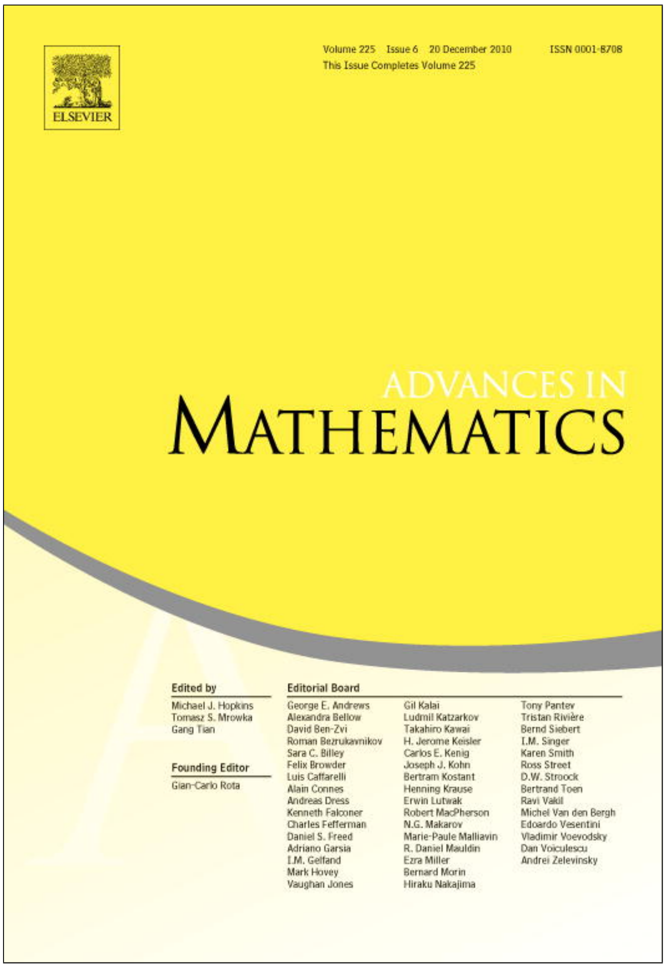
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Friezes

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Abstract

The construction of friezes is motivated by the theory of cluster algebras. It gives, for each acyclic quiver, a family of integer sequences, one for each vertex. We conjecture that these sequences satisfy linear recursions if and only if the underlying graph is a Dynkin or an Euclidean (affine) graph. We prove the "only if" part, and show that the "if" part holds true for all non-exceptional Euclidean graphs, leaving a finite number of finite number of cases to be checked. Coming back to cluster algebras, the methods involved allow us to give formulas for the cluster variables in case A_m and $\tilde{A}_m$; the novelty is that these formulas use 2 by 2 matrices over the semiring of Laurent polynomials generated by the initial variables (which explains simultaneously positivity and the Laurent phenomenon). One tool involved consists of the SL_2-tilings of the plane, which are particular cases of T-systems of Mathematical Physics.

Keywords: Rationality; Periodicity; Dynkin diagrams; Euclidean diagrams; Cluster algebras; Positivity; Laurent phenomenon; SL_2 tilings; Frieze patterns

1. Introduction

In this paper, we classify certain integer sequences by means of Dynkin and Euclidean (also called affine) diagrams. These sequences are constructed with the use of a new mathematical

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tool, called *friezes*. Their construction is motivated by the theory of cluster algebras of Fomin and Zelevinsky and was shown to two of the authors by Philippe Caldero some years ago.

Our work is motivated by a well-known example, shown in Section 2, that of the Kronecker quiver (that is, the quiver with two points and two arrows from one to the other). Defining a frieze for this quiver yields the Fibonacci numbers of even rank. This observation led us to the question of determining those quivers (or, more generally, those Cartan matrices) for which the sequences of numbers obtained are rational, that is, satisfy some linear recursion, as do the Fibonacci numbers. Note that this is a finiteness condition, since a sequence satisfies a linear recursion if and only if the space spanned by its translates is finite dimensional.

One of our main results, stated in Theorems 1 and 2, answers this question by giving a classification result *à la* Cartan–Killing of the integer sequences appearing in a frieze. This classification is somewhat similar to the theorems of Gabriel and Dlab–Ringel classifying the representation-finite and the tame quivers [21,15], and to the theorem of Fomin–Zelevinsky, classifying the cluster algebras of finite type [20]. Namely, we show that if the sequences of integers associated to a given quiver, or Cartan matrix, satisfy a linear recursion, then the Cartan matrix is Dynkin or Euclidean. Conversely, we prove that if the Cartan matrix is Dynkin, or else Euclidean but not exceptional, then the frieze gives a rational sequence of numbers. Thus, our classification is not complete, but almost, and what remains to be proved is just the rationality for the exceptional Euclidean Cartan matrices, which are finitely many. This will be done elsewhere. Added in proof: In a recent article, B. Keller and S. Scherotzke have completed this task; they give also a new proof for the rationality of the frieze associated Euclidean friezes.

A byproduct of the proof of Theorem 1 is a simple proof of one implication, in the acyclic case, in the main result of Fomin and Zelevinsky [20]: if the quiver is acyclic and if the number of cluster variables is finite, then the underlying graph is Dynkin (Corollary 1 in Section 4).

For the proof of Theorem 2, we use an object that we call SL_2-tiling. This is similar to an object considered by Coxeter and Conway [12,10,11] (see also [29,3]), called by them *frieze patterns*. Note that the Coxeter–Conway frieze patterns are simultaneously friezes (in our terminology) of type A_m, and also *partial* SL_2-tilings. Moreover, a variant of SL_2-tilings appear in Mathematical Physics under then name T-systems [25]: more precisely, a T-system of type A_1 is equivalent to the data of two independent SL_2-tilings (this is explained in [13] Section 2.2). The values of an SL_2-tiling are given in Theorem 3.

Next, we turn to the task of obtaining formulas for cluster variables in the Dynkin and Euclidean cluster algebras. Cluster algebras were introduced by Fomin and Zelevinsky [19,20] in order to explain the connection between the canonical basis of a quantized enveloping algebra and total positivity for algebraic groups. Since then, they turned out to have important ramifications in several areas of mathematics. Roughly speaking, a cluster algebra is an integral domain with a possibly infinite set of distinguished generators (the cluster variables) grouped into (overlapping) clusters of the same finite cardinality and computed recursively from an initial cluster. By construction, every cluster variable can be uniquely expressed as a rational function of the elements of any given cluster. The Laurent phenomenon, established in [18] asserts that these rational functions are Laurent polynomials with integral coefficients. It was conjectured by Fomin and Zelevinsky that these coefficients are positive: this is the positivity conjecture.

In the second part of the paper, using friezes, we derive computational tools for the cluster variables without going through the recursive process. We give, in Section 8, Theorems 4 and 5, explicit and simple formulas, involving only matrix products, for all cluster variables (or all but finitely many cluster variables) for cluster algebras without coefficients of Dynkin type $\mathbb{A}$

(or Euclidean type $\tilde{\mathbb{A}}$, respectively), explaining simultaneously the Laurent phenomenon and the positivity for initial acyclic clusters.

The positivity conjecture and direct formulas for the cluster variables are known in special cases only, see, for instance, [20,9,27,30,8,28,26,31]. Our approach however is novel and gives, instead of summation formulas, only products of 2×2 matrices over the Laurent polynomial semiring over $\mathbb{N}$.

The ideas and results of the present paper have already been used by several authors: see, for instance, [13,5,1,16,2]. In particular, it is shown in [1] that frieze functions allow to construct all transjective cluster variables for acyclic cluster algebras, and even, in the Euclidean case, to construct all cluster variables.

For cluster algebras and cluster categories, we refer the reader to [19,20,6] and for linear recurrences and rational series, we refer to [17,5].

2. An introductory example

We give here an example which motivated our work and illustrates the constructions and the theorems. The experienced reader may skip this section. This example is not new and appears in many articles, see [18] (Ex. 3.2 with $a = b = 1$), [32,26,9,14] (the renormalized A_1 Q-system (1.4) with $r = 1$).

Given a quiver Q (in other words, a directed graph with possibly multiple edges), which we assume to be acyclic, let V be its set of vertices. Define for each v in V a sequence $v(n)$ by the initial condition $v(0) = 1$ and the recursion

$$v(n+1) = \frac{1}{v(n)}\left(1 + \prod_{v \to w} w(n) \prod_{w \to v} w(n+1)\right).$$

The fact that these equations define uniquely the sequences $v(n)$ follows from the acyclicity of the graph.

The previous recursion formula may be represented by defining the *frieze* associated to the quiver: it is an infinite graph with set of vertices $V \times \mathbb{N}$ and edges $(v, n) \to (w, n)$ if $v \to w$ is in Q, and edges $(v, n) \to (w, n+1)$ if $v \leftarrow w$ in Q. Then the sequence $v(n)$ labels the vertex (v, n) of the frieze, $l(v, n) = v(n)$ say, and the recursion reads

$$l(v, n+1) = \frac{1}{l(v, n)}\left(1 + \prod_{(w,i) \to (v,n+1)} l(w, i)\right),$$

with the initial conditions $l(v, 0) = 1$. Note that only $i = n$ or $i = n + 1$ may occur in the product. As an example take the Kronecker quiver, with two vertices and two edges from one to the other. The frieze is represented in Fig. 1, together with the labels.

It turns out that the numbers $1, 1, 2, 5, 13, 34, \ldots$ are the Fibonacci numbers of even rank F_{2n}, if one defines $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. This may be proved for instance by using the identity

$$\begin{vmatrix} F_{2n+4} & F_{2n+2} \\ F_{2n+2} & F_{2n} \end{vmatrix} = 1,$$

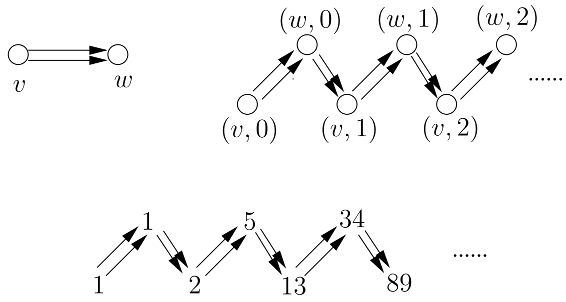


Fig. 1. Kronecker quiver and frieze.

which is a consequence of the fact that the Fibonacci numbers of even rank satisfy the recursion $F_{2n+4} = 3F_{2n+2} - F_{2n}$, as is well known.

Actually, as mentioned before, we shall also deal with more general friezes: the initial values $v(0)$ will be variables. In the example of the Kronecker quiver, we may take $u_0 = a$, $u_1 = b$ and the recursion $u_{n+2} = \frac{1+u_{n+1}^2}{u_n}$, which shortcuts the frieze: $u_0, u_2, u_4, \ldots$ label the vertices $(v,0), (v,1), (v,2), \ldots$ and $u_1, u_3, u_5 \ldots$ the vertices $(w,0), (w,1), (w,2) \ldots$. Then one has also a linear recursion, generalizing the recursion for Fibonacci numbers of even rank:

$$u_{n+2} = \frac{a^2+b^2+1}{ab} u_{n+1} - u_n.$$

Moreover, $u_n = \frac{1}{a^{n-1}b^{n-2}}(1,b)M^{n-2}\binom{1}{b}$, where

$$M = \begin{pmatrix} a^2+1 & b \\ b & b^2 \end{pmatrix}.$$

A summation formula for u_n has already been given by Caldero and Zelevinsky [9], Th. 4.1. See also [14], where an explicit matrix formulation is given.

For quivers of type A_m and $\tilde{A}_m$, we shall obtain these kinds of formulas, which explain simultaneously the Laurent phenomenon (the denominator is a monomial) of Fomin and Zelevinsky, and the positivity of the formulas.

3. Friezes associated to Cartan matrices

Recall that a *Cartan matrix* $C = (C_{ij})_{1 \leqslant i,j \leqslant d}$ is defined by the following properties:

(i) $C_{ij} \in \mathbf{Z}$;
(ii) $C_{ii} = 2$;
(iii) $C_{ij} \leqslant 0$, if $i \neq j$;
(iv) $C_{ij} \neq 0 \Leftrightarrow C_{ji} \neq 0$.

The *simple graph associated to* C has set of vertices $\{1, \ldots, d\}$ and an undirected edge $\{i, j\}$ if $i \neq j$ and $C_{ij} \neq 0$. The Cartan matrix is completely described by its *diagram*, which is the previous graph, with *valuations* on it; the couple $(|C_{ij}|, |C_{ji}|)$ is represented on the edge $\{i, j\}$ (see Fig. 2).

