GALOIS COVERINGS AND CHANGE OF RINGS

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Dedicated to the memory of Andrzej Skowroński.

Abstract. Given finite dimensional algebras A, B over an algebraically closed field and a morphism between them, we study when exhaustive Galois coverings of A define exhaustive Galois coverings of B. We focus on the particular cases where B is a quotient or a subalgebra of A. We apply our results to trivial extension algebras.

Introduction

Covering techniques, introduced in the eighties by Gabriel and his school, see [Rie80, BG82, Gab81], are among the best known ones in the representation theory of finite dimensional algebras. They consist, if one is given an algebra whose representation theory is hard to study, in constructing another one, called its covering and much easier to handle, such that the study of the module category of the original algebra can be reduced to that of the covering. This, of course, requires a functor from the module category of the covering to that of the original algebra. Clearly, this method is especially efficient when the latter functor is dense, that is, every indecomposable module over the original algebra can be described using one over the covering. Often, coverings are constructed using the action of a group on a locally bounded category, they are then called Galois coverings. Covering techniques were especially useful in the study of the representation-finite algebras, see [BG82] or [Gab81], but also of tame algebras, see, for instance [DS87].

The origin of this paper lies in the observation that several quotients of algebras with exhaustive Galois coverings also have exhaustive Galois coverings. More generally, we consider the situation where we have two finite dimensional algebras A, B over an algebraically closed field $\Bbbk$ with a morphism between them, and a Galois covering $\tilde{A}$ of A, and try to construct a Galois covering of B which is compatible with that of A. We look specifically at two cases, the one where B is a quotient of A and the one where it is a subalgebra of A. In the first case, we identify a compatibility condition between the covering $\tilde{A}$ and the morphism $A \to B$, which we call liftability, see 2.1 below. Our main theorem is the following.

Theorem. Let A be a finite dimensional algebra over an algebraically closed field, $F\colon \tilde{A} \to A$ a Galois covering with group G and I a liftable ideal. If $B = A/I$, then there exists an induced Galois covering $F'\colon \tilde{B} \to B$ with the same group G such that we have a commutative square of locally bounded categories and functors

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & \tilde{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ A & \longrightarrow & B \end{array}$$

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where the horizontal arrows are projection functors. Moreover, if $\tilde{A}$ is exhaustive, then so if $\tilde{B}$.

In case B is a subalgebra of A, we have to make an additional hypothsesis on B, which we express by saying that B is well-behaved, see 3.1 below. Under this hypothesis and assuming that the class of morphisms of B is liftable in the sense seen before, we are also able to construct an induced covering of B, compatible with that of A, this is the proposition in 3.2 below.

We then proceed to apply our results to the case of trivial extension algebras. Indeed, a trivial extension algebra has a well-known Galois covering with group $\mathbb{Z}$, called its repetitive covering, see [HW83]. We prove the following: let C be a finite dimensional algebra and E a $C-C$-bimodule, then any ideal I of $A = C \ltimes E$ contained in E is liftable and the induced covering of $B = A/I$ coincides with the repetitive covering of B. Similarly, $B = C \ltimes I$ is a well-behaved subalgebra of A whose class of morphisms is liftable, and its induced covering again coincides with the repetitive covering. As was to be expected, the situation gets especially nice when I is a direct summand of E, leading us to an application to partial relation extensions, see [ABD+19].

We describe the contents of the paper. After a brief section 1 devoted to recalling the basic notions and results needed later on, we consider in section 2 the case where B is a quotient of A by a liftable ideal and in section 3 the case where B is a well-behaved subalgebra of A with liftable morphisms. Sections 4 and 5 are devoted to the application to trivial extensions.

1. Preliminaries

1.1. **Categories and modules.** Throughout, $\mathbb{k}$ denotes an algebraically closed field. We recall that a $\mathbb{k}$-*category* A is a category whose morphism sets $A(x, y)$ are $\mathbb{k}$-vector spaces such that the composition is bilinear, see [BG82, (2.1)]. We denote by A_o the object class of A. A $\mathbb{k}$-category A is *locally bounded* if

(a) For any $x \in A_o$, the algebra $A(x, x)$ is local.
(b) Distinct objects are not isomorphic.
(c) For any $x \in A_o$, we have $\sum_{y\in A_o} \dim_{\mathbb{k}} A(x, y) < \infty$ and $\sum_{y\in A_o} \dim_{\mathbb{k}} A(y, x) < \infty$.

A functor between locally bounded categories is $\mathbb{k}$-*linear* if it induces a $\mathbb{k}$-linear map between the morphism sets.

As shown in [BG82, (2.1)], every locally bounded catgory A is of the form $A \cong \mathbb{k}Q/I$, where $\mathbb{k}Q$ is the path category of a locally finite quiver Q, and I is an admissible ideal of $\mathbb{k}Q$. A basic finite dimensional $\mathbb{k}$-algebra A can be considered as a locally bounded category: fix a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents, take as object set $A_o = \{1, 2, \dots, n\}$ and for $i, j \in A_o$, let $A(i, j) = e_i A e_j$, the compositions being induced by the multiplication of A.

Let A be a locally bounded category, a (finite dimensional) *right* A-*module* M is a $\mathbb{k}$-linear functor from A to the category of finite dimensional vector spaces. We denote their category by $\operatorname{mod} A$, and let $\operatorname{ind} A$ be a full subcategory of $\operatorname{mod} A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable A-modules. The Auslander-Reiten quiver of A is denoted by $\Gamma(\operatorname{mod} A)$. For further notions of representation theory, we refer the reader to [ASS06, ARS95].

1.2. **Galois coverings.** Let $\tilde{A}$, A be locally bounded categories. A $\mathbb{k}$-linear functor $F\colon \tilde{A} \to A$ is *covering functor* [BG82, (3.1)] if, for any $a, b \in A_o$, the maps

$$\bigoplus_{z\in F^{-1}(a)} \tilde{A}(x, z) \to A(Fx, a) \quad \text{and} \quad \bigoplus_{z\in F^{-1}(b)} \tilde{A}(y, x) \to A(b, Fx)$$

induced by F are bijective for any $x \in \tilde{A}_0$.

Let $F\colon \tilde{A} \to A$ be a covering functor and G a group of automorphisms of $\tilde{A}$. Then F is called a *Galois covering* [Gab81, (3.1)] if:

(a) G acts freely on $\tilde{A}$.
(b) F is G-invariant: $Fg = F$ for every $g \in G$.
(c) F is surjective on objects and G acts transitively of the fibre $F^{-1}a$ for each $a \in A_o$.

The following notation is used for the action of G on $\tilde{A}$: if $g \in G$ and $x \in \tilde{A}$ then gx denotes the action of g on x, and if u is a morphism in $\tilde{A}$ then ${}^g u$ denotes the action of g on u.

