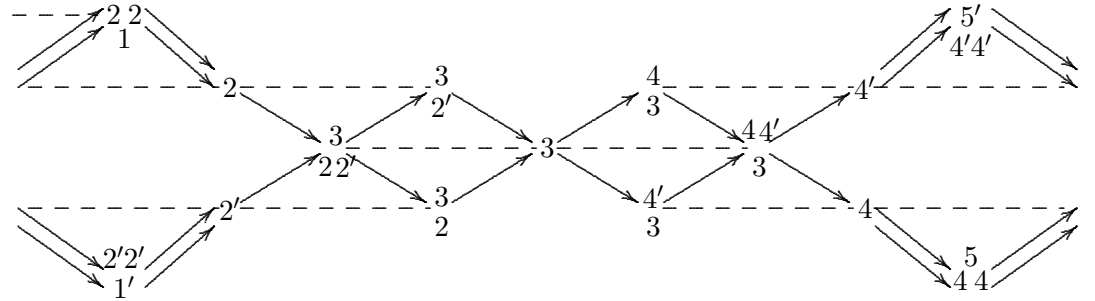
Let k be a field, and A be the radical square zero k -algebra given by the quiver



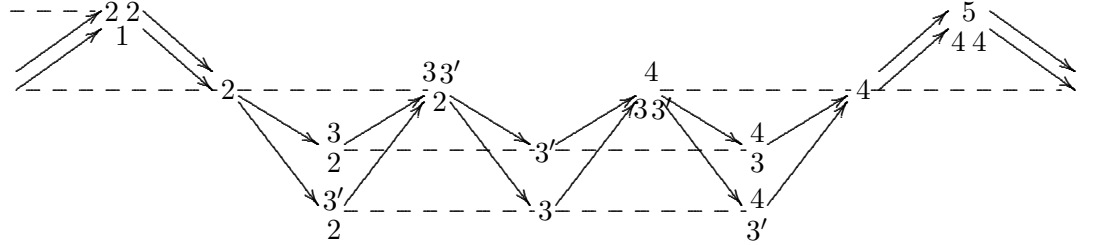
Hence, A is a strict lura algebra. We let the group $\mathbb{Z}/2\mathbb{Z}$ act on A , where the only non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ fixes the point 3, permutes the points x and x' (for $x \in \{1, 2, 4, 5\}$) and the arrows ξ and ξ' (for $\xi \in \{\alpha, \beta, \gamma, \lambda, \sigma, \mu\}$). Then, by [19, 2.3], R is the radical square zero k -algebra given by the quiver



According to our Theorem 1.2, R is also a strict lura algebra. We now draw the unique non-semiregular faithful component of $\Gamma(\text{mod } A)$.



(where indecomposable modules are represented by their Loewy series and horizontal dotted lines describe the Auslander-Reiten translation). The unique non-semiregular faithful component of $\Gamma(\text{mod } R)$ is

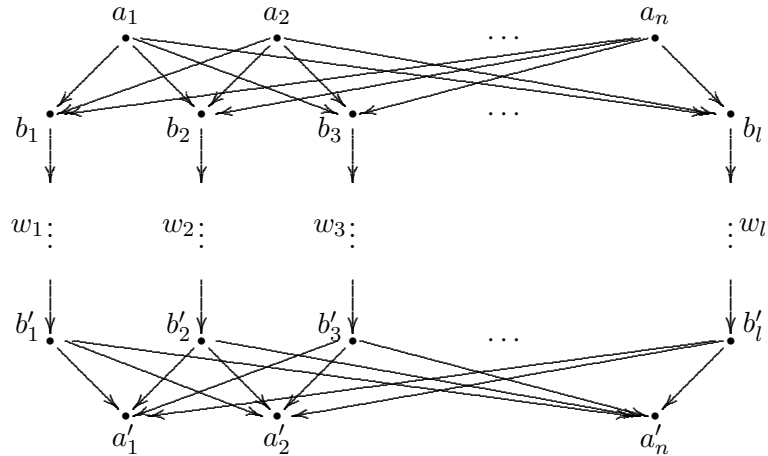


Notice that $\Gamma(\text{mod } A)$ has two postprojective (or preinjective) components, while $\Gamma(\text{mod } R)$ has only one.

6 Skew toupie algebras.

Throughout this section, all algebras are finite dimensional over an algebraically closed field, thus are bound quiver algebras.