If $|C_{ij}| = 1$, it is omitted. In this manner, Cartan matrices are equivalent to diagrams.

Fig. 2. Cartan diagram.

Without loss of generality, we consider only *connected* Cartan matrices; this means that the underlying graph is connected. Consider some fixed acyclic orientation of this graph; if the edge $\{i, j\}$ of this graph is oriented from i to j, we write $i \to j$. For each $j = 1, \dots, d$, we define a sequence $a(j, n)$, $n \in \mathbf{N}$, by the formula, for all $j = 1, \dots, d$:

$$a(j,n)a(j,n+1) = 1 + \left(\prod_{j \to i} a(i,n)^{|C_{ij}|}\right)\left(\prod_{i \to j} a(i,n+1)^{|C_{ij}|}\right) \tag{1}$$

and the initial conditions $a(j, 0) = 1$. The data of these d sequences is called the *frieze* associated to the Cartan matrix and the given acyclic orientation of its graph. This generalizes the construction of Section 2. The acyclicity of the orientation ensures that these sequences are well defined. Moreover, they have clearly coefficients in $\mathbf{Q}_+^*$. Now, it is a consequence of the Laurent phenomenon of Fomin and Zelevinsky that the coefficients are actually positive integers; see [18]; see also [24].

Recall that a sequence $(a_n)_{n \in \mathbf{N}}$ of complex numbers *satisfies a linear recurrence* if for some $k \geqslant 1$, some $\alpha_1, \dots, \alpha_k$ in $\mathbf{C}$, one has: for all n in $\mathbf{N}$, $a_{n+k} = \alpha_1 a_{n+k-1} + \dots + \alpha_k a_n$. Equivalently, the series $\sum_{n \in \mathbf{N}} a_n x^n$ is *rational*, that is, it is the quotient of two polynomials in $\mathbf{C}[x]$; we say also that the sequence (a_n) is rational. We say that a frieze is *rational* if the sequences $a(i, n)$ are all rational for $i = 1, \dots, d$.

For Cartan matrices and the associated matrices, we refer to [22].

Conjecture. *A frieze associated to a Cartan matrix with some acyclic orientation is rational if and only if the Cartan matrix is of Dynkin or Euclidean type.*

One direction of the conjecture is completely solved by the following result.

Theorem 1. *Let C be a connected Cartan matrix with some acyclic orientation. Suppose that the associated frieze is rational. If the sequences are all bounded, then C is of Dynkin type. If they are not all bounded, then C is of Euclidean type.*

Note that since we assume that the Cartan matrix is connected, the sequences are all simultaneously bounded or unbounded. We prove Theorem 1 in Section 4.

We prove the opposite direction of the conjecture in all cases, except for the exceptional Euclidean cases. Actually, we prove more than rationality of the sequences. For this, recall that a series $S = \sum_{n \in \mathbf{N}} a_n x^n \in \mathbf{N}[[x]]$ is called $\mathbf{N}$-*rational* if it satisfies one of the two equivalent conditions (this equivalence is a particular case of the Kleene–Schützenberger theorem, see [5]):

(i) S belongs to the smallest subsemiring of $\mathbf{N}[[x]]$ closed under the operation $T \to T^* = \sum_{n \in \mathbf{N}} T^n$ (which is defined if T has zero constant term);
(ii) for some matrices $\lambda \in \mathbf{N}^{1 \times d}$, $M \in \mathbf{N}^{d \times d}$, $\gamma \in \mathbf{N}^{d \times 1}$, one has: for all n in $\mathbf{N}$, $a_n = \lambda M^n \gamma$.

We then say that the sequence (a_n) is $\mathbf{N}$-*rational*.

A result which we need later in this article is that **N**-rational sequences are closed under Hadamard product; this is the coefficientwise product $((a_n), (b_n)) \mapsto (a_n b_n)$. This follows easily from (ii) by tensoring the matrices. Note that if K is any commutative semiring, we may define K-rational series in exactly the same way, by replacing above **N** by K. See [5].

Theorem 2. *If a Cartan matrix is of Dynkin or Euclidean type, but not one of the exceptional Euclidean types, then the sequences of each associated frieze are* **N***-rational, and in particular, rational.*

Recall that therefore these sequences satisfy linear recursions. Our proof allows to compute these recursions, by using well-known methods, see [17,5]. Theorem 2 will be proved in Section 7. Note that in the Dynkin case, this is an immediate consequence of the finiteness of the set of cluster variables, proved in [20]. Indeed, the construction of the frieze is a particular case of the mutations of Fomin and Zelevinsky, by performing mutations only on sources. See for example [24].

In order to prove Theorem 2, we introduce objects which we call *SL_2-tiling of the plane*. This is a mapping $t : \mathbf{Z}^2 \mapsto \mathbf{N}$ such that for any i, j in $\mathbf{Z}$,

$$\begin{vmatrix} t(i,j) & t(i,j+1) \\ t(i+1,j) & t(i+1,j+1) \end{vmatrix} = 1.$$

An example is given below; the 1's are boldfaced, since they will play a special role in the sequel; we have not represented the numbers above them. The perfect squares in the tiling will be explained further in the article: see Lemma 4.

										1	**1**	**1**	**1**	
									1	**1**	2	3	4	
									1	2	5	8	11	
									1	3	8	13	18	
				…		…		**1**	**1**	2^2	11	18	25	…
								1	2	9	5^2	41	57	
								1	3	14	39	8^2	89	
			1	**1**	**1**	**1**	**1**	**1**	4	19	53	87	11^2	
1	**1**	**1**	**1**	2	3	4	5	6	5^2	119	332	545	758	
1	2	3	4	9	14	19	24	29	121	24^2	1607	2368	3669	…
					…		…				…			

We call *frontier* a bi-infinite sequence

$$\dots x_{-3} x_{-2} x_{-1} x_0 x_1 x_2 x_3 \dots \tag{2}$$

with $x_i \in \{x, y\}$. It is called *admissible* if there are arbitrarily large and arbitrarily small i's such that $x_i = x$, and arbitrarily large and arbitrarily small j's such that $x_j = y$; in other words, none of the two sequences $(x_n)_{n \geqslant 0}$ and $(x_n)_{n \leqslant 0}$ is ultimately constant. Each frontier may be embedded into the plane: the x (resp. y) determine the horizontal (resp. vertical) edges of a bi-infinite discrete path: x (resp. y) corresponds to a segment of the form $[(a,b),(a+1,b)]$ (resp.

$[(a,b),(a,b+1)]$). The vertices of the path (that is, the endpoints of the previous segments) get the label 1.

An example is the frontier

$$\dots yxxxyxxxxxyyyxyyyyxyxxx\dots$$

corresponding to the 1's in the above SL_2-tiling.

We prove below that an admissible frontier, embedded into the plane, may be extended to a unique SL_2-tiling. For this, we need the following notation. Let

$$M(x)=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \text{and} \quad M(y)=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

We extend M into a homomorphism from the free monoid generated by x and y into the group $SL_2(\mathbf{Z})$.

Given an admissible frontier, embedded in the plane as explained previously, let $(u,v)\in\mathbf{Z}^2$. Then we obtain a finite word, which is a factor of the frontier, by projecting the point (u,v) horizontally and vertically onto the frontier. We call this word the *word* of (u,v). It is illustrated in the figure below, where the word of the point M is $yyyxxyx$:

$$\begin{array}{cccccccccccc}
 & & & & & & & & & & & \cdot\\
 & & & & & & & & & & \cdot & \\
 & & & & & & & & & \cdot & & \\
 & & & & & & \mathbf{1} & \mathbf{1} & \mathbf{1} & & & \\
 & & & & \mathbf{1} & \mathbf{1} & \mathbf{1} & | & & & & \\
 & & & & \mathbf{1} & & & | & & & & \\
 & & & & \mathbf{1} & & & | & & & & \\
 & & & \mathbf{1} & \mathbf{1} & - & - & M & & & & \\
 & & \cdot & & & & & & & & & \\
 & \cdot & & & & & & & & & & \\
\cdot & & & & & & & & & & &
\end{array}$$

Note that such a word always begins by y and ends by x. We define the word of a point only for points below the frontier; for points above, the situation is symmetric and we omit it.

Theorem 3. *Given an admissible frontier, there exists a unique SL_2-tiling of the plane t extending the embedding of the frontier into the plane. It is defined, for any point (u,v) below the frontier, with associated word $x_1x_2\dots x_{n+1}$, where $n\geqslant 1$, $x_i\in\{x,y\}$, by the formula*

$$t(u,v)=(1,1)M(x_2)\cdots M(x_n)\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{3}$$

Theorem 3 will be proved in Section 5. In the formula, note that the first and last letter of the word are omitted. An instance of the formula, for the tiling above, is

$$14 = (1,1)\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^3\begin{pmatrix}1\\ 1\end{pmatrix},$$

since the word corresponding to 14 in the figure is y^2xy^3x.

Theorem 3 will be further generalized, by considering tilings with variables. Indeed, in Section 8, we consider frontiers with variables, instead of 1's as in Theorem 3. Then we show that one obtains an SL_2-tiling whose values lie in the semiring of Laurent polynomials over $\mathbf{N}$ in these variables.

4. Proof of Theorem 1

The proof of Theorem 1 requires a precise study of the asymptotics of a sequence which satisfies a linear recursion. This will allow us to produce for each quiver satisfying the hypothesis of the theorem an additive or a sub-additive function of the underlying graph, which implies that the graph is Dynkin or Euclidean.

Given two sequences of positive real numbers (a_k) and (b_k), we shall write $a_k \approx b_k$ to express the fact that for some positive constant C, one has $\lim_{k\to\infty} a_k/b_k = C$.

Lemma 1. *Let $a(j,n)$, for $j = 1,\dots,d$, be d unbounded sequences of positive integers, each satisfying a linear recurrence. There exist an integer $p \geqslant 1$, real numbers $\lambda(j,l) \geqslant 1$ and integers $d(j,l) \geqslant 0$, for $j = 1,\dots,d$ and $l = 0,\dots,p$, and a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ of nonnegative integers, such that*:

(i) *for every $j = 1,\dots,d$ and every $l = 0,\dots,p$, $a(j, pn_k + l) \approx \lambda(j,l)^{n_k} n_k^{d(j,l)}$*;
(ii) *for every $j = 1,\dots,d$, there exists $l = 0,\dots,p$ such that $\lambda(j,l) > 1$ or $d(j,l) \geqslant 1$*;
(iii) *for every $j = 1,\dots,d$, $\lambda(j,0) = \lambda(j,p)$ and $d(j,0) = d(j,p)$.*

This lemma may be well known to the specialist of linear recurrent sequences. Since we could not find a precise reference, we give a proof below.

Proof. Step 1. Recall that each sequence $(a_n)_{n\in\mathbf{N}}$ satisfying a linear recurrence has a unique expression, called the *exponential polynomial*, of the form

$$a_n = \sum_{i=1}^{k} P_i(n)\lambda_i^n, \tag{4}$$

for n large enough, where $P_i(n)$ is a nonzero polynomial in n and the λ_i are distinct nonzero complex numbers; see e.g. [5] or [17]. We call the λ_i the *eigenvalues* of the sequence a_n. The *degree* of λ_i is $deg(P_i)$. Note that if the a_n are all positive integers, then at least one of its eigenvalues has modulus $|\lambda_i| \geqslant 1$. The *principal part* of the above exponential polynomial is

$$n^D \sum_j \alpha_j \lambda_j^n, \tag{5}$$

where the sum is restricted to those j with $|\lambda_j|$ maximum, $D = deg(P_j)$ maximum for these j, and α_j is the coefficient of n^D in P_j. We call these λ_j the *dominating eigenvalues*, *dominating* *modulus* their modulus and D the *maximum degree*. Note that if λ_i is not a dominating eigenvalue, then either λ_i has modulus strictly smaller than the maximum modulus, or its modulus is the maximum modulus, but its degree is strictly smaller than D. For further use, note that if Eq. (5) is the principal part of (a_n), then the principal part of the sequence a_{n+H} is

$$n^D \sum_j \alpha_j \lambda_j^H \lambda_j^n. \tag{6}$$

Note also that for any $p \geqslant 1$ and $l \geqslant 0$, the eigenvalues of the sequence $(a_{np+l})_{n\in\mathbf{N}}$ are p-th powers of eigenvalues of (a_n): it suffices, to see it, to replace n by $np+l$ in Eq. (4).

Step 2. Consider the subgroup G of $\mathbf{C}^*$ generated by the eigenvalues of the d sequences $a(j,n)$. It is a finitely generated abelian group and therefore, by the fundamental theorem of finitely generated abelian groups, there exists $p \geqslant 1$ such that the subgroup G_1 generated by the p-th powers of any set of elements of G is a free abelian group.