To a covering functor $F\colon \tilde{A} \to A$, we associate its *pushdown functor* [BG82, (3.2)] $F_\lambda\colon \operatorname{mod} \tilde{A} \to \operatorname{mod} A$ as follows: to an $\tilde{A}$-module M, we assign the A-module $F_\lambda M$ defined by

$$(F_\lambda M)(a) = \bigoplus_{x \in F^{-1}a} M(x)$$

for every $a \in A_o$ and, if $\alpha\colon a \to b$ is a morphism on A and, for each $x \in F^{-1}a$, $y \in F^{-1}b$, α_{yx} denotes the morphism in $\tilde{A}(x,y)$ such that $\alpha = \sum_y F(\alpha_{yx})$, then $F_\lambda M(\alpha)\colon F_\lambda M(a) \to F_\lambda M(b)$ sends $(m_x)_x$, which lies in $\oplus_{x\in F^{-1}a} M(x)$ to $\left(\sum_x M(\alpha_{yx})(m_x)\right)_y$, which lies in $\oplus_{y \in F^{-1}b} M(y)$.

For properties of F_λ, we refer to [BG82, (3.2)]. A Galois covering $F\colon \tilde{A} \to A$ is called *exhaustive* when the pushdown functor F_λ is dense [DS87].

1.3. **Trivial extensions.** Let C be a finite dimensional algebra and ${}_C E_C$ a finite dimensional $C-C$-bimodule equipped with an associative and bilinear product $E \otimes_C E \to E$ denoted as $e \otimes e' \mapsto ee'$ (for $e, e' \in E$). The *split extension* of C by E is the $\Bbbk$-algebra with underlying vector space $A = C \oplus E$ and multiplication defined for $(c,e),(c',e') \in A$ by

$$(c,e)\,(c',e') = (cc', ce' + ec' + ee')\,.$$

If E is nilpotent for its product, then A is called a *split-by-nilpotent extension* and, if $E^2 = 0$, it is called a trivial extension and denoted by $A = C \ltimes E$, see [ACT08].

Assume that A is the trivial extension of C by the $C-C$-bimodule E. The *repetitive category* $\hat{A}$ of $A = C \ltimes E$ is the locally bounded category with objects set $\tilde{A}_o = A_o \times \mathbb{Z}$ and, for any $(a,i),(b,j) \in \hat{A}_o$, we have

$$\hat{A}((a,i),(b,j)) = \begin{cases} C(a,b) & \text{if } j = i \\ E(a,b) & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\hat{A}$ can be represented as an algebra of matrices as

$$\hat{A} = \begin{pmatrix} \ddots & \ddots & & & 0 \\ \ddots & C_{i-1} & 0 & & \\ \ddots & E_i & C_i & 0 & \\ & 0 & E_{i+1} & C_{i+1} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}$$

where matrices have only finitely many nonzero elements, $C_i = C$ and $E_i = E$ for all $i \in \mathbb{Z}$, all remaining coefficients are zero, addition is the usual matrix addition and multiplication is induced from that of C, the $C-C$-bimodule structure of E and the zero maps $E \otimes_C E \to 0$. The identity maps $C_i \to C_{i+1}$, $E_i \to E_{i+1}$ induce an automorphism φ of $\hat{A}$. The orbit category $\hat{A}/\langle\varphi\rangle$ inherits from $\hat{A}$ a $\Bbbk$-algebra structure, easily seen to be isomorphic to A. The projection functor $F\colon \hat{A} \to A$ is a Galois covering with infinite cyclic group generated by φ, see [HW83]. It is called the *repetitive Galois covering.* For simplicity, $\hat{A}$ is called exhaustive whenever so is the repetitive covering. Notice that we depart from standard notation due a more general context. Two cases have been studied extensively, those where $E = {}_C DC_C$, where $D = \mathrm{Hom}_{\mathbb{k}}(-, \mathbb{k})$ is the standard duality, see for instance [HW83, AS93] and those where $E = \mathrm{Ext}^2_C(DC, C)$, with C a tilted algebra, see [ABS08].

2. Quotients

2.1. **Liftable sets.** Let A, B be finite dimensional algebras and $f\colon A \to B$ a surjective algebra morphism. Assume that there exist two complete sets of pairwise orthogonal primitive idempotents of A and B, respectively, such that f maps the ones in A onto the ones in B. Looking at A and B as locally bounded categories by means of these sets, then f can be considered as a functor.

Note that the image under f of any complete set of pairwise orthogonal idempotents of A is a complete set of pairwise orthogonal idemptents of B. Moreover if the former consists of primitive idempotents, then the latter consists of idempotents which are zero or primitive. Indeed, for any primitive idempotent $e \in A$, the surjective algebra morphism f induces a surjective algebra morphism $eAe/\mathrm{rad}(eAe) \to f(e)Bf(e)/\mathrm{rad}(f(e)Bf(e))$ whose domain is a skew field, hence $f(e)$ is zero or is a primitive idempotent.

The objective of this section is the following. Let $F\colon \tilde{A} \to A$ be a Galois covering with group G. We wish to construct a Galois covering $F'\colon \tilde{B} \to B$ with the same group G such that $\tilde{B}$ is a quotient of the locally bounded category $\tilde{A}$ and the projection functor $\tilde{f}\colon \tilde{A} \to \tilde{B}$ is compatible with f, in the sense that we have a commutative diagram of locally bounded categories and functors

$$\begin{array}{ccc} \tilde{A} & \overset{\tilde{f}}{\dashrightarrow} & \tilde{B} \\ {\scriptstyle F}\downarrow & & \vdots\, {\scriptstyle F'} \\ A & \xrightarrow{f} & B \end{array}$$

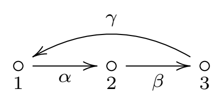
We need a compatibility condition between F and f, which we call liftability. A *morphism set* $\mathcal{M}$ in A is defined by the data, for each pair $a, b \in A_o$, of a $\mathbb{k}$-subspace $\mathcal{M}(a, b)$ of $A(a, b)$. For instance, any ideal in A is a morphism set.

Definition. Let $F\colon \tilde{A} \to A$ be a Galois covering with group G. A morphism set $\mathcal{M}$ in A is called F*-liftable* if, for any $x, y \in \tilde{A}_o$ and $u \in \mathcal{M}(Fx, Fy)$, then, if $(u_g)_{g\in G} \in \oplus_{g\in G}\tilde{A}(x, gy)$ is such that $u = \sum_{g\in G} F(u_g)$, we have $F(u_g) \in \mathcal{M}(Fx, Fy)$.

In the sequel, the Galois covering F with respect to which liftability is defined will always be clear from the context. Therefore we simply say "liftable" instead of "F-liftable".

Because of the G-invariance of G, liftability is equivalent to the condition obtained by replacing "for any" by "there exists" in its definition.

Example. *Let A be given by the quiver*



bound by $\mathrm{rad}^2 A = 0$ *and* $F\colon \tilde{A} \to A$ *be the Galois covering with group* $\mathbb{Z}$ *given by the infinite quiver*

$$\cdots \to \underset{1}{\circ} \xrightarrow{\alpha} \underset{2}{\circ} \xrightarrow{\beta} \underset{3}{\circ} \xrightarrow{\gamma} \underset{1}{\circ} \xrightarrow{\alpha} \underset{2}{\circ} \xrightarrow{\beta} \underset{3}{\circ} \xrightarrow{\gamma} \underset{1}{\circ} \xrightarrow{\alpha} \cdots$$

also bound by $\operatorname{rad}^2 \tilde{A} = 0$*. The two-sided ideal* $A\gamma A$ *generated by the arrow* γ *is clearly liftable.*

In this example, it suffices, in order to check the liftability of $A\gamma A$, to look only at its generator γ. This is a general fact.