Let n, l be two positive integers. We define the *complete bipartite quiver* Q_l^n to have as its only points n sources a_1, \dots, a_n and l sinks b_1, \dots, b_l and, for each pair (i, j) with $1 \leq i \leq n$, $1 \leq j \leq l$, there is an arrow $a_i \rightarrow b_j$ and these are the only arrows of Q_l^n . A *skew toupie quiver* Q is defined as follows: it consists of a complete bipartite quiver Q_l^n , its opposite quiver $(Q_l^n)^{op}$ as well as l disjoint paths w_1, \dots, w_l from the sinks of Q_l^n to the sources of $(Q_l^n)^{op}$. Thus, a skew toupie is a quiver of the form:



An algebra $R = kQ/I$ is called a *skew toupie algebra* if its quiver Q is a skew toupie quiver. Our objective in this section is to exhibit a family of skew toupie algebras which are

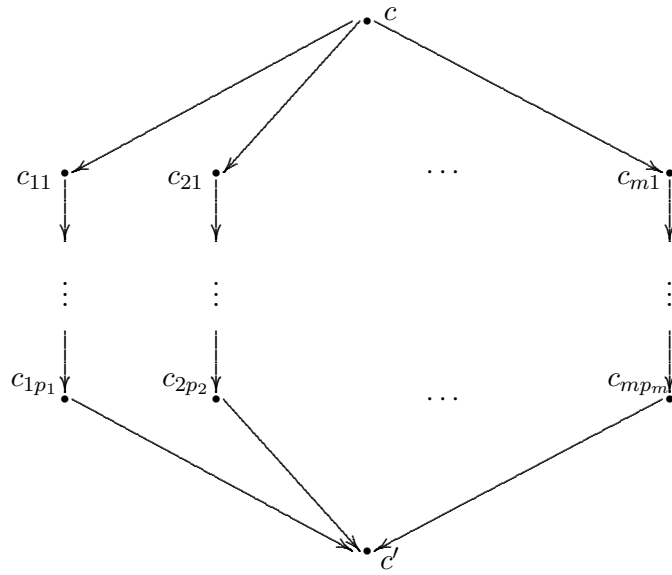
laura, and even weakly shod.

If, above, Q has exactly one source and one sink (that is, $n = 1$), then Q is a toupie quiver, and $R = kQ/I$ is a toupie algebra, as defined and studied in [9].

PROPOSITION 6.1 *Let $R = kQ/I$ be a skew toupie algebra. Then R is a weakly shod algebra provided:*

- (1) *The ideal I is generated by all possible commutativity relations. In this case, R is tilted.*
- (2) *The ideal I is monomial, and generated by at least one subpath of each of the w_i . In this case, R is tilted if and only if each path w_i is bound by exactly one relation.*
- (3) *The ideal I is generated by the sums of all paths from each source to each sink, and moreover $nl \in \{2, 3\}$ or the length of each w_i does not exceed one. In this case, R is canonical if $l = 3$, and tilted otherwise.*

Proof. We consider the toupie quiver Q' with $m = nl$ branches



For each i with $1 \leq i \leq m$, we set $i' = i + l$ if $i + l \leq m$ and $i' = i + l - m$ if $i + l > m$. We suppose that, for each i with $1 \leq i \leq m$, we have $p_i = p_{i'}$ and, if $i \leq l$, we set $p_i = \ell(w_i) + 1$, where $\ell(w_i)$ denotes the length of the path w_i . We finally denote by γ_i the path $c_{i_1} \rightarrow c_{i_2} \rightarrow \dots \rightarrow c_{i_{p_i}}$, for $1 \leq i \leq m$.

We define on Q' an action of the cyclic group $\mathbb{Z}/n\mathbb{Z} = \langle \sigma \rangle$ as follows: we set $\sigma(c) = c$, $\sigma(c') = c'$ and, for each pair (i, j) , with $1 \leq i \leq m$ and $1 \leq j \leq p_i$, $\sigma(c_{ij}) = c_{i'j}$ where i' is as above. We let σ have the induced action on the arrows. This defines indeed an action on Q' (and hence on the path algebra kQ') because of our assumption on the p_i .

Let $A = kQ'/I'$, where I' is an admissible ideal of one of the following forms:

- (1) I' is generated by all possible commutativity relations.
- (2) I' is a monomial ideal generated by at least one subpath of each of the γ_i . Moreover, for each i such that $1 \leq i \leq m$, the path γ_i is isomorphic to $\gamma_{i'}$, as full convex subcategories of A (again, i' is as above).
- (3) I' is generated by the sum of all paths from c to c' and moreover $m \in \{2, 3\}$ or $p_i \leq 2$ for all i .

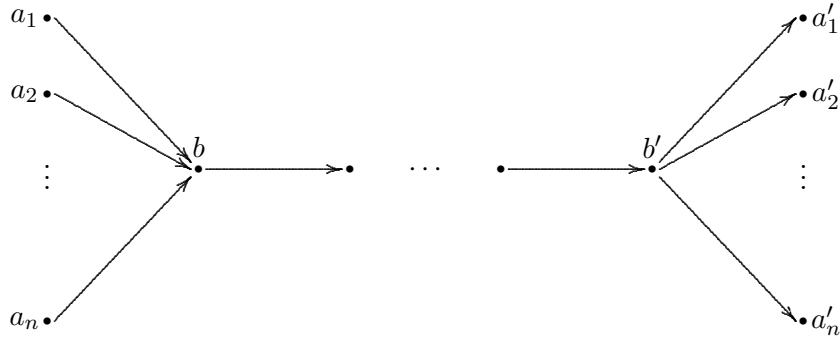
Clearly, the action of $\mathbb{Z}/n\mathbb{Z}$ on kQ' leaves invariant the ideal I' . Thus, by [19, 2.1], $\mathbb{Z}/n\mathbb{Z}$ acts on A .

