HOCHSCHILD COHOMOLOGY OF PARTIAL RELATION EXTENSIONS

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Abstract. We show how to compute the low Hochschild cohomology groups of a partial relation extension algebra.

1. Introduction

Cluster-tilted algebras appeared as a gift from the theory of cluster algebras to representation theory. These are finite-dimensional algebras which are endomorphism algebras of tilting objects in the cluster category [10]. This class of algebras was much investigated, see for example [1, 2, 7, 8, 11, 12, 15, 16], and [5, 6, 4] for results on their Hochschild cohomology. Among the main results is that every cluster-tilted algebra can be written as trivial extension of a tilted algebra by a bimodule called the relation bimodule [1]. This explains why many features of tilted algebras are retained by cluster-tilted algebras. In particular, complete slices of tilted algebras embed as what is called local slices in cluster-tilted algebras [2]. However, unlike tilted algebras, cluster-tilted algebras are not characterized by the existence of local slices. In an effort to find a larger class of algebras having local slices, the authors of [3] introduced what are called partial relation extensions which, because of the existence of local slices, share many properties with cluster-tilted algebras.

This paper is devoted to the study of the low Hochschild cohomology groups of partial relation extensions. We now state our main theorem. Let C be a triangular algebra of global dimension at most 2, and assume that the relation bimodule $E = \mathrm{Ext}^2_C(DC, C)$ splits as a direct sum of two C-C-bimodules $E = E' \oplus E''$. Then the trivial extension $B = C \ltimes E'$ is called a *partial* relation extension, while $\widetilde{C} = C \ltimes E$ is called *the* relation extension of C. Further, given an algebra A and an A-A-bimodule M, we denote by $\mathrm{H}^i(A, M)$ the i-th Hochschild cohomology group of A with coefficients in M and we set $\mathrm{H}^i(A, A) = \mathrm{HH}^i(A)$. Finally, we denote by $\mathcal{E}(M, A)$ the set of all A-A-bimodule morphisms $f : M \to A$ such that $xf(y) + f(x)y = 0$, for all $x, y \in M$. With this notation our main theorem reads as follows.

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Theorem A. *There exist short exact sequences of k-vector spaces*

$$0 \longrightarrow \mathrm{H}^0(B,E') \longrightarrow \mathrm{HH}^0(B) \xrightarrow{\varphi^0} \mathrm{HH}^0(C) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^1(B,E') \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi^1} \mathrm{HH}^1(C) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^0(\widetilde{C},E'') \longrightarrow \mathrm{HH}^0(\widetilde{C}) \xrightarrow{\varphi^0} \mathrm{HH}^0(B) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^1(\widetilde{C},E'') \oplus \mathcal{E}(E'',B) \longrightarrow \mathrm{HH}^1(\widetilde{C}) \xrightarrow{\varphi^1} \mathrm{HH}^1(B) \longrightarrow 0.$$

We recall that, if C is tilted, then $\widetilde{C}$ is cluster-tilted and the partial relation extension B is then a quotient (and a subalgebra) of a cluster-tilted algebra.

The techniques we use are those of [6] and [4]. In fact several of our proofs follow directly from results of [4], which suggests that the latter hold in greater generality than originally considered.

As a consequence of this, we give another realization of the group $\mathrm{HH}^1(B)$ as the amalgamated sum of two morphisms.

The paper is organized as follows. After a preliminary section 2, we prove our main theorem in section 3. Section 4 is devoted to corollaries and examples.

2. Preliminaries

Throughout this paper, k denotes an algebraically closed field, all algebras are finite-dimensional over k and have an identity. Given an algebra C, we denote by $C^e = C \otimes_k C^{op}$ its enveloping algebra. If Q is a quiver, we denote by kQ its path algebra. For a point i of Q, let e_i be the primitive idempotent of kQ corresponding to the stationary path at i. We refer the reader to [9, 14] for general notions and results of representation theory.

