FULL EMBEDDINGS OF ALMOST SPLIT SEQUENCES OVER SPLIT-BY-NILPOTENT EXTENSIONS

IBRAHIM ASSEM AND DAN ZACHARIA

ABSTRACT. Let R be a split extension of an artin algebra A by a nilpotent bimodule ${}_AQ_A$, and let M be an indecomposable non-projective A-module. We show that the almost split sequences ending with M in mod A and mod R coincide if and only if $\mathrm{Hom}_A(Q, \tau_A M) = 0$ and $M \otimes_A Q = 0$.

Introduction

While studying the representation theory of the trivial extension $T(A)$ of an artin algebra A by its minimal injective cogenerator bimodule DA, Tachikawa [12] and Yamagata [13] have shown that, if A is hereditary, then the Auslander-Reiten quiver of A fully embeds in the Auslander-Reiten quiver of $T(A)$. This result was generalised by Hoshino in [7] who has shown that, if A is an artin algebra and M is an indecomposable non-projective A-module, then the almost split sequences ending with M in mod A and mod $T(A)$ coincide if and only if the projective dimension of M, and the injective dimension of the Auslander-Reiten translate $\tau_A M$ of M in mod A, do not exceed 1. This enabled him to prove that the trivial extension of a tilted algebra of Dynkin type is representation-finite. A similar result was obtained by Happel when considering the embedding of mod A inside the derived category of bounded complexes over mod A (see [6](I.4.7), p. 38). Our objective in this note is to try to understand the results of Hoshino, Tachikawa and Yamagata in the following more general context. Let A and R be two artin algebras such that there exists a split surjective algebra morphism $R \to A$ whose kernel Q is contained in the radical of R : we then say that R is a split extension of A by the nilpotent bimodule Q, or simply a split-by-nilpotent extension (see [2, 5, 9]). We ask when does an almost split sequence in mod A embed as an almost split sequence in mod R, and show the following generalisation of Hoshino's result.

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Theorem. *Let R be the split extension of an artin algebra A by a nilpotent bimodule Q, and M be an indecomposable non-projective A-module. The following conditions are equivalent:*

(a) *The almost split sequences ending with M in* mod A and mod R *coincide.*
(b) $\tau_A M \cong \tau_R M$.
(c) $\mathrm{Hom}_A(Q, \tau_A M) = 0$ *and* $M \otimes_A Q = 0$.

The paper is organised as follows. In section (1), we construct an exact sequence relating the Auslander-Reiten translates of M in mod A and mod R. In section (2), we prove our theorem, from which we deduce several consequences and end the paper with some examples.

1. Preliminary results

Throughout this note, we use freely and without further reference properties of the module categories and the almost split sequences as can be found, for instance, in [4, 10]. We assume that A and R are two artin algebras such that R is a split extension of A by a (nilpotent) bimodule ${}_AQ_A$. This means that we have a split short exact sequence of abelian groups

$$0 \to Q \xrightarrow{\iota} R \xrightarrow{\pi} A \to 0$$

where $\iota : q \mapsto (0, q)$ is the inclusion of Q as a two-sided ideal of $R = A \oplus Q$, and the projection (algebra) morphism $\pi : (a, q) \mapsto a$ has as section the inclusion morphism $\sigma : a \mapsto (a, 0)$. If M is an A-module, we have a canonical R-linear epimorphism $p_M : M \otimes_A R \to M$ given by $m \otimes (a, q) \mapsto ma$ which is minimal [2](1.1). Moreover, if P is a projective cover of the A-module M, then $P \otimes_A R$ is a projective cover of M when the latter is viewed as an R-module. In particular, the indecomposable projective R-modules are all induced modules of the form $P \otimes_A R$, where P is an indecomposable projective A-module (see [2]).

