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Glueings of tilted algebras

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Abstract

Let A be a basic and connected finite-dimensional algebra over an algebraically closed field. We show that if A has all its indecomposable projectives (or injectives) lying in a component of the Auslander–Reiten quiver consisting entirely of postprojective (or preinjective, respectively) modules in the sense of Auslander and Smalø then A is a finite enlargement in the postprojective (or preinjective, respectively) components of a finite set of tilted algebras having complete slices in these components. We call such an algebra A a left (or right, respectively) glued algebra and study some of its homological properties in particular in the case where A is itself a tilted algebra.

1. Introduction

The starting impetus for this work was the desire to link the theory of preprojective (called here postprojective, see 2.2) and preinjective partitions, as initiated by Auslander and Smalø in [7], with the theory of tilted algebras, as initiated by Happel and Ringel in [14].

In [11, 12], the second author has studied and characterised the components of the Auslander–Reiten quiver of an artin algebra which consists entirely of postprojective (or preinjective, respectively) modules in the sense of Auslander and Smalø. Such a component is called there a π-component (or an ι-component, respectively). Also, an algebra A having all its indecomposable projective (or injective, respectively) modules lying in π-components (or in ι-components, respectively) can be characterised by the property that the injective (or projective, respectively) dimension of almost all (that is, all but at most finitely many non-isomorphic) indecomposable A-modules is at most one (see 2.2). However, our approach here is quite different. We introduce the notion

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of left (or right, respectively) glued algebra, which is, roughly speaking, a finite enlargement in the postprojective (or preinjective) components of a finite set of tilted algebras having complete slices in these components (see 3.1 for details). Our first theorem is as follows.

Theorem. (a) *An algebra A is left glued if and only if* id $M = 1$ *for almost all indecomposable A-modules M.*

(b) *An algebra A is right glued if and only if* pd $M = 1$ *for almost all indecomposable A-modules M.*

As a consequence, we establish an existence result for regular cotilting (or tilting) modules over a left glued (or right glued, respectively) algebra (see 3.6). We also deduce from this theorem that a representation-infinite algebra A is concealed (in the sense of [20]) if and only if it is both a left and a right glued algebra or, equivalently, if and only if both the projective and the injective dimensions of almost all its indecomposable modules are equal to one (see 3.4). This latter equivalence has also been shown by Skowroński in [23], using different techniques. This leads us to consider the case when a representation-infinite algebra A is such that pd $M > 1$ and id $M = 1$ (or id $M > 1$ and pd $M = 1$) for almost all indecomposable A-modules M. Clearly, such an algebra A is left, but not right glued (or right, but not left glued, respectively). We shall define notions of left (or right) extremal subsection and reduced left (or right, respectively) extremal subsection for such an algebra (see 4.1 and 4.3 for the definitions). This will allow us to obtain necessary and also sufficient conditions for a representation-infinite algebra A to satisfy the above property. If, in particular, A is a tilted algebra, the underlying graphs of the left (or right) extremal subsection and the reduced left (or right, respectively) extremal subsection are respectively equal to the left (or right, respectively) type of A, as defined in [2]. We then obtain the following.

Proposition. *Let A be a representation-infinite tilted algebra.*

(a) pd $M = 2$ *and* id $M = 1$ *for almost all M in* ind A *if and only if A is a left glued algebra and its reduced right type is a disjoint union of Dynkin graphs.*

(b) pd $M = 1$ *and* id $M = 2$ *for almost all M in* ind A *if and only if A is a right glued algebra and its reduced left type is a disjoint union of Dynkin graphs.*

In [2], the same techniques yield a similar result in case A is a tilted algebra which is not necessarily left or right glued.

Our paper is organised as follows. In Section 2, we fix our notation, recall briefly some results and prove some lemmata that will be used in the sequel. Section 3 is devoted to the description of left and right glued algebras and their Auslander–Reiten components. Finally, in Section 4 we discuss the notions of extremal subsections, then prove the above proposition.

2. Postprojective and preinjective partitions and components

2.1. Notation. All algebras in this paper are basic, connected, associative, finite-dimensional algebras with identities over a fixed algebraically closed field k, and all modules are finitely generated right modules. Following [10], we shall sometimes equivalently consider an algebra as a k-linear category. For an algebra A, we denote by mod A its module category, and by ind A a full subcategory of mod A consisting of a complete set of representatives of the isomorphism classes of indecomposable objects in mod A. We shall use freely and without further reference properties of mod A, the Auslander–Reiten translations $\tau = \mathrm{DTr}$ and $\tau^{-1} = \mathrm{TrD}$, and the Auslander–Reiten quiver $\Gamma(\mathrm{mod}\,A)$ of A as can be found, for instance, in [4, 5, 19]. A path $x_0 \to x_1 \to \cdots \to x_m$ in $\Gamma(\mathrm{mod}\,A)$ is called *sectional* if $\tau x_{i+1} \neq x_{i-1}$ for all $0 < i < m$ and a full subquiver Σ of $\Gamma(\mathrm{mod}\,A)$ is called a *subsection* if any path in Σ is sectional [9]. A subsection is *maximal* if it is not properly contained in another subsection. We shall always identify a point of $\Gamma(\mathrm{mod}\,A)$ with the corresponding object of ind A, thus a component $\Gamma(\mathrm{mod}\,A)$ with the corresponding full subcategory of ind A. Given a full subcategory $\mathscr{C}$ of ind A, we denote by add $\mathscr{C}$ the additive full subcategory of mod A generated by $\mathscr{C}$. Given an A-module M, we denote by pd M its projective dimension and by id M its injective dimension. Given a k-linear functor $F: \mathrm{mod}\,A \to \mathrm{mod}\,k$, we denote by $l(F)$ its *length*, that is, $l(F) = \sum \dim_k F(M)$, where the sum is taken over all M in ind A. Thus, $l(F) < \infty$ if and only if $F(M) = 0$ for almost all (that is, all but at most finitely many non-isomorphic) indecomposable modules M.

2.2. For an algebra A, let $\underline{\underline{P}}_0, \ldots, \underline{\underline{P}}_n, \ldots, \underline{\underline{P}}_\infty$ denote its (unique) postprojective partition, and let $\underline{\underline{I}}_0, \ldots, \underline{\underline{I}}_n, \ldots, \underline{\underline{I}}_\infty$ denote its (unique) preinjective partition, as defined by Auslander–Smalø [7]. Following [13], we use the term *postprojective* rather than the original *preprojective*: we believe it is more suggestive. An A-module M is called *postprojective* if all its indecomposable summands lie in $\bigcup_{i<\infty} \underline{\underline{P}}_i$, and *preinjective* if all its indecomposable summands lie in $\bigcup_{i<\infty} \underline{\underline{I}}_i$.