2.1. **Hochschild cohomology.** Let C be an algebra and E a C-C-bimodule which is finite-dimensional over k. The *Hochschild complex* is the complex

$$0 \to E \xrightarrow{b^1} \mathrm{Hom}_k(C,E) \xrightarrow{b^2} \cdots \to \mathrm{Hom}_k(C^{\otimes i},E) \xrightarrow{b^{i+1}} \mathrm{Hom}_k(C^{\otimes(i+1)},E) \to \cdots$$

where, for each $i>0$, $C^{\otimes i}$ denotes the i-fold tensor product of C with itself over k. The map $b^1 \colon E \to \mathrm{Hom}_k(C,E)$ is defined by $(b^1x)(c) = cx - xc$ for $x \in E,\ c \in C$, and b^{i+1} is defined by

$$(b^{i+1}f)(c_0 \otimes \cdots \otimes c_i) = c_0 f(c_1 \otimes \cdots \otimes c_i) + \sum_{j=1}^{i} (-1)^j f(c_0 \otimes \cdots \otimes c_{j-1}c_j \otimes \cdots \otimes c_i) + (-1)^{i+1} f(c_0 \otimes \cdots \otimes c_{i-1})c_i$$

for a k-linear map $f : C^{\otimes i} \to E$ and elements $c_0, \cdots, c_i$ in C.

The i-th cohomology group of this complex is called the *i-th Hochschild cohomology group* of C with coefficients in E, and is denoted by $\mathrm{H}^i(C, E)$. If ${}_C E_C = {}_C C_C$, then we write $\mathrm{HH}^i(C) = \mathrm{H}^i(C, C)$.

The first Hochschild cohomology group has the following concrete description. Let $\mathrm{Der}(C, E)$ be the vector space of all *derivations*, that is, k-linear maps $d : C \to E$ such that, for $c, c' \in C$, we have

$$d(cc') = cd(c') + d(c)c'.$$

A derivation d is *inner* if there exists $x \in E$ such that $d = [x, -]$. Letting $\mathrm{Inn}(C, E)$ denote the subspace of all inner derivations, we have $\mathrm{H}^1(C, E) \cong \mathrm{Der}(C, E)/\mathrm{Inn}(C, E)$.

A derivation $d : C \to E$ is called *normalized* if, for any primitive orthogonal idempotent e_i in a complete set $\{e_1, \ldots, e_n\}$, we have $d(e_i) = 0$ for all i. Let $\mathrm{Der}_0(C, E)$ be the subspace of $\mathrm{Der}(C, E)$ of the normalized derivations, and $\mathrm{Inn}_0(C, E) = \mathrm{Der}_0(C, E) \cap \mathrm{Inn}(C, E)$. Then we also have $\mathrm{H}^1(C, E) \cong \mathrm{Der}_0(C, E)/\mathrm{Inn}_0(C, E)$.

2.2. **The Hochschild projection maps.** Let C be a finite-dimensional algebra and E a finitely generated C-C-bimodule equipped with an associative C-C-bimodule morphism $E \otimes_C E \to E$, $e \otimes e' \mapsto ee'$. The *split extension* of C by E is the k-algebra B which has the additive structure of $C \oplus E$ and whose product is defined by

$$(c, e)(c', e') = (cc', ce' + ec' + ee').$$

If $E^2 = 0$, then B is the *trivial extension* of C by E, which we denote by $B = C \ltimes E$.

In the special case where C is a triangular algebra of global dimension at most 2, and E is the relation bimodule $E = \mathrm{Ext}^2_C(DC, C)$, the trivial extension $C \ltimes E$ is called the *relation extension of C*. If E splits as a direct sum of two C-C-bimodules $E = E' \oplus E''$, then the trivial extension $B = C \ltimes E'$ is called a *partial relation extension of C*.

Given a split extension B of C by E, there is an exact sequence of vector spaces

$$0 \longrightarrow E \xrightarrow{\ i\ } B \underset{q}{\overset{p}{\rightleftarrows}} C \longrightarrow 0,$$

where $p\colon (c, x) \mapsto c$ and $i\colon e \mapsto (0, e)$. Then p is an algebra morphism which has a section $q : c \mapsto (c, 0)$.

Given a k-linear morphism $f\colon B^{\otimes n} \to B$, we have a k-linear morphism $pfq^{\otimes n}\colon C^{\otimes n} \to C$. It is shown in [4, Corollary 2.2] that the assignment $[f] \mapsto [pfq^{\otimes n}]$ defines a k-linear map $\varphi^n\colon \mathrm{HH}^n(B) \to \mathrm{HH}^n(C)$, called the n-th *Hochschild projection morphism.*

3. Main result

This section is devoted to the proof of Theorem A.

