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GENERALIZED TILTED ALGEBRAS OF TYPE $\mathbb{A}_n$

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Let k be a commutative field, and A a finite-dimensional k-algebra (which we assume associative and with identity). All A-modules will be finite-dimensional right A-modules. By mod A we denote the category of finite-dimensional right A-modules. Homomorphisms will always be written on the opposite side of the scalars.

Following [5], a module T_A is called a tilting module provided the following properties are satisfied:

(T1) There is a short exact sequence $0 \to P_A \to Q_A \to T_A \to 0$ with P, Q projective (thus, $\mathrm{pd}\, T_A \leq 1$) ,

(T2) $\mathrm{Ext}^1_A(T,T) = 0$,

(T3) There is a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T', T'' direct sums of summands of T_A.

If A is hereditary, and T_A a tilting module, then $B = \mathrm{End}\, T_A$ is called a tilted algebra [5]. We shall introduce the notion of a going down tilting series and that of a generalized tilted algebra (see §1). These are, roughly speaking, those finite-dimensional k-algebras which can be reached from a hereditary algebra by a finite number of applications of the tilting process. The aim of this paper is to give a complete classification of those generalized tilted algebras which are of type $\mathbb{A}_n$ (that is, which can be reached from all path algebras $k\Delta$, where the underlying graph $\overline{\Delta}$ is $\mathbb{A}_n$). In fact, we shall prove:

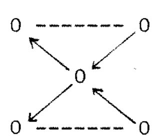
<u>Theorem</u>: A finite-dimensional k-algebra A is a generalized tilted algebra of type $\mathbb{A}_n$ if and only if the bounden quiver $(Q, (\rho_\alpha)_{\alpha \in I})$ of A satisfies the following conditions:

(i) $\overline{Q}$ is a tree,

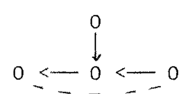
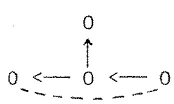
(ii) Every point in Q has at most four neighbours,

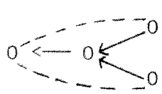
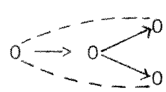
(iii) All relations ρ_α are of length two,

(iv) If a point has four neighbours, then



is a full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$,

(v) If a point has three neighbours, then  or  is a full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$,

(vi) There is no full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$ of the forms  and  (where the dotted lines indicate zero relations).

The main tool in proving the necessity of these conditions is the observation that most local properties of the Auslander-Reiten quiver of an algebra remain unchanged when applying the tilting process. In the converse part, we actually construct the tilting series starting from an algebra satisfying the stated conditions and reaching a hereditary algebra of type $\mathbb{A}_n$ by generalizing the so-called APR-tilts [2].

This answers for a subclass of algebras a question in [2] which algebras can be tilted to a hereditary algebra. Observe that the theorem immediately implies that generalized tilted algebras can be of arbitrary global dimension (see (2.5)). This answers a question of M. Auslander.

Note that a particular class of generalized tilted algebras of type $\mathbb{A}_n$ was studied in [6].

For the convenience of the reader, we have collected in §1 the material we need from the general theory of tilting modules. §2 then contains the proof of the theorem.

§1. Preliminaries and notations

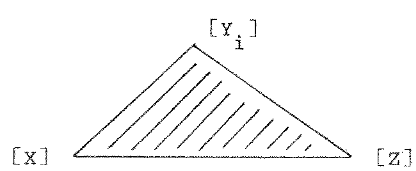
(1.1) We shall use the letter Q to denote a quiver, $\overline{Q}$ its underlying graph. Points of Q will be denoted by small letters such as $a, b, \ldots, i, j, \ldots$. Relations in Q will be denoted by the letter ρ , we shall always assume the paths entering the relations to be the minimal paths satisfying the relations.

Recall that any finite-dimensional k-algebra with $A/\text{rad } A$ being a product of copies of k is given by a bounden quiver

$(Q, (\rho_\alpha)_{\alpha \in I})$ [4]. We shall denote by $S(a)$, $S(i)$, ... the simple A-modules corresponding to the points a, i, ... of Q. We shall denote by $P(a)$ (respectively $I(a)$) the indecomposable projective (respectively injective) such that $P(a)/\mathrm{rad}\, P(a) = S(a)$ (respectively $\mathrm{soc}\, I(a) = S(a)$).

(1.2) We shall use freely the properties of Auslander-Reiten sequences and irreducible maps such as can be found, for instance, in [1] or [4]. Recall that the Auslander-Reiten quiver Γ_A of the algebra A is defined to have as points the isomorphism classes $[M]$ of indecomposable A-modules, and there is an arrow $[M] \to [N]$ provided there exists an irreducible map $M \to N$. Note that this quiver is endowed with the (partially defined) Auslander-Reiten translation $\tau = DTr$ and has the following property: if for some vertex z, τz is defined, then the set of end points of arrows $\tau z \to y$ coincides with the set of starting points of arrows $y \to z$ and this set is finite.

One may also regard the Auslander-Reiten quiver as part of a two-dimensional simplicial complex, with edges both the underlying edges of the arrows, as well as additional edges $[X] \to [Z]$ for each Auslander-Reiten sequence $0 \to X \to Y \to Z \to 0$ and with triangles of the form



in case Y_i is an indecomposable direct summand of Y in the Auslander-Reiten sequence $0 \to X \to Y \to Z \to 0$. Thus we have a topological structure on this simplicial complex. We refer the reader to [7].