In [7], Auslander and Smalø studied the algebras A such that all submodules of A_A are postprojective, or, equivalently, such that there are no non-zero morphisms from a module in $\underline{\underline{P}}_\infty$ to A_A. In [11, 12], the second author has given a description of the components of the Auslander–Reiten quiver of such an algebra containing projective modules. The following theorem was proved (in [7, 11, 12]) in the more general case where A is an artin algebra.

Theorem. (a) *The following conditions are equivalent for an algebra A:*

(1) *Any component of $\Gamma(\mathrm{mod}\,A)$ containing a projective module consists only of postprojective modules.*

(2) $l(\mathrm{Hom}_A(-, A)) < \infty$.

(3) $l(\mathrm{Hom}_A(-, M)) < \infty$ *for all postprojective modules M.*

(4) *Any component Γ of $\Gamma(\text{mod } A)$ containing a projective module satisfies:*

(4.1) *almost all modules in Γ lie in the τ-orbit of a projective module; and*

(4.2) *at most finitely many modules in Γ belong to oriented cycles.*

(5) *For every postprojective module M, the set of all X in* ind A *such that there exists a sequence of irreducible morphisms $X \to \cdots \to M$, is finite.*

(6) id $M \leq 1$ *for almost all M in ind A.*

(7) *For every postprojective module M, the set of all X in* ind A *such that* $\text{Hom}_A(X, M) \neq 0$, *consists only of postprojective modules.*

(b) *The following conditions are equivalent for an algebra A:*

(1) *Any component of $\Gamma(\text{mod } A)$ containing an injective module consists only of preinjective modules.*

(2) $l(\text{Hom}_A(DA, -)) < \infty$.

(3) $l(\text{Hom}_A(M, -)) < \infty$ *for all preinjective modules M.*

(4) *Any component Γ of $\Gamma(\text{mod } A)$ containing an injective module satisfies:*

(4.1) *almost all modules in Γ lie in the τ-orbit of an injective module; and*

(4.2) *at most finitely many modules in Γ belong to oriented cycles.*

(5) *For every preinjective module M, the set of all X in* ind A *such that there exists a sequence of irreducible morphisms $M \to \cdots \to X$, is finite.*

(6) pd $M \leq 1$ *for almost all M in* ind A.

(7) *For every preinjective module M, the set of all X in* ind A such that $\text{Hom}_A(M, X) \neq 0$, *consists only of preinjective modules.* □

2.3. Corollary. *Let A be an algebra.*

(a) $l(\text{Hom}_A(-, A)) < \infty$ *if and only if there exists a component Γ of $\Gamma(\text{mod } A)$ such that $\Gamma = \bigcup_{i<\infty} \underline{P}_i$. In particular, Γ contains all projectives.*

(b) $l(\text{Hom}_A(DA, -)) < \infty$ *if and only if there exists a component Γ of $\Gamma(\text{mod } A)$ such that $\Gamma = \bigcup_{i<\infty} \underline{I}_i$. In particular, Γ contains all injectives.*

Proof. We shall only prove (a), since the proof of (b) is dual. The sufficiency follows from the fact that in Theorem 2.2(a), (2) implies (1). For the necessity, we note that, by [6, (1.8)], any non-zero morphism $f \in \text{Hom}_A(M, A)$, with M an A-module, is a sum of compositions of irreducible morphisms. Since A is connected, we infer that there is a component Γ of $\Gamma(\text{mod } A)$ containing all projectives. Using the fact that in Theorem 2.2(a), (1) implies (2), we deduce that Γ consists only of postprojective modules. In fact, we have the equality $\Gamma = \bigcup_{i<\infty} \underline{P}_i$ since, given a module M in $\bigcup_{i<\infty} \underline{P}_i$, there exists, by [7, (8.3)], a sequence of irreducible morphisms leading from a projective to M, so that M lies in Γ. □

2.4. Definition [11]. A component Γ of the Auslander–Reiten quiver of an algebra is called a *π-component* if

(i) almost all modules in Γ lie in the τ-orbit of a projective module; and

(ii) at most finitely many modules in Γ belong to oriented cycles.

Dually, Γ is called an *ι-component* if

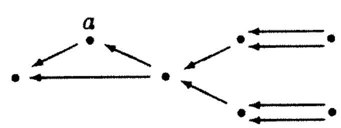
(i) almost all modules in Γ lie in the τ-orbit of an injective module; and

(ii) at most finitely many modules in Γ belong to oriented cycles.

Thus, if an algebra A satisfies $l(\mathrm{Hom}_A(-, A)) < \infty$ (or $l(\mathrm{Hom}_A(DA, -)) < \infty$) then, by Corollary 2.3, $\Gamma(\mathrm{mod}\, A)$ has a π-component (or an ι-component, respectively) containing all projectives (or all injectives, respectively).

Related notions are those of *postprojective* and *preinjective components*. We recall that a component Γ of $\Gamma(\mathrm{mod}\, A)$ is a postprojective (or preinjective) component if it contains no oriented cycles and any module in it lies in the τ-orbit of a projective (or of an injective, respectively). Clearly, postprojective components are π-components, and preinjective components are ι-components. The converse, however, is not true, as is shown in the following example.

Example. Let A be the radical square zero algebra given by the quiver



Then $\Gamma(\mathrm{mod}\, A)$ has the shape shown in Fig. 1, where one has to identify the two copies of the simple module $S(a)$ at the point a and the horizontal dotted lines denote the Auslander–Reiten translations. The component containing (all) projectives is a π-component but not a postprojective component.