Proposition 1.1. *Let M be an indecomposable A-module, P_0 be its projective cover in* mod A, P *be the projective cover of $P_0 \otimes_A Q$ in* mod A, *and $p_M : M \otimes_A R \to M$ be the canonical epimorphism. Then there exists an exact sequence of A-modules*

$$0 \to \tau_A M \oplus \mathrm{Hom}_A(Q, \tau_A M) \xrightarrow{u} \tau_R M \to P \otimes_A \mathrm{D}R \to \mathrm{Ker}\ (p_M \otimes \mathrm{D}R) \to 0.$$

Proof. We start with a minimal projective presentation of M in mod A

$$P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

which yields, by [2](1.3), a minimal projective presentation in mod R

$$P_1 \otimes_A R \stackrel{f_1\otimes R}{\longrightarrow} P_0 \otimes_A R \stackrel{f_0\otimes R}{\longrightarrow} M \otimes_A R \to 0.$$

Applying the Nakayama functor $- \otimes_R \mathrm{D}R$, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \tau_R(M\otimes_A R) & \longrightarrow & P_1\otimes_A R\otimes_R \mathrm{D}R & \longrightarrow & P_0\otimes_A R\otimes_R \mathrm{D}R & \longrightarrow & M\otimes_A R\otimes_R \mathrm{D}R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \tau_R(M\otimes_A R) & \longrightarrow & P_1\otimes_A \mathrm{D}R & \longrightarrow & P_0\otimes_A \mathrm{D}R & \longrightarrow & M\otimes_A \mathrm{D}R & \longrightarrow & 0
\end{array}$$

We need to compute $\tau_R M$ and, for this purpose, we need a minimal projective presentation of M in mod R

$$\bar{P}_1 \to \bar{P}_0 \to M \to 0.$$

It is clear that $\bar{P}_0 \cong P_0 \otimes_A R$ and that we have a commutative diagram with exact rows in mod R

$$\begin{array}{ccccccc}
P_1 \otimes_A R & \xrightarrow{f_1\otimes R} & P_0 \otimes_A R & \xrightarrow{f_0\otimes R} & M \otimes_A R & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p_M} & & \\
\overline{P}_1 & \longrightarrow & \overline{P}_0 & \xrightarrow{p_M(f_0\otimes R)} & M & \longrightarrow & 0
\end{array}$$

In order to compute $\bar{P}_1$, we consider the short exact sequence of R-modules

$$0 \to \Omega^1_R M \to P_0 \otimes_A R \stackrel{p_M(f_0\otimes R)}{\longrightarrow} M \to 0$$

as an exact sequence of A-modules. We have an isomorphism of A-modules $P_0 \otimes_A R \cong P_0 \oplus (P_0 \otimes_A Q)$ and, as A-linear maps, we have $p_M = [1\ 0]$ and $f_0 \otimes R = \begin{bmatrix} f_0 & 0 \\ 0 & f_0 \otimes Q \end{bmatrix} : P_0 \oplus (P_0 \otimes_A Q) \to M \oplus (M \otimes_A Q)$. Therefore $p_M(f_0 \otimes Q) = [f_0\ 0]$ and we have an isomorphism of A-modules

$$\Omega^1_R M = \mathrm{Ker}\,[f_0\ 0] \cong \Omega^1_A M \oplus (P_0 \otimes_A Q).$$

Let P be the projective cover of $P_0 \otimes_A Q$ in mod$A$. We have a projective cover morphism in mod R

$$P \otimes_A R \stackrel{p}{\longrightarrow} P_0 \otimes_A Q.$$

Since P_0 is projective and ${}_AQ_R$ is a subbimodule of ${}_AR_R$, then $P_0 \otimes_A Q$ is a submodule of $P_0 \otimes_A R$ when viewed as R-modules. Let $\bar{f}$ be the R-linear map defined by the composition $P \otimes_A R \xrightarrow{p} P_0 \otimes_A Q \hookrightarrow P_0 \otimes_A R$. We thus have a commutative diagram with exact rows in mod R

$$\begin{array}{ccccccc}
P_1 \otimes_A R & \xrightarrow{f_1 \otimes R} & P_0 \otimes_A R & \xrightarrow{f_0 \otimes R} & M \otimes_A R & \longrightarrow & 0 \\
\downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p_M} & & \\
(P_1 \oplus P) \otimes_A R & \xrightarrow{[f_1 \otimes R \ \ \overline{f}]} & P_0 \otimes_A R & \xrightarrow{p_M (f_0 \otimes R)} & M & \longrightarrow & 0
\end{array}$$

Applying $- \otimes_R \mathrm{D}R$, we obtain a commutative diagram with exact rows in mod R