(1.3) Assume now that A is a finite-dimensional k-algebra, and T_A a tilting module with $\mathrm{End}\, T_A = B$. We consider two full subcategories of $\mathrm{mod}\, A$: $\mathcal{T}(T_A)$ which is the full subcategory of all modules generated by T_A (or, equivalently, of all modules M_A such that $\mathrm{Ext}^1_A(T,M) = 0$) and $\mathcal{F}(T_A)$, which is the full subcategory of all modules cogenerated by τT_A (or, equivalently, of all modules M_A such that $\mathrm{Hom}_A(T,M) = 0$). Always the pair $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ forms a torsion theory for $\mathrm{mod}\, A$.

There are two corresponding full subcategories of $\mathrm{mod}\, B$ defined by:

$$\mathcal{X} = \mathcal{X}(T_A) = \{N_B \mid N_B \otimes {}_BT_A = 0\}$$

and $\mathcal{Y} = \mathcal{Y}(T_A) = \{N_B \mid \mathrm{Tor}^B_1(N_B, {}_BT_A) = 0\}$. Then we have the following:

<u>Theorem of Brenner-Butler</u>: Let T_A be a tilting module with $\mathrm{End}\, T_A = B$. Then also ${}_BT$ is a tilting module, and $A = \mathrm{End}\, {}_BT$, canonically. Moreover the subcategories $\mathcal{T}(T_A)$ and $\mathcal{Y}(T_A)$ are equivalent under the restrictions of the functors $\mathrm{Hom}_A(T,-)$ and $-\otimes_B T_A$ which are mutually inverse to each other, and similarly, the subcategories $\mathcal{F}(T_A)$ and $\mathcal{X}(T_A)$ are equivalent under the restrictions of the functors $\mathrm{Ext}^1_A(T,-)$ and $\mathrm{Tor}^B_1(-, T)$ which are again mutually inverse to each other. $\square$

For the proof, we refer the reader to [5].

(1.4) Given a hereditary algebra A, a going-down tilting series from A, $(A_i, T_{A_i})_{i \in \mathbb{N}}$ consists of a family of algebras A_i, and a family of tilting modules T_{A_i} such that:

(1) $A_o = A$,

(2) $A_{i+1} = \operatorname{End} T_{A_i}$,

(3) The induced torsion theories $(\mathcal{X}(T_{A_i}), \mathcal{Y}(T_{A_i}))$ are all splitting.

A finite-dimensional algebra B will be called a generalized tilted algebra if there exists a hereditary algebra A, a going-down tilting series $(A_i, T_{A_i})_{i \in \mathbb{N}}$ from A, and an $m \in \mathbb{N}$ such that $B = A_m$. B will be called generalized tilted of type $\mathbb{A}_n$ if A is the path algebra of a quiver whose underlying graph is $\mathbb{A}_n$.

2. Proof of the theorem

(2.1) Let us start by defining two sets of properties that will be used later: here A simply denotes a finite-dimensional k-algebra.

Properties (γ): The Auslander-Reiten quiver Γ_A of A satisfies the following:

(γ_1) Γ_A is simply connected,

(γ_2) There are at most two irreducible maps with prescribed domain or codomain,

(γ_3) If P_A is projective, with indecomposable radical R, then there is at most one irreducible map of codomain R. Dually, if I_A is injective with $I/\mathrm{Soc}\ I$ indecomposable, then there is at most one irreducible map of domain $I/\mathrm{Soc}\ I$.

Properties (κ): The bounden quiver $(Q, (\rho_\alpha)_{\alpha \in I})$ of A satisfies the following:

(κ_1) $\overline{Q}$ is a tree,

(κ_2) Every point has at most four neighbours,

(κ_3) All relations ρ_α in Q are of length two,

(κ_4) If a point has four neighbours, then

$$\begin{tikzcd} 0 \arrow[rr, dash, dashed] & & 0 \arrow[dl] \\ & 0 \arrow[ul] \arrow[dl] & \\ 0 \arrow[rr, dash, dashed] & & 0 \arrow[ul] \end{tikzcd}$$ is a full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$,

(κ_5) If a point has three neighbours, then $\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[rr, dash, dashed, bend right] & 0 \arrow[l] & 0 \arrow[l] \end{tikzcd}$

or $\begin{tikzcd} & 0 & \\ 0 \arrow[rr, dash, dashed, bend right] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \end{tikzcd}$ is a full subquiver of $(Q,(\rho_\alpha)_{\alpha \in I})$,

(κ_6) There is no full subquiver of $(Q,(\rho_\alpha)_{\alpha \in I})$ of one of

the forms $\begin{tikzcd} 0 \arrow[drr, dash, dashed, bend left] & & \\ & 0 \arrow[ul] \arrow[dl] & 0 \arrow[l] \\ 0 \arrow[urr, dash, dashed, bend right] & & \end{tikzcd}$ or $\begin{tikzcd} 0 \arrow[dr] \arrow[drr, dash, dashed, bend left] & & \\ & 0 \arrow[r] & 0 \\ 0 \arrow[ur] \arrow[urr, dash, dashed, bend right] & & \end{tikzcd}$

(where $\overline{Q}$ denotes the underlying graph of Q and the zero relations are indicated by dotted lines.)

The following lemma is crucial for the proof:

Lemma: Let A be a finite-dimensional algebra of finite representation type satisfying the properties (γ_2) and (γ_3) , then, for every indecomposable M_A, the set of all (isomorphism classes) of indecomposable modules N_A such that there exists a non-zero map $N \to M$, but no non-zero map $N \to \tau M$, is the union of two full linear subquivers of Γ_A intersecting at $[M]$.