2.5. The following result [11, (6.7)] relates the preceding notions.

Proposition. *Let A be an algebra, and let Γ be a component of $\Gamma(\mathrm{mod}\, A)$.*

(a) *If Γ is a π-component and contains no injective module, then Γ is a postprojective component.*

(b) *If Γ is an ι-component and contains no projective module, then Γ is a preinjective component.* □

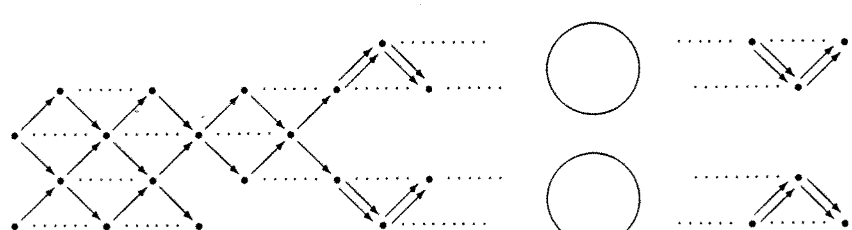


Fig. 1

2.6. For tilting theory, we refer the reader to [1, 19]. In particular, we recall that tilted algebras are characterised by the existence of complete slices in a component of their Auslander–Reiten quiver, called *connecting component* [19, (4.2)]. A tilted algebra has at most two connecting components and, if it has two, then it is a concealed algebra [20, Lecture 2]. We shall also need the following equivalent characterisation, obtained independently by Liu and Skowroński [18, 22]: let Γ be a component of the Auslander–Reiten quiver of an algebra A, a *section* Σ in Γ is a connected full subquiver of Γ such that

(1) Σ contains no oriented cycles;
(2) Σ meets each τ-orbit in Γ exactly once;
(3) Σ is convex in Γ, that is, any path in Γ with endpoints in Σ lies entirely in Σ; and
(4) for each arrow $M \to N$ in Γ, if M is in Σ, either N or τN is in Σ and, if N is in Σ, either M or $\tau^{-1}M$ is in Σ.

Thus, a complete slice in a connecting component Γ of a tilted algebra is an example of a section in Γ. We have the following result.

Theorem [18, 22]. *An algebra A is tilted if and only if $\Gamma(\text{mod}\, A)$ has a component Γ with a faithful section Σ such that $\text{Hom}_A(M, \tau N) = 0$ for all M, N in Σ. In this situation, Σ is a complete slice and Γ is a connecting component of $\Gamma(\text{mod}\, A)$.* □

2.7. Lemma. *Let A be an algebra, and Γ be a component of $\Gamma(\text{mod}\, A)$.*
(a) *If Γ is a π-component containing a complete slice, then Γ is a postprojective component.*
(b) *If Γ is an ι-component containing a complete slice, then Γ is a preinjective component.*

Proof. We shall only prove (a), since the proof of (b) is dual. If Γ contains a complete slice Σ, this slice is a section in Γ. By [16, (3.2)], Γ may be embedded in $\mathbb{Z}\Sigma$. In particular, Γ contains no oriented cycles. Hence the π-component Γ is actually a postprojective component. □

2.8. Lemma. *Let A be an algebra, and Γ be a component of $\Gamma(\text{mod}\, A)$.*
(a) *If Γ is a postprojective component containing all projectives but no injective, then Γ contains a complete slice, hence A is a tilted algebra.*
(b) *If Γ is a preinjective component containing all injectives but no projective, then Γ contains a complete slice, hence A is a tilted algebra.*

Proof. We shall only prove (a), since the proof of (b) is dual. Observe that Γ is right stable, that is, $\tau^{-t}M \neq 0$ for all $t \geq 0$ and M in Γ. Consider the maximal subsection determined by the indecomposable projectives which correspond to sources in the ordinary quiver of A, that is, let Σ be the set of all M in Γ such that there is a path in Γ from M to some indecomposable projective, and any such path is sectional. It is easily seen that Σ satisfies the conditions of 2.6 and thus is a complete slice. □

2.9. Lemma. *Let A be a tilted algebra, and Γ be a component of $\Gamma(\text{mod } A)$.*

(a) *If Γ is a π-component containing all indecomposable projectives, then Γ is a connecting postprojective component.*

(b) *If Γ is an ι-component containing all indecomposable injectives, then Γ is a connecting preinjective component.*

Proof. We shall only prove (a), since the proof of (b) is dual. By [24, (7.7)], a tilted algebra always has a postprojective component Γ', which is therefore a π-component containing some indecomposable projectives. Since Γ contains all indecomposable projectives, we must have $\Gamma = \Gamma'$. Assume that Γ is not a connecting component. By Lemma 2.8, it must contain at least one injective. But then it must be a connecting component, a contradiction. □

3. Left and right glued algebras

3.1. In this section, we shall give a constructive characterisation of the algebras satisfying the equivalent conditions of Theorem 2.2(a) or (b), and describe the components of their Auslander–Reiten quiver. We shall prove that such an algebra is in fact a finite enlargement of a direct product of tilted algebras.

Definition. (a) Let $B_1, \ldots, B_t$ be representation-infinite tilted algebras having complete slices $\Sigma_1, \ldots, \Sigma_t$ respectively, in the postprojective components and no injectives in these components, $B = B_1 \times \cdots \times B_t$ and C be a representation-finite algebra. An algebra A is called *a left glueing of $B_1, \ldots, B_t$ by C along the slices $\Sigma_1, \ldots, \Sigma_t$* or, more briefly, to be a *left glued algebra* if $A = C$ or:

(LG1) each of $B_1, \ldots, B_t$ and C is a full convex subcategory of A and any object in A belongs to one of these subcategories;

(LG2) no projective A-module is a proper successor of the union $\Sigma_1 \cup \cdots \cup \Sigma_t$, considered as embedded in ind A; and

(LG3) ind B is cofinite in ind A, that is, almost all indecomposable A-modules are also B-modules.

(b) Let $B_1, \ldots, B_t$ be representation-infinite tilted algebras having complete slices $\Sigma_1, \ldots, \Sigma_t$ respectively, in the preinjective components and no projectives in these components, $B = B_1 \times \cdots \times B_t$ and C be a representation-finite algebra. An algebra A is called a *right glueing of $B_1, \ldots, B_t$ by C along the slices $\Sigma_1, \ldots, \Sigma_t$* or, more briefly, to be a *right glued algebra* if $A = C$ or:

(RG1) each of $B_1, \ldots, B_t$ and C is a full convex subcategory of A, and any object in A belongs to one of these subcategories;

(RG2) no injective A-module is a proper predecessor of the union $\Sigma_1 \cup \cdots \cup \Sigma_t$, considered as embedded in ind A; and

(RG3) ind B is cofinite in ind A.