Consider the sequence $(a(j, pn+l))_{n\in\mathbf{N}}$, with $j = 1, \ldots, d$ and $l = 0, \ldots, p$. The eigenvalues of the sequence $(a(j, pn+l))_{n\in\mathbf{N}}$ are p-th powers of the eigenvalues of a_n (this follows easily from Eq. (4)). Denote by $Z_{j,l}$ the set of its dominating eigenvalues and

$$Z = \bigcup_{1\geqslant j\geqslant d,\, 0\geqslant l\geqslant p} Z_{j,l}.$$

Then $Z \subset G_1$. In particular, no quotient z/z' with $z, z' \in Z$, is a nontrivial root of unity, since G_1 is a torsion-free group. Note also that $Z_{j,l}$ is nonempty, since the sequences $a(j, pn+l)$ are positive.

Step 3. Suppose that we have proved the lemma for the d sequences $a'(j,n) = a(j, n+ph)$ for some nonnegative integer h. Denote by $\lambda(j,l), d(j,l), (n_k)$ the corresponding numbers and sequences. Then we have $a'(j, pn_k + l) \approx \lambda(j,l)^{n_k} n_k^{d(j,l)}$. Thus $a(j, p(n_k+h)+l) \approx \lambda(j,l)^{n_k} n_k^{d(j,l)}$. This shows that the lemma then holds for the d sequences $a(j,n)$: we simply replace $(n_k)_{k\in\mathbf{N}}$ by $(n_k+h)_{k\in\mathbf{N}}$. We choose h below.

Step 4. The principal part of the sequence $a(j, pn+l)$ is of the form

$$n^{d(j,l)} \sum_{z\in Z_{j,l}} \alpha_z z^n. \tag{7}$$

Consider the sequence $(\sum_{z\in Z_{j,l}} \alpha_z z^n)_{n\in\mathbf{N}}$. Since no quotient of distinct elements of $Z_{j,l}$ is a root of unity, we see by the theorem of Skolem–Mahler–Lech (see [5], Th. 4.1 or [17], Th. 2.1) that the previous sequence has only finitely many zeros. Hence, for some h, there is no zero for $n \geqslant h$. We may choose the same h for each $j = 1, \ldots, d$ and $l = 0, \ldots, p$.

By Step 3 and Eq. (6) with $H = ph$, we may therefore assume that the principal part of $a(j, pn+l)$ is Eq. (7), with $\sum_{z\in Z_{j,l}} \alpha_z \neq 0$.

Step 5. Choose some $z(j,l)$ in $Z_{j,l}$. The complex numbers $z/z(j,l)$, for $z \in Z_{j,l}$, $j = 1, \ldots, d$ and $l = 0, \ldots, p$ have modulus 1 and generate a subgroup of G_1, which by Step 2 is a finitely generated free abelian subgroup of $\mathbf{C}^*$.

Observe that if a finite set E of complex numbers of modulus 1 generates a free abelian subgroup of $\mathbf{C}^*$, then there is a strictly increasing sequence $(n_k)_{k\in\mathbf{N}}$ of nonnegative integers such that: for every $e \in E$, $\lim_{k\to\infty} e^{n_k} = 1$. This follows from Kronecker's simultaneous approximation theorem applied to a basis of the previous free abelian group (write the basis as $\exp(2i\pi x)$, for a finite set of $\mathbf{Q}$-linear independent real numbers x, see [23], Th. 442).

Thus we may assume the existence of n_k with $z^{n_k} \sim_{k\to\infty} z(j,l)^{n_k}$ for any $j = 1,\dots,d$ and $l = 0,\dots,p$.

Step 6. Going back to the principal part Eq. (7), we see that

$$n_k^{d(j,l)} \sum_{z\in Z_{j,l}} \alpha_z z^{n_k} \sim_{k\to\infty} n_k^{d(j,l)} z(j,l)^{n_k} \sum_{z\in Z_{j,l}} \alpha_z$$

since the last sum is nonzero.

Now, since the eigenvalues of $a(j, pn+l)$ which are not in $Z(j,l)$ have modulus strictly smaller than that of $z(j,l)$, or have the same modulus but smaller degree, and since $a(j,n) > 0$, we obtain that

$$a(j, pn_k + l) \approx \lambda(j,l)^{n_k} n_k^{d(j,l)},$$

with $\lambda(j,l) = |z(j,l)|$.

Step 7. In order to prove (ii), we use the fact that $a(j,n)$ is unbounded; hence, for any $j = 1,\dots,d$, there exists $l = 0,\dots,p-1$ such that $a(pn+l)$ is unbounded. Then at least one of its eigenvalues is of modulus > 1, and $|z(j,l)| > 1$, or otherwise, they have all modulus $\leqslant 1$ and some $d(j,l)$ must be > 1.

For (iii), note that $a(j, p(n+1)) = a(j, pn+p)$, hence the sequences $a(j,pn)$ and $a(j,pn+p)$ have the same eigenvalues and maximum degrees. Thus $|z(j,0)| = |z(j,p)|$ and $d(j,0) = d(j,p)$. □

Before proving Theorem 1, we must recall some facts about additive and subadditive functions of diagrams. Let $C = (C_{ij})_{1\leqslant i,j\leqslant d}$ be a Cartan matrix. An *additive* (resp. *subadditive*) *function for* C is a function $f : \{1,\dots,d\} \to \mathbf{R}_+^*$ such that for any $j = 1,\dots,d$, one has $2f(j) = \sum_{i\neq j} f(i)|C_{ij}|$ (resp. $2f(j) \geqslant \sum_{i\neq j} f(i)|C_{ij}|$). Note that, by the properties of a Cartan matrix, this may equivalently be rewritten as $\sum_i f(i)C_{ij} = 0$ (resp. $\sum_i f(i)C_{ij} \geqslant 0$).

The results we need is the following.

Theorem. *A Cartan matrix C is of Euclidean type if and only if there exists an additive function for C; it is of Dynkin type if and only if there exists a subadditive function for C which is not additive.*

The second part of this theorem is due to Vinberg [33] and the first to Berman, Moody and Wonenburger [4]. Both results were proved by Happel, Preiser and Ringel [22], under the assumption that the function takes integer values, although this assumption was unnecessary. We need the generalization involving real valued functions, which thus holds by the proof of [22], Theorem, p. 286.

The idea of the proof of Theorem 1 is as follows: we show first that if the sequences of the frieze are rational and bounded, then there exists a subadditive function which is not additive for the diagram. Hence by the theorem above, the diagram is of Dynkin type. The subadditive function is obtained by multiplying p consecutive values of each sequence, p being a common period, and then taking their logarithm. The recurrence relations Eq. (1) imply that this logarithm is a subadditive function which is not additive.

In the case where the sequences are unbounded, the proof is similar by replacing each sequence by its principal part. However, the proof is more technical, since the growth of a rational sequence is in general not of exponential type. Lemma 1 allows to bypass this difficulty.

Proof of Theorem 1. Case 1. The sequences $a(j,n)$, $j=1,\dots,d$, are all bounded. Since they are integer-valued, they take only finitely many values. Since they satisfy linear recursions, they are ultimately periodic. Let p be a common period and let n_0 be such that each sequence is purely periodic for $n \geqslant n_0$.

Let $b(j) = \prod_{n_0 \leqslant n < n_0+p} a(j,n)$. Note that $b(j) > 1$. Indeed, if $a(j,n) = 1$, then $a(j,n+1) > 1$ by Eq. (1); moreover, each $a(j,n)$ is a positive integer. We have, since $a(j,n_0) = a(j,n_0+p)$,

$$
\begin{aligned}
b(j)^2 &= \Bigl(\prod_{n_0 \leqslant n < n_0+p} a(j,n)\Bigr)\Bigl(\prod_{n_0 \leqslant n < n_0+p} a(j,n+1)\Bigr)\\
&= \prod_{n_0 \leqslant n < n_0+p} a(j,n)a(j,n+1)\\
&= \prod_{n_0 \leqslant n < n_0+p} \Bigl(1 + \Bigl(\prod_{j\to i} a(i,n)^{|C_{ij}|}\Bigr)\Bigl(\prod_{i\to j} a(i,n+1)^{|C_{ij}|}\Bigr)\Bigr)
\end{aligned}
$$

by Eq. (1). Thus

$$
\begin{aligned}
b(j)^2 &> \prod_{n_0 \leqslant n < n_0+p} \Bigl(\Bigl(\prod_{j\to i} a(i,n)^{|C_{ij}|}\Bigr)\Bigl(\prod_{i\to j} a(i,n+1)^{|C_{ij}|}\Bigr)\Bigr)\\
&= \Bigl(\prod_{j\to i}\prod_{n_0 \leqslant n < n_0+p} a(i,n)^{|C_{ij}|}\Bigr)\Bigl(\prod_{i\to j}\prod_{n_0 \leqslant n < n_0+p} a(i,n+1)^{|C_{ij}|}\Bigr)\\
&= \Bigl(\prod_{j\to i} b(i)^{|C_{ij}|}\Bigr)\Bigl(\prod_{i\to j} b(i)^{|C_{ij}|}\Bigr)\\
&= \prod_{i\neq j} b(i)^{|C_{ij}|}.
\end{aligned}
$$

Taking logarithms, we obtain

$$
2\log\bigl(b(j)\bigr) > \sum_{i\neq j} \log\bigl(b(i)\bigr)|C_{ij}|
$$

and we have a subadditive function which is not additive, since $b(j) > 1$.

Case 2. We assume now that some sequence $a(j,n)$ is unbounded. Then by Eq. (1) and the connectedness of the underlying graph of the Cartan matrix, they are all unbounded.

We show, by using Lemma 1, that there exists an additive function for the Cartan matrix. We use freely the notations of this lemma. Define

$$b(j,n) = a(j,n)a(j,n+1)\cdots a(j,n+p-1).$$

Then

$$b(j,pn_k) \approx \lambda(j,0)^{n_k} n_k^{d(j,0)} \cdots \lambda(j,p-1)^{n_k} n_k^{d(j,p-1)} \approx \lambda(j)^{n_k} n_k^{d(j)} \tag{8}$$

where $\lambda(j) = \lambda(j,0)\cdots\lambda(j,p-1)$ and $d(j) = d(j,0) + \cdots + d(j,p-1)$. Now

$$\begin{aligned} b(j,pn_k)^2 = {} & a(j,pn_k)a(j,pn_k+1)a(j,pn_k+1)a(j,pn_k+2)\cdots \\ & a(j,pn_k+p-1)a(j,pn_k). \end{aligned}$$

By the lemma, $a(j,pn_k) \approx a(j,pn_k+p)$. Thus

$$b(j,pn_k)^2 \approx \prod_{0\leqslant l<p} a(j,pn_k+l)a(j,pn_k+l+1).$$

Using Eq. (1), we obtain

$$b(j,pn_k)^2 \approx \prod_{0\leqslant l<p} \left(1 + \left(\prod_{j\to i} a(i,pn_k+l)^{|C_{ij}|}\right)\left(\prod_{i\to j} a(i,pn_k+l+1)^{|C_{ij}|}\right)\right).$$

Let

$$u_k = \left(\prod_{j\to i} a(i,pn_k+l)^{|C_{ij}|}\right)\left(\prod_{i\to j} a(i,pn_k+l+1)^{|C_{ij}|}\right).$$

If u_k is unbounded when $k\to\infty$, then by (i) in Lemma 1, there exists i with: either $j\to i$, and $\lambda(i,l)>1$ or $d(i,l)\geqslant 1$; or $j\leftarrow i$, and $\lambda(i,l+1)>1$ or $d(i,l+1)\geqslant 1$. Then $\lim_{k\to\infty} u_k = \infty$ and $u_k \approx 1+u_k$. Otherwise, u_k is bounded and by Lemma 1, u_k is constant, therefore $u_k \approx 1+u_k$. Thus in both cases, $1+u_k \approx u_k$.