Lemma. *Let* $F\colon \tilde{A} \to A$ *be a Galois covering. A two-sided ideal* I *in* A *is liftable if and only if there exists a generating set* R *of* I *such that*

$$R \subseteq \bigcup_{x,y\in\tilde{A}_o} F(\tilde{A}(x,y))\,.$$

Proof. Necessity is easy. Indeed, let I be liftable and set

$$R = \bigcup_{g\in G,\ x,y\in\tilde{A}_o} \{F(u_g) \mid u \in I(Fx,Fy)\}$$

where, as before, $(u_g)_{g\in G} \in \oplus_{g\in G}\tilde{A}(x,gy)$ is such that $u = \sum_{g\in G} F(u_g)$. Then $R \subseteq \cup_{x,y\in\tilde{A}_o} F(\tilde{A}(x,y))$. Because I is liftable, $R \subseteq I$ and moreover, every element of I is a sum of compositions of elements of R.

We prove sufficiency. Assume I satisfies the stated condition, choose $x, y \in \tilde{A}_o$ and $u \in I(Fx,Fy)$. Then there exists a finite family of pairs of morphisms of A, denoted as $((\alpha_r,\beta_r))_{r\in R}$ such that, for every $r \in R$, there exist

- an object $a_r \in A_o$ such that $\alpha_r \in A(Fx,a_r)$ and
- an object $b_r \in A_o$ such that $\beta_r \in A(b_r,Fy)$

so that we have $u = \sum_{r\in R} \beta_r r\alpha_r$.

For $r \in R$, take $x_r \in F^{-1}a_r$ and let $(\alpha_{r,g})_{g\in G} \in \oplus_{g\in G}\tilde{A}(x,gx_r)$ be such that $\alpha_r = \sum_{g\in G} F(\alpha_{r,g})$. Because of the hypothesis on R, r is the image under F of a morphism in $\tilde{A}$. Because the fibre $F^{-1}a_r$ equals the G-orbit of x_r and F is G-invariant, there exist $y_r \in F^{-1}b_r$ and $\tilde{r} \in \tilde{A}(x_r,y_r)$ such that $r = F(\tilde{r})$. Let also $(\beta_{r,g})_{g\in G} \in \oplus_{g\in G}\tilde{A}(y_r,gy)$ be defined by $\beta_r = \sum_{g\in G} F(\beta_{r,g})$. Then, for each $r \in R$,

$$\begin{aligned} \beta_r r\alpha_r &= (\sum_{g\in G} F(\beta_{r,g}))F(\tilde{r})(\sum_{g\in G} F(\alpha_{r,g})) \\ &= \sum_{g,h\in G} F(\beta_{r,h})F(\tilde{r})F(\alpha_{r,g}) \\ &= \sum_{g,h\in G} F({}^{g}\beta_{r,h}\,{}^{g}\tilde{r}\,\alpha_{r,g})\,. \end{aligned}$$

Setting $g = kh^{-1}$, or equivalently $k = gh$, we get

$$\beta_r r\alpha_r = \sum_{k\in G} F(\sum_{h\in G} {}^{kh^{-1}}\beta_{r,h}\,{}^{kh^{-1}}\tilde{r}\,\alpha_{r,kh^{-1}})$$

where the sum in parenthesis lies in $\tilde{A}(x,ky)$. Thus we have

$$\begin{aligned} u &= \sum_{r\in R} \beta_r r\alpha_r \\ &= \sum_{r\in R}\sum_{k\in G} F(\sum_{h\in G} {}^{kh^{-1}}\beta_{r,h}\,{}^{kh^{-1}}\tilde{r}\,\alpha_{r,kh^{-1}}) \\ &= \sum_{k\in G} F(\sum_{h\in G,\,r\in R} {}^{kh^{-1}}\beta_{r,h}\,{}^{kh^{-1}}\tilde{r}\,\alpha_{r,kh^{-1}})\,. \end{aligned}$$

Now, u can be written as $u = \sum_{k\in G} F(u_k)$, with $(u_k)_{k\in G} \in \oplus_{k\in G}\tilde{A}(x,ky)$. Because F is a Galois covering, this implies

$$u_k = \sum_{h\in G,\,r\in R} {}^{kh^{-1}}\beta_{r,h}\,{}^{kh^{-1}}\tilde{r}\,\alpha_{r,kh^{-1}}\,.$$

Then

$$F(u_k) = \sum_{h\in G, r\in R} F(\beta_{r,h})F(\tilde{r})F(\alpha_{,kh^1})$$

belongs to $I(Fx, Fy)$ because $F(\tilde{r}) = r \in I(Fa_r, Fb_r)$. □

2.2. **Lifted ideal.** We proceed to lift a liftable ideal to the covering.

Lemma. *Let I be a liftable ideal and, for each $x, y \in \tilde{A}_o$, let*

$$\tilde{I}(x,y) = \{u \in \tilde{A}(x,y) \mid Fu \in I(Fx, Fy)\}\,.$$

Then $\tilde{I} = \cup_{x,y\in\tilde{A}_o}\tilde{I}(x,y)$ is an ideal in $\tilde{A}$ such that $F(\tilde{I}) = I$ and, for each $x, y \in \tilde{A}_o$, the map $\oplus_{g\in G}\tilde{I}(x, gy) \to I(Fx, Fy)$ induced by F is bijective.

Proof. The definition of $\tilde{I}$ says that it is the preimage of I under the functor F. Because I is an ideal in A, so is $\tilde{I}$ in $\tilde{A}$. Moreover, $F(\tilde{I}) \subseteq I$. Because I is liftable, the map $\oplus_{g\in G}\tilde{I}(x, gy) \to I(Fx, Fy)$ induced by F is surjective. It is bijective because F is a Galois covering. □

The ideal $\tilde{I}$ will be called the *lifting* of I.

2.3. **The induced covering.** We prove that a Galois covering $F\colon \tilde{A} \to A$ and a liftable ideal I of A induce together a Galois covering $F'\colon \tilde{A}/\tilde{I} \to A/I$ compatible with F.

Proposition. *Let $F\colon \tilde{A} \to A$ be a Galois covering with group G and I a liftable ideal of A. Then there exists an ideal $\tilde{I}$ of $\tilde{A}$ and a Galois covering $F'\colon \tilde{A}/\tilde{I} \to A/I$ with group G such that the square*

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & \tilde{A}/\tilde{I} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ A & \longrightarrow & A/I \end{array}$$

where the horizontal arrows are the projections, commutes.