3.1. We start with a criterion for the surjectivity of the first Hochschild projection morphism.

Lemma 3.1. *Let B be a trivial extension of C by E. The Hochschild projection morphism $\varphi^1\colon \mathrm{HH}^1(B) \to \mathrm{HH}^1(C)$ is surjective if and only if, for each derivation d of C, there exists a k-linear map $\alpha\colon E \to E$ such that*

$$\begin{array}{rcll} x\,d(c) & = & \alpha(x)\,c - \alpha(xc), & \qquad \text{(C1)} \\ d(c)\,x & = & c\,\alpha(x) - \alpha(cx), & \qquad \text{(C2)} \end{array}$$

for $x \in E,\ c \in C$.

Proof. This is a reformulation of [4, Corollary 3.6 (b)] in case $n = 1$ taking into account that the third condition in loc.cit. is void in this case. □

3.2. From now on, we assume that C is a triangular algebra of global dimension two.

Lemma 3.2. *Let $E = E' \oplus E''$ be a decomposition of the C-C-bimodule $E = \mathrm{Ext}^2_C(DC, C)$ and $B = C \ltimes E'$ be a partial relation extension. Then $\varphi^1\colon \mathrm{HH}^1(B) \to \mathrm{HH}^1(C)$ is surjective.*

Proof. Let $\alpha\colon C \to C$ be a derivation and p', q' be respectively the canonical projection and inclusion between E and E'. Applying Lemma 3.1 above to $\widetilde{C} = C \ltimes E$, there exists a k-linear map $\alpha\colon E \to E$ which satisfies conditions (C1) and (C2) of the lemma. Let $\alpha' = p'\,\alpha\,q'\colon E' \to E'$. This is a k-linear morphism. Then for all $x \in E',\ c \in C$, we have

$$\alpha'(x)\,c - \alpha'(xc) = (p'\,\alpha'\,q')(x)\,c - (p'\,\alpha'\,q')(xc).$$

Considering x and xc as elements of E, this expression can be written as

$$p'\alpha(x)\,c - p'\alpha(xc) = p'[\alpha(x)\,c - \alpha(xc)],$$

because p' is a morphism of C-C-bimodules. Now, because of Lemma 3.1, we have $\alpha(x)\,c - \alpha(xc) = x\,d(c)$ inside E. Since $x \in E'$ and $d(c) \in C$, we have $x\,d(c) \in E'$. Hence $p'(x\,d(c)) = x\,d(c)$. This shows that $\alpha'(x)\,c - \alpha'(xc) = x\,d(c)$ as required. The second relation (C2) is proven in the same way. □

3.3. We prove the exactness of the first sequence of our main theorem. Here and in the sequel, we keep the notation of Lemma 3.2, that is, we have a direct sum decomposition $E = E' \oplus E''$ and $B = C \ltimes E'$.

Lemma 3.3. *There exists a short exact sequence of vector spaces*

$$0 \longrightarrow \mathrm{H}^0(B, E') \longrightarrow \mathrm{HH}^0(B) \longrightarrow \mathrm{HH}^0(C) \longrightarrow 0.$$

Proof. Because C is triangular, its center $Z(C)$ is equal to k, and hence the bimodule E' is (trivially) symmetric over $Z(C)$, that is, for every $e' \in E'$ and $z \in Z(C)$ we have $z\,e' = e'\,z$. On the other hand, $\mathrm{HH}^0(B) = Z(B)$, $\mathrm{HH}^0(C) = Z(C)$ and φ^0 is the restriction to $Z(B)$ of the projection $p\colon B \to C$. Thus φ^0 maps the identity of B to the identity of C, hence it is a nonzero morphism. Because $Z(C) = k$, it is surjective, and its kernel is the subspace of E' consisting of all elements which are central in B. Thus $\operatorname{Ker}\varphi^0 = E' \cap Z(B)$.

We claim that $E' \cap Z(B) \cong \operatorname{Hom}_{B^e}(B, E')$. Indeed, if $f \in \operatorname{Hom}_{B^e}(B, E')$ then $f(1) \in E' \cap Z(B)$ because f is a morphism of B-B-bimodules. On the other hand, if $x \in E' \cap Z(B)$, then the map $f_x \colon B \to E'$ defined by $1 \mapsto x$ is a morphism of B-B-bimodules, because x is central. It is easily seen that these two maps are inverses to each other. This establishes the claim which implies that $\operatorname{Ker}\varphi^0 \cong \operatorname{Hom}_{B^e}(B, E) = \mathrm{H}^0(B, E')$ as desired. □

3.4. The following statement is necessary for the proof of Lemma 3.5.

Lemma 3.4. $\mathcal{E}(E', C) = 0$.