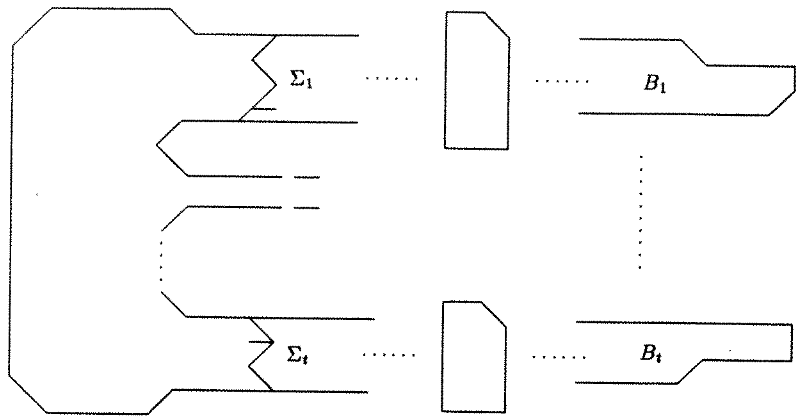


Fig. 2

In particular, any representation-finite algebra is both left and right glued. If a representation-infinite algebra A is left glued, then $\Gamma(\text{mod } A)$ has the shape shown in Fig. 2.

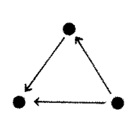
The algebra C being an arbitrary representation-finite algebra, the component of $\Gamma(\text{mod } A)$ containing $\Sigma_1 \cup \cdots \cup \Sigma_t$ may contain periodic modules and oriented cycles: it is actually a π-component containing all the projective A-modules (Definition 2.4). On the other hand, the injective A-modules are either injective B-modules or belong to the π-component containing the Σ_i. Consequently, the ordinary quiver of A is the union of the quivers of $B_1, \ldots, B_t$ and C together with some additional arrows of the form $x \to y$, with x in the quiver of some B_i, and y in the quiver of C. In particular, a left glued algebra A may be written as a lower triangular matrix algebra

$$A \cong \begin{pmatrix} C & 0 \\ M & B \end{pmatrix}$$

where M is a B-C-bimodule. Also, it is easily seen that A is epivalent (representation-equivalent) to B so that A is tame if and only if so is each of the tilted algebras $B_1, \ldots, B_t$. Dual comments can be made on right glued algebras.

Example. The algebra of the example in 2.4 is the left glueing of two copies of the Kronecker algebra, given by the quiver

by the representation-finite radical square zero algebra given by the quiver



along the slices consisting of the indecomposable projective modules.

3.2. The main result of this section asserts that the left glued (or right glued) algebras coincide with those satisfying the equivalent conditions of Theorem 2.2(a) (or Theorem 2.2(b), respectively).

Theorem. (a) *An algebra A is a left glued algebra if and only if* id $M = 1$ *for almost all non-isomorphic indecomposable A-modules M.*

(b) *An algebra A is a right glued algebra if and only if* pd $M = 1$ *for almost all non-isomorphic indecomposable A-modules M.*

Proof. We shall only prove (a), since the proof of (b) is dual. We clearly may assume that A is representation-infinite. Suppose first that A is a left glueing of $B_1, \ldots, B_t$ by C along the slices $\Sigma_1, \ldots, \Sigma_t$. We may assume each of the slices Σ_i of $\Gamma(\text{mod } B_i)$ to be fully embedded in $\Gamma(\text{mod } A)$: this indeed follows from (LG3) and the fact that we can replace Σ_i by $\tau^{-m}\Sigma_i$ for any $m > 0$. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$. Since each Σ_i lies in the postprojective component of $\Gamma(\text{mod } B_i)$, almost all indecomposable B-modules (where $B = B_1 \times \cdots \times B_t$) are successors of Σ. On the other hand, (LG2) says that no projective A-module is a proper successor of Σ. Therefore, each projective A-module has at most finitely many predecessors. Consequently, $l(\text{Hom}_A(-, A)) < \infty$, which, by Theorem 2.2, is equivalent to the condition that id $M = 1$ for almost all non-isomorphic indecomposable A-modules M.

Conversely, suppose that id $M = 1$ for almost all non-isomorphic indecomposable A-modules M. It follows from Theorem 2.2 that $l(\text{Hom}_A(-, A)) < \infty$. Since A is connected, it follows from Corollary 2.3 that all indecomposable projective A-modules belong to the same component Γ of $\Gamma(\text{mod } A)$, which is even a π-component. Let $\{e_1, \ldots, e_m, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A ordered so that the indecomposable injective A-module $\text{D}(Ae_i)$ belongs to Γ if and only if $1 \le i \le m$. Let $e = e_1 + \cdots + e_m$, $P = eA$, $B = \text{End}(1 - e)A$ and $C = \text{End}(eA)$. Thus, $\text{D}P \in \text{add } \Gamma$. By Theorem 2.2, this implies that $l(\text{Hom}_A(-, \text{D}P)) < \infty$. Therefore at most finitely many non-isomorphic indecomposable A-modules M satisfy

$$\text{Hom}_A(P, M) \cong \text{D}\text{Hom}_A(M, \text{D}P) \neq 0$$

and so ind B is cofinite in ind A.

Let Γ' be the translation subquiver of $\Gamma(\text{mod } B)$ consisting of the indecomposable B-modules which, when considered as A-modules, belong to Γ. By [11, (7.4)], Γ' is a (finite) union of π-components $\Gamma_1, \ldots, \Gamma_t$. By construction, none of the Γ_i contains an injective module. Consequently, by Proposition 2.5, $\Gamma_1, \ldots, \Gamma_t$ are postprojective components. For each i, let Σ_i be a maximal subsection in Γ_i chosen so that it embeds fully in Γ and has no successor which is a projective A-module. We shall denote by B_i the support algebra of Σ_i.

Since each projective B_i-module is a (not necessarily proper) predecessor of Σ_i, there exists a monomorphism $0 \to B_{iB_i} \to U_{B_i}$, where $U_{B_i} \in \text{add } \Sigma_i$, so that Σ_i is faithful in mod B_i. Since postprojective components are standard (by [19, (2.4)(11), p. 80]), we have that $\mathrm{Hom}_{B_i}(U, \tau V) = 0$ for all U, V in Σ_i. By 2.6, Σ_i is a complete slice in $\Gamma(\mathrm{mod}\, B_i)$ and B_i is a tilted algebra.

There remains to show (LG1). By construction, each of the connected algebras B_i is a full subcategory of A and any object in A belongs to C or to one of $B_1, \ldots, B_t$. Further, for any arrow $\alpha: x \to y$ in the ordinary quiver of A, with y in the quiver of B_i, then x must belong to the quiver of B_i as well. Consequently, C is also convex and the theorem is proven. □

We note that, by Corollary 2.3, the unique π-component of a left glued algebra containing all the projectives consists of all the postprojective modules.