Proof. Let C denote the full subcategory of A whose object set is $C_o = \{a \in A_o \mid e_a \notin I(a,a)\}$. We identify A/I with the quotient category C/I, that is, the quotient of C whose morphism sets are $(C/I)(a,b) = C(a,b)/I(a,b)$ for $a, b \in C_o$. Let $\tilde{I}$ be the lifting of I, as constructed in 2.2, and $\tilde{C}$ the full subcategory of $\tilde{A}$ with object set $\tilde{C}_o = \{x \in \tilde{A}_o \mid e_x \notin \tilde{I}(x,x)\}$. Then we can identify similarly $\tilde{A}/\tilde{I}$ with the quotient category $\tilde{C}/\tilde{I}$.

Let $x \in \tilde{C}_o$, then, because of the definition of $\tilde{I}$ as the preimage of I under F, the morphism e_{Fx} $(= Fe_x)$ does not belong to $I(Fx, Fx)$. and so $Fx \in C_o$. This shows that the functor $F\colon \tilde{A} \to A$ induces a functor $F_1\colon \tilde{C} \to C$. Moreover, because $F\tilde{I} = I$, see the lemma in 2.2, the functor F_1 induces a functor $F'\colon \tilde{A}/\tilde{I} \to A/I$. We now prove that F' is a Galois covering with group G.

Because of the definition of $\tilde{I}$ and because F is G-invariant, the ideal $\tilde{I}$ is G-stable. In particular, $\tilde{C}_o$ is G-stable. Therefore the free action of G on $\tilde{A}$ induces a free action of G on $\tilde{C}$. Because $\tilde{I}$ is G-stable, this action induces in turn a free action of G on $\tilde{A}/\tilde{I}$.

Because F is G-invariant and F_1, F' are induced by F, we get that F_1 and F' are G-invariant.

Let $a \in (A/I)_o$. Then $a \in A_o$ and $e_a \notin I(a,a)$. Because F is a Galois covering, there exists $x \in \tilde{A}_o$ such that $Fx = a$ and the fibre $F^{-1}a$ is actually the G-orbit of x. Let $g \in G$. Then $e_{gx} = {}^{g}e_x \notin \tilde{I}(gx, gx)$ because $\tilde{I}$ is G-stable and $gx \notin \tilde{C}_o$ so that $gx \in (\tilde{A}/\tilde{I})_o$. Finally, $F'(gx) = F(gx) = a$. That is, the fibre $(F')^{-1}a$ equals the G-orbit of x.

There remains to prove that F' is a covering functor. Let $x, y \in (\tilde{A}/\tilde{I})_o$, then we have a commutative diagram with exact rows.

$$\begin{array}{ccccccc}
\bigoplus_{g\in G} \tilde{I}(x,gy) & \longrightarrow & \bigoplus_{g\in G} \tilde{A}(x,gy) & \longrightarrow & \bigoplus_{g\in G} \tilde{A}/\tilde{I}(x,gy) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
I(Fx,Fy) & \longrightarrow & A(Fx,Fy) & \longrightarrow & A/I(Fx,Fy) & \longrightarrow & 0
\end{array}$$

where the vertical arrows are induced by F. Now the vertical arrow in the middle is bijective, because F is a covering functor, and the one on the left is also bijective, because of the lemma in 2.2. Therefore, the dotted arrow on the right is bijective.

We have established that F' is a Galois covering with group G. The commutativity of the square in the statement follows from the very construction of F'. □

2.4. **Pushdowns and changes of rings.** The module category over a locally bounded $\Bbbk$-category admits tensor products. This indeed follows from general results, see, for instance, [Fre64, p. 84]. The exactness of the pushdown functor implies the following easy "locally bounded version" of Watts theorem.

Lemma. *Let A be a finite dimensional algebra and $F\colon \tilde{A} \to A$ a Galois covering. Then the associated pushdown functor F_λ is isomorphic to the functor $- \otimes_{\tilde{A}} A$.*

Proof. We first give a left $\tilde{A}$-module structure to the algebra A. The functor F_λ induces an algebra morphism

$$\varphi\colon \tilde{A} \simeq \operatorname{End} \tilde{A}_{\tilde{A}} \to \operatorname{End} F_\lambda(\tilde{A}_{\tilde{A}}) \simeq \operatorname{End} A_A \simeq A\,.$$

We may then define, for $\tilde{a} \in \tilde{A}$ and $a \in A$

$$\tilde{a} \cdot a = \varphi(\tilde{a})a\,.$$

Let now M be an arbitrary $\tilde{A}$-module and

$$P_1 \to P_0 \to M \to 0$$

a projective resolution in $\operatorname{mod} \tilde{A}$. Applying the right exact functor F_λ and $- \otimes_{\tilde{A}} A$, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
F_\lambda(P_1) & \longrightarrow & F_\lambda(P_0) & \longrightarrow & F_\lambda(M) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \\
P_1 \otimes_{\tilde{A}} A & \longrightarrow & P_0 \otimes_{\tilde{A}} A & \longrightarrow & M \otimes_{\tilde{A}} A & \longrightarrow & 0
\end{array}$$

because, if e_x is an idempotent in $\tilde{A}$, then $F_\lambda(e_x\tilde{A}) = e_{Fx}A$, see [BG82, (3.2)]. This implies the statement. □

2.5. **Proof of our theorem.** We are now able to state and prove our theorem.

Theorem. *Let A be a finite dimensional algebra, $F\colon \tilde{A} \to A$ a Galois covering with group G and I a liftable ideal of A. If $B = A/I$, then there exists a Galois covering $F'\colon \tilde{B} \to B$ with group G and a commutative square of locally bounded categories and functors*

$$\begin{array}{ccc}
\tilde{A} & \longrightarrow & \tilde{B} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\
A & \longrightarrow & B
\end{array}$$

where the horizontal arrows are the projections. Moreover, if F is exhaustive, then so is F'.

Proof. Let $\tilde{I}$ be constructed as in the lemma in 2.2, and $\tilde{B} = \tilde{A}/A$ as in the proposition in 2.3. Then the first statement follows directly from that proposition, so we just have to prove the second one. Because $\tilde{B} = \tilde{A}/\tilde{I}$, we have a change of rings functor $- \otimes_{\tilde{A}} \tilde{B} \colon \operatorname{mod} \tilde{A} \to \operatorname{mod} \tilde{B}$, hence a diagram of module categories and functors

$$\begin{array}{ccc} \operatorname{mod} \tilde{A} & \xrightarrow{-\otimes_{\tilde{A}} \tilde{B}} & \operatorname{mod} \tilde{B} \\ {\scriptstyle F_\lambda}\downarrow & & \downarrow{\scriptstyle F'_\lambda} \\ \operatorname{mod} A & \xrightarrow{-\otimes_A B} & \operatorname{mod} B \end{array}$$

This diagram commutes because $F_\lambda \simeq - \otimes_{\tilde{A}} A$ and $F'_\lambda \simeq - \otimes_{\tilde{B}} B$, see the lemma in 2.4, so both compositions in the diagram equal $- \otimes_{\tilde{A}} B$. If F is exhaustive then F_λ is dense. Because $B = A/I$, the functor $- \otimes_A B$ is also dense so $(- \otimes_A B)F_\lambda$ is dense. Hence so is F'_λ and we are done. □

2.6. **Consequences.** We list some consequences of the theorem.