Proof. Let $f \in \mathcal{E}(E', C)$ and define $\overline{f} \colon B \to C$ by $\overline{f}(c, x) = f(x)$, for $(c, x) \in B = C \oplus E'$. Clearly, $\overline{f}|_C = 0$. We claim that $\overline{f}$ is a derivation. Let $(c, x),\ (c', x') \in B$. Then

$$\begin{aligned}(c,x)\overline{f}(c',x') + \overline{f}(c,x)(c',x') &= (c,x)f(x') + f(x)(c',x') \\ &= (cf(x') + f(x)c',\ xf(x') + f(x)x') \\ &= (cf(x') + f(x)c', 0),\end{aligned}$$

because $f \in \mathcal{E}(E', C)$. On the other hand, f is a morphism of C-C-bimodules, hence

$$\begin{aligned}(c,x)\overline{f}(c',x') + \overline{f}(c,x)(c',x') &= (f(cx' + xc'), 0) \\ &= \overline{f}(cc', cx' + xc') = \overline{f}((c,x)(c',x')).\end{aligned}$$

This completes the proof that $\overline{f}$ is a derivation. Now let $\gamma \colon x \to y$ be an arrow in the quiver of B which does not belong to the quiver of C. Then γ is a generator of E' as a C-C-bimodule. We have

$$\overline{f}(\gamma) = \overline{f}(e_x \gamma e_y) = e_x \overline{f}(\gamma) e_y$$

because $e_x, e_y \in C$ imply $\overline{f}(e_x) = 0$ and $\overline{f}(e_y) = 0$. This shows that $\overline{f}$ maps $e_x E e_y$ to $e_x C e_y$. Moreover the existence of a new arrow $\gamma \colon x \to y$ implies that there exists a path from y to x inside C (in fact a relation), see [3, Corollary 2.2.1]. But C is triangular, hence $e_x C e_y = 0$. This shows that $\overline{f} = 0$ and hence $f = 0$. □

3.5. We are now able to prove the exactness of the second sequence of our main theorem.

Lemma 3.5. *There exists a short exact sequence of vector spaces*

$$0 \longrightarrow \mathrm{H}^1(B, E') \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi^1} \mathrm{HH}^1(C) \longrightarrow 0.$$

Proof. Because of Lemma 3.2, the projection morphism φ^1 is surjective. Moreover, we have seen in the proof of Lemma 3.3 that E' is symmetric over $Z(C)$. We apply [4, Theorem 4.4] taking into account that $\mathcal{E}(E', C) = 0$, by Lemma 3.4. □

Remark. Recall that, by [4, Proposition 4.8], we have

$$\mathrm{H}^1(B,E') = \mathrm{H}^1(C,E') \oplus \mathrm{End}_{C^e}E.$$

3.6. We now continue the proof of our main theorem. Recall that $E = E' \oplus E''$ and $\widetilde{C} = C \ltimes E$.

Lemma 3.6. *The morphism* $\varphi^1 \colon \mathrm{HH}^1(\widetilde{C}) \to \mathrm{HH}^1(B)$ *is surjective.*

Proof. Because of [3, Lemma 2.1.1], the morphism φ^1 is well-defined. Let $d\colon C \to C$ be a derivation and p'', q'' be respectively the canonical projection and inclusion morphisms between E and E''. Because of Lemma 3.1, there exists $\alpha\colon E \to E$ which satisfies conditions (C1) and (C2). Consider the k-linear map $\alpha'' = p''\,\alpha\,q''\colon E'' \to E''$. Let $x'' \in E''$ and $c \in C$.

$$\begin{aligned} \alpha''(x'')\,c - \alpha''(x''c) &= p''\,\alpha\,q''(x'')\,c - p''\,\alpha\,q''(x''c) \\ &= p''\,\alpha(0,x'')\,c - p''\,\alpha(0,x''c) \\ &= p''[\alpha(0,x'')\,c - \alpha(0,x''c)], \end{aligned}$$

because p'' is a morphism of bimodules. Now, in E'' we have $\alpha(0,x'')\,c - \alpha(0,x''c) = (0,x'')d(c)$ because α satisfies condition (C1). On the other hand, $p''[(0,x'')d(c)] = p''(0,x'')\,d(c) = x''\,d(c)$. We have thus proved that

$$\alpha''(x'')\,c - \alpha''(x''c) \;=\; x''\,d(c).$$

Hence α'' satisfies condition (C1). The proof of condition (C2) is similar. □

3.7. The next lemma is needed for the proof of exactness of the third and the fourth sequence of our main theorem.

Lemma 3.7. $E''Z(B) = Z(B)E'' = 0$. *In particular, the bimodule* E'' *is symmetric over* $Z(B)$.