Dually, the set of all (isomorphism classes) of indecomposable modules L_A such that there exists a non-zero map $M \to L$, but no non-zero map $\tau^{-1}M \to L$, is the union of two full linear subquivers of Γ_A intersecting at $[M]$.

Proof: We start by constructing two linear subquivers of Γ_A intersecting at $[M]$. Assume first that M is not projective, then, by (γ_2), the Auslander-Reiten sequence ending in M has at most two terms, thus there exist indecomposables E_1, F_1 such that

$$0 \longrightarrow \tau M \xrightarrow{\begin{pmatrix} g_1' \\ f_1' \end{pmatrix}} F_1 \oplus E_1 \xrightarrow{(g_1 f_1)} M \longrightarrow 0$$

is an Auslander-Reiten sequence. If M is projective, then, again by (γ_2), there are at most two irreducible maps into M which we denote again by $g_1 : F_1 \to M$ and $f_1 : E_1 \to M$.

Now, assume that $E_1 \neq 0$, then if E_1 is not projective, there exists an Auslander-Reiten sequence

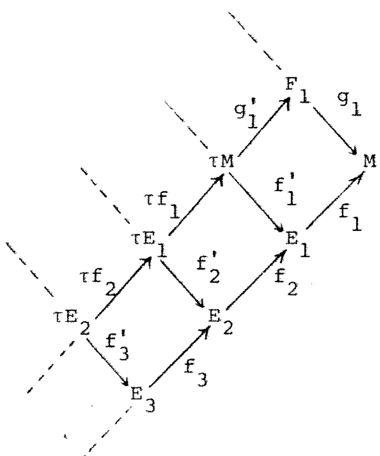
$$0 \longrightarrow \tau E_1 \xrightarrow{\begin{pmatrix} \tau f_1 \\ f_2' \end{pmatrix}} E_2' \oplus E_2 \xrightarrow{(f_1' \; f_2)} E_1 \longrightarrow 0$$

where $E_2' = \tau M$. Thus we have defined $f_2 : E_2 \to E_1$. If on the other hand E_1 is not projective, then, by (γ_3), M cannot be projective, thus there are at most two irreducible maps with codomain E_1, one of which is $f_1' : \tau M \to E_1$. If there is another irreducible map into E_1 we define this map to be f_2. Otherwise the construction stops.

Inductively, if E_i has been defined (and is non-zero) for all $i \leq t$, and E_t is not projective, there exists an Auslander-Reiten sequence with at most two middle terms ending in E_t:

$$0 \longrightarrow \tau E_t \xrightarrow{\begin{pmatrix}\tau f_t \\ f'_{t+1}\end{pmatrix}} \tau E_{t-1} \oplus E_{t+1} \xrightarrow{(f'_t \ f_{t+1})} E_t \longrightarrow 0 .$$

If E_t is projective, $\operatorname{rad} E_t = \tau E_{t-1} \oplus E_{t+1}$, and we define $f_t : E_{t+1} \to E_t$ to be the inclusion map. The process stops when $E_t = 0$ or there is only one irreducible map into E_{t-1}. The set $(E_t)_t$ together with M defines by construction a linear subquiver of Γ_A (indeed, (γ_3) ensures that there is no branching), and will be denoted by $L(f_1)$.



Similarly g_1 defines a linear subquiver $L(g_1)$.

Now let N be an indecomposable such that there exists a non-zero map $h : N \to M$, but no non-zero map $N \to \tau M$. We claim that $[N] \in L(f_1) \cup L(g_1)$.

Either $h : N \to M$ is an isomorphism, or else it factors through the right almost split map $(g_1\ f_1) : F_1 \oplus E_1 \to M$. Thus we may assume that there exists a non-zero map $h_1 : N \to E_1$ such that $h = f_1 h_1$.

Thus, assume that h_1 is not an isomorphism, then it factors through $(f_1'\ f_2) : \tau M \oplus E_2 \to E_1$. But it cannot factor through τM, by hypothesis. Hence there exists a non-zero map $h_2 : N \to E_2$ such that $h_1 = f_2 h_2$.

Inductively, either $N \approx E_i$ for some $i \leq t$, or else, by the same argument, there exists an $h_{t+1} : N \to E_{t+1}$ such that $h_t = f_{t+1} h_{t+1}$.

Since for some m we have $E_m = 0$, it follows that $[N] \in L(f_1)$.

Similarly, if there exists a non-zero map $h_1' : N \to F_1$ such that $h = g_1 h_1'$, we have $[N] \in L(g_1)$.

On the other hand, if $E_t \in L(f_1)$ $(E_t \neq 0)$, then the map $f_1 f_2 \ldots\ldots f_t : E_t \to M$ is non-zero. This can be proved by an eas argument, assuming that t is minimal with $f_1 \ldots f_t = 0$ and provi that then f_1 is a monomorphism, thus getting a contradiction.

Similarly, if $F_t \in L(g_1)$ $(F_t \neq 0)$, the map $g_1 g_2 \ldots g_t$: $F_t \to M$ is non-zero. This proves the first half of the lemma, the second half is dual. □

(2.2) The following proposition is the first step towards the proof of the theorem.

Proposition: Let B be a generalized tilted algebra of type $\mathbb{A}_n$, then B satisfies the properties (γ) and (κ).

Proof: We shall prove this proposition by induction on the length of the going down tilting series needed to reach B. Since all the statements are trivial for $\mathbb{A}_n$, we let A be a generalized tilted algebra satisfying (γ) and (κ), and T_A a tilting module such that $B = \operatorname{End} T_A$. Then:

(γ_1) The Auslander-Reiten quiver Γ_B of B is simply connected.