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \tau_R(M \otimes_A R) & \xrightarrow{j} & P_1 \otimes_A \mathrm{D}R & \xrightarrow{f_1 \otimes \mathrm{D}R} & P_0 \otimes_A \mathrm{D}R & \xrightarrow{f_0 \otimes \mathrm{D}R} & M \otimes_A \mathrm{D}R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p_M \otimes \mathrm{D}R} & & \\
0 & \longrightarrow & \tau_R M & \longrightarrow & (P_1 \oplus P) \otimes_A \mathrm{D}R & \xrightarrow{[f_1 \otimes \mathrm{D}R \ \ \overline{f} \otimes \mathrm{D}R]} & P_0 \otimes_A \mathrm{D}R & \xrightarrow{(p_M (f_0 \otimes R)) \otimes \mathrm{D}R} & M \otimes_R \mathrm{D}R & \longrightarrow & 0
\end{array}$$

where u is induced by passing to the kernels. Since the composition $\begin{bmatrix}1\\0\end{bmatrix} j$ is a monomorphism, so is u. On the other hand, the above diagram induces the following two commutative diagrams in mod R, where the rows are short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_R(M \otimes_A R) & \xrightarrow{j} & P_1 \otimes_A \mathrm{D}R & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle u'} & & \\
0 & \longrightarrow & \tau_R M & \longrightarrow & (P_1 \oplus P) \otimes_A \mathrm{D}R & \longrightarrow & Y & \longrightarrow & 0
\end{array}$$

where $X = \mathrm{Im}(f_1 \otimes \mathrm{DR})$, $Y = \mathrm{Im}[f_1 \otimes \mathrm{DR} \ \bar{f} \otimes \mathrm{DR}]$, and u' is induced by passing to the cokernels, and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & P_0 \otimes_A \mathrm{D}R & \xrightarrow{f_0 \otimes \mathrm{D}R} & M \otimes_A \mathrm{D}R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p_M \otimes \mathrm{D}R} & & \\
0 & \longrightarrow & Y & \longrightarrow & P_0 \otimes_A \mathrm{D}R & \longrightarrow & M \otimes_R \mathrm{D}R & \longrightarrow & 0
\end{array}$$

Applying the snake lemma to the second diagram yields that u' is a monomorphism, and that Coker $u' \cong$ Ker $(p_M \otimes \mathrm{DR})$. Applying the snake lemma to the first diagram yields a short exact sequence

$$0 \longrightarrow \text{Coker } u \longrightarrow P \otimes_A \mathrm{D}R \longrightarrow \text{Coker } u' \longrightarrow 0 \, .$$

Hence, we have a short exact sequence of R-modules

$$0 \to \text{Coker } u \to P \otimes_A \mathrm{D}R \to \text{Ker } (p_M \otimes \mathrm{D}R) \to 0.$$

On the other hand, [2] (2.1) gives

$$\tau_R(M \otimes_A R) \cong \mathrm{Hom}_A(R, \tau_A M) \cong \tau_A M \oplus \mathrm{Hom}_A(Q, \tau_A M)$$

where the second isomorphism is an isomorphism of A-modules. Hence we have a short exact sequence of A-modules.

$$0 \to \tau_A M \oplus \mathrm{Hom}_A(Q, \tau_A M) \xrightarrow{u} \tau_R M \to \text{Coker } u \to 0$$

the proposition follows at once. □

Remark. It follows from the proof of the proposition that we have a short exact sequence of R-modules

$$0 \to \tau_R(M \otimes_A R) \to \tau_R M \to \text{Coker } u \to 0$$

Corollary 1.2. *For every indecomposable A-module M, the A-module $\tau_A M$ is a submodule of $\tau_R M$.* □

The above corollary was shown in a more general setting in [3] (4.2). In fact, one can easily prove that, if A is a quotient of R and M is an indecomposable A-module, then we have a commutative diagram in mod R

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tau_A M & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & {\scriptstyle f}\big\downarrow & & \big\downarrow & & \big\| & & \\
0 & \longrightarrow & \tau_R M & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are the almost split sequences ending with M in mod A and mod R, respectively. It would be interesting to know whether f, when considered as an A-linear map, coincides with our embedding $\tau_A M \to \tau_R M$.

Corollary 1.3. *Assume $M \otimes_A Q = 0$, then we have*

(a) $P_0 \otimes_A Q = 0$, *and*
(b) $\tau_R M \cong \tau_A M \oplus \mathrm{Hom}_A(Q, \tau_A M)$, *as A-modules.*

Proof. (a) If $M \otimes_A Q = 0$, then $M \otimes_A R = M$ so $\Omega^1_R(M \otimes_A R) = \Omega^1_R M = \Omega^1_A M \oplus (P_0 \otimes_A Q)$. Let P' the projective cover of $\Omega^1_R(M \otimes_A R)$. By [2](1.3), $P' \otimes_A R \cong P_1 \otimes_A R$ as R-modules, so $P' \cong P_1$ by [2](1.2). Therefore top $\Omega^1_R(M \otimes_A R) =$ top $\Omega^1_A M$ in mod A. Hence $P_0 \otimes_A Q = 0$.