Proof. We must prove that, for each $z \in Z(B)$ and each $e \in E''$, we have $ze = ez = 0$. Now $z \in Z(B)$ is a linear combination of nonzero cycles in B. Each one of these cycles contains exactly one arrow in the quiver of B which is not in the quiver of C (for, if it contains none, then it lies in C which is triangular, and if it contains more than one, then it lies in $E^2 = 0$.) On the other hand, $e \in E''$ is a linear combination of nonzero paths, each containing at least one generator of E'', that is, an arrow of $\widetilde{C}$ which is not in B. Therefore each path appearing in ze or ez contains at least two arrows in $\widetilde{C}$ which are not in C. Then $ze = ez = 0$ because $E^2 = 0$. □

3.8. Proof of the main theorem. We have proved the existence of the first two exact sequences in lemmata 3.3 and 3.5. The sequence

$$0 \longrightarrow \mathrm{H}^0(\widetilde{C},E'') \longrightarrow \mathrm{HH}^0(\widetilde{C}) \xrightarrow{\varphi^0} \mathrm{HH}^0(B) \longrightarrow 0$$

is exact because of Lemma 3.7 and [4, Lemma 4.1] while the exactness of the sequence

$$0 \longrightarrow \mathrm{H}^1(\widetilde{C},E'') \oplus \mathcal{E}(E'',B) \longrightarrow \mathrm{HH}^1(\widetilde{C}) \xrightarrow{\varphi^1} \mathrm{HH}^1(B) \longrightarrow 0$$

follows from lemmata 3.6 and 3.7 and [4, Theorem 4.4]. □

Remark. (a) In contrast to the second sequence, the term $\mathcal{E}(E'', B)$ in the fourth sequence does not usually vanish. We refer to Example 4.2 below.

(b) Moreover, because of [4, Proposition 4.8], we have

$$\mathrm{H}^1(\widetilde{C}, E'') \cong \mathrm{H}^1(B, E'') \oplus \mathrm{End}_{B^e} E''.$$

4. Corollaries and examples

In this last section, we deduce some consequences of our main result and give a couple of examples.

4.1. In our first corollary, we give another description of the group $\mathrm{HH}^1(B)$. Because we shall deal at the same time with several bimodule projections and inclusions, we introduce the following notation. Let X, Y be bimodules overt the same algebra, then if there exists a natural inclusion from X to Y, it will be denoted by $q_{YX} \colon X \to Y$ and similarly, if there exists a natural projection from Y to X, say, it will be denoted by $p_{XY} \colon Y \to X$.

We define a natural morphism $\eta \colon \mathrm{H}^1(\widetilde{C}, E) \to \mathrm{H}^1(B, E')$. Recall that $\mathrm{H}^1(\widetilde{C}, E) = \mathrm{Ext}^1_{C^e}(\widetilde{C}, E)$ while $\mathrm{H}^1(B, E') = \mathrm{Ext}^1_{C^e}(B, E')$. Let the exact sequence

$$\underline{e} \qquad 0 \longrightarrow E \longrightarrow X \longrightarrow \widetilde{C} \longrightarrow 0$$

represent an element of $\mathrm{Ext}^1_{C^e}(\widetilde{C}, E)$, and consider the inclusion morphism $q_{\widetilde{C}B} \colon B \to \widetilde{C}$ and the projection $p_{E'E} \colon E \to E'$. Then the exact sequence

$$p_{E'E}\ \underline{e}\ q_{\widetilde{C}B} \qquad 0 \longrightarrow E' \longrightarrow Y \longrightarrow B \longrightarrow 0$$

represents an element of $\mathrm{Ext}^1_{C^e}(B, E')$. We set $\eta \colon [\underline{e}] \mapsto [p_{E'E}\ \underline{e}\ q_{\widetilde{C}B}]$.