We deduce that

$$\begin{aligned} b(j,pn_k)^2 &\approx \prod_{0\leqslant l<p} \left(\prod_{j\to i} a(i,pn_k+l)^{|C_{ij}|}\right)\left(\prod_{i\to j} a(i,pn_k+l+1)^{|C_{ij}|}\right) \\ &\approx \prod_{0\leqslant l<p} \left(\prod_{j\to i} \lambda(i,l)^{n_k|Cij|} n_k^{d(i,l)|C_{ij}|}\right)\left(\prod_{i\to j} \lambda(i,l+1)^{n_k|Cij|} n_k^{d(i,l+1)|C_{ij}|}\right) \\ &\approx \left(\prod_{j\to i}\prod_{0\leqslant l<p} \lambda(i,l)^{n_k|Cij|} n_k^{d(i,l)|C_{ij}|}\right)\left(\prod_{i\to j}\prod_{0\leqslant l<p} \lambda(i,l+1)^{n_k|Cij|} n_k^{d(i,l+1)|C_{ij}|}\right). \end{aligned}$$

Since $d(i,0) = d(i,p)$ and $\lambda(i,0) = \lambda(i,p)$, we obtain

$$b(j,pn_k)^2 \approx \left(\prod_{j\to i} \lambda(i)^{n_k|Cij|} n_k^{d(i)|C_{ij}|}\right)\left(\prod_{i\to j} \lambda(i)^{n_k|Cij|} n_k^{d(i)|C_{ij}|}\right)$$
$$\approx \prod_{i\neq j} \lambda(i)^{n_k|Cij|} n_k^{d(i)|C_{ij}|}.$$

Thus by Eq. (8),

$$\lambda(j)^{2n_k} n_k^{2d(j)} \approx \prod_{i\neq j} \lambda(i)^{n_k|Cij|} n_k^{d(i)|C_{ij}|}.$$

Therefore, since n_k tends to infinity with k, for $j = 1, \dots, d$,

$$\lambda(j)^2 = \prod_{i\neq j} \lambda(i)^{|Cij|}$$

and

$$2d(j) = \sum_{i\neq j} d(i)|C_{ij}|.$$

If the $d(j)$ are all positive, we have the additive function $d(j)$. If one of them is 0, then they are all 0, by connectedness of the graph and the above equation. In this case, $d(j,l) = 0$ for any j and l. Thus (ii) in the lemma ensures that for any j, some $\lambda(j,l) > 1$ and therefore $\lambda(j) > 1$. Taking logarithms, we find

$$2\log\bigl(\lambda(j)\bigr) = \sum_{i\neq j} \log\bigl(\lambda(i)\bigr)|C_{ij}|$$

and we have therefore an additive function. □

As a consequence of Theorem 1, we obtain a particular case of one implication of the finite type classification of Fomin and Zelevinsky [20]. Note that here the technical Lemma 1 is not necessary and that only the first part of the proof of Theorem 1 (the simpler one) is necessary.

Corollary 1. *Let A be a cluster algebra with initial acyclic seed $(\mathbb{X}, Q)$. If A has only finitely many distinct cluster variables, then Q is a quiver of Dynkin type.*

Proof. Since friezes are obtained by particular mutations, the frieze corresponding to Q takes only finitely many distinct numerical values, the latter being obtained from the cluster variables by evaluating all indeterminates at 1. In this case, the sequences of the frieze are necessarily periodic because of the recurrence (see Eq. (1) above). It then follows from the first part of the proof of Theorem 1 that Q is a Dynkin quiver. □

5. Proof of Theorem 3

Step 1. We prove first that the function t given by Eq. (3) is an SL_2-tiling of the plane. It is enough to show that for any $(u, v) \in \mathbf{Z}^2$, the determinant of the matrix $\left(\begin{smallmatrix} t(u,v) & t(u,v+1) \\ t(u+1,v) & t(u+1,v+1) \end{smallmatrix}\right)$ is equal to 1.

By inspection of the figure below, where $k, l \geqslant 0$ and $w = x_1 \cdots x_n$, $n \geqslant 0$ and $x_i \in \{x, y\}$,

$$
\begin{array}{cccccccccccc}
 & & & & & & & & & & \mathbf{1} & \mathbf{1} \\
 & & & & & & & & & & . & | \\
 & & & & & & & & & & . & | \\
 & & & & & & & & & & . & | \\
 & & & & & & & & & & \mathbf{1} & | \\
 & & & & & & & & & \mathbf{1} & \mathbf{1} & | \\
 & & & & & & & & . & & | & | \\
 & & & & & & & w & & & | & | \\
 & & & & & & . & & & & | & | \\
 & & & & & \mathbf{1} & & & & & | & | \\
\mathbf{1} & . & . & . & \mathbf{1} & \mathbf{1} & - & - & - & - & (u, v) & (u, v+1) \\
\mathbf{1} & - & - & - & - & - & - & - & - & - & (u+1, v) & (u+1, v+1)
\end{array}
$$

it is seen that the words associated to the four points (u, v), $(u, v + 1)$, $(u + 1, v)$ and $(u + 1, v + 1)$ are respectively of the form ywx, $ywxy^l x$, $yx^k ywx$ and $yx^k ywxy^l x$. Let $M = M(x_1 \dots x_n)$. Moreover, denote by $S(A)$ the sum of the coefficients of any matrix A. Then $t(u, v) = S(M)$, $t(u, v+1) = S(MM(x)M(y)^l)$, $t(u, v+1) = S(M(x)^k M(y)M)$ and moreover $t(u+1, v+1) = S(M(x)^k M(y)MM(x)M(y)^l)$. A straightforward computation, which uses the fact that $\det(M) = 1$, then shows that $t(u, v)t(u + 1, v + 1) - t(u, v + 1)t(u + 1, v) = 1$.

Step 2. Clearly $t(u, v) > 0$ for any $(u, v) \in \mathbf{Z}^2$. Then it is easily deduced, by induction on the length of the word associated to (u, v), that $t(u, v)$ is uniquely defined by the SL_2 condition. This proves that the tiling is unique.

6. Properties of SL_2-tilings

Note that it follows from the work of Fomin and Zelevinsky [20] that the friezes of Dynkin type are periodic, so for the proof of Theorem 2, in this case, we have nothing to do. For the Euclidean case, we need to study several properties of SL_2-tilings. The first property shows that the sequences appearing on the tiling satisfy a linear recursion provided that the frontier of the tiling is periodic. Then it will be seen in Section 7.1 that each frieze of type $\tilde{A}_m$ may be simulated by an SL_2-tiling with purely periodic frontier, determined by the orientation of the quiver. For friezes of type $\tilde{D}_m$, things are analogue, however much more involved. Noticing that on the frieze there is a duplication of the numbers (due to the fork in the graph $\tilde{D}_m$), one is lead to study special SL_2-tilings where perfect square appear: see the example in Section 3. Then the proof (Section 7.2) is similar to the case of A_m, although much more technical. For the other Cartan matrices of Euclidean type, excluding the exceptional ones, one is reduced to the two previous cases (Section 7.3).

6.1. Rays and periodic frontiers

Given a mapping $t : \mathbf{Z}^2 \to R$, a point $M \in \mathbf{Z}^2$ and a nonzero vector $v \in \mathbf{Z}^2$, we consider the sequence $a_n = t(M + nv)$. Such a sequence will be called a *ray associated to* t. We call M the *origin* of the ray and v its *directing vector*. The ray is *horizontal* if $v = (1, 0)$, *vertical* if $v = (0, -1)$ and *diagonal* if $v = (1, -1)$.

We say that the frontier Eq. (2) is *ultimately periodic* if for some $p \geqslant 1$, called a *period*, and some $n_0, n_0' \in \mathbf{Z}$, one has:

(i) for $n \geqslant n_0$, $x_n = x_{n+p}$;
(ii) for $n \leqslant n_0'$, $x_n = x_{n-p}$.

Corollary 2. *If the frontier in Theorem* 3 *is ultimately periodic, then each ray associated to* t, *and whose directing vector is of the form* (a, b) *with* $ab \leqslant 0$, *is* $\mathbf{N}$*-rational.*

It is a standard consequence (see [17,5]) that one may compute linear recursions for each ray, if the periodic frontier is given.

Proof. Step 1. The points $M + nv$ are, for n large enough, all above or all below the frontier, since the frontier is admissible and by the hypothesis on the directing vector. Since rationality is not affected by changing a finite number of values, we may, by symmetry, suppose that they are all below.

Step 2. Let w_n be the word associated to the point $M + nv$. By ultimate periodicity of the frontier, there exist an integer $q \geqslant 1$ and words $v_0, \ldots, v_{q-1}, u_0', \ldots, u_{q-1}', u_0, \ldots, u_{q-1}$ such that for any $i = 0, \ldots, q-1$ and for n large enough, $w_{i+nq} = u_i'^n v_i u_i^n$.

Step 3. It follows from Theorem 3 that for some 2×2 matrices M_i', N_i, M_i over $\mathbf{N}$, one has for any $i = 0, \ldots, q-1$ and n large enough, $a_{i+nq} = \lambda_i M_i'^n N_i M_i^n \gamma_i$, where $\lambda_i \in \mathbf{N}^{1\times 2}$, $\gamma_i \in \mathbf{N}^{2\times 1}$.

Step 4. Since rational series over $\mathbf{N}$ are closed under Hadamard product, each series $\sum_{n\in\mathbf{N}} a_{i+nq} x^n$ is rational over $\mathbf{N}$; indeed, such a series is by the formula in Step 3 an $\mathbf{N}$-linear combination of products of Hadamard products of two $\mathbf{N}$-rational series. Therefore

$$\sum_{n\in\mathbf{N}} a_n x^n = \sum_{i=0,\ldots,q-1} x^i \left(\sum_{n\in\mathbf{N}} a_{i+nq} \left(x^q\right)^n \right)$$

is also $\mathbf{N}$-rational. □

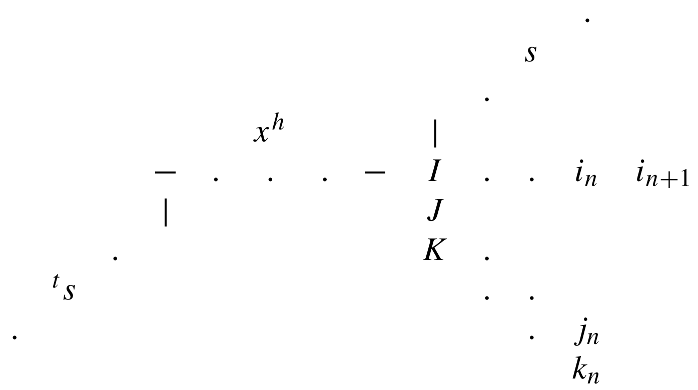
6.2. Symmetric frontiers, perfect squares and quadratic relations

Given a finite or infinite word w on the alphabet $\{x, y\}$, we call *transpose* of w, and denote it by ${}^t w$ the word obtained by reversing it and exchanging x and y. For instance, ${}^t(xyyxy) = xyxxy$. If w is a right infinite word, then its transpose is a left infinite word.

Lemma 2. *Consider an admissible frontier of the form* ${}^t syx^h ys$, *embedded in the plane, with* $h \in \mathbf{N}$. *Let* I, J, K *be the points of the plane defined as follows*: I *corresponds to the point between* x^h *and* y *on the frontier*; J (*resp.* K) *is immediately below* I (*resp.* J); *see the figure.*

Let i_n (resp. j_n, k_n) be the horizontal (resp. diagonal) ray of origin I (resp. J, K) of the SL_2-tiling corresponding to the frontier. Then for any $n \in \mathbf{N}$,

$$j_n = (h+1)i_n^2 \quad \textit{and} \quad k_n + 1 = (h+1)i_n i_{n+1}.$$



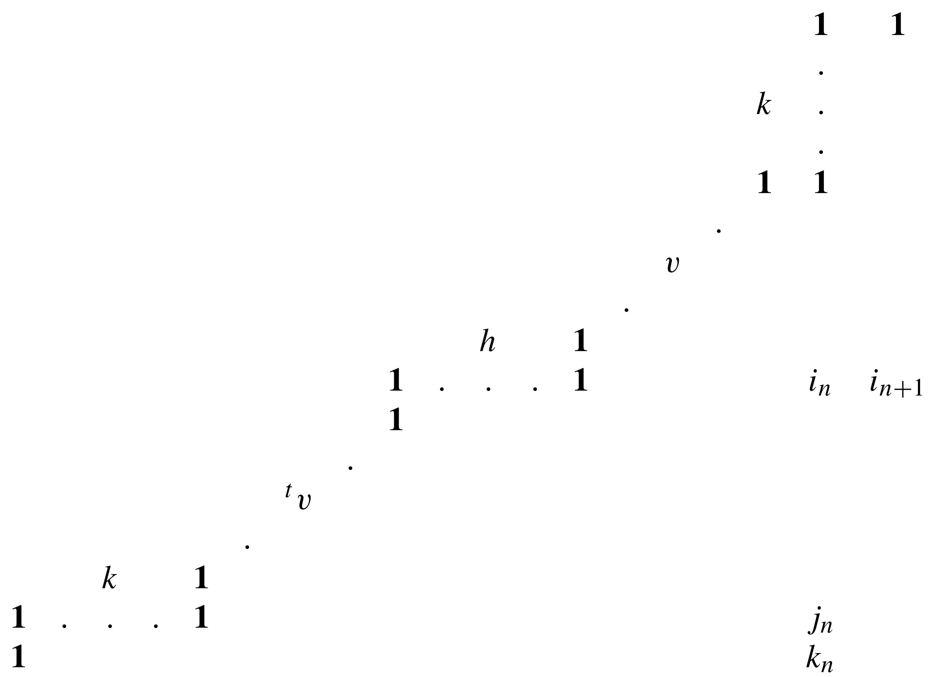
Proof. Step 1. We have

$$M(yx^h y) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} h+1 & h \\ h+2 & h+1 \end{pmatrix}.$$

The quadratic form associated to this matrix (which is not symmetric) is therefore

$$(a,b) \begin{pmatrix} h+1 & h \\ h+2 & h+1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (h+1)a^2 + (2h+2)ab + (h+1)b^2 = (h+1)(a+b)^2.$$

Step 2. The words associated to i_n, j_n, i_{n+1}, k_n are respectively of the form yvx, $y\,{}^t v y x^h y v x$, $yvxy^k x$, $yx^k y\,{}^t v y x^h y v x$; see the figure.