It also follows from the proof of the theorem that the subcategories $B_1, \ldots, B_t$ (and hence C) of A are uniquely determined: indeed, $B_1, \ldots, B_t$ are the connected components of B, which are determined by the condition that its injective modules (are successors of the slices Σ_i thus must) embed in ind A as those indecomposable injectives which do not belong to the unique π-component of $\Gamma(\mathrm{mod}\, A)$.

3.3. Corollary. (a) *A left glued algebra having no injective postprojective module is tilted with a complete slice in the postprojective component.*

(b) *A right glued algebra having no projective preinjective module is tilted with a complete slice in the preinjective component.*

Proof. We shall only prove (a), since the proof of (b) is dual. Let A be left glued. It has a unique π-component Γ consisting of all postprojective modules. By hypothesis, Γ contains no injective hence, by Proposition 2.5, it is a postprojective component. Since moreover Γ contains all projectives, the result follows from Lemma 2.8. □

3.4. As a first consequence of Theorem 3.2, we obtain the following new characterisation of concealed algebras (compare with [23]).

Proposition. *Let A be a representation-infinite algebra. The following conditions are equivalent*:

(a) *A is a concealed algebra.*

(b) *A is both a left glued and a right glued algebra.*

(c) pd $M = 1$ *and* id $M = 1$ *for almost all M in* ind A.

Proof. Assume that A is a concealed algebra. All projective A-modules belong to the postprojective component. In particular, each projective has at most finitely many predecessors so that $l(\mathrm{Hom}_A(-, A)) < \infty$. By Theorem 2.2, we have id $M \leq 1$ for almost all M in ind A. Similarly, pd $M \leq 1$ for almost all M in ind A. We have shown that (a) implies (c).

Since (c) implies (b) by Theorems 2.2 and 3.2, there remains to show that (b) implies (a). Since A is a right glued algebra, all injective A-modules lie in an ι-component Γ. We claim that Γ contains no projective A-modules. Indeed, if this is not the case, there exists a projective module P_A in Γ whose radical has support in (at least) one of the glued tilted algebras $B_1, \ldots, B_t$. Hence P has infinitely many predecessors which implies (by Theorem 2.2) that $l(\mathrm{Hom}_A(-, A)) = \infty$, a contradiction to the fact that A is also a left glued algebra. This shows our claim. Since Γ is an ι-component containing all injectives but no projective, it follows from 2.5 and Lemma 2.8 that Γ is a preinjective component and contains a complete slice.

Dually, A has a postprojective component containing all projectives but no injective, hence containing a complete slice.

Since A has complete slices in two distinct components, it follows from [20, Lecture 2], that A is concealed. □

3.5. Let A be a left glued algebra. We shall now describe the components of $\Gamma(\mathrm{mod}\, A)$. As we have seen, it has a unique π-component Γ containing all indecomposable projective and postprojective A-modules. Moreover, if Γ contains no injective, then Γ is itself a postprojective component and A is a tilted algebra. Let now Γ' be a component other than Γ. It follows from our description of left glued algebras that Γ' is the image of a full embedding inside $\Gamma(\mathrm{mod}\, A)$ of a component of $\Gamma(\mathrm{mod}\, B_i)$, for some $1 \le i \le t$, containing no projective module. Since B_i is a tilted algebra, it follows from [17] that Γ' is of one of the following types:

(i) a preinjective component;
(ii) a stable tube;
(iii) of type $\mathbb{Z}\mathbb{A}_\infty$; or
(iv) obtained from (ii) or from (iii) by finitely many coray insertions.

A similar description can be given for the components of a right glued algebra. In this case, they are of the following types:

(i) the unique ι-component containing all the indecomposable preinjectives;
(ii) a postprojective component;
(iii) a stable tube;
(iv) of type $\mathbb{Z}\mathbb{A}_\infty$; or
(v) obtained from (iii) or from (iv) by finitely many ray insertions.

3.6. We shall also deduce from Theorem 3.2 the following result on the existence of regular cotilting (or tilting) modules over left (or right, respectively) glued algebras. A module over a left (or right) glued algebra will be called *regular* if none of its indecomposable summands belongs to the π-component or to a preinjective component (or to the ι-component or to a postprojective component, respectively).

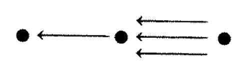
Proposition. (a) *A left glued algebra has a regular cotilting module if and only if it is a wild tilted algebra with at least 3 non-isomorphic simple modules, and having no injective postprojective module.*

(b) *A right glued algebra has a regular tilting module if and only if it is a wild tilted algebra with at least 3 non-isomorphic simple modules, and having no projective preinjective module.*

Proof. We shall only prove (a), since the proof of (b) is dual. Assume that the left glued algebra A has a regular cotilting module T_A. We first claim that there is no injective postprojective A-module. Indeed, if this is not the case and I is an injective postprojective module, then I must be cogenerated by T (see, for instance, [1, (1.6)]) that is, there exist $m > 0$ and a monomorphism $I \to T^{(m)}$. Since I is injective, such a monomorphism splits, so that the postprojective module I is a summand of T. This contradiction to our assumption on T establishes our claim.

By Corollary 3.3, A is a tilted algebra with a complete slice in the postprojective component. Since a regular cotilting module induces a torsion theory $(\mathscr{T}, \mathscr{F})$ with both $\mathscr{T}$ and $\mathscr{F}$ containing infinitely many non-isomorphic indecomposables, it follows from [3, Theorem B], that A is a wild algebra with at least 3 non-isomorphic simple modules. Conversely, assume that A is a wild tilted algebra with at least 3 non-isomorphic simple modules, and having no injective postprojective A-module. By Corollary 3.3, A has a complete slice in the postprojective component and no injective in that component. Moreover, there exists a wild hereditary algebra H with at least 3 non-isomorphic simple modules and a tilting module U_H without postprojective direct summands such that $A = \operatorname{End} U_H$. By [15, (2.1)], each component Γ of $\Gamma(\operatorname{mod} H)$ contains a complete right cone $\mathscr{C}_\Gamma$, closed under successors, and entirely contained in the class $\mathscr{F}(U) = \{X_H \mid \operatorname{Hom}_H(U, X) = 0\}$. By [8, 21], H has a regular (co)tilting module V. We clearly may assume that all the summands of V belong to the cones $\mathscr{C}_\Gamma$. But this implies that $T_A = \operatorname{Ext}^1_H(U, V)$ is a regular cotilting A-module. □

We remark that both conditions in the statement are necessary. Indeed, the radical square zero algebra A given by the quiver



is a wild tilted algebra with 3 non-isomorphic simple modules, which is left (but not right) glued and has an injective postprojective module. Clearly, A has no regular cotilting module.