Corollary. *Under the assumptions of the theorem, if we assume moreover that G acts freely on $\operatorname{ind} \tilde{A}$, then:*

(a) G acts freely on $\operatorname{ind} \tilde{B}$.
(b) F'_λ sends any almost split sequence in $\operatorname{mod} \tilde{B}$ to an almost split sequence in $\operatorname{mod} B$.
(c) F'_λ induces an isomorphism between $\Gamma(\operatorname{mod} \tilde{B})/G$ and a subquiver of $\Gamma(\operatorname{mod} B)$ consisting of connected components. If F is exhaustive, then $\Gamma(\operatorname{mod} \tilde{B})/G \simeq \Gamma(\operatorname{mod} B)$.
(d) If moreover $\tilde{A}$ is locally support-finite, then $\tilde{B}$ is tame if and only if so is B.

Proof. (a) Indeed, because $\tilde{B} = \tilde{A}/\tilde{I}$ and $\tilde{I}$ is G-stable, $\operatorname{mod} \tilde{B}$ is canonically isomorphic of a G-stable full subcategory of $\operatorname{mod} \tilde{A}$, closed under direct sums and summands.

(b) This follows from [Gab81, (3.6)] and (a).

(c) This follows from [Gab81, (3.6)] and (a), and also the theorem in 2.5.

(d) Because $\operatorname{mod} \tilde{B}$ is a full subcategory of $\operatorname{mod} \tilde{A}$ closed under direct sums and summands, $\tilde{A}$ being locally support-finite implies $\tilde{B}$ being locally support-finite. Hence, because of (a) and [DLS86], we have that $\tilde{B}$ is tame if and only if so is B. □

3. Subalgebras

3.1. **Well-behaved subalgebras.** We now consider a situation in some sense dual to that of section 2. We let A, B be finite dimensional algebras and $f \colon B \to A$ an injective algebra morphism, that is, B is isomorphic to a subalgebra of A. We let $F \colon \tilde{A} \to A$ be a Galois covering with group G and wish to construct another Galois covering $F'' \colon \tilde{B} \to B$ with the same group G which is compatible with F, that is, there exists a functor $\tilde{f} \colon \tilde{B} \to \tilde{A}$ and a commutative diagram of locally bounded categories and functors

$$\begin{array}{ccc} \tilde{B} & \xrightarrow{\tilde{f}} & \tilde{A} \\ {\scriptstyle F''}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow{f} & A \end{array}$$

Because subalgebras are harder to handle than quotients, we need additional assumptions.

Definition. A subalgebra B of a finite dimensional algebra A is called *well-behaved* if:

(a) For every $a \in A_o$, we have $e_a \in B_o$, and
(b) For every $a, b \in A_o$, the inclusion $B \to A$ induces an injective $\Bbbk$-linear map

$$\frac{\operatorname{rad} B(a,b)}{\operatorname{rad}^2 B(a,b)} \to \frac{\operatorname{rad} A(a,b)}{\operatorname{rad}^2 A(a,b)}\,.$$

For instance, if A is a split extension of B by some bimodule, then it follows from [ACT08, (1.5)] that B is a well-behaved subalgebra of A.

We fix some notation. Let $F\colon \tilde{A} \to A$ be a Galois covering with group G. Because of [LM07, (4.4)], see also [BG82, (2.1)], there exist a bound quiver (Q, I), a presentation $\eta\colon \Bbbk Q/I \to A$, inducing a bijection $Q_0 \to A_o$, a Galois covering $\pi\colon (\tilde{Q}, \tilde{I}) \to (Q, I)$ of bound quivers with group G, and a presentation $\tilde{\eta}\colon \Bbbk\tilde{Q}/\tilde{I} \to \tilde{A}$ inducing a bijection $\tilde{Q}_0 \to \tilde{A}_o$, such that F equals the composition

$$\tilde{A} \xrightarrow[\sim]{\tilde{\eta}^{-1}} \Bbbk\tilde{Q}/\tilde{I} \xrightarrow{\pi^*} \Bbbk Q/I \xrightarrow[\sim]{\eta} A$$

where $\pi^*\colon \Bbbk\tilde{Q}/\tilde{I} \to \Bbbk Q/I$ is induced by π.

Lemma. *Assume B is a well-behaved subalgebra of A. If there exists a subquiver Q' of Q such that $B \simeq \Bbbk Q'/(I \cap \Bbbk Q')$, then the morphism set $\mathcal{B} = \bigcup_{a,b \in B_o} B(a,b)$ is liftable.*

Proof. Let $I' = I \cap \Bbbk Q'$ and identify $\Bbbk Q'/I'$ with $\{u + I \mid u \in \Bbbk Q'\}$. For each pair of objects $a, b \in B_o$ and morphism $u \in B(a,b)$, there exist paths $u_1, \ldots, u_m$ in Q' and scalars $\lambda_1, \ldots, \lambda_m$ such that

$$u = \sum_{i=1}^{m} \lambda_i\,(u_i + I)\,.$$

Assume $x \in F^{-1}a$, $y \in F^{-1}b$ then there exist, for each i with $1 \leqslant i \leqslant m$, a unique $g_i \in G$ and a unique path $\tilde{u}_i$ in $\tilde{Q}$ from x to ${}^{g_i}y$ such that $\pi(\tilde{u}_i) = u_i$. Then

$$u = \sum_{i=1}^{m} \lambda_i F(\tilde{u}_i + \tilde{I})\,.$$

Because F is a Galois covering, letting $(v_g)_{g\in G} \in \oplus_{g\in G}\tilde{A}(x, gy)$ be such that $u = \sum_{g\in G} F(v_g)$, we have, for every $g \in G$

$$v_g = \sum_{g_i = g} \lambda_i\,(\tilde{u}_i + \tilde{I})\,,$$

where the sum is taken over all i such that $g_i = g$. Therefore

$$F(v_g) = \sum_{g_i = g} \lambda_i\,(u_i + I)\,.$$

Because u_i is a path in Q', we have $u_i + I \in B$. Hence $F(v_g) \in B(a,b)$. □

3.2. **Lifting of well-behaved subalgebras.** Let B be a well-behaved subalgebra of A and $F\colon \tilde{A} \to A$ a Galois covering with group G such that the morphism set $\mathcal{B} = \bigcup_{a,b\in B_o} B(a,b)$ is liftable. We define the *lifting* $\tilde{B}$ of B to be the (non-full) subcategory of $\tilde{A}$ such that $\tilde{B}_o = \tilde{A}_o$ and, for $x, y \in \tilde{A}_o$,

$$\tilde{B}(x,y) = \{u \in \tilde{A}(x,y) \mid Fu \in B(Fx, Fy)\}\,.$$

Because $\tilde{B}$ is a subcategory of $\tilde{A}$, the functor F induces a functor $F''\colon \tilde{B} \to B$ by restriction and corestriction.

Proposition. *With the above notation, $F''\colon \tilde{B} \to B$ is a Galois covering with group G such that the square*

$$\begin{array}{ccc} \tilde{B} & \hookrightarrow & \tilde{A} \\ {\scriptstyle F''}\downarrow & & \downarrow{\scriptstyle F} \\ B & \hookrightarrow & A \end{array}$$

where the horizontal functors are inclusions, commute.