Step 3. Thus $i_n = (1,1)M(v)\binom{1}{1}$ and $j_n = (1,1)M({}^t v y x^h y v)\binom{1}{1}$. Let $\binom{a}{b} = M(v)\binom{1}{1}$. Then $i_n = a+b$ and $j_n = (1,1)M({}^t v)M(yx^h y)M(v)\binom{1}{1} = (a,b)M(yx^h y)\binom{a}{b}$, since $M({}^t v) = {}^t M(v)$. Thus by Step 1, $j_n = (h+1)(a+b)^2 = (h+1)i_n^2$.

Step 4. Let $M(v) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then

$$\begin{aligned} k_n &= (1,1)M\bigl(x^k y\, {}^t v y x^h y v\bigr)\begin{pmatrix}1\\1\end{pmatrix} \\ &= \begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}p & r\\ q & s\end{pmatrix}\begin{pmatrix}h+1 & h\\ h+2 & h+1\end{pmatrix}\begin{pmatrix}p & q\\ r & s\end{pmatrix}. \end{aligned}$$

Furthermore, $i_n = p+q+r+s$ and

$$\begin{aligned} i_{n+1} &= (1,1)M\bigl(vxy^k\bigr)\begin{pmatrix}1\\1\end{pmatrix} \\ &= (1,1)\begin{pmatrix}p & q\\ r & s\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}. \end{aligned}$$

A straightforward computation then shows that $(h+1)i_n i_{n+1} - k_n = ps - rq$. Since $\det(M(v)) = 1$, the lemma is proved. $\square$

Remark. Denote by k'_n the value of the tiling in the point immediately to the right of the value j_n. It is easily shown that one has also $k'_n - 1 = (h+1)i_n i_{n+1}$. Therefore, the 2×2 matrix $\begin{pmatrix} j_n & k'_n \\ k_n & j_{n+1}\end{pmatrix}$, which appears as a connected submatrix of the tiling, encodes a pythagorean triple: indeed, $(j_{n+1} + j_n, j_{n+1} - j_n, k_n + k'_n)$ is such a triple, because $(j_{n+1} - j_n)^2 + (k_n + k'_n)^2 = (h+1)^2(i_{n+1}^2 - i_n^2)^2 + (h+1)^2(2i_n i_{n+1})^2 = (h+1)^2(i_{n+1}^2 + i_n^2)^2 = (j_{n+1} + j_n)^2$. See for example the tiling given in Section 3 and its submatrices $\begin{pmatrix} 1 & 3 \\ 1 & 2^2\end{pmatrix}$, representing the triple $(5,3,4)$, or $\begin{pmatrix} 2^2 & 11 \\ 9 & 5^2\end{pmatrix}$, representing the triple $(29,21,20)$ (here $h=0$).

Lemma 3. *Let $h, h' \in \mathbf{N}$. Consider a frontier of the form $f = s'xy^{h'}xwyx^h ys$, where $w \in \{x,y\}^*$, such that ${}^t s = s'xy^{h'}xw$ and ${}^t s' = wxyx^h ys$. Let $P_0, \ldots, P_k$ be the points corresponding to w, with k the length of w, and $b(j,n)$ be the diagonal rays of origin P_j, for $j = 0, \ldots, k$. Let I, J, K, I', J', K' be the points defined as follows: I (resp. I') corresponds to the point on the frontier between x^h and y (resp. x and $y^{h'}$), J (resp. K) is immediately below I (resp. J), J' (resp. K') is immediately to the right of I' (resp. of J'). Let i_n (resp. i'_n, resp. j_n, k_n, j'_n, k'_n) denote the horizontal (resp. vertical, resp. diagonal) ray of origin I (resp. I', resp. J, K, J', K'). Then $j_n = (h+1)i_n^2$, $k_n + 1 = (h+1)i_n i_{n+1}$, $j'_n = (h'+1)i'^2_n$, $k'_n + 1 = (h'+1)i'_n i'_{n+1}$. Moreover, for $j = 1, \ldots, k-1$, one has, with $w = x_1 \cdots x_k$: for any $n \in \mathbf{N}$, $b(j,n)b(j,n+1) = 1 + B$, where*

$$B = \begin{cases} b(j-1,n+1)b(j+1,n) & \text{if } x_j x_{j+1} = xx, \\ b(j-1,n+1)b(j+1,n+1) & \text{if } x_j x_{j+1} = xy, \\ b(j-1,n)b(j+1,n) & \text{if } x_j x_{j+1} = yx, \\ b(j-1,n)b(j+1,n+1) & \text{if } x_j x_{j+1} = yy. \end{cases}$$

Note that the points given in the lemma need not be all distinct. The lemma is illustrated by the following figure, where $h = 1$, $h' = 0$, and $w = xyxx$; in bold are represented the 1's corresponding to the factors $xy^{h'}x = xx$ and $yx^h y = yxy$ of the frontier.

$$
\begin{array}{lllllllllll}
 & & & & & & & & & 1 & 1 \\
 & & & & & & & & & 1 & 2 \\
 & & & \cdots & & & & & & 1 & 3 \\
 & & & & & & & & 1 & 1 & 4 \\
 & & & & & & & & 1 & 2 & 9 \\
 & & & & & & & & \mathbf{1} & 3 & 14 \\
 & & & & & & & \mathbf{1} & \mathbf{1}(I) & 4 & 19 \\
 & & & & & 1(P_2) & 1(P_3) & \mathbf{1}(P_4) & 2\cdot 1^2(J) & 9 & 43 \\
 & & \mathbf{1} & \mathbf{1}(I') & \mathbf{1}^2(J'=P_0) & 1(K'=P_1) & 2 & 3 & 7(K) & 2\cdot 4^2 & 153 \\
 & 1 & 1 & 2 & 3 & 2^2 & 9 & 14 & 33 & 151 & 2\cdot 19^2 \\
 & 1 & 2 & 5 & 8 & 11 & 5^2 & 39 & & & \\
1 & 1 & 3 & 8 & & & & & & & \\
 & & & & & & & & \cdots & &
\end{array}
$$

Proof. This follows from Lemma 1, and its symmetrical statement, together with the inspection of the figure below, which shows the four different possible configurations.

$$
\begin{array}{lllllll}
P_{j-1} & P_j & P_{j+1} & & & & \\
 & . & . & . & & & \\
 & & . & . & . & & \\
 & & & & b(j-1,n) & b(j,n) & b(j+1,n) \\
 & & & & & b(j-1,n+1) & b(j,n+1)
\end{array}
$$

$$
\begin{array}{llllll}
 & P_{j+1} & & & & \\
P_{j-1} & P_j & . & & & \\
 & . & . & . & & \\
 & & . & . & b(j+1,n) & \\
 & & & b(j-1,n) & b(j,n) & b(j+1,n+1) \\
 & & & & b(j-1,n+1) & b(j,n+1)
\end{array}
$$

$$
\begin{array}{lllll}
P_j & P_{j+1} & & & \\
P_{j-1} & . & . & & \\
 & . & . & . & \\
 & & . & b(j,n) & b(j+1,n) \\
 & & & b(j-1,n) & b(j,n+1)
\end{array}
$$

$$
\begin{array}{lllll}
P_{j+1} & & & & \\
P_j & . & & & \\
P_{j-1} & . & . & & \\
 & . & . & b(j+1,n) & \\
 & & . & b(j,n) & b(j+1,n+1) \\
 & & & b(j-1,n) & b(j,n+1)
\end{array}
$$

□

Lemma 4. *Let h, h' in* $\mathbf{N}$ *and w in $\{x, y\}^*$. Then there exists a periodic frontier $f = s'xy^{h'}xwyx^hys$ satisfying the hypothesis of Lemma* 3.

Proof. Let indeed

$$f = {}^{\infty}\big(wxy^h x\, {}^t wxy^{h'}x\big)\big(wyx^h y\, {}^t wyx^{h'}y\big)^{\infty}. \tag{9}$$

In other words, we take $s = ({}^t wyx^{h'}ywyx^h y)^{\infty}$ and $s' = {}^{\infty}(xy^{h'}xwxy^h x\, {}^t w)$. Then ${}^t s = {}^{\infty}(xy^h x\, {}^t wxy^{h'}xw) = {}^{\infty}(xy^{h'}xwxy^h x\, {}^t w)xy^{h'}xw = s'xy^{h'}xw$. Similarly ${}^t s' = wyx^h ys$. $\square$

7. Proof of Theorem 2

We completely omit the case of Dynkin diagrams, since Theorem 2 in this case follows immediately from the finiteness of the set of cluster variables, see [20].

7.1. The case $\tilde{A}_m$

Let $1, \ldots, m+1$ be the vertices of the graph $\tilde{A}_m$, with edges $\{j, j+1\}$, $j = 1, \ldots, m+1$, with $j+1$ taken $mod.m+1$. An acyclic orientation being given, let $x_j = x$ if the orientation is $j \to j+1$ and $x_j = y$ if it is $j \leftarrow j+1$. Let $a(j, n)$ be the sequences of the frieze, $j = 1, \ldots, m+1$. We extend the notation $a(j, n)$ to $j \in \mathbf{Z}$ by taking $j\ mod.m+1$. Let w be the word $x_1 \cdots x_{m+1}$, which encodes the orientation. Then ${}^{\infty}w^{\infty}$ is an admissible frontier; indeed, x and y appear both in w, since the orientation is acyclic. Embed this frontier into the plane and denote by P_j, with $j \in \mathbf{Z}$, the successive points of this embedding, in such a way that P_j corresponds to the point between x_{j-1} and x_j, with j taken $mod.m+1$. Let t be the tiling given by Theorem 3. Because of the periodicity of the frontier, the diagonal ray $b(j, n)$ of origin P_j, $j \in \mathbf{Z}$, depends only on the class of $j\ mod.m$.

We claim that the ray $b(j, n)$ is equal to $a(j, n)$, for $j = 1, \ldots, m+1$. This is true for $n = 0$, since both are equal to 1. It is enough to show that $b(j, n)$ satisfies Eq. (1). Fix $j = 1, \ldots, m+1$. We have four cases according to the relative positions of P_{j-1}, P_j, P_{j+1} (see the figure in the proof of Lemma 3).