(b) Clearly, $P_0 \otimes_A Q = 0$ implies $P = 0$. The result follows. □

Corollary 1.4. *Let $e \in A$ be idempotent. The projective A-module eA is projective in* mod R *if and only if $eQ = 0$.*

Proof. If $M = eA$ is a projective R-module, then $M \otimes_A R = eR$ is a projective R-module with the same top as eA. Consequently, $eR = eA$ and hence $eQ = 0$. Conversely, $M \otimes_A Q = eQ = 0$ implies by (1.3) above that $\tau_R M \cong \tau_A M \oplus \mathrm{Hom}_A(Q, \tau_A M) = 0$. □

We have the following interesting consequence of [2] (2.1).

Corollary 1.5. *Let M be an indecomposable A-module such that pd $M = 1$. Then*

(a) $\mathrm{Hom}_A(Q, \tau_A M) \cong \mathrm{Tor}^A_1(M, \mathrm{D}Q)$ *as A-modules.*

(b) *If Q_A is injective, then $\tau_R(M \otimes_A R) \cong \tau_A M$.*

Proof.

(a) Let $0 \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of M. The A-module decomposition $\mathrm{D}R = \mathrm{D}A \oplus \mathrm{D}Q$ yields a commutative diagram with exact rows and columns in mod A

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}^A_1(M, \mathrm{D}Q) & \longrightarrow & P_1 \otimes_A \mathrm{D}Q & \longrightarrow & P_0 \otimes_A \mathrm{D}Q & \longrightarrow & M \otimes_A \mathrm{D}Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau_R(M \otimes_A R) & \longrightarrow & P_1 \otimes_A \mathrm{D}R & \longrightarrow & P_0 \otimes_A \mathrm{D}R & \longrightarrow & M \otimes_A \mathrm{D}R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau_A M & \longrightarrow & P_1 \otimes_A \mathrm{D}A & \longrightarrow & P_0 \otimes_A \mathrm{D}A & \longrightarrow & M \otimes_A \mathrm{D}A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

An easy calculation shows that the left column splits in mod A. The result follows from [2] (2.1).

(b) Since Q is injective, $\mathrm{D}Q$ is projective. Hence $\mathrm{Tor}^A_1(M, \mathrm{D}Q) = 0$ and the statement follows. □

2. The main result

In this section, we let $\mathcal{C}_A$ denote the full subcategory of mod A consisting of all the indecomposable A-modules M having the property that $\tau_A M \cong \tau_R M$. Corollary (1.4) characterises the objects of $\mathcal{C}_A$ which are indecomposable projective A-modules. Our main theorem below characterises those which are not projective.

Theorem 2.1. *Let M be an indecomposable non-projective A-module. The following conditions are equivalent:*

(a) *The almost split sequences ending with M in* mod A *and in* mod R *coincide.*
(b) *M is in $\mathcal{C}_A$.*
(c) $\mathrm{Hom}_A(Q, \tau_A M) = 0$ *and* $M \otimes_A Q = 0$.
(d) $\mathrm{Hom}_A(Q, \tau_A M) = 0$ *and* $\mathrm{Hom}_A(M, \mathrm{D}Q) = 0$.
(e) $M \otimes_A Q = 0$ *and* $Q \otimes_A \mathrm{Tr} M = 0$.
(f) $\mathrm{Hom}_A(M, \mathrm{D}Q) = 0$ *and* $Q \otimes_A \mathrm{Tr} M = 0$.
(g) *If $P_1 \xrightarrow{f} P_0 \to M \to 0$ is a minimal projective presentation of M, then $f \otimes Q$ and $Q \otimes f^t$ are epimorphisms.*

Proof. (a) implies (b) trivially.

(b) implies (a). Let $0 \to \tau_R M \xrightarrow{f} E \xrightarrow{g} M \to 0$ be an almost split sequence in mod R. We claim that it is almost split in mod A. First, it does not split in mod A, since then we would have $E \cong M \oplus \tau_A M \cong M \oplus \tau_R M$ implying that it splits in mod R. If $h : L \to M$ is an A-linear map which is not a retraction in mod A, then h is also R-linear and it is not a retraction in mod R. Hence there exists an R-linear map $h' : L \to E$ such that $h = gh'$. Since h' is R-linear, it is also A-linear.