Indeed, assume this is not the case, we would then have in Γ_B two oriented paths w_1, w_2 from the point M_1 to the point M_2 which are not homotopic. We can of course assume that w_1 and w_2 are minimal with this property. Thus M_1 is injective and M_2 is projective and w_1, w_2 have no common arrow. Since M_2 is projective, it belongs to $\mathcal{Y}(T_A)$. Hence modules lying on the paths w_1 and w_2 are in $\mathcal{Y}(T_A)$.

Applying the functor $- \otimes {}_BT_A$, we obtain two paths $\bar{w}_1$ and $\bar{w}_2$ in Γ_A, from $M_1 \otimes {}_BT_A$ to $M_2 \otimes {}_BT_A$ (since A is of finite representation type). Now, since M_2 is projective, $M_2 \otimes {}_BT_A$ is an indecomposable summand T' of T and, since M_1 is an injective B-module lying in $\mathcal{Y}(T_A)$, $M_1 \otimes {}_BT_A = I_A$ is an injective A-module (by the corollary (2.4) of [5]).

Since Γ_A is simply connected, $\bar{w}_1$ and $\bar{w}_2$ are necessarily homotopic. But the injectivity of I then implies that the paths $\bar{w}_1$ and $\bar{w}_2$ cannot be minimal, that is, there exists an irreducible map $I \to X$ such that, for $i = 1,2$, $\bar{w}_i$ is the composition of

$I \to X$ and of a path from X to T'. Obviously the minimality of w_1 and w_2 implies that $X \notin \mathcal{T}(T_A)$.

On the other hand, $I \in \mathcal{T}(T_A)$, and $I \to X$ is an irreducible map, thus an epimorphism of I onto a direct summand of $I/\mathrm{Soc}\, I$ (since I is injective). The class $\mathcal{T}(T_A)$ is closed under quotien so $X \in \mathcal{T}(T_A)$, a contradiction.

(γ_3) Let P_B be a projective B-module of indecomposable radical R. Then there is at most one irreducible map of codomain R. Dually, if I_B is an injective B-module with $I/\mathrm{Soc}\, I$ indecomposable, then there is at most one irreducible map of domain $I/\mathrm{Soc}\, I$.

Indeed, let P_B be projective with indecomposable radical R such that there are two irreducible maps of domains X_1, X_2 into R.

$$\begin{tikzcd} X_1 \arrow[dr] & & \\ & R \arrow[r] & P \\ X_2 \arrow[ur] & & \end{tikzcd}$$

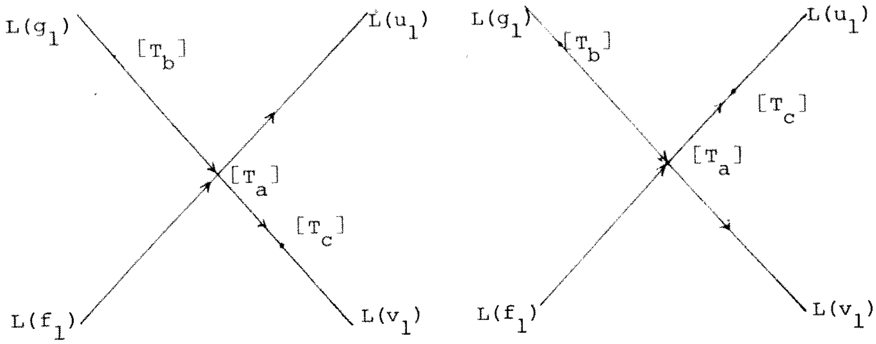
Since P_B is projective, $P \in \mathcal{Y}(T_A)$, hence R, X_1, X_2 all belong to $\mathcal{Y}(T_A)$. Apply the functor $- \otimes {}_BT_A$ to return to mod A. P_B being projective, $P_B \otimes {}_BT_A = T'_A$ is an indecomposable summand of T. Since the functor $- \otimes {}_BT_A$ is exact in $\mathcal{Y}(T_A)$, $R \otimes T \to T'$ is a monomorphism. Hence the maps $k_i : X_i \otimes T \to R \otimes T \to T'$ $(i = 1,2)$ are non-zero. Now there is no non-zero map $R \otimes T \to \tau T'$. For, either T' is projective, and then $\tau T' = 0$ or else T' is not projective, and then we have $\mathrm{Hom}_A(R \otimes T, \tau T') \approx D\,\mathrm{Ext}^1_A(T', R \otimes T) = 0$ (by lemma (2.5) of [5]).

Lemma (2.1) shows then that $R \otimes T$ belongs to one of the two linear subquivers determined by T', say to $L(f_1)$. Thus $X_1 \otimes T$, $X_2 \otimes T$, by the same reason, belong to $L(f_1)$, and hence there exists a non-zero map from $X_1 \otimes T$ to $X_2 \otimes T$ (or from $X_2 \otimes T$ to $X_1 \otimes T$). But then, returning to mod B via the functor $\mathrm{Hom}_A(T, -)$, we obtain a non-zero map from X_1 to X_2 (or from X_2 to X_1), an absurdity. This shows the first half of (γ_3), the second half is dual.

We shall now show the properties (κ) for B. We first take a closer look at the correspondence between indecomposable summands of T and points of the quiver Q of B.