They correspond to the four possible values of the couple (x_{j-1}, x_j):

$$(x, x), \quad (x, y), \quad (y, x), \quad (y, y).$$

By definition of w, these four cases correspond to the four possible orientations:

$$j-1 \to j \to j+1, \quad j-1 \to j \leftarrow j+1,$$
$$j-1 \leftarrow j \to j+1, \quad j-1 \leftarrow j \leftarrow j+1.$$

Thus, by Eq. (1), they correspond to the four induction formulas $a(j, n+1) = \frac{1+A}{a(j,n)}$, where A takes one of the four possible values:

$$a(j-1, n+1)a(j+1, n), \quad a(j-1, n+1)a(j+1, n+1),$$
$$a(j-1, n)a(j+1, n), \quad a(j-1, n)a(j+1, n+1).$$

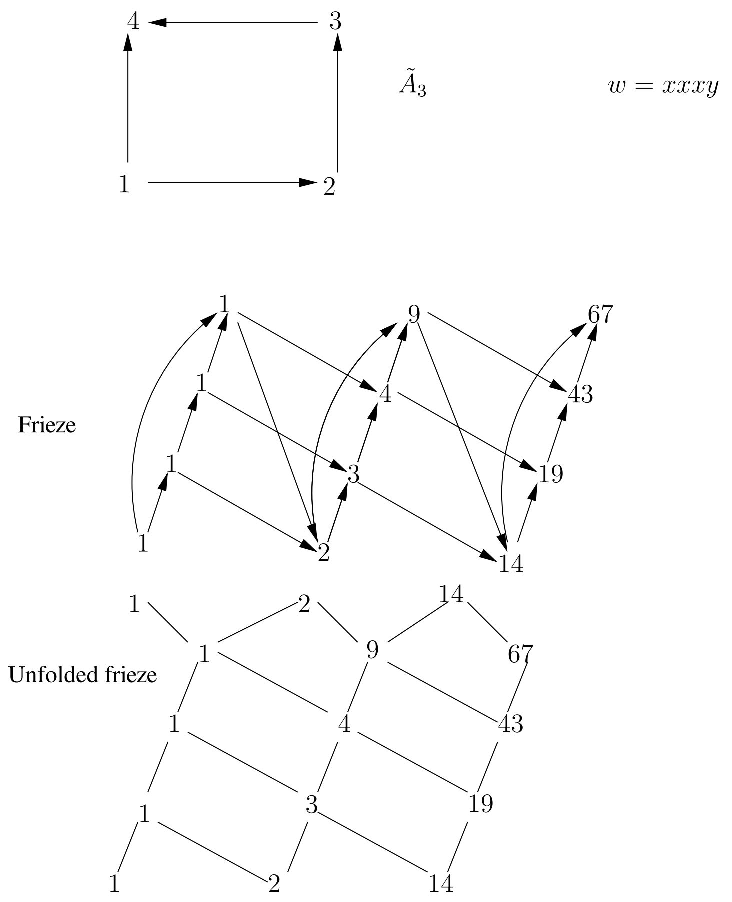


Fig. 3. Quiver $\tilde{A}_3$ and frieze.

Regarding the tiling, these four cases correspond to the four possible configurations, shown in the same figure. Hence, by the SL_2-condition, they correspond to the four induction formulas for $b(j,n)$: $b(j,n+1) = \frac{1+B}{b(j,n)}$, where B takes one of the four possible values:

$$b(j-1,n+1)b(j+1,n), \quad b(j-1,n+1)b(j+1,n+1),$$
$$b(j-1,n)b(j+1,n), \quad b(j-1,n)b(j+1,n+1).$$

This concludes the proof, by using Corollary 1.

The proof is illustrated in the tiling below and in Fig. 3, for a specific orientation of $\tilde{A}_3$.

$$\begin{array}{cccccccccccccc}
 & & & & & & & & & & & & 1 & \\
 & & & & & \dots & & & & 1 & 1 & 1 & 1 & \\
 & & & & & & 1 & 1 & 1 & 1 & 2 & 3 & 4 & \\
 & & & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 9 & 14 & 19 & \\
1 & 1 & 1 & 1 & 2 & 3 & 4 & 9 & 14 & 19 & 43 & 67 & & \dots \\
1 & 2 & 3 & 4 & 9 & 14 & 19 & 43 & 67 & & & & & \\
 & & & & \dots & & & & & & & & &
\end{array}$$

Fig. 4. Quiver $\tilde{D}_m$.

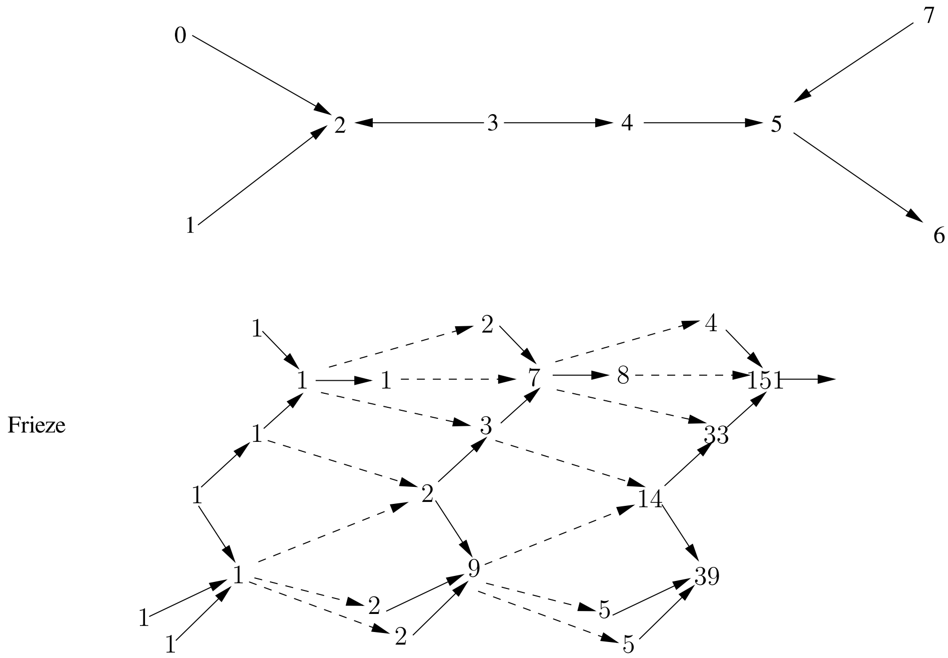


Fig. 5. Quiver $\tilde{D}_7$.

7.2. *The case* $\tilde{D}_m$

We consider an orientation of $\tilde{D}_m$, of the form shown in Fig. 4, where the orientations of the edges $\{i, i+1\}$ for $i = 2, \ldots, m-3$ are arbitrary. The other cases, which differ from the case considered here by changing the orientation of the forks, are similar.

An example is shown in Fig. 5. The reader may recognize that this frieze is encoded in the tiling shown in Section 6.2 after the statement of Lemma 3.

Step 1. We define a word w, which encodes the orientation, as follows: $w = x_1 x_2 \cdots x_{m-3}$, with

$$x_i = \begin{cases} x & \text{if } i \to i+1, \\ y & \text{if } i \leftarrow i+1. \end{cases}$$

Note that by our choice in Fig. 4, $x_1 = x$. In the example, $w = xyxx$. We consider now the frontier given by Lemma 4, with $h' = 0$ and $h = 1$; see Eq. (9). The associated rays $b(j, n)$ are all rational by Corollary 2.

Step 2. Taking the notations of Lemma 3, we have for any $n \in \mathbf{N}$:

$$j'_n = b(0, n), \qquad k'_n = b(1, n), \qquad k_n = b(m-3, n+1).$$

For j'_n and k'_n this follows from $h' = 0$ and therefore $J' = P_0$, $K' = P_1$, hence j'_n (resp. k'_n) and $b(0,n)$ (resp. $b(1,n)$) are diagonal rays with the same origin. Moreover w is of length k in Lemma 3 and here of length $m-3$, hence $k = m-3$. Since $h = 1$, we have the configuration shown below.

$$\begin{array}{ll} & 1 \\ 1 & 1(I) \\ 1(P_{m-3}) & (J) \\ & (K) \end{array}$$

This implies $k_n = b(m-3, n+1)$.

Step 3. In accordance with Lemma 3, we have for $j = 1, \ldots, k$,

$$x_j = \begin{cases} x & \text{if } [P_{j-1}, P_j] \text{ is horizontal,} \\ y & \text{if } [P_{j-1}, P_j] \text{ is vertical.} \end{cases}$$

Step 4. We define $m+1$ sequences $a'(j,n)$ for $j = 0, \ldots m$. First, for any $n \in \mathbf{N}$,

$$a'(0,n) = a'(0,n) = a'(1,n) = i'_n.$$

Now, for $j = 2, \ldots, m-2$,

$$a'(j,n) = b(j-1,n).$$

Furthermore,

$$a'(m-1,n) = \begin{cases} i_n & \text{if } n \text{ is even,} \\ 2i_n & \text{if } n \text{ is odd.} \end{cases}$$

Finally, for $n \geqslant 1$,

$$a'(m,n) = \begin{cases} i_{n-1} & \text{if } n \text{ is even,} \\ 2i_{n-1} & \text{if } n \text{ is odd,} \end{cases}$$

with $a'(m,0) = 1$.

Observe that for any $n \in \mathbf{N}$, $a'(m-1,n)a'(m,n+1) = 2i_n^2$. Moreover, since the sequences i_n, i'_n and $b(j,n)$ are rays, they are rational. Hence, so are the sequences $a'(j,n)$ (for $a'(m-1,n)$ and $a'(m,n)$, this follows from standard constructions on rational sequences).

Step 5. In order to end the proof, it is enough to show that $a(j,n) = a'(j,n)$. First note that $a(j,0) = a'(j,0)$, as is easily verified. Thus it suffices to show that $a'(j,n)$ satisfies the same recursion formula as $a(j,n)$, that is, Eq. (1).

Step 6. We have

$$\begin{aligned} a(0,n+1)a(0,n) &= i'_{n+1}i'_n \quad \text{by Step 4} \\ &= 1 + k'_n \quad \text{by Lemma 3} \\ &= 1 + b(1,n) \quad \text{by Step 2} \\ &= 1 + a'(2,n) \quad \text{by Step 4.} \end{aligned}$$

Similarly

$$a'(1,n+1)a'(1,n) = 1 + a'(2,n).$$

This is the good recursion for $a'(0,n)$ and $a'(1,n)$ since, by Fig. 4 and Eq. (1),

$$a(0,n+1)a(0,n) = 1 + a(2,n),$$
$$a(1,n+1)a(1,n) = 1 + a(2,n).$$

Step 7. Let

$$n' = \begin{cases} n & \text{if } x_2 = x, \\ n+1 & \text{if } x_2 = y. \end{cases}$$

We have by Step 4, $a'(2,n+1)a'(2,n) = b(1,n+1)b(1,n)$. Looking at the figures below, where we use Step 3:

Case $n' = n$, $x_2 = x$,

$$\begin{array}{llllll} P_0 & P_1 & P_2 & b(1,n) & b(2,n) \\ & & & b(0,n+1) & b(1,n+1) \end{array}$$

Case $n' = n+1$, $x_2 = y$,

$$\begin{array}{llll} & P_2 & b(1,n) & b(2,n+1) \\ P_0 & P_1 & b(0,n+1) & b(1,n+1) \end{array}$$

we see that this is equal to $1 + b(0,n+1)b(2,n')$. Using Step 2, Step 4, Lemma 3 then Step 4 again, we obtain

$$\begin{aligned} b(0,n+1)b\bigl(2,n'\bigr) &= j'_{n+1}a'\bigl(3,n'\bigr) \\ &= i'^{2}_{n+1}a'\bigl(3,n'\bigr) \\ &= a'(0,n+1)a'(1,n+1)a'\bigl(3,n'\bigr). \end{aligned}$$

Thus $a'(2,n+1)a'(2,n) = 1 + a'(0,n+1)a'(1,n+1)a'(3,n')$. This corresponds to Eq. (1) for $(2,n)$, that is, $a(2,n+1)a(2,n) = 1 + a(0,n+1)a(1,n+1)a(3,n')$.

Step 8. Let

$$n'' = \begin{cases} n & \text{if } x_{m-3} = y, \\ n+1 & \text{if } x_{m-3} = x. \end{cases}$$

We have

$$\begin{aligned} a'(m-2,n+1)a'(m-2,n) &= b(m-3,n+1)b(m-3,n) \quad \text{by Step 4} \\ &= 1 + b\bigl(m-4,n''\bigr)j_n, \end{aligned}$$

where the second equality follows from Figs. 6 and 7, which use Step 3:

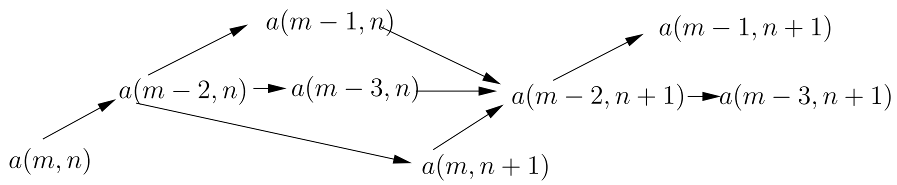


Fig. 6. Frieze in the case $n''=n$.

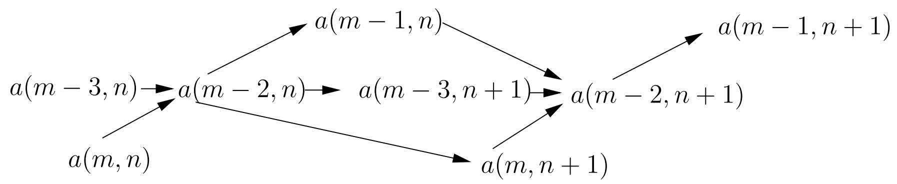


Fig. 7. Frieze in the case $n''=n+1$.