(b) implies (c). Let $u : \tau_A M \oplus \mathrm{Hom}_A(Q, \tau_A M) \to \tau_R M$ be as in (1.1). Since u is injective and $\tau_A M \cong \tau_R M$, it follows that $\mathrm{Hom}_A(Q, \tau_A M) = 0$ and that u is an isomorphism between the R-modules $\tau_R(M \otimes_A R)$ and $\tau_R M$. But $\tau_R(M \otimes_A R) \cong \tau_R M$ means $M \otimes_A R = M$, hence $M \otimes_A Q = 0$.

(c) implies (b). This follows from (1.3).

The equivalence of (c) with (d),(e) and (f) follows from the canonical isomorphisms $M \otimes_A Q \cong \mathrm{D}\mathrm{Hom}_A(M, \mathrm{D}Q)$ and $Q \otimes_A \mathrm{Tr} M \cong \mathrm{D}\mathrm{Hom}_A(Q, \tau_A M)$. The equivalence of (e) and (g) follows from the facts that $M \otimes_A Q \cong \mathrm{Coker}\ (f \otimes Q)$ and $Q \otimes_A \mathrm{Tr} M \cong \mathrm{Coker}\ (Q \otimes f^t)$. $\square$

Corollary 2.2. (a) *If $0 \to L \to M \to N \to 0$ is an exact sequence in* mod A, *with L and N in $\mathcal{C}_A$, then every indecomposable non-projective summand of M is in $\mathcal{C}_A$.*

(b) *If $f : M \to N$ is irreducible in* mod A *and if N is in $\mathcal{C}_A$, then f is irreducible in* mod R.

(c) *If* $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t$ *is a sectional path in the Auslander-Reiten quiver of* A *consisting of modules in* $\mathcal{C}_A$*, then this is a sectional path in the Auslander-Reiten quiver of* R*.*

Proof. (a) Applying $-\otimes_A Q$ to the given sequence yields an exact sequence

$$L \otimes_A Q \to M \otimes_A Q \to N \otimes_A Q \to 0$$

which shows that $M \otimes_A Q = 0$. On the other hand, there exists an injective module I_A such that we have a short exact sequence

$$0 \to \tau_A L \to \tau_A M \oplus I \to \tau_A N \to 0.$$

Applying $\mathrm{Hom}_A(Q, -)$, we obtain an exact sequence

$$0 \to \mathrm{Hom}_A(Q, \tau_A L) \to \mathrm{Hom}_A(Q, \tau_A M) \oplus \mathrm{Hom}_A(Q, I) \to \mathrm{Hom}_A(Q, \tau_A N)$$

hence $\mathrm{Hom}_A(Q, \tau_A M) = 0$.

(b) and (c) follow trivially from the theorem. □

We now deduce (and generalise) Hoshino's result: in fact, let $\hat{A}$ denote the repetitive algebra of A (as defined in [8]) then there exist quotients of $\hat{A}$ which are split extensions of A by the bimodule $Q = \oplus_{i=1}^{n} (\mathrm{D}A)^{\otimes i}$ for some $n \geq 1$. We have the following Corollary.

Corollary 2.3. *Assume that* $Q = (\mathrm{D}A)^n$ *for some* $n \geq 1$ *or that* $Q = \oplus_{i=1}^{n} (\mathrm{D}A)^{\otimes i}$ *for some* $n \geq 1$*. Then*

(a) M *is in* $\mathcal{C}_A$ *if and only if* $\mathrm{pd}\ M \leq 1$ *and* $\mathrm{id}\ \tau_A M \leq 1$.