Now, it follows from the main result of [9] that, in each of these cases, A is a weakly shod algebra. Furthermore, in the case (1) it is always tilted, while, in the case (2), it is tilted if and only if each path γ_i is bound by exactly one relation, and, finally, in the case (3), it is canonical if and only if $m = 3$ and $l > 1$ (thus $l = 3$) and tilted in all the other cases.

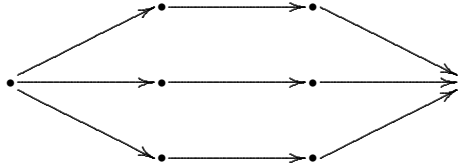
By [19, 2.3], we get $A[\mathbb{Z}/n\mathbb{Z}]^b = R$, as given in the statement of our proposition. The assertion now follows from Theorem 1.2. \square

6.1 Remarks and examples.

- (a) If $l = 1$, then R has as quiver the following tree

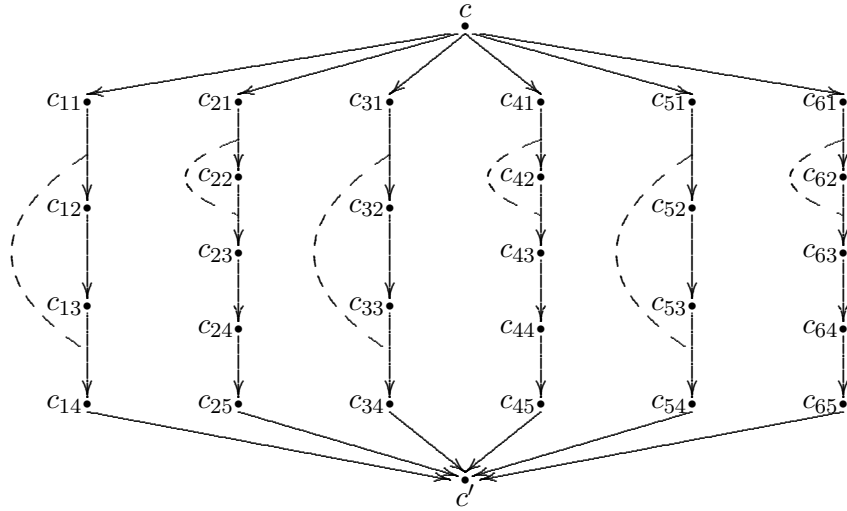


- (b) The tubular canonical algebra given by the quiver

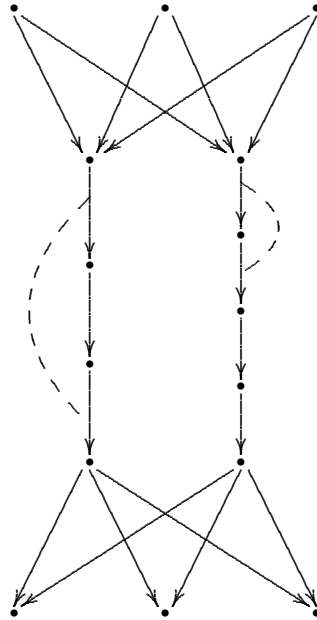


bound by the ideal generated by the sum of all paths from the source to the sink provides an example of a (skew) toupie algebra which is quasi-tilted, but not tilted.

(c) Let A be given by the toupie quiver



bound by the relations denoted by the shown dotted lines. We define an action of $\mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle$ as follows: $\sigma(c) = c$, $\sigma(c') = c'$, $\sigma(c_{1i}) = c_{3i}$, $\sigma(c_{3i}) = c_{5i}$ for $1 \leq i \leq 4$, $\sigma(c_{2j}) = c_{4j}$, $\sigma(c_{4j}) = c_{6j}$ for $1 \leq j \leq 5$. In this case, R is given by the quiver



bound by the relations denoted by the shown dotted lines. According to 6.1 or, directly, by the main result of [16], R is tilted.

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I. ASSEM; DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE, QUÉBEC, CANADA, J1K 2R1.

E-mail address: ibrahim.assem@usherbrooke.ca

M. LANZILOTTA; CENTRO DE MATEMÁTICA (CMAT), IGUÁ 4225, UNIVERSIDAD DE LA REPÚBLICA, CP 11400, MONTEVIDEO, URUGUAY.

E-mail address: marclan@cmat.edu.uy

M. J. REDONDO; INSTITUTO DE MATEMÁTICA, UNIVERSIDAD NACIONAL DEL SUR, AV. ALEM 1253, (8000) BAHÍA BLANCA, ARGENTINA.

E-mail address: mredondo@criba.edu.ar