4. Homological properties of left and right glued algebras

4.1. We have seen in Proposition 3.4 that a representation-infinite algebra A is such that pd $M = 1$ and id $M = 1$ for almost all non-isomorphic indecomposable A-modules M if and only if it is a concealed algebra, or, equivalently, it is both left and right glued. It is thus natural to consider the representation-infinite algebras which satisfy the properties: (a) pd $M > 1$ and id $M = 1$ for almost all M in ind A; or (b) pd $M = 1$ and id $M > 1$ for almost all M in ind A.

Suppose, for instance, that A satisfies property (a) above. By Theorem 3.2, A is a left glued algebra and hence contains a unique π-component Γ. Moreover, Γ contains an injective module, since otherwise, by Corollary 3.3, Γ is a connecting postprojective component and all modules in Γ have projective dimension one, a contradiction. In order to say more on the representation-infinite algebras satisfying properties (a) and (b), we shall need to introduce graphical invariants for the left glued algebras with injective postprojective modules and, dually, for the right glued algebras with projective preinjective modules.

Let A be a representation-infinite left glueing of $B_1, \ldots, B_t$ by C and Γ be its π-component. As observed above, we can assume that Γ contains an injective or, equivalently, that C is non-zero.

Fix an index i, $1 \leq i \leq t$. It follows from the description of the ordinary quiver of A given in Definition 3.1 that there exists a point c in the quiver of C such that there exists an arrow ending at c and having source in B_i. That is, there exists an injective A-module with one of its socle factors in ind B_i. Let $\mathscr{J}_i$ be the set of all injective A-modules I such that there exists an irreducible morphism $I \to K$ with K in ind B_i and such that there is no injective in Γ which is a proper successor of those summands of $I/\mathrm{soc}\, I$ which belong to ind B_i. Clearly, $\mathscr{J}_i$ is non-empty by our description of A. Finally, let $\mathscr{J}$ be the union of the $\mathscr{J}_i$'s.

Let Σ be the subsection of Γ consisting of the modules M such that there exists a path in Γ of length at least one from some injective in $\mathscr{J}$ to M, and any such path is sectional. We shall call Σ the *right extremal subsection of* Γ.

Observe that Σ as defined above is generally not connected. Also, there exists a projective module which is a proper successor of Σ if and only if there exists an $m > 0$ such that $\tau^{-m}\Sigma$ is not a maximal subsection.

Dually, let A be a representation-infinite right glueing of $B_1, \ldots, B_t$ by C and Γ be its unique ι-component. As observed above, we can assume that Γ contains a projective or, equivalently, that C is non-zero. For a fixed index i, $1 \leq i \leq t$, let $\mathscr{P}_i$ be the set of all the projective A-modules P such that there exists an irreducible morphism $K \to P$, with K in ind B_i and such that there is no projective in Γ which is a proper predecessor of those summands of rad P which belong to ind B_i. Finally, let $\mathscr{P}$ be the union of the $\mathscr{P}_i$'s.

Let Σ be the subsection of Γ consisting of the modules M such that there exists a path in Γ of length at least one from M to some projective in $\mathscr{P}$, and any such path is sectional. We shall call Σ the *left extremal subsection of* Γ.

Again, Σ is generally not connected. Also, there exists an injective module which is a proper predecessor of Σ if and only if there exists an $m > 0$ such that $\tau^{-m}\Sigma$ is not a maximal subsection,

Example. In the example below Definition 2.4, the right extremal subsection consists of two copies of the Kronecker quiver, each of which consisting of the projective module at one of the two sources, and its radical. In this example, no projective is a proper successor of the right extremal subsection.

4.2. We now recall the notion of type of a tilted algebra (see, for instance, [1, (5.1)]). If A is a tilted algebra, there exists a finite connected quiver without oriented cycles

Σ and a tilting module T over the path algebra $k\Sigma$ of Σ such that $A = \operatorname{End} T$. Clearly, this is equivalent to the requirement that $\Gamma(\operatorname{mod} A)$ contains a complete slice whose underlying quiver is isomorphic to Σ^{op}. The quiver Σ is generally not uniquely determined by A, but two different quivers Σ and Σ' whose path algebras tilt to A have the same underlying graph and can be deduced from each other by an admissible change of orientation (that is, a sequence of reflections). The underlying graph $\bar{\Sigma}$ of a quiver Σ whose path algebra tilts to A is called *the type of A*.

Lemma. (a) *Let A be a representation-infinite left glueing of $B_1, \ldots, B_t$ by the non-zero algebra C, and let Σ be its right extremal subsection. If no projective is a proper successor of Σ, the underlying graph of Σ is the disjoint union of the types of $B_1, \ldots, B_t$.*

(b) *Let A be a representation-infinite right glueing of $B_1, \ldots, B_t$ by the non-zero algebra C, and let Σ be its left extremal subsection. If no injective is a proper predecessor of Σ, the underlying graph of Σ is the disjoint union of the types of $B_1, \ldots, B_t$.*

Proof. Follows directly from the respective definitions. □

It is worthwhile to observe that if A satisfies the conditions of the lemma above, then its representation type is determined by the underlying graph of its extremal subsection. Thus, if A is a left glued algebra with an injective postprojective module and such that no projective is a successor of the right extremal subsection Σ, then A is tame if and only if the underlying graph of Σ is a disjoint union of euclidean graphs. Dually, if A is a right glued algebra with a projective preinjective module and such that no injective is a predecessor of the left extremal subsection Σ, then A is tame if and only if the underlying graph of Σ is a disjoint union of euclidean graphs.

4.3. In order to state our next result, we shall introduce the notion of reduced right (or left, respectively) extremal subsection of a left glued algebra having an injective postprojective module (or of a right glued algebra having a projective preinjective module, respectively).

Let A be a representation-infinite left glueing of $B_1, \ldots, B_t$ by the non-zero representation-finite algebra C, and let Σ be its right extremal subsection. We define the *reduced right extremal subsection* to be the full convex subquiver of Σ obtained by deleting all the sources. We should observe that the sources of Σ correspond to socle factors of injective A-modules.