Let T_a be the indecomposable summand of T corresponding under $\mathrm{Hom}\,(T_A,-)$ to the projective B-module corresponding to the point a of Q, then there are at most two irreducible maps f_1, g_1 of codomains T_a and at most two irreducible maps u_1, v_1 of domain T_a, and these determine (at most) four linear subquivers intersecting at T_a, namely $L(f_1)$, $L(g_1)$, $L(u_1)$ and $L(v_1)$. Let T_b be the indecomposable summand of T corresponding to the point b of Q: then if $\mathrm{Hom}_A(T_a, T_b) \neq 0$, $[T_b] \in L(u_1)$ or $[T_b] \in L(v_1)$, or if $\mathrm{Hom}_A(T_b, T_a) \neq 0$, and then $[T_b] \in L(f_1)$ or $[T_b] \in L(g_1)$.

Assume thus that b and c are two neighbours of a in Q. Then either $[T_b]$, $[T_a]$, $[T_c]$ are colinear or not; we represent these two cases by the pictures:

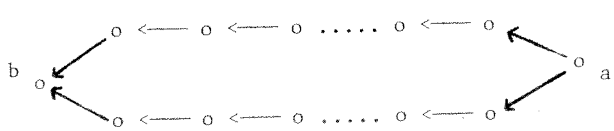


in the first case, the map $T_b \to T_a \to T_c$ is non-zero, indeed both $[T_b]$ and $[T_a]$ belong to a linear subquiver determined by an irreducible map of codomain T_c. In the second case, $T_b \to T_a \to T_c$ is zero, since $[T_b]$ is not in the linear subquiver determined by an irreducible map of codomain T_c on which $[T_a]$ lies. After the general remarks, we turn to the proof of (κ), we shall keep throug the same notations.

(κ_1) $\overline{Q}$ is a tree:

If $\overline{Q}$ is not a tree, then there is a full subquiver Q' of which is a (non-oriented) cycle. Since the quiver Q has no orient cycle, Q' always contains at least one source and one sink. Then consider two cases for Q':

Case (1): There is no commutativity relation on Q': indeed assume that Q' is a commutative cycle



then $\mathrm{Hom}_B(P(b), P(a)) \neq 0$, and it follows that in Γ_A, the module lies on one of the two linear subquivers determined by T_a. Now this is a commutative cycle, hence T_b should in fact lie on both linear subquivers determined by T_a. This, however, is impossible. Hence there are no commutativity relations in Q, that is, all relations are zero relations.

Case (2): There is a zero-relation on at least one of the non-oriented paths from the source a to the sink b : then there are two irreducible maps with domain $P(b)$ and codomains M_1 and M_2, and rad $P(a)$ is decomposable, with indecomposable summands R_1 and R_2. Since Γ_A is path connected, there exists a path w_i from M_i to R_i $(i = 1,2)$. However w_1 and w_2 are clearly non-homotopic, since they factor over exactly one indecomposable summand of rad $P(a)$, a contradict

(κ_2) Every point of the quiver Q of B has at most four neighb

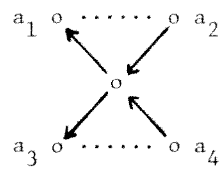
Indeed, let a be a point of Q, then, if b is a neighbour of a, $[T_b]$ has to lie on one of the four linear subquivers intersecting at $[T_a]$.

(κ_3) All relations ρ_α in Q are of length two.

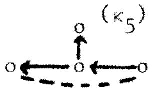
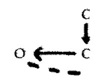
Indeed, let $\underset{a_1}{\circ} \leftarrow \underset{a_2}{\circ} \leftarrow \underset{a_3}{\circ} \leftarrow \cdots \leftarrow \underset{a_r}{\circ}$ be a zero relation

with $r \geq 3$, then, passing to the corresponding summands T_{a_i} $(1 \leq i \leq r)$, we see that necessarily $r = 3$.

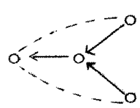
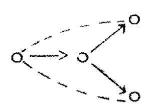
(κ_4) If a point a_o has four neighbours a_1, a_2, a_3, a_4, then we have a full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$ of the form



Indeed, considering the corresponding summands T_{a_i} $(0 \leq i \leq$ necessarily, each of $[T_{a_1}]$, $[T_{a_2}]$, $[T_{a_3}]$, $[T_{a_4}]$, lies on a differer linear subquiver determined by $[T_{a_o}]$. We then just need to apply t remarks preceding the proof of (κ_2).

 (κ_5) If a point a has three neighbours a_1, a_2, a_3, then  is a full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$.

For, necessarily, two of the $[T_{a_i}]$ $(0 \leq i \leq 3)$ are colinear with while the third one is not.

(κ_6) There is no full subquiver of $(Q, (\rho_\alpha)_{\alpha \in I})$ of one of the forms  or  .

This follows from the proof of (κ_5).

(γ_2) Let X_B be indecomposable, there are at most two irreducible maps with domain X, and two irreducible maps with codomain X.

Observe that for any indecomposable X_B and point a of the quiver Q of B we have $\dim X_a \leq 1$.

(i) Assume first that X is projective. Then, by (κ_1), rad X_B has at most two indecomposable summands, therefore there are at most two irreducible maps with codomain X. If X is not injective, consider the Auslander-Reiten sequence starting with X,

$$0 \longrightarrow X \longrightarrow \bigoplus_{i=1}^{r} Y_i \longrightarrow \tau^{-1}X \longrightarrow 0$$

with the Y_i indecomposables. Let $X = P(a)$ for some point a of Q, then $(Y_i)_a \neq 0$. But now $\dim (\tau^{-1}X)_a \leq 1$ implies that $r \leq 2$. Similarly if X_B is injective.