Case $n''=n$, $x_{m-3}=y$,

$$\begin{array}{llll} P_{m-3} & J & b(m-3,n) & j_n \\ P_{m-4} & & b(m-4,n) & b(m-3,n+1) \end{array}$$

Case $n''=n+1$, $x_{m-3}=x$,

$$\begin{array}{lllll} P_{m-4} & P_{m-3} & J & b(m-3,n) & j_n \\ & & & b(m-4,n+1) & b(m-3,n+1) \end{array}$$

By Lemma 3, $j_n=2i_n^2$ and by Step 4, $2i_n^2=a'(m-1,n)a'(m,n+1)$. Furthermore, $b(m-4,n'')=a'(m-3,n'')$. Thus

$$a'(m-2,n+1)a'(m-2,n)=1+a'\big(m-3,n''\big)a'(m-1,n)a'(m,n+1).$$

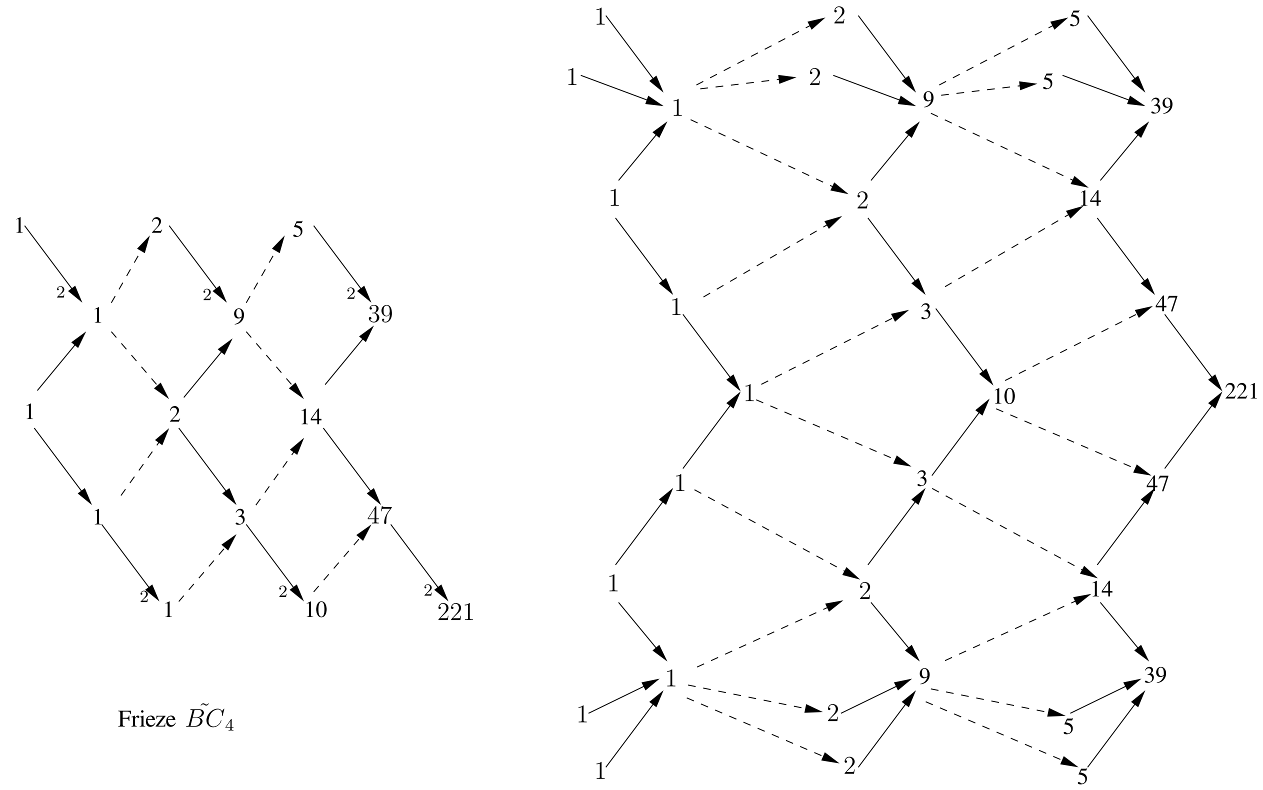
This equality corresponds to Eq. (1) for $a(m-2,n)$, if we look at the two Figs. 6 and 7, representing the frieze in the two cases $n''=n$, $x_{n-3}=y$ and $n''=n+1$, $x_{n-3}=x$.

Step 9. We have

$$\begin{aligned} a'(m-1,n+1)a,(m-1,n) &= 2i_ni_{n+1} \quad \text{by Step 4} \\ &= 1+k_n \quad \text{by Lemma 3} \\ &= 1+b(m-3,n+1) \quad \text{by Step 2} \\ &= 1+a'(m-2,n+1) \quad \text{by Step 4,} \end{aligned}$$

in accordance with Eq. (1), which gives $a(m-1,n+1)a(m-1,n)=1+a(m-2,n+1)$. Moreover, for $n\geqslant 1$,

$$\begin{aligned} a'(m,n+1)a'(m,n) &= 2i_ni_{n-1} \quad \text{by Step 4} \\ &= 1+k_{n-1} \quad \text{by Lemma 3} \\ &= 1+b(m-3,n) \quad \text{by Step 2} \\ &= 1+a'(m-2,n) \quad \text{by Step 4,} \end{aligned}$$



Frieze $\tilde{BC}_4$

Simulation by frieze$\tilde{D}_{10}$

Fig. 8. Frieze simulation.

in accordance with Eq. (1), which gives $a(m,n+1)a(m,n) = 1 + a(m-2,n)$, noting that for $n = 0$: $a'(m,1)a'(m,0) = 2i_0$ (by Step 4) $= 2$ and $1 + a'(m-2,0) = 1 + 1 = 2$.

7.3. Cases $\tilde{B}_m$, $\tilde{C}_m$, $\tilde{B}C_m$, $\tilde{B}D_m$, $\tilde{C}D_m$.

Each frieze in these cases is reduced to a frieze of type $\tilde{A}$ or $\tilde{D}$. Precisely: $\tilde{B}_m$ is reduced to $\tilde{D}_{m+2}$; $\tilde{C}_m$ is reduced to $\tilde{A}_{2m}$; $\widetilde{BC}_m$ is reduced to $\tilde{D}_{2m+2}$; $\widetilde{BD}_m$ is reduced to $\tilde{D}_{m+1}$; $\widetilde{CD}_m$ is reduced to $\tilde{D}_{2m}$.

We give no formal proof, but an example; it should convince the reader. In this example, we show how a frieze of type $\widetilde{BC}_4$ can be simulated by a frieze of type $\tilde{D}_{10}$. See Fig. 8.

8. Friezes and tilings with variables

As mentioned in the introduction, if we replace the initial values $a(j,0)$ in Section 3 by commuting variables, and keep the recurrence of Eq. (1) unchanged, we obtain friezes of variables. The variables therefore obtained are usual cluster variables, in the sense of Fomin and Zelevinsky. It is well known that all cluster variables, but finitely many of them, can be obtained in this way (those not obtained in this way being the cluster variables corresponding to the exceptional objects, in the corresponding cluster category, lying in tubes or $\mathbb{Z}A_\infty$ components).

Likewise, we generalize the SL_2-tilings; these are simply fillings of the discrete plane by elements of a ring R such that each 2×2 connected minor is of determinant 1. We generalize Theorem 3 by putting variables on the frontier.

8.1. Case $\tilde{A}_m$

We call (*generalized*) *frontier* a bi-infinite sequence

$$\ldots x_{-2}a_{-2}x_{-1}a_{-1}x_0a_0x_1a_1x_2a_2x_3a_3\ldots \tag{10}$$

where $x_i \in \{x, y\}$ and a_i are variables, for any $i \in \mathbf{Z}$. It is called *admissible* if there are arbitrarily large and arbitrarily small i's such that $x_i = x$, and similarly for y; in other words, none of the two sequences $(x_n)_{n\geqslant 0}$ and $(x_n)_{n\leqslant 0}$ is ultimately constant. The a_i's are called the *variables* of the frontier. Each frontier may be embedded into the plane: the variables label points in the plane, and the x (resp. y) determine a bi-infinite discrete path, in such a way that x (resp. y) corresponds to a segment of the form $[(a,b),(a+1,b)]$ (resp. $[(a,b),(a,b+1)]$). For example, corresponding to the frontier $\ldots a_{-2}xa_{-1}xa_0ya_1ya_2xa_3\ldots$ is given below:

$$\begin{array}{cccccccccc}
 & & & & & & & & & \cdot \\
 & & & & & & & & \cdot & \\
 & & & & & & & \cdot & & \\
 & & & & & a_2 & a_3 & & & \\
 & & & & & a_1 & & & & \\
 & & & a_{-2} & a_{-1} & a_0 & & & & \\
 & & \cdot & & & & & & & \\
 & \cdot & & & & & & & & \\
 \cdot & & & & & & & & &
\end{array}$$

Formally we do as follows. We define a partial function f from $\mathbf{Z}^2$ into the semiring of Laurent polynomials over $\mathbf{N}$ generated by the variable, defined up to translation, as follows: fix some $(k,l) \in \mathbf{Z}^2$ and $i \in \mathbf{Z}$; then $f(k,l) = a_i$; moreover, if $x_{i+1}x_{i+2}\ldots x_p$ labels the discrete path from (k,l) to (k',l'), then $f(k',l') = a_p$; furthermore, if $x_px_{p+1}\ldots x_i$ labels the path from (k',l') to (k,l), then $f(k',l') = a_{p-1}$.

We see below that an admissible frontier, embedded into the plane, may be extended to an SL_2-tiling. For this, we need the following notation. Let

$$M(a,x,b) = \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \quad \text{and} \quad M(a,y,b) = \begin{pmatrix} b & 0 \\ 1 & a \end{pmatrix}.$$

Note that these matrices reduce to the matrices $M(x)$ and $M(y)$ when the variables a and b are set to 1.

Given an admissible frontier, embedded in the plane as explained previously, let $(u,v) \in \mathbf{Z}^2$. Then we obtain a finite word, which is a factor of the frontier, by projecting the point (u,v) horizontally and vertically onto the frontier. We call this word the *word* of (u,v). It is illustrated in the figure below, where the word of the point $M = (k,l)$ is $a_{-3}ya_{-2}ya_{-1}ya_0xa_1xa_2ya_3xa_4$:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & \cdot \\
 & & & & & & & & & \cdot & \\
 & & & & & & & & \cdot & & \\
 & & & & & a_3 & a_4 & a_5 & & & \\
 & & & a_0 & a_1 & a_2 & | & & & & \\
 & & & a_{-1} & & & | & & & & \\
 & & & a_{-2} & & & | & & & & \\
 & & a_{-4} & a_{-3} & - & - & M & & & & \\
 & \cdot & & & & & & & & & \\
\cdot & & & & & & & & & &
\end{array}
$$

We define the word of a point only for points below the frontier; for points above, things are symmetric and we omit this case. We call *denominator* of the point M the product of the variables of its word, excluding the two extreme ones. In the example, its denominator is $a_{-2}a_{-1}a_0a_1a_2a_3$.

Theorem 4. *Given an admissible frontier, there exists a unique SL_2-tiling t of the plane, with values in the semiring of Laurent polynomials over $\mathbf{N}$ generated by the variables lying on the frontier, extending the embedding of the frontier into the plane. It is defined, for any point (u, v) below the frontier, with associated word $a_0x_1a_1x_2 \dots x_{n+1}a_{n+1}$, where $n \geqslant 1$ and $x_i \in \{x, y\}$, by the formula*

$$t(u,v) = \frac{1}{a_1a_2 \dots a_n}(1, a_0)M(a_1, x_2, a_2)M(a_2, x_3, a_3) \cdots M(a_{n-1}, x_n, a_n)\begin{pmatrix} 1 \\ a_{n+1} \end{pmatrix}. \tag{11}$$

In order to prove the theorem, we need two lemmas, where R denotes some commutative ring. We extend the notation $M(a, x, b)$ and $M(a, y, b)$ for a, b in R.

Lemma 5.

(i) *Let $A \in R^{2\times 2}$, $\lambda, \lambda' \in R^{1\times 2}$, $\gamma, \gamma' \in R^{2\times 1}$, and define $p = \lambda A\gamma$, $q = \lambda A\gamma'$, $r = \lambda' A\gamma$, $s = \lambda' A\gamma'$. Then* $\det\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = \det(A)\det\left(\begin{smallmatrix} \lambda \\ \lambda' \end{smallmatrix}\right)\det(\gamma, \gamma')$.