Proof. By definition, $\tilde{B}$ is a G-stable subcategory of $\tilde{A}$. In particular, the free action of G on $\tilde{A}$ induces a free action of G on $\tilde{B}$. Moreover, for this action, F'' is G-invariant. Because $\tilde{A}_o = \tilde{B}_o$, $A_o = B_o$ and F'' is restricted from F, the fibres of objects in B coincide with the G-orbits in $\tilde{B}_o$. Let $x, y \in \tilde{A}_o$. Because F'' is restricted from F, the following square, where the vertical arrows are induced from F and the horizontal ones are inclusions, commutes

$$\begin{array}{ccc} \bigoplus_{g\in G} \tilde{B}(x, gy) & \hookrightarrow & \bigoplus_{g\in G} \tilde{A}(x, gy) \\ \downarrow & & \downarrow \\ B(Fx, Fy) & \hookrightarrow & A(Fx, Fy) \end{array}$$

Because the vertical right arrow is bijective, the left one is injective. Let $u \in B(Fx, Fy)$. We have assumed that u is liftable. That is, if $(u_g)_g \in \oplus_{g\in G}\tilde{A}(x, gy)$ is such that $u = \sum_{g\in G} F(u_g)$ then we have $F(u_g) \in B(Fx, Fy)$. But this implies $u_g \in \tilde{B}(x, y)$ due to the definition of $\tilde{B}$. This shows that the vertical left arrow is surjective and hence bijective. Therefore F'' is a covering functor, and thus a Galois covering.

The commutativity of the square in the statement follows directly from the definitions of $\tilde{B}$ and F''. □

4. Trivial extensions and quotients

4.1. **The setting.** Let C be a finite dimensional algebra, E a $C-C$-bimodule and $A = C \ltimes E$. If I is a $C-C$-subbimodule of E, then I is an ideal of A. Indeed, if $(0, x) \in I$ and $(c, e) \in A$ then $(c, e)(0, x) = (0, cx) \in I$ and similarly $(0, x)(c, e) \in I$. Because the converse is obvious, the $C-C$-subbimodules of E coincide with the two-sided ideals of A contained in E. Such subbimodules determine new trivial extensions.

Lemma. *Let $A = C \ltimes E$ and I a $C-C$-subbimodule of E, then $A/I \simeq C \ltimes E/I$.*

Proof. We have a short exact sequence

$$0 \to E/I \to A/I \to C \to 0\,.$$

Moreover, the direct sum decomposition $A = C \oplus E$ as $C-C$-bimodules and the fact that $I \subseteq E$ imply that $A/I \simeq C \oplus E/I$ as $C-C$-bimodules. □

From now on, let I be an ideal of A contained in E. We set $B = A/I$. Each of A, B has a repetitive covering. We wish to compare the repetitive covering $\hat{B} \to B$ with the covering constructed in the proposition of 2.3 starting from the repetitive covering $\hat{A} \to A$ and the injective algebra morphism $A \to B$.

4.2. **Liftability.** We prove that any subbimodule of E is liftable with respect to the repetitive covering $\hat{A} \to A$.

Lemma. *Let $A = C \ltimes E$ and I a $C-C$-subbimodule of E. Then I is liftable with respect to the repetitive covering of A.*

Proof. Let $F\colon \hat{A} \to A$ denote the repetitive covering, and $a, b \in A_o$. Because of the definition of $\hat{A}$, we have $E(a,b) = \hat{A}((a,0),(b,1))$ and the map $\hat{A}((a,0),(b,1)) \to A(a,b)$ induced from F is the inclusion morphism $E(a,b) \hookrightarrow A(a,b)$. Let $u \in I(a,b)$. Because $I \subseteq E$, the element u of $I(a,b)$ defines a morphism $v \in \hat{A}((a,0),(b,1))$ such that $u = F(v)$. □

4.3. **Lifting the quotient.** With the above notation, let $\hat{I}$ denote the lifting of I as defined in 2.2. As shown in the proposition in 2.3, we have a Galois covering $F'\colon \hat{A}/\hat{I} \to A/I$ with the same group $\mathbb{Z}$ as $F\colon \hat{A} \to A$.

Proposition. *With the above notation, the induced Galois covering $F'\colon \hat{A}/\hat{I} \to A/I = B$ coincides with the repetitive covering $\hat{B} \to B$.*

Proof. We first show that $\hat{B} = \hat{A}/\hat{I}$. Let $(a,i),(b,j) \in \hat{A}_o$ and $u \in \hat{A}((a,i),(b,j))$. In order for u to be nonzero, we must have $j \in \{i, i+1\}$. If $j = i$, then $\hat{A}((a,i),(b,i)) = C(a,b)$ and the image Fu of u is u itself viewed as element of $C(a,b)$. If, on the other hand, $j = i+1$, then $\hat{A}((a,i),(b,i+1)) = E(a,b)$ and Fu is again u viewed as element of $E(a,b)$. It follows from the construction of $\hat{I}$ in 2.2 that $u \in \hat{I}((a,i),(b,i+1))$ if and only if $Fu = u \in I(a,b)$. Using the matrix representation of $\hat{A}$, the ideal I can be written as

$$\hat{I} = \begin{pmatrix} \ddots & & & & 0 \\ \ddots & 0 & & & \\ & I & 0 & & \\ & & I & 0 & \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

so that, clearly

$$\hat{A}/\hat{I} \cong \begin{pmatrix} \ddots & & & & 0 \\ \ddots & C & & & \\ & E/I & C & & \\ & & E/I & C & \\ 0 & & & \ddots & \ddots \end{pmatrix} \cong \hat{B}$$

as algebras and therefore as categories. This establishes our statement. That the functor $F'\colon \hat{A}/\hat{I} \to B$ coincides with the repetitive covering $\hat{B} \to B$ follows from its construction, see the proposition in 2.3. □

4.4. **Exhaustivity.** Combining the proposition in 4.3 with the theorem in 2.5, we get the following corollary.

Corollary. *With the above notation, if the repetitive covering $\hat{A} \to A$ is exhaustive, then so is the repetitive covering $\hat{B} \to B$.*

4.5. **Auslander-Reiten quivers.** Using the corollaries in 4.4 and 2.6 yields the next statement.

Corollary. *With the above notation, if $\mathbb{Z}$ acts freely on* $\operatorname{ind}\hat{A}$*, then:*

(a) $\mathbb{Z}$ *acts freely on* $\operatorname{ind}\hat{B}$.

(b) The pushdown functor F'_λ *sends every almost split sequence in* $\operatorname{mod}\hat{B}$ *to an almost split sequence in* $\operatorname{mod} B$.

(c) F'_λ *induces an isomorphism between* $\Gamma(\operatorname{mod}\hat{B})/\mathbb{Z}$ *and a subquiver of* $\Gamma(\operatorname{mod} B)$ *consisting of connected components. If* $\hat{A}$ *is exhaustive, then this subquiver equals* $\Gamma(\operatorname{mod} B)$.