(ii) Finally, assume X is neither projective nor injective, and consider an Auslander-Reiten sequence

$$0 \longrightarrow X \longrightarrow \bigoplus_{i=1}^{r} Y_i \longrightarrow \tau^{-1}X \longrightarrow 0$$

with the Y_i indecomposables. Thus $\dim \operatorname{Hom}_B(X, \tau^{-1}X) = r-1$.

a) If $\tau^{-1}X \in \mathcal{Y}(T_A)$, we have $X, Y_i \in \mathcal{Y}(T_A)$ and hence $\dim \operatorname{Hom}_A(X \otimes {}_BT_A, \tau^{-1}X \otimes {}_BT_A) = r-1$. The induction hypothesis implies that $r \leq 2$.

b) Dually, $r \leq 2$ if $X \in \mathcal{X}(T_A)$.

c) If $X \in \mathcal{Y}(T_A)$ and $\tau^{-1}X \in \mathcal{X}(T_A)$, we can assume that Y_1 is projective and another summand of $\bigoplus_{i=1}^{r} Y_i$ is injective, otherwise we can get a contradiction to a) or b) (namely, if the middle term has no projective summand, and $r \geq 3$, there are at least three irreducible maps with codomain $X \in \mathcal{Y}(T_A)$, dually, there must be an injective summand). Assume the injective summand is not Y_1, say $Y_1 = P(a)$ and $Y_2 = I(b)$. The existence of the non-zero map $X \to I(b)$ implies that $X_b \neq 0$, and, since X is a direct summand of $\operatorname{rad} P(a)$, we have $P(a)_b \neq 0$. But we also have $I(b)_b \neq 0$. Therefore $(\tau^{-1}X)_b \neq 0$, which contradicts the fact that $\tau^{-1}X$ is a direct summand of $I(b)/\operatorname{Soc} I(b)$. Hence Y_1 is projective-injective. Now let Q' be the support of Y_1 (that is, the set of those points a of Q such that $(Y_1)_a \neq 0$). Since $\overline{Q}$ is a tree, Q' is of the form $b\ \circ \leftarrow \circ \leftarrow \ldots \leftarrow \circ \leftarrow \circ\ a$ and there is no relation on Q'. Furthermore, for any point c of Q not in Q' there is no non-zero path from c to Q' or from Q' to c. Therefore $\operatorname{rad} P(a)$ is indecomposable and defined on Q'. Also $I(b)/\operatorname{Soc} I(b)$ is indecomposable and defined on Q'. Thus the given Auslander-Reiten sequence contains only modules defined on Q' and is in fact given as follows

$$\begin{tikzcd} & X/S(a) \arrow[dr] & \\ X \arrow[ur] \arrow[r] & Y_1 = P(a) = I(b) \arrow[r] & \tau^{-1}X \end{tikzcd}$$

where $S(a)$ is the simple module corresponding to the point a. Since $X/S(a)$ is indecomposable we get an Auslander-Reiten-sequence with two middle terms.

This completes the proof of the Proposition. $\square$

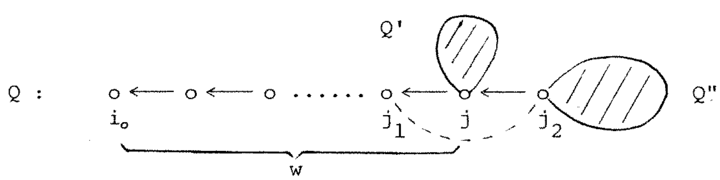
(2.3) We shall now prove the theorem.

<u>Theorem:</u> The finite-dimensional algebra B is generalized tilted of type $\mathbb{A}_n$ if and only if the bounden quiver $(Q,(\rho_\alpha)_{\alpha\in I})$ of B satisfies (κ) .

<u>Proof:</u> By Proposition (2.2), we know that if B is generalized tilted of type $\mathbb{A}_n$, its bounden quiver $(Q, (\rho_\alpha)_{\alpha\in I})$ satisfies (κ) . Conversely, let B be such that $(Q, (\rho_\alpha)_{\alpha\in I})$

satisfies (κ), we shall show that B is generalized tilted of type $\mathbb{A}_n$, where n denotes the number of points in Q.

We can assume that there is a simple sink i_o in Q, that is, a sink with only one arrow going in, otherwise we form the opposite algebra. And it is easily seen that B is generalized tilted of type Δ, for any quiver Δ, if and only if B^{op} is generalized tilted of type Δ. Now let $\bar{w}$ be the minimal non-oriented path in Q starting from i_o and ending at the middle point j of a zero-relation. By applying reflection functors [3], which are in fact given by tilting modules, and if necessary passing to the opposite algebra, we can in fact assume that we have the following situation



where Q', Q'' may be empty, and w is the linear oriented path from j to i_o. Let r be the number of points in w, and choose a numbering of the points of Q such that $i_o = 1, \ldots j = r$. We shall construct a module T_B such that:

a) T_B is a tilting module,

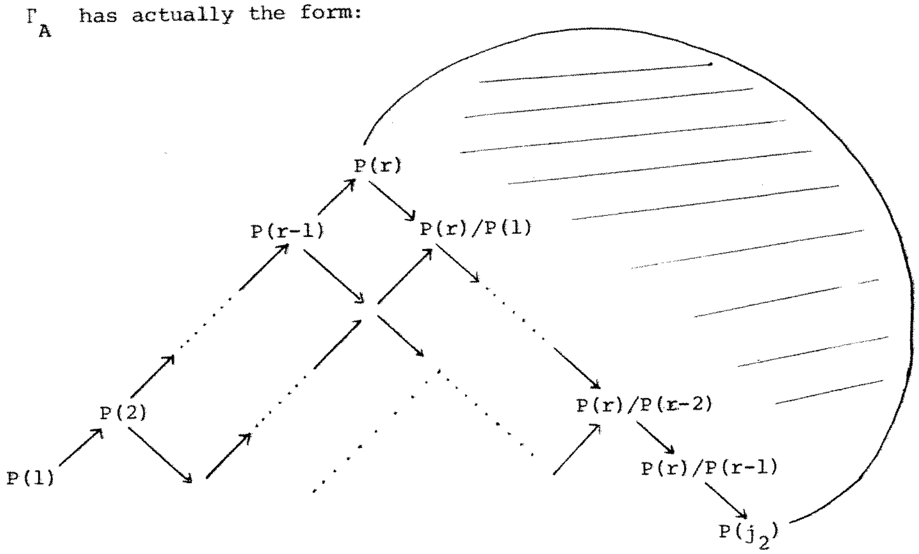
b) The torsion theory $(\mathcal{T}(T_B), \mathcal{F}(T_B))$ is splitting,