(b) *If* A *is hereditary, then all the indecomposable non-projective* A*-modules are in* $\mathcal{C}_A$*. Hence the Auslander-Reiten quiver of* A *fully embeds in the Auslander-Reiten quiver of* R*.*

(c) *If* A *is tilted, and if* M_A *is an indecomposable module lying on a complete slice, then* M *lies in* $\mathcal{C}_A$*.*

(d) A *is concealed if and only if all but at most finitely many isomorphism classes of indecomposable* A*-modules are in* $\mathcal{C}_A$*.*

Proof. (a) We know by [10], p. 74, that $\mathrm{pd}\, M \leq 1$ if and only if $\mathrm{Hom}_A(\mathrm{D}A, \tau_A M) = 0$ while $\mathrm{id}\ \tau_A M \leq 1$ if and only if $M \otimes_A \mathrm{D}A \cong \mathrm{D}\mathrm{Hom}_A(M, A) = 0$. If $Q = (\mathrm{D}A)^n$, the result follows at once. If $Q = \oplus_{i=1}^{n} (\mathrm{D}A)^{\otimes i}$, then $M \otimes_A \mathrm{D}A = 0$ implies $M \otimes_A (\mathrm{D}A)^{\otimes i} = 0$ for all $i \geq 1$, and the adjunction isomorphism implies $\mathrm{Hom}_A((\mathrm{D}A)^{\otimes i}, \tau_A M) \cong \mathrm{Hom}_A((\mathrm{D}A)^{\otimes (i-1)}, \mathrm{Hom}_A(\mathrm{D}A, \tau_A M)) = 0$ for all $i \geq 1$.

(b) and (c) follow directly from (a).
(d) follows from (a) and [1](3.4) (see also [11](3.3)). □

Remark. It is worthwhile to observe that, if $Q = \mathrm{D}A$, there exist split extensions of A which are not trivial extensions, as is shown by the following example due to K. Yamagata (private communication). Let A be a symmetric algebra, and $R = A \oplus \mathrm{D}A$ with the multiplication induced by the multiplication of A and the structural isomorphism ${}_AA_A \cong {}_A\mathrm{D}A_A$.

Corollary 2.4. *If M is an indecomposable non-projective A-module, then $\tau_{\hat{A}}M \cong \tau_A M$ if and only if $\mathrm{pd}\ M \leq 1$ and $\mathrm{id}\ \tau_A M \leq 1$.* □

Clearly, if gl.dim.$A < \infty$, then the above corollary can also be understood in terms of the derived category of bounded complexes over mod A (see [6](I.4.7) p. 38). We also deduce the following consequence (compare with [13](4.1)).

Corollary 2.5. *Assume that $Q = (\mathrm{D}A)^n$ for some $n \geq 1$ or that $Q = \oplus_{i=1}^n(\mathrm{D}A)^{\otimes i}$ for some $n \geq 1$. The following conditions are equivalent:*
(a) *A is hereditary*
(b) *Every irreducible morphism in mod A is irreducible in mod R.*
(c) *Every almost split sequence in mod A is almost split in mod R.*

Proof. (a) implies (b). Let $M \to N$ be irreducible in mod$A$. If N is not projective, then we are done by (2.2)(b). If N is projective, so is M and we have an almost split sequence in mod A

$$0 \to M \to N \oplus L \to \tau_A^{-1}M \to 0$$

since M is not injective. Thus $\tau_A^{-1}M$ is in $\mathcal{C}_A$ and the statement follows
(b) implies (c) trivially.
(c) implies (a). Every indecomposable non-projective A-module M is in $\mathcal{C}_A$, hence $\mathrm{Hom}_A(Q, \tau_A M) = 0$. Consequently $\mathrm{Hom}_A(\mathrm{D}A, \tau_A M) = 0$, thus $\mathrm{pd}\ M \leq 1$ and A is hereditary. □

Remarks. (a) If Q is as in (2.3) and (2.5), no projective A-module is projective in mod$R$. Indeed, for any idempotent $e \in A$, we have $e\mathrm{D}A = \mathrm{D}(Ae) \neq 0$, hence $eQ \neq 0$ and we apply (1.4).
(b) Assume $Q = {}_AA_A$, then no indecomposable A-module lies in $\mathcal{C}_A$. Indeed, if M lies in $\mathcal{C}_A$, then $M \cong M \otimes_A A = 0$.

We now turn our attention to one-point extensions. Let k be a commutative field, B be a finite dimensional basic k-algebra and $R = B[X]$ be the one-point extension of B by the B-module X. Let $A = B \times k$ and, letting a denote the extension point, let Q be the $R-R$- bimodule generated by the arrows from a to the quiver of B. It is easily seen that R is a split extension of A by Q, that $Q_A \cong X_A$ while $\mathrm{D}({}_AQ) \cong S(a)^t$ for some $t \geq 1$, where $S(a)$ denotes the simple module corresponding to the point a. We have the following corollary (compare [10] p. 88).