4.6. **Selfinjective trivial extensions.** If E equals the minimal injective cogenerator ${}_C DC_C$, then the trivial extension $A = C \ltimes DC$ is selfinjective.

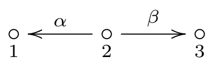
Corollary. *Let* C *be derived equivalent to a tame hereditary or to a tubular algebra,* $A = C \ltimes DC$, I *a* $C-C$*-subbimodule of* E *and* $B = A/I$. *Then*

(a) $\hat{B}$ *is tame and the repetitive covering* $\hat{B} \to B$ *is exhaustive.*

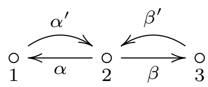
(b) The pushdown functor induces a quiver isomorphism $\Gamma(\operatorname{mod}\hat{B})/\mathbb{Z} \simeq \Gamma(\operatorname{mod} B)$.

Proof. Because of [AS93, Theorem B], the hypothesis implies that $\hat{A}$ is tame and exhaustive. Therefore $\hat{B}$ is tame because $\hat{A}$ is tame and $\hat{B} = \hat{A}/\hat{I}$. It is exhaustive because of the corollary in 4.4. Moreover, in this case, $G = \mathbb{Z}$ acts freely on $\operatorname{ind}\hat{A}$. Therefore, because of the corollary in 4.5, it acts freely on $\operatorname{ind}\hat{B}$ and the pushdown functor associated to the repetitive covering induces the required quiver isomorphism. □

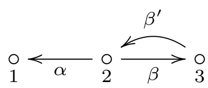
Example. *Let* C *be the path algebra of the quiver*



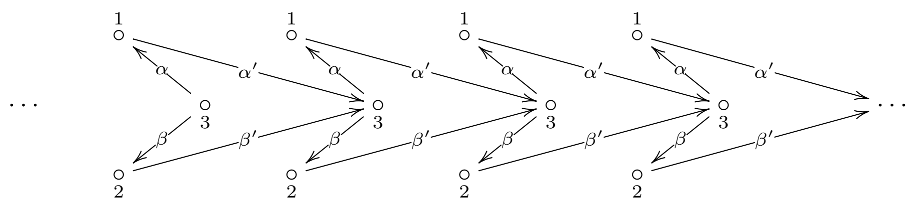
Then the trivial extension $A = C \ltimes DC$ *is given by the quiver*



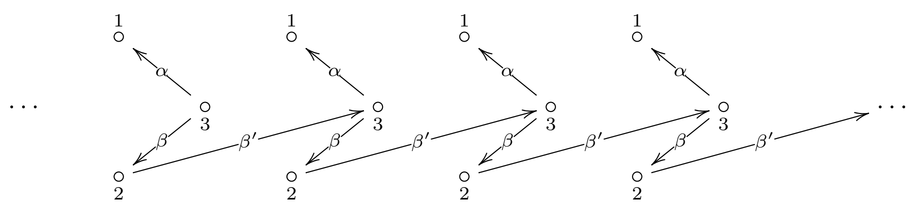
bound by $\alpha\alpha' = \beta\beta'$ *and* $\operatorname{rad}^3 A = 0$. *Let* $I = C\alpha' C$ *be generated by the arrow* α', *then* I *is a subbimodule of* DC *and* $B = A/I$ *is given by the quiver*



bound by $\beta\beta' = 0$. *The repetitive category* $\hat{A}$ *is given by the quiver*



bound by $\alpha\alpha' = \beta\beta'$ *and* $\operatorname{rad}^3\hat{A} = 0$. *Similarly,* $\hat{B}$ *is given by the quiver*



bound by $\beta\beta' = 0$. Here, C is hereditary of Dynkin type therefore $\hat{A}$ is exhaustive, hence so is $\hat{B}$.

5. Trivial extensions and subalgebras

5.1. **The setting.** We now apply he construction of Section 3. Let, as before, $A = C \ltimes E$ and E' a $C-C$-subbimodule of E. Then $B = C \ltimes E'$ can be identified to a subalgebra of A such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & E & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the exact rows are trivial extensions and the vertical arrows are inclusions, commutes.

Lemma. *Let $A = C \ltimes E$ and E' a $C - C$-subbimodule of E. Then $B = C \ltimes E'$ is a well-behaved subalgebra of A such that the morphism set $\mathcal{B} = \bigcup_{a,b \in B_o} B(a,b)$ is liftable with respect to the repetitive covering.*

Proof. Indeed, B as well as A have the same primitive idempotents as C. The second condition of well-behavedness is satisfied because of [ACT08, (2.5)]. Thus, B is well-behaved.

Let $a, b \in B_o$, then $B(a,b) = C(a,b) \oplus E'(a,b)$. If $u \in C(a,b)$, then it is clearly liftable. If $u \in E'(a,b)$, then it is liftable because E' is an ideal of A contained in E and because of the lemma in 4.2. □

5.2. **Repetitive covering.** The lemma in 5.1 shows that the proposition in 3.2 can be applied to construct a Galois covering $F'' \colon \tilde{B} \to B$ with group $\mathbb{Z}$.

Proposition. *Let $A = C \ltimes E$. Consider a $C - C$-subbimodule E' of E and let $B = C \ltimes E'$. Then the repetitive covering $\hat{B} \to B$ coincides with the induced covering constructed in the proposition in 3.2.*

Proof. In order to show that $\tilde{B} = \hat{B}$, because the two categories have the same objects, we just need to check that they have the same morphisms.

Let $a, b \in C_o$ and $i, j \in \mathbb{Z}$. If $i = j$, then $\hat{A}((a,i),(b,j)) = C(a,b)$, that is, the map $\hat{A}((a,i),(b,i)) \to A(a,b)$ induced by F takes values in $C(a,b)$. Because C is a subalgebra of B, we get $\tilde{B}((a,i),(b,i)) = C(a,b)$ for all $a, b \in C_o$ and $i \in \mathbb{Z}$. If $j = i+1$, then $\hat{A}((a,i),(b,i+1)) = E(a,b)$ and the map $\hat{A}((a,i),(b,i+1)) \to A(a,b)$ is the inclusion $E(a,b) \to A(a,b)$. Because of the definition of $\tilde{B}$, we have $\tilde{B}((a,i),(b,i+1)) = E'(a,b)$ for all $a, b \in C_o$ and $i \in \mathbb{Z}$. Finally, if $j \notin \{i, i+1\}$, then $\hat{A}((a,i),(b,j)) = 0$ and hence $\tilde{B}((a,i),(b,j)) = 0$ for all $a, b \in C_o$ and $i, j \in \mathbb{Z}$ such that $j \notin \{i, i+1\}$. This proves that $\tilde{B} = \hat{B}$.

Because the functor $F'' \colon \tilde{B} \to B$ of the proposition in 3.2 is obtained by restriction and corestriction from the repetitive covering $\hat{A} \to A$, it equals the repetitive covering $\hat{B} \to B$. □

5.3. **Direct summands.** We now consider the intersection of the situations of 4.3 and 5.2, namely, we assume that $A = C \ltimes E$ and $E = E' \oplus E''$ as $C-C$-bimodules. We set $B = C \ltimes E'$. Then we have a trivial extension

$$0 \longrightarrow E'' \xrightarrow{\ \iota\ } A \underset{\sigma}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

where ι is the inclusion, π is the projection and σ is such that $\pi\sigma = \mathrm{id}_B$. For a proof of this statement, we refer to [ABD+19, (2.1.1)]. This situation actually happens when we deal with partial relation extensions, see later in [ABD+19].