c) The bounden quiver $(Q_A, (\rho_\alpha)_{\alpha \in I'})$ of $A = \operatorname{End} T_B$ satisfies again (κ),

d) There is exactly one zero-relation less.

This means that we can apply the prescribed process to A and obtain finally a quiver satisfying (κ) but with no zero relations, hence of type $\mathbb{A}_n$, and the assertion will be proved. Define $T_i = P(r)/P(i)$ for $1 \le i \le r-1$ and $T_i = P(i)$ for $r \le i \le n$. Then obviously the T_i for $1 \le i \le r-1$ are indecomposable. Let $T_B = \bigoplus_{i=1}^{n} T_i$. We shall show that T_B satisfies the required conditions. But first note that the Auslander-Reiten quiver Γ_A has actually the form:

Γ_A :



We also note that by construction, and by the assumption on $(Q, (\rho_\alpha)_{\alpha \in I})$, we have that $P(r)/P(r-1)$ is a direct summand of rad $P(j_2)$.

a) T_B is a tilting module.

Indeed:

(T_1) pd $T_B \le 1$:

We show that for every indecomposable summand T_i of T_B, pd $T_i \le 1$. If $r \le i \le n$, then pd $T_i = 0$. If $1 \le i \le r-1$,

$T_i = P(r)/P(i)$ has the projective resolution

$$0 \longrightarrow P(i) \longrightarrow P(r) \longrightarrow T_i \longrightarrow 0$$

and then $\operatorname{pd} T_i = 1$.

(T2) $\operatorname{Ext}^1_B(T,T) = 0$:

It clearly suffices to consider $\operatorname{Ext}^1_B(T_i,T_j)$ with $1 \leq i \leq r-1$ and $r \leq j \leq n$. Now, since $\operatorname{pd} T_B \leq 1$ $\operatorname{Ext}^1_B(T_i,T_j) = D\operatorname{Hom}_B(T_j,\tau T_i)$ (by Corollary (2.5) of [5]). But $\operatorname{Hom}_B(T_j,\tau T_i) = 0$ by construction, since the support of τT_i is in fact $w \setminus \{r\}$.

(T3) There exists a short exact sequence $0 \to B_B \to T' \to T'' \to 0$ with T', T'' direct sums of direct summands of T.

It suffices to show the existence of such a short exact sequence for every indecomposable projective P_B. If $P_B = P(i)$, $r \leq i \leq n$, there is nothing to show, while if $P_B = P(i)$ with $1 \leq i \leq r-1$, the required sequence is, as in (T1),

$$0 \longrightarrow P(i) \longrightarrow P(r) \longrightarrow P(r)/P(i) \longrightarrow 0 .$$

b) The torsion theory $(\mathcal{T}(T_B), \mathcal{F}(T_B))$ is splitting.

Indeed, let X_B be an indecomposable not in $\mathcal{F}(T)$. Then $\operatorname{Hom}_B(T,X) \neq 0$, and so there is a $r \leq j \leq n$ with $X_j \neq 0$. Now $\operatorname{Ext}^1_B(T,X) = \bigoplus_{i=1}^{r-1} \operatorname{Ext}^1_B(T_i,X) = \bigoplus_{i=1}^{r-1} D\operatorname{Hom}_B(X, \tau T_i) = 0$ by construction. Hence $X_B \in \mathcal{T}(T_B)$.

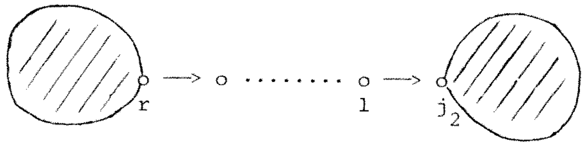
In fact, we have

$$\mathcal{F}(T_B) = \{X_B \mid X_j = 0 \text{ for all } r \leq j \leq n\}$$

and $\quad T(T_B) = \{X_B \mid X_j \neq 0 \text{ for some } r \leq j \leq n\}$.

c) The bounden quiver of $A = \operatorname{End} T_B$ satisfies (κ):

Indeed, Q_A has the following form

Q_A'  Q_A''

where $\bar{Q}_A' = \bar{Q}'$, $\bar{Q}_A'' = \bar{Q}''$ and all arrows of Q', Q'' are reversed. For, the points lying inside Q_A' or Q_A'' correspond to indecomposable projectives, and thus, we must only reverse the arrows. Obviously there are no maps from $P(r)/P(i)$ $(1 \leq i \leq r-1)$ to projectives corresponding to points in Q' and similarly no maps from projectives corresponding to points in Q'' to $P(r)/P(i)$ $(1 \leq i \leq r-1)$. On the other hand, all maps from projectives corresponding to points in Q' to $P(r)/P(i)$ $(1 \leq i \leq r-1)$ factor over $P(r)$, and all maps from $P(r)/P(i)$ $(1 \leq i \leq r-1)$ to projectives corresponding to points in Q'' must factor over $P(j_2)$. This shows that Q_A has the above form.