Let also $\psi \colon \mathrm{HH}^1(\widetilde{C}, E) \to \mathrm{HH}^1(\widetilde{C})$ denote the kernel of the Hochschild projection morphism $\varphi^1 \colon \mathrm{HH}^1(\widetilde{C}) \to \mathrm{HH}^1(C)$.

Corollary 4.1. (a) *The morphism η is surjective and*

$$\mathrm{Ker}\,\eta = \mathrm{H}^1(\widetilde{C}, E'') \oplus \mathcal{E}(E'', B).$$

(b) $\mathrm{HH}^1(B)$ *is the amalgamated sum of the morphisms* $\psi \colon \mathrm{HH}^1(\widetilde{C}, E) \to \mathrm{HH}^1(\widetilde{C})$ *and* $\eta \colon \mathrm{HH}^1(\widetilde{C}, E) \to \mathrm{H}^1(B, E')$.

Proof. Because $p_{C\widetilde{C}} = p_{CB} p_{B\widetilde{C}}$ and $q_{\widetilde{C}C} = p_{\widetilde{C}B} p_{BC}$, the right square of the diagram below commutes.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^1(\widetilde{C}, E') & \xrightarrow{\psi} & \mathrm{HH}^1(\widetilde{C}) & \xrightarrow{\varphi^1} & \mathrm{HH}^1(C) & \longrightarrow & 0 \\
 & & \downarrow \eta & & \downarrow \varphi^1 & & \| & & \\
0 & \longrightarrow & \mathrm{H}^1(B, E') & \xrightarrow{\zeta} & \mathrm{HH}^1(B) & \xrightarrow{\varphi^1} & \mathrm{HH}^1(C) & \longrightarrow & 0
\end{array}$$

where all φ^1 are Hochschild projection morphisms, and the two rows are exact because of [4, Theorem B] and our main theorem above. We claim that the left square also commutes.

Let $[\underline{e}] \in \mathrm{Ext}^1_{C^e}(\widetilde{C}, E) = \mathrm{H}^1(\widetilde{C}, E)$. Then, by definition we have $\eta([\underline{e}]) = [p_{E'E}\ \underline{e}\ q_{\widetilde{C}B}]$. The morphism ζ is induced from the long exact cohomology sequence, hence $\zeta\eta([\underline{e}]) = \zeta([p_{E'E}\ \underline{e}\ q_{\widetilde{C}B}]) = [q_{BE'}\ p_{E'E}\ \underline{e}\ q_{\widetilde{C}B}]$. On the other side of the square, we have similarly $\varphi^1\psi([\underline{e}]) = [p_{B\widetilde{C}}\ q_{\widetilde{C}E}\ \underline{e}\ q_{\widetilde{C}B}]$. It thus suffices to show that $p_{B\widetilde{C}}\ q_{\widetilde{C}E} = q_{BE'}\ p_{E'E}$, and this follows from the fact that the image of $p_{B\widetilde{C}}\ q_{\widetilde{C}E} \colon E \to \widetilde{C} \to B$ is E'. This establishes our claim.

Applying the snake lemma and the surjectivity of $\varphi^1 \colon \mathrm{HH}^1(\widetilde{C}) \to \mathrm{HH}^1(B)$, see our main theorem, we get that η is surjective and $\mathrm{Ker}\,\eta \cong \mathrm{Ker}\,\varphi^1 = \mathrm{H}^1(\widetilde{C}, E'') \oplus \mathcal{E}(E'', B)$. This proves (a), while (b) follows at once from the commutative diagram with exact rows. □

4.2. In the next corollary, we need the algebra structure of $\mathrm{HH}^*(C) = \oplus_{n\geq 0}\mathrm{HH}^n(C)$. Let $\zeta \in \mathrm{HH}^s(C)$ and $\xi \in \mathrm{HH}^t(C)$ be represented by cocycles $f \in \mathrm{Hom}_k(C^{\otimes s}, C)$ and $g \in \mathrm{Hom}_k(C^{\otimes t}, C)$, then the *cup product* $\zeta \smile \xi$ is the cohomology class of the map $f \times g \in \mathrm{Hom}_k(C^{\otimes (s+t)}, C)$ defined by

$$(f \times g)(c_1 \otimes \cdots \otimes c_{s+t}) = f(c_1 \otimes \cdots \otimes c_s)g(c_{s+1} \otimes \cdots \otimes c_{s+t}).$$

With this product, $\mathrm{HH}^*(C)$ becomes a graded commutative and associative ring called the *Hochschild cohomology algebra*. It is shown in [4, Theorem 1] that if B is a split extension of C, then the Hochschild projection morphisms φ^n induce an algebra morphism $\varphi^* \colon \mathrm{HH}^*(B) \to \mathrm{HH}^*(C)$.

