EXTENDING TILTING MODULES TO ONE-POINT EXTENSIONS BY PROJECTIVES

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1. INTRODUCTION

Let k be a commutative field, and A be a finite dimensional k-algebra. We denote by mod A the category of finitely generated left A-modules. Throughout this paper, we say that an object T in mod A is a tilting A-module if

(a) The projective dimension of T is finite,
(b) $\operatorname{Ext}_A^i(T,T)=0$ for all $i>0$, and
(c) There exists an exact sequence $0 \longrightarrow A_A \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0$, where all T^i are direct sums of direct summands of T.

We say that a tilting module is multiplicity-free if, in an indecomposable direct sum decomposition of T, all the summands are pairwise non-isomorphic. The right orthogonal $T^\perp$ (see [1]) of a multiplicity-free tilting module T is the full subcategory of mod$A$ defined by:

$$T^\perp = \{X \in \operatorname{mod}A \mid \operatorname{Ext}_A^i(T,X)=0 \text{ for all } i>0\}$$

A partial order on a full set $\mathcal{T}_A$ of representatives of the isomorphism classes of multiplicity-free tilting A-modules is defined as follows: for $T,T' \in \mathcal{T}_A$, we set $T \leq T'$ provided that $T^\perp \subseteq T'^\perp$ (see, [10]). The Hasse quiver $\overrightarrow{\mathcal{K}}_A$ of this poset (partially ordered set) has been characterised in [9].

Our objective in this paper is to compare the posets corresponding to two algebras in the following situation: Let B be any finite dimensional k-algebra, and A be the one-point extension of B by a projective B-module. Denoting by e_B the identity of B, the B-A-bimodule $U=e_BA$ induces two adjoint functors $\mathcal{R}=U\otimes_A - : \operatorname{mod}A \longrightarrow \operatorname{mod}B$ and $\mathcal{E}=\operatorname{Hom}_B(U,-) : \operatorname{mod}B \longrightarrow \operatorname{mod}A$ which are easily seen to satisfy $\mathcal{R}\mathcal{E}\cong id_{\operatorname{mod}B}$. We can now state our main theorem.

Theorem. *Let B be a finite dimensional k-algebra, P_0 be a projective B-module, and $A=B[P_0]$. Then the functors* $\mathcal{R} : \operatorname{mod}A \longrightarrow \operatorname{mod}B$

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and $\mathcal{E}$: mod$B \longrightarrow$ mod$A$ *induce respectively morphisms of posets* $r : \mathcal{T}_A \longrightarrow \mathcal{T}_B$ *and* $e : \mathcal{T}_B \longrightarrow \mathcal{T}_A$ *such that* $re = id_{\mathcal{T}_B}$.

Moreover, e *induces a full embedding of the quiver* $\overrightarrow{\mathcal{K}}_B$ *into the quiver* $\overrightarrow{\mathcal{K}}_A$*, whose image is closed under successors and such that distinct connected components of* $\overrightarrow{\mathcal{K}}_B$ *map to distinct connected components of* $\overrightarrow{\mathcal{K}}_A$.

We point out that, under the maps r and e, the tilting modules of projective dimension (at most) one are mapped to tilting modules of projective dimension (at most) one.

Further, in the hereditary case, if T is a tilting A-module, then $\operatorname{End} rT$ is representation-finite whenever $\operatorname{End} T$ is and, if M is a tilting B-module, then $\operatorname{End} e(M)$ is a one-point extension of $\operatorname{End} M$.

As we shall see, most statements in the theorem fail if we drop the assumption that the module P_0 is projective.

We now describe the contents of the paper. Sections 1 and 2 are devoted to studying properties of the functors $\mathcal{R}$ and $\mathcal{E}$. Section 3 contains the construction of the maps r and e. In section 4, we prove our theorem, and deduce some of its consequences. Finally, in section 5, we consider statements relevant to endomorphism algebras.

2. Extensions and restriction