Dually, let A be a representation-infinite right glueing of $B_1, \ldots, B_t$ by the non-zero representation-finite algebra C, and let Σ be its left extremal subsection. We define the *reduced left extremal subsection* to be the full convex subquiver of Σ obtained by deleting all the sinks. Again, we observe that the sinks of Σ correspond to radical summands of projective A-modules.

For instance, in the example below Definition 2.4 the reduced right extremal subsection consists of the disjoint union of two quivers with one point and no arrow (the point corresponds to the projective indecomposable at a source in the quiver of A).

Proposition. *Let A be a representation-infinite algebra.*

(a) (i) *If A is a left glueing of $B_1, \dots, B_t$ by the non-zero representation-finite algebra C such that no projective is a proper successor of the right extremal subsection, and its reduced right extremal subsection is a union of Dynkin quivers, then* pd $M > 1$ *and* id $M = 1$ *for almost all M in* ind A.

(ii) *If* pd $M > 1$ *and* id $M = 1$ *for almost all M in* ind A, *then A is a left glued algebra containing an injective postprojective module and its reduced right extremal subsection is a union of Dynkin quivers.*

(b) (i) *If A is a right glueing of $B_1, \dots, B_t$ by the non-zero representation-finite algebra C such that no injective is a proper predecessor of the left extremal subsection, and its reduced left extremal subsection is a union of Dynkin quivers, then* pd $M = 1$ *and* id $M > 1$ *for almost all M in* ind A.

(ii) *If* pd $M = 1$ *and* id $M > 1$ *for almost all M in* ind A, *then A is a right glued algebra containing an projective preinprojective module and its reduced left extremal subsection is a union of Dynkin quivers.*

Proof. We shall only prove (a), since the proof of (b) is dual.

(i) Let A be as in the statement, and $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ be its right extremal subsection where, for each i, Σ_i is a connected component of Σ. By Lemma 4.2, the underlying graph $\bar{\Sigma}$ of Σ is the type of $B_1 \times \cdots \times B_t$. Let us assume that, for each i, $\bar{\Sigma}_i$ is the type of B_i.

Since A is left glued, we have id $M = 1$ for almost all M in ind A (by Theorem 3.2). There remains to show that pd $M > 1$ for almost all M in ind A. Since pd $M > 1$ if and only if $\mathrm{Hom}_A(\mathrm{D}A, \tau M) \neq 0$ (see, for instance, [19, (2.4)(1), p. 74]), it suffices to show that $\mathrm{Hom}_A(\mathrm{D}A, N) \neq 0$ for almost all N in ind A. Let us fix an index i, $1 \le i \le t$. For each source S in Σ_i, there exists an indecomposable injective A-module I and an irreducible epimorphism $I \to S$.

Let $T = \bigoplus\{M \mid M \in \Sigma_i\}$ be the tilting B_i-module given by the subsection Σ_i. Then $H = \mathrm{End}\, T$ is a hereditary algebra and, for each source S in Σ_i, the H-module $S' = \mathrm{Hom}_{B_i}(T, S)$ is simple projective. Let U denote the direct sum of all sources in Σ_i and set $U' = \mathrm{Hom}_{B_i}(T, U)$. Then U' is a direct sum of simple projective H-modules so $U' = eH$ for some non-zero idempotent $e \in H$. The hereditary algebra $H' = \mathrm{End}(1-e)H$ has for type the full convex subquiver Σ_i' of Σ_i obtained by dropping the summands of U. That is, Σ_i' is the (disjoint union of the connected) component(s) of the reduced right extremal subsection of A, hence is a (disjoint union of) Dynkin quiver(s), so that H' is representation-finite. This implies that $\mathrm{Hom}_H(U', X) \neq 0$ for almost all X in ind H.

Let $(\mathscr{T}, \mathscr{F})$ denote the torsion theory induced by T in mod B_i, that is, $\mathscr{T}$ is the class of all B_i-modules generated by T, while $\mathscr{F}$ is the class of all B_i-modules M such that $\mathrm{Hom}_{B_i}(T, M) = 0$. It is easily seen that $\mathscr{F}$ consists of all the proper predecessors of Σ_i in $\Gamma(\mathrm{mod}\, B_i)$, hence contains at most finitely many non-isomorphic indecomposables B_i-modules. On the other hand, by the Brenner–Butler theorem

$$\mathrm{Hom}_{B_i}(U, M) \cong \mathrm{Hom}_H(U', \mathrm{Hom}_{B_i}(T, M))$$

for any M in $\mathscr{T}$. This implies that $\mathrm{Hom}_{B_i}(U, M) \neq 0$ for almost all M in ind B_i.

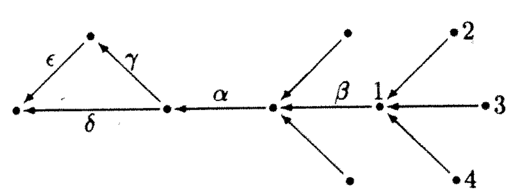
Since, for any indecomposable summand S of U, there exists an indecomposable injective A-module I and an irreducible epimorphism $I \to S$, we deduce that $\mathrm{Hom}_A(\mathrm{D}A, N) \neq 0$ for almost all N in ind B_i. This being true for each i, we also have $\mathrm{Hom}_A(\mathrm{D}A, N) \neq 0$ for almost all N in $\mathrm{ind}(B_1 \times \cdots \times B_t)$. Since $\mathrm{ind}(B_1 \times \cdots \times B_t)$ is cofinite in ind A, we infer that $\mathrm{Hom}_A(\mathrm{D}A, N) \neq 0$ for almost all N in ind A, which concludes the proof of (*i*).

(ii) Suppose that pd $M > 1$ and id $M = 1$ for almost all M in ind A. By Theorems 2.2 and 3.2, A is a left glueing of, say, representation-infinite tilted algebras $B_1, \ldots, B_t$ by the representation-finite algebra C. Moreover, it follows from 4.1 that A has an injective postprojective module. Let $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_t$ be the right extremal subsection of A where, for each i, $\bar{\Sigma}_i$ is the type of B_i (see Lemma 4.2), and let $\Sigma' = \Sigma'_1 \cup \cdots \cup \Sigma'_t$ be the reduced right extremal subsection where, for each i, Σ'_i is a full convex subquiver of Σ_i.