Corollary 4.2. *Let C be a tilted algebra and $B = C \ltimes E'$ a partial relation extension. Then the algebra morphism $\varphi^* \colon \mathrm{HH}^*(B) \to \mathrm{HH}^*(C)$ is surjective and there exists an exact sequence*

$$0 \longrightarrow K \longrightarrow \mathrm{HH}^*(B) \xrightarrow{\varphi^*} \mathrm{HH}^*(C) \longrightarrow 0,$$

where $K = \mathrm{H}^0(B, E') \oplus \mathrm{H}^1(B, E') \oplus (\oplus_{n\geq 2}\, \mathrm{HH}^n(B))$.

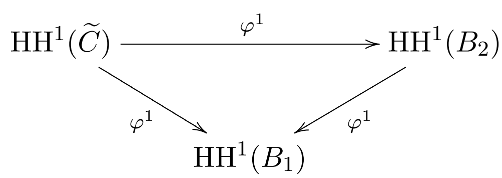
Proof. Because C is tilted, we have $\mathrm{HH}^*(C) = 0$ for all $n \geq 2$, see [13]. We apply the exact sequences of Lemmata 3.3 and 3.5. □

4.3. We recall that in [3, Remark 2.1.2] was defined a poset $\underline{P}$ of partial relation extensions. Let C be a triangular algebra of global dimension two, and $\widetilde{C} = C \ltimes E$ its relation extension. Then a partial relation extension $B_1 = C \ltimes E_1$ is said to be smaller than $B_2 = C \ltimes E_2$ if E_1 is a direct summand of E_2. This defines a partial order on the set of all partial relation extensions of C, and we denote this poset by $\underline{P}$. Note that this poset has a unique minimal element C and a unique maximal element $\widetilde{C}$.

We give another realization of the poset $\underline{P}$. Assume $B_1 \leq B_2$ in $\underline{P}$, where $B_1 = C \ltimes E_1$ and $B_2 = C \ltimes E_2$. There exists a C-C-bimodule E_1' such that $E_2 = E_1 \oplus E_1'$. Using the same proof as in [3, Lemma 2.1.1], we get that $B_2 = B_1 \ltimes E'_1$. This implies the existence of a Hochschild projection morphism $\varphi^1 \colon \mathrm{HH}^1(B_2) \to \mathrm{HH}^1(B_1)$. We are now able to define the poset $\underline{\mathrm{HH}^1}$. Its elements are the first Hochschild cohomology groups $\mathrm{HH}^1(B)$ with B a partial relation extension of C. We say that $\mathrm{HH}^1(B_1)$ is smaller than $\mathrm{HH}^1(B_2)$ whenever there exists a Hochschild projection morphism $\varphi^1 \colon \mathrm{HH}^1(B_2) \to \mathrm{HH}^1(B_1)$.

Corollary 4.3. (a) *The posets* $\underline{P}$ *and* $\underline{\mathrm{HH}^1}$ *are isomorphic.*
(b) *The map* $\dim \mathrm{HH}^1(-) \colon \underline{P} \to \mathbb{N}$ *is a morphism of posets.*

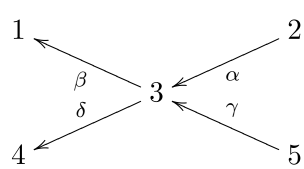
Proof. Statement (a) is clear from the respective definitions of our posets. In order to prove (b), we assume that $B_1 = C \ltimes E_1$ and $B_2 = C \ltimes E_2$ are partial relation extensions, with B_1 smaller than B_2. We must prove that $\dim \mathrm{HH}^1(B_1) \leq \dim \mathrm{HH}^1(B_2)$. Consider the diagram of Hochschild projection morphisms.