(ii) *Let $a, b_1, \dots, b_k, b \in R$, $\lambda' = (1, a)M(b_1, x, b_2) \cdots M(b_{k-1}, x, b_k)M(b_k, y, b)$ and $\lambda = (1, b_k)$. Then* $\det\left(\begin{smallmatrix} \lambda' \\ \lambda \end{smallmatrix}\right) = b_1 \cdots b_k b$.

Proof. (i) This follows since $\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda \\ \lambda' \end{smallmatrix}\right)A(\gamma, \gamma')$.

(ii) Let $N = M(b_1, x, b_2) \cdots M(b_{k-1}, x, b_k)$. Then $N = \left(\begin{smallmatrix} b_1 \cdots b_{k-1} & u \\ 0 & b_2 \cdots b_k \end{smallmatrix}\right)$ and $(1, a)N = (b_1 \cdots b_{k-1}, u + ab_2 \cdots b_k)$. Thus

$$\lambda' = (1, a)N\begin{pmatrix} b & 0 \\ 1 & b_k \end{pmatrix} = \big(bb_1 \cdots b_{k-1} + u + ab_2 \cdots b_k, ub_k + ab_2 \cdots b_{k-1}b_k^2\big).$$

It follows that

$$\det\begin{pmatrix} \lambda' \\ \lambda \end{pmatrix} = (bb_1 \cdots b_{k-1} + u + ab_2 \cdots b_k)b_k - \big(ub_k + ab_2 \cdots b_{k-1}b_k^2\big) = bb_1 \cdots b_k. \qquad \square$$

Lemma 6. *Let $A \in R^{2\times 2}$ and $a, b_1, \dots, b_k, b, c, c_1, \dots, c_l, d \in R$, $k, l \geqslant 1$. Define*

$$p = (1, b_k)A\begin{pmatrix} 1 \\ c_1 \end{pmatrix},$$

$$q = (1, b_k)AM(c, x, c_1)M(c_1, y, c_2)\cdots M(c_{l-1}, y, c_l)\begin{pmatrix} 1 \\ d \end{pmatrix},$$

$$r = (1, a)M(b_1, x, b_2)\cdots M(b_{k-1}, x, b_k)M(b_k, y, b)A\begin{pmatrix} 1 \\ c_1 \end{pmatrix},$$

$$s = (1, a)M(b_1, x, b_2)\cdots M(b_{k-1}, x, b_k)M(b_k, y, b)AM(c, x, c_1)M(c_1, y, c_2)\cdots M(c_{l-1}, y, c_l)\begin{pmatrix} 1 \\ d \end{pmatrix}.$$

Then $\det\left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) = b_1 \cdots b_k b c c_1 \cdots c_l \det(A)$.

Proof. Let $\lambda = (1, b_k)$, $\gamma = \left(\begin{smallmatrix} 1 \\ c_1 \end{smallmatrix}\right)$, $\gamma' = M(c, x, c_1)M(c_1, y, c_2)\cdots M(c_{l-1}, y, c_l)\left(\begin{smallmatrix} 1 \\ d \end{smallmatrix}\right)$, $\lambda' = (1, a)M(b_1, x, b_2)\cdots M(b_{k-1}, x, b_k)M(b_k, y, b)$. By Lemma 5,

$$\det\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \det(A)\det\begin{pmatrix} \lambda \\ \lambda' \end{pmatrix}\det(\gamma, \gamma').$$

By Lemma 5 again, $\det\left(\begin{smallmatrix} \lambda \\ \lambda' \end{smallmatrix}\right) = -b_1 \cdots b_k b$ and symmetrically, $\det(\gamma, \gamma') = -cc_1 \cdots c_l$, which ends the proof. $\square$

Proof of Theorem 4. We prove that t given by Eq. (3) is an SL_2-tiling of the plane. It is enough to show that for any $(u, v) \in \mathbf{Z}^2$, the determinant of the matrix $\left(\begin{smallmatrix} t(u,v) & t(u,v+1) \\ t(u+1,v) & t(u+1,v+1) \end{smallmatrix}\right)$ is equal to 1.

By inspection of the figure below, where $k, l \geqslant 1$ and $a, b_1, \dots, b_k, b, c, c_1, \dots, c_l, d$ are in R and $w = x_1 e_1 \cdots e_{n-1} x_n$, $n \geqslant 0$, with $e_i \in R$ and $x_i \in \{x, y\}$,

$$\begin{array}{cccccccccccc}
 & & & & & & & & & & c_l & d \\
 & & & & & & & & & & . & | \\
 & & & & & & & & & & . & | \\
 & & & & & & & & & & . & | \\
 & & & & & & & & & & c_2 & | \\
 & & & & & & & & & c & c_1 & | \\
 & & & & & & & & . & & | & | \\
 & & & & & & & w & & & | & | \\
 & & & & & & . & & & & | & | \\
 & & & & & b & & & & & | & | \\
b_1 & . & . & . & b_{k-1} & b_k & - & - & - & - & (u, v) & (u, v+1) \\
a & - & - & - & - & - & - & - & - & - & (u+1, v) & (u+1, v+1)
\end{array}$$

it is seen that the words associated to the four points (u,v), $(u,v+1)$, $(u+1,v)$ and $(u+1,v+1)$ are respectively of the forms $b_k ybwcxc_1$, $b_k ybwcxc_1 yc_2\cdots yc_l xd$, $ayb_1x\cdots b_{k-1}xb_k ybwcxc_1$ and $ayb_1x\cdots b_{k-1}xb_k ybwcxc_1yc_2\cdots yc_lxd$.

Let $A = M(b,x_1,e_1)M(e_1,x_2,e_2)\cdots M(e_{n-1},x_n,c)$ and $D = be_1e_2\cdots e_{n-1}c$. Then define p,q,r,s as in Lemma 6. Thus we have $t(u,v)=\frac{p}{D}$, $t(u,v+1)=\frac{q}{Dc_1\cdots c_l}$, $t(u+1,v)=\frac{r}{b_1\cdots b_kD}$ and $t(u+1,v+1)=\frac{s}{b_1\cdots b_kDc_1\cdots c_l}$. Therefore by Lemma 6, $\Delta = t(u,v)t(u+1,v+1) - t(u,v+1)t(u+1,v) = \frac{1}{b_1\cdots b_kD^2c_1\cdots c_l}(ps-rq) = \frac{b_1\cdots b_kbcc_1\cdots c_l}{b_1\cdots b_kD^2c_1\cdots c_l}\det(A)$. Now $\det(A) = be_1^2e_2^2\cdots e_{n-1}^2c$ and $D^2 = b^2e_1^2e_2^2\cdots e_{n-1}^2c^2$. Therefore $\Delta = 1$. $\square$

Corollary 3. *The sequences of a frieze of type $\tilde{A}_m$, whose initial values are variables, are rational over the semiring of Laurent polynomials generated by these variables.*

The proof is quite analogue to the proof in Section 7.1. Observe that if one follows the lines of the proof, one may recover the formula for the sequence associated to frieze of the Kronecker quiver, as given in Section 2. In particular, one verifies that

$$M(a,x,b)M(b,y,a) = M,$$

with the notations in Section 2.

8.2. *Partial tilings and explicit formulas in case A_m*

We call *partial SL_2-tiling of the plane* a filling of a subset of $\mathbf{Z}^2$ such that each connected 2×2 submatrix is of determinant 1. We construct some of these partial tilings. Note that our construction below is somewhat equivalent to the construction of frieze patterns of Coxeter [12] and [7].

We begin with a construction which we call the *cross construction*: consider a finite word $w = a_0x_1a_1\ldots x_na_n$, where the x_i's are either x or y, and where the a_i's are invertible elements of some commutative semiring R (in our applications R is either $\mathbb{N}$ or a semiring of Laurent polynomials over $\mathbb{N}$). We call *transpose* of w the word ${}^tw = a_ny_n\ldots a_1y_1a_0$ where $y_i = {}^tx_i$, see Section 6.2. Let r be the number of x's in w and s be the number of y's. Recall that a *polyomino* is a finite subset of the discrete plane $\mathbb{Z}\times\mathbb{Z}$. It is called *convex* if for any two points A, B lying in the polyomino and which have either same x coordinate or same y coordinate, the whole discrete segment from A to B is also in the polyomino.

We associate to w a convex polyomino P_w, together a partial function from P_w into R, as follows. P_w is the smallest convex polyomino such that: it contains the points $(-r-1,1)$ and $(-1,s+1)$ together with the path from one to another determined by w; it contains the points $(1,-r-1)$ and $(s+1,-1)$ together with the path from one to another determined by ${}^tw$; it contains the points $(0,s+1)$ and $(s+1)$ together with the diagonal path (of directing vector $(1,-1)$) from one to another; it contains the points $(-r-1,0)$ and $(0,-r-1)$ together with the diagonal path from one to another; the border of P_w is the union of these 4 paths; the partial mapping is defined on this border, is determined by the two first paths (as in Section 8.1), and maps each point of the two diagonal paths onto 1.

As an example consider the $w = aybycxdxexfxgyhyiyj$ and its transpose ${}^t w = jxixhxgyfyeydycxbxa$. The origin of the plane is at the intersection of the asterisks lines, which represent the coordinate axis.

$$\begin{array}{cccccccccccc}
 & & & & j & 1 & & & & & & \\
 & & & & i & * & 1 & & & & & \\
 & & & & h & * & & 1 & & & & \\
c & d & e & f & g & * & & & 1 & & & \\
b & & & & & * & & P & & 1 & & \\
a & & & & & * & & & & & 1 & \\
1 & * & * & * & * & * & * & * & * & * & * & 1 \\
 & 1 & & & & * & & & & c & b & a \\
 & & 1 & & & * & & & & d & & \\
 & & & 1 & & * & & & & e & & \\
 & & & & 1 & * & & & & f & & \\
 & & & & & 1 & j & i & h & g & &
\end{array}$$

Theorem 5. *There is unique partial SL_2-tiling defined on the polyomino P_w which extends the partial function on the border defined above.*

The proof of this result is a straightforward generalization of the proofs given so far, so we leave the details to the reader. The only new thing is the definition of the word associated to each point in the polyomino constructed above. Note that it has naturally four non-disjoint components, separated by the axis.

For the north-west component, including the negative part of the x axis and the nonnegative part of the y axis the word is defined by projection on the frontier, exactly as it has been done in Section 8.1.

For the north-east component, one projects horizontally the point on the north-west frontier, and vertically on the south-east frontier: this give two words and the actual word is given by intersecting them. We do it on an example, which should be explicit enough. Consider the point denoted by P above. Then the horizontal projection gives the point corresponding to the variable b on the north-west frontier, and the vertical projection gives the point corrresponding to the variable i on the south-east frontier. Then, the word associated to P is by definition the word $bycxdxexfxgyhyi$; its denominator is, similarly to Section 8.1, the product of all the variables of the word, except the extreme ones. Thus, for P, it is $cdefgh$.

Then the value of the tiling at P is

$$\frac{1}{cdefgh}(1,b)M(c,x,d)M(d,x,e)M(e,x,f)M(f,x,g)M(g,y,h)\begin{pmatrix} i \\ 1 \end{pmatrix},$$

similarly to Eq. (11), except that the column matrix in the product is changed:

For the two other components, things are symmetric.

For the example, we show this tiling below in the case where all variables are set equal to 1.

				1	*1*						
				1	2	*1*					
				1	3	2	*1*				
1	1	1	1	1	4	3	2	*1*			
1	2	3	4	5	21	16	11	6	*1*		
1	3	5	7	9	38	29	20	11	2	*1*	
1	4	7	10	13	55	42	29	16	3	2	*1*
	1	2	3	4	17	13	9	5	1	1	1
		1	2	3	13	10	7	4	1		
			1	2	9	7	5	3	1		
				1	5	4	3	2	1		
					1	1	1	1	1		

The corresponding frieze is then easily constructed; indeed, the word coding the south-east frontier is the transpose of the word coding the frontier; thus one continues the cross-construction with this new word. The tiling obtained has only to be repeated indefinitely; it is then seen to be periodic and its period is 2 plus the length of the frontier w (as follows already from the work of Fomin and Zelevinsky). If the frontier is an anti-palindrome, that is, ${}^t w = w$, then the period is the half of this number.

As a consequence of Theorem 5, we get the following well-known result (see [8] for instance).

Corollary 4. *The Laurent phenomenon and the positivity conjecture hold for cluster algebras of type A_m with acyclic initial clusters.*

We stress that our approach also demonstrates that the Laurent phenomenon and the positivity conjecture hold for all, but finitely many, cluster variables in cluster algebras of type $\tilde{A}_m$ with acyclic initial clusters. Recently, Assem, and Dupont showed in [1] how to extend our approach to show that these conjectures hold true for all clusters variables in cluster algebras of type $\tilde{A}_m$ with acyclic initial clusters.

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