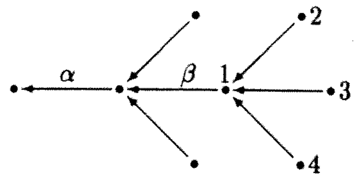
Assume that, for some i, Σ'_i is not a (disjoint union of) Dynkin quiver(s) and let H' be the endomorphism algebra of the module $\bigoplus\{M \mid M \in \Sigma'_i\}$. Then H' is representation-infinite so that there exist infinitely many non-isomorphic indecomposable B_i-modules $(L_\lambda)_{\lambda \in \Lambda}$ such that $\mathrm{Hom}_{B_i}(S_1 \oplus \cdots \oplus S_m, L_\lambda) = 0$ for all $\lambda \in \Lambda$, where $S_1, \ldots, S_m$ are all sources in Σ_i. We claim that this implies $\mathrm{Hom}_A(\mathrm{D}A, L_\lambda) = 0$ for all $\lambda \in \Lambda$. Indeed, let I be an indecomposable injective A-module and consider the left minimal almost split morphism $f: I \to I/\mathrm{soc}\, I$. If $I/\mathrm{soc}\, I$ has no summand in mod B_i, clearly $\mathrm{Hom}_A(I, L_\lambda) = 0$ since any non-zero morphism would factor through f, an absurdity. If $I/\mathrm{soc}\, I$ has a summand in mod B_i, this summand must be isomorphic to one of $S_1, \ldots, S_m$. Thus, $\mathrm{Hom}_A(I, L_\lambda) \neq 0$ implies $\mathrm{Hom}_A(S_j, L_\lambda) \neq 0$ for some $1 \leq j \leq m$, a contradiction to our assumption on the family $(L_\lambda)_{\lambda \in \Lambda}$. This shows that we indeed have $\mathrm{Hom}_A(\mathrm{D}A, L_\lambda) = 0$ for all $\lambda \in \Lambda$, or equivalently, that $\mathrm{pd}(\tau^{-1} L_\lambda) \leq 1$ for all $\lambda \in \Lambda$, a contradiction to the hypothesis. □

The next example shows that the hypothesis in Proposition 4.3(a)(i) that there is no projective which is a successor of the reduced right extremal subsection of A is essential.

Example. Let A be the algebra given by the quiver



bound by $\beta\alpha = 0$, $\alpha\gamma = 0$, $\alpha\delta = 0$ and $\gamma\varepsilon = 0$. Then A is a representation-infinite left glueing of the tilted algebra given by the quiver



bound by $\beta\alpha = 0$, by the representation-finite algebra given by the quiver

Also, the reduced right extremal subsection is given by the quiver

whose underlying graph is the Dynkin graph $\mathbb{A}_3$. However, none of the injective postprojective A-modules maps into any of the projectives corresponding to the points 1, 2, 3 and 4 in the quiver of A. Hence, there exist infinitely many non-isomorphic indecomposable modules M_A such that $\mathrm{Hom}_A(\mathrm{D}A, \tau M) = 0$ or, equivalently, such that $\mathrm{pd}\, M = 1$. We should notice that the projective modules considered are all successors of the right extremal subsection.

4.4. Let A be a left (or right) glued algebra. In the rest of this section, we shall be interested in the case where A itself is a tilted algebra.

If A a is representation-infinite left glued tilted algebra, it follows from Corollary 3.3 that $\Gamma(\mathrm{mod}\, A)$ has a unique postprojective component Γ which is moreover a connecting component. We shall now recall from [2, (1.2), (2.4)] the notions of *right type* and *reduced right type* of A. If Γ contains no injective module, the right type and the reduced right type of A are defined to be both equal to the empty graph if A is concealed, or to be both equal to the type of A (see Lemma 4.2) if A is not concealed. If Γ contains an injective module, and Σ and Σ' denote respectively the right extremal subsection and the reduced right extremal subsection of A, then the right type and the reduced right type of A are defined to be equal to the underlying graphs $\bar{\Sigma}$ of Σ and $\bar{\Sigma}'$ of Σ', respectively.

Dually, if A is a representation-infinite right glued tilted algebra, $\Gamma(\mathrm{mod}\, A)$ has a unique preinjective component Γ which is a connecting component. We define the *left type* and *reduced left type* of A as follows. If Γ contains no projective module, the left type and the reduced left type of A are defined to be both equal to the empty graph if A is concealed, or to be both equal to the type of A if A is not concealed. If Γ contains a projective module, and Σ and Σ' denote respectively the left extremal subsection and the reduced left extremal subsection of A, then the left type and the reduced left type of A are defined to be equal to the underlying graphs $\bar{\Sigma}$ of Σ and $\bar{\Sigma}'$ of Σ', respectively.

Proposition. *Let A be a representation-infinite tilted algebra.*

(a) pd $M = 2$ *and* id $M = 1$ *for almost all M in* ind A *if and only if A is a left glued algebra and its reduced right type is a union of Dynkin graphs.*

(b) pd $M = 1$ *and* id $M = 2$ *for almost all M in* ind A *if and only if A is a right glued algebra and its reduced left type is a union of Dynkin graphs.*

Proof. We shall only prove (a), since the proof of (b) is dual. We first recall that the global dimension of A is at most 2, so that pd $M > 1$ if and only if pd $M = 2$. The necessity follows from Proposition 4.3(a)(ii). For the sufficiency, we first observe that there exists an injective postprojective A-module, since otherwise the type of the tilted algebra A would be a Dynkin graph, contradicting the assumption that A is representation-infinite. Finally, since A is a tilted algebra, no projective is a successor of its right extremal subsection. The statement then follows from Proposition 4.3(a)(i). □

4.5. Examples. (i) From every Dynkin diagram, one can construct a left glued algebra satisfying the conditions of the Proposition. Indeed, let Δ be any Dynkin diagram, and $\tilde{\Delta}$ be the corresponding euclidean diagram. We orient $\tilde{\Delta}$ in such a way that the unique point in $\tilde{\Delta}$ which is not in Δ is the unique sink of $\tilde{\Delta}$, and let B denote the path algebra of the resulting quiver. We then let A be the one-point coextension of B by the simple B-module corresponding to the unique sink. The algebra A is as required.

(ii) In [3], a torsion-theoretical characterisation was given to tilted algebras which if tame, are representation-finite or one-parametric and, if wild, are such that one of the end algebras is zero and the other is hereditary with two non-isomorphic simple modules. If such a tilted algebra is representation-infinite, it is clearly left or right glued. If moreover it is not concealed, its reduced right or left type, respectively, is a disjoint union of Dynkin quivers, so that the above theorem applies in this case.

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