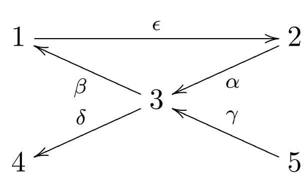
Because the maps φ^1 are induced by the inclusions and projections, this diagram is commutative. It follows from our main theorem that the morphisms $\mathrm{HH}^1(\widetilde{C}) \to \mathrm{HH}^1(B_2)$ and $\mathrm{HH}^1(\widetilde{C}) \to \mathrm{HH}^1(B_1)$ are surjective. Therefore the morphism $\mathrm{HH}^1(B_2) \to \mathrm{HH}^1(B_1)$ is surjective and $\dim \mathrm{HH}^1(B_1) \leq \dim \mathrm{HH}^1(B_2)$ as required. □

4.4. We end the paper with a couple of examples. In both examples, C is a tilted algebra, so that $\widetilde{C}$ is cluster-tilted.

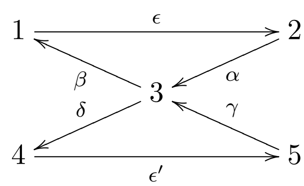
Example 4.1. Let C be given by the quiver



bound by the relations $\alpha\beta = 0$ and $\gamma\delta = 0$. Then $\mathrm{HH}^1(C) = 0$. Let B be the partial relation extension of C given by the quiver

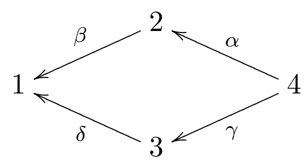


bound by the relations $\alpha\beta = \beta\epsilon = \epsilon\alpha = 0$ and $\gamma\delta = 0$. Then $\mathrm{HH}^1(B) = k$. Finally, the relation extension $\widetilde{C}$ is given by the quiver

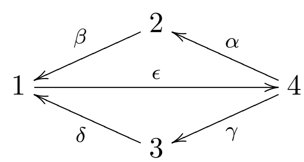


bound by the relations $\alpha\beta = \beta\epsilon = \epsilon\alpha = 0$ and $\gamma\delta = \delta\epsilon' = \epsilon'\gamma = 0$. Clearly, $\mathrm{HH}^1(\widetilde{C}) = k^2$.

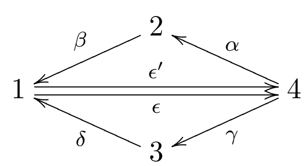
Example 4.2. Let C be given by the quiver



bound by $\alpha\beta = 0$ and $\gamma\delta = 0$. Let B be the partial relation extension given by



bound by $\alpha\beta = 0$ and $\gamma\delta = \delta\epsilon = \epsilon\gamma = 0$. The relation extension $\widetilde{C}$ is given by



bound by $\alpha\beta = \beta\epsilon' = \epsilon'\alpha = 0$ and $\gamma\delta = \delta\epsilon = \epsilon\gamma = 0$. Then $\mathrm{HH}^1(C) = k$ and $\mathrm{HH}^1(\widetilde{C}) = k^3$. In order to compute $\mathrm{HH}^1(B)$, observe first that ${}_C E'_C$ has a simple top (the arrow ϵ) hence is indecomposable and so $\mathrm{End}_{C^e} E' = k$. We claim that $\mathrm{H}^1(C, E') = 0$. It suffices to prove that $\mathrm{Der}_0(C, E') = 0$. Let ξ be an arrow from x to y in C, and $d\colon C \to E'$ a derivation. Then $d(\xi) = d(e_x \xi e_y) = e_x d(\xi) e_y$, thus corresponds to a path from x to y in B passing through ϵ and parallel to the arrow $\xi \in \{\alpha, \beta, \gamma, \delta\}$. There is no such nonzero path. Therefore $\mathrm{H}^1(C, E') = 0$ and so $\mathrm{H}^1(B, E') = k$. It follows that $\mathrm{HH}^1(B) = k^2$.

In this example we have $\mathcal{E}(E'', B) \neq 0$. Indeed $E = E' \oplus E'' = <\epsilon> \oplus <\epsilon'>$. Let $f\colon E'' \to B$ be the morphism defined by $f(\epsilon') = \epsilon$. Then, for each $x \in E''$ we have $x\epsilon = 0 = \epsilon x$ because $x\epsilon, \epsilon x \in E^2 = 0$. In particular $xf(\epsilon') + f(x)\epsilon' = 0$ and thus f is a nonzero element of $\mathcal{E}(E'', B)$.

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