As shown in the proof of the lemma in 4.3, the $C-C$-subbimodule E'' of E determines an ideal $\hat{E}''$ of $\hat{A}$ defined by:

$$\hat{E}'' = \begin{pmatrix} \ddots & & & & 0 \\ \ddots & 0 & & & \\ & E'' & 0 & & \\ & & E'' & 0 & \\ 0 & & & \ddots & \ddots \end{pmatrix}$$

so that $\hat{A}/\hat{E}'' \simeq \hat{B}$. We show that the trivial extension $A = B \ltimes E''$ lifts to a trivial extension $\hat{A} = \hat{B} \ltimes \hat{E}''$ and this lifting is compatible with the repetitive coverings $F' \colon \hat{B} \to B$ and $F \colon \hat{A} \to A$.

Corollary. *Let $A = C \ltimes E$. Consider a decomposition $E = E' \oplus E''$ of $C-C$-bimodules and let $B = C \ltimes E'$. Then we have a trivial extension $\hat{A} = \hat{B} \ltimes \hat{E}''$ together with functors $\hat{\pi} \colon \hat{A} \to \hat{B}$, $\hat{\sigma} \colon \hat{B} \to \hat{A}$ such that $\sigma F' = F\hat{\sigma}$ and $\pi F = F'\hat{\pi}$.*

Proof. Applying the proposition in 4.3 to $\pi \colon A \to B$ yields a commutative square

$$\begin{array}{ccc} \hat{A} & \xrightarrow{\hat{\sigma}} & \hat{B} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ A & \xrightarrow{\pi} & B \end{array}$$

and applying the proposition in 5.2 to $\sigma \colon B \to A$ yields another commutative square

$$\begin{array}{ccc} \hat{B} & \xrightarrow{\hat{\sigma}} & \hat{A} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow{\sigma} & A \end{array}$$

Moreover, the constructions of $\hat{\pi}, \hat{\sigma}$ in these lemmata give immediately that $\hat{\pi}\hat{\sigma} = \mathrm{id}_{\hat{B}}$. Because it is clear that $\hat{E}''$ is the kernel of $\hat{\pi}$, this gives the required trivial extension. □

As an immediate consequence, the exhaustiveness of F implies that of F'.

5.4. **The selfinjective case.** In this case, $E = DC$. It seems to be well-known that DC is indecomposable as a $C-C$-bimodule, and even a brick, whenever C is triangular. However, we were not able to find it in the literature and therefore we give here a proof.

Lemma. *Let C be a connected and triangular algebra, then the endomorphism algebra of the $C-C$-bimodule DC is $\Bbbk$. In particular, ${}_C DC_C$ is indecomposable.*

Proof. Since $\dim_{\Bbbk}(C)$ is finite then so is $\dim_{\Bbbk}(C \otimes_{\Bbbk} C^{\mathrm{op}})$. Therefore, the algebras $\mathrm{End}_{C\otimes_{\Bbbk} C^{\mathrm{op}}}(DC)$ and $\mathrm{End}_{C\otimes_{\Bbbk} C^{\mathrm{op}}}(C)$ are isomorphic. Note that $\mathrm{End}_{C\otimes_{\Bbbk} C^{\mathrm{op}}}(C) \cong \{c_0 \in C \mid cc_0 = c_0 c \text{ for all } c \in C\} = Z(C)$. Now, because C is triangular, a given $c \in Z(C)$ must satisfy

$$c = \sum_{i \in C_o} e_i c = \sum_{i \in C_o} c e_i = \sum_{i \in C_o} e_i c e_i \in \prod_{i \in C_o} \Bbbk \cdot e_i \,,$$

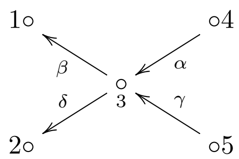
and the family of scalars $(\lambda_i)_{i \in C_o}$ such that $c = \sum_{i \in C_o} \lambda_i e_i$ must be constant because C is connected and, for $i, j \in C_o$ and $x \in e_i C e_j \backslash \{0\}$ one has $\lambda_i x = cx = xc = \lambda_j x$. Accordingly, $Z(C) = \Bbbk \cdot 1_C$. □

5.5. **Partial relation extensions.** Let C be a triangular algebra of global dimension at most two and $E = \mathrm{Ext}^2_C(DC, C)$ with its natural $C-C$-bimodule structure. Then $A = C \ltimes E$ is called a *relation extension.* This class of algebras was much investigated, see, for instance [ABS08, ABS09, AGST16, ABD$^+$19], because of its connection with cluster algebras: indeed, if C is a tilted algebra, then its relation extension is cluster tilted, and every cluster tilted algebra is of this form. If $A = C \ltimes E$ is a relation extension and $E = E' \oplus E''$ as $C - C-$-bimodules, then $B = C \ltimes E'$ is called a *partial relation extension*, see [ABD$^+$19]. We obtain in this case the following result due to Sanchez Mc Millan.

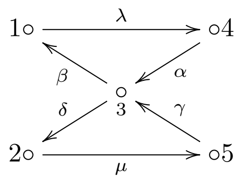
Corollary ([SMM17])**.** *Let C be a tilted algebra and assume $E = \mathrm{Ext}^2_C(DC, C)$ decomposes as $E = E' \oplus E''$ as $C - C$-bimodule. Then the repetitive covering of $B = C \ltimes E'$ is exhaustive.*

Proof. Indeed, $A = C \ltimes E$ is cluster tilted. Because of the main result of [ABS09], the repetitive covering of A is exhaustive. Applying the corollary in 5.3 yields our statement. □

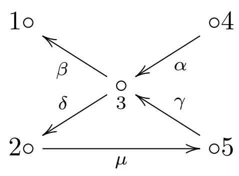
Example. *Let C be the tilted algebra of type $\mathbb{A}_5$ given by the quiver*



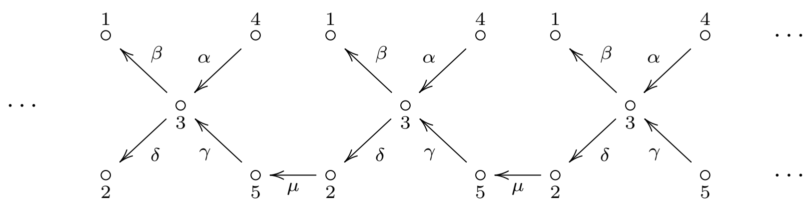
bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Its relation extension $A = C \ltimes E$ is the cluster tilted algebra



bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$. In this case, $E = C\lambda C \oplus C\mu C$. Letting $E' = C\mu C$, we get $B = C \ltimes E'$ given by the quiver



with the inherited relations. Its repetitive covering is given by the quiver



with the lifted relations. According to the corollary, it is exhaustive.

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