Corollary 2.6. *Let $R = B[X]$ and M be an indecomposable non-projective B-module.*

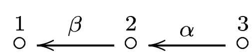
(a) $\tau_B M \cong \tau_R M$ *if and only if* $\mathrm{Hom}_B(X, \tau_B M) = 0$. *In particular, if every indecomposable summand of X is in $\mathcal{C}_A$, then* $\mathrm{Ext}^1_B(X, X) = 0$.

(b) *If $\tau_B M$ is not a successor of X, then $\tau_B M \cong \tau_R M$. In particular, if N is not a successor of X, then $\tau_B N \cong \tau_R N$.*

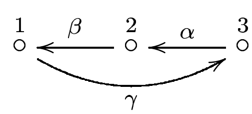
Proof. (a) We have $M \otimes_A Q \cong \mathrm{DHom}_A(M, \mathrm{D}Q) \cong \mathrm{DHom}_A(M, S(a)^t) = 0$. Therefore M is in $\mathcal{C}_A$ if and only if $\mathrm{Hom}_B(X, \tau_B M) = \mathrm{Hom}_A(Q, \tau_A M) = 0$. The second statement follows from the isomorphism $\mathrm{Ext}^1_B(X, X) \cong \mathrm{D}\overline{\mathrm{Hom}}_B(X, \tau_B X) = \mathrm{D}\overline{\mathrm{Hom}}_A(Q, \tau_A Q)$.

(b) If $\tau_B M \not\cong \tau_R M$, then $\mathrm{Hom}_B(X, \tau_B M) \neq 0$ so $\tau_B M$ is a successor of X. The second statement follows from the fact that, if $\tau_B N$ is a successor of X, then so is N. □

Examples. (a) Let k be a commutative field, and A be the finite dimensional k-algebra given by the quiver



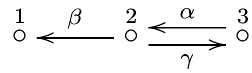
bound by $\alpha\beta = 0$. The algebra R given by the quiver



bound by $\alpha\beta = 0, \beta\gamma = 0, \gamma\alpha = 0$ is the split extension of A by the two-sided ideal A generated by γ. A k-basis of Q is the set $\{\gamma\}$ so that $Q_A = S(3)$ and $\mathrm{D}({}_AQ) = S(1)$.

Here, every irreducible morphism (or almost split sequence) in mod A remains irreducible (or almost split, respectively) in mod R, even though A is not hereditary.

(b) Let A be as in (a), and R be given by the quiver



bound by $\alpha\beta = 0, \gamma\alpha\gamma\alpha = 0$. Here R is the split extension of A by the two-sided ideal Q generated by γ. A k-basis of Q is the set $\{\gamma, \alpha\gamma, \gamma\alpha, \alpha\gamma\alpha, \gamma\alpha\gamma, \alpha\gamma\alpha\gamma\}$. We have $Q_A = \left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)^2 \oplus S(3)^2$ and $\mathrm{D}({}_AQ) = \left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)^3$, where $\left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)$ denotes the uniserial module of length two with top $S(3)$ and socle $S(2)$. We claim that $S(2)$ is not in $\mathcal{C}_A$. Indeed, consider the minimal projective resolution of $S(2)_A$

$$0 \to e_1A \to e_2A \to S(2) \to 0.$$

Applying $- \otimes_A Q$, we obtain an exact sequence

$$e_1Q \to e_2Q \to S(2) \otimes_A Q \to 0.$$

Since $e_1Q = 0$, we have $S(2) \otimes_A Q \cong e_2Q = \left(\begin{smallmatrix}3\\2\end{smallmatrix}\right) \oplus S(3) \neq 0$. On the other hand, $S(3)$ lies in $\mathcal{C}_A$. Indeed, we have $\mathrm{Hom}_A(Q, \tau_A S(3)) = \mathrm{Hom}_A(\left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)^2 \oplus S(3)^2, S(2)) = 0$ and also $\mathrm{Hom}_A(S(3), \mathrm{D}Q) = \mathrm{Hom}_A(S(3), \left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)^3) = 0$.

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Ibrahim Assem
Département de mathématiques et d'informatique
Université de Sherbrooke
Sherbrooke, Québec, J1K 2R1, Canada
ibrahim.assem@dmi.usherb.ca

Dan Zacharia
Department of Mathematics
Syracuse University
Syracuse, NY 13244, USA.
zacharia@mailbox.syr.edu