Next, the relations which took place inside Q' and Q'' remain the same: suppose there was a zero relation ending at r and starting in Q' at some point s , then we get obviously a zero relation $s \circ \to \circ \to \circ r$ in Q_A. If the zero-relation ending at r started in Q'' at t , we get a zero relation $1 \circ \to \underset{j_2}{\circ} \to \circ\, t$. We now claim there are no new relations. Such new relations can only start in Q' and end at some point $1 \leq i \leq r-$ or else start at some point $1 \leq i \leq r-1$ and end in Q''. Suppose

s is a point in Q' such that $\mathrm{Hom}_B(P(s),(P(r)/P(i)) = 0$, but $\mathrm{Hom}_B(P(s), P(r)) \neq 0$. Thus there is a non-zero map $f : P(s) \to P(r)$ which maps into the kernel of $P(r)/P(i)$, and this is impossible. Finally, if t is a point in Q'' such that $\mathrm{Hom}_B(P(r)/P(i),P(t)) = 0$ for some $1 \leq i \leq r-1$ but $\mathrm{Hom}_B(P(j_2), P(t)) \neq 0$, we again have that $\mathrm{Hom}_B(P(r), P(t)) = 0$ and this gives one of the zero-relations we have discussed before.

d) If m is the number of relations in Q , then the number of relations in Q_A is $m-1$.

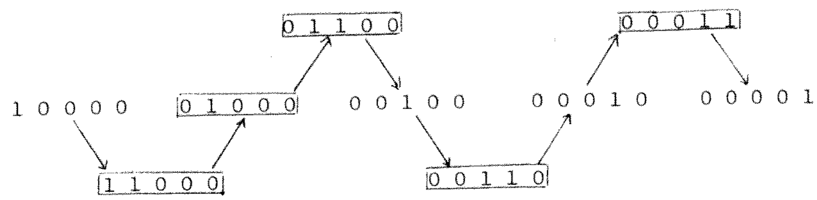
Indeed, c) shows that the number of relations in Q_A is not greater than m , but now by construction there is no relation between $r-1$ and j_2, that is, we have removed exactly one relation.

This completes the proof of the theorem. $\square$

(2.4) Proposition (2.2) and Theorem (2.3) obviously imply:

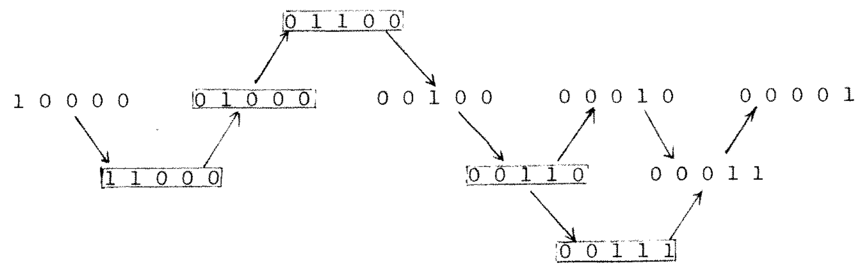
<u>Corollary</u>: If B is a generalized tilted algebra of type $\mathbb{A}_n$, then B satisfies (γ) . $\square$

(2.5) We now want to illustrate the result of Theorem (2.3) on a particular example. We write down the tilting series using the method of the proof in the theorem. We start with the bounden quiver $\circ \leftarrow \circ \leftarrow \circ \leftarrow \circ \leftarrow \circ$, with corresponding Auslander-Reiten quiver, where the indecomposable modules are denoted by their dimension types. The summands of the tilting module are encircled:



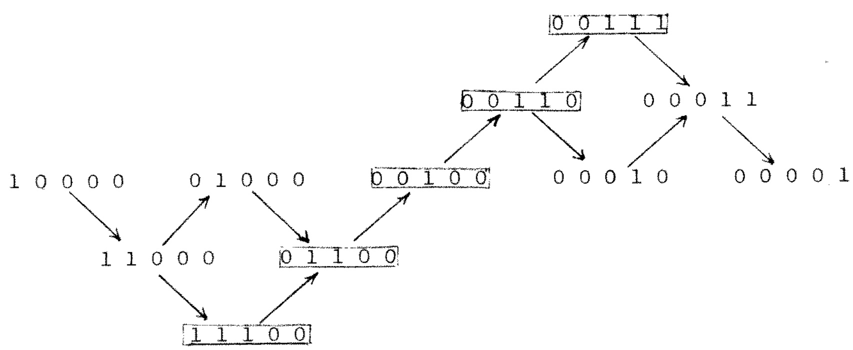
Then the endomorphism ring of the chosen tilting module is given by

$\circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ$ with Auslander-Reiten quiver:



Thus the endomorphism ring of the chosen tilting module is given by

$\circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ$ with Auslander-Reiten quiver:



It is then clear that we get in the next step the quiver

$\mathbb{A}_5 : \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ \longleftarrow \circ$.

Observe that the theorem immediately implies that generalized tilted algebras can have arbitrary global dimension. Namely, let A be the algebra $k\mathbb{A}_n/(\mathrm{rad}\, k\,\mathbb{A}_n)^2$, where in $\mathbb{A}_n$ we choose the orientation $o \longleftarrow o \longleftarrow o \;.....\; o \longleftarrow o$. Then, obviously, $\mathrm{gl.\,dim}\, A = n-1$, and A is generalized tilted of type $\mathbb{A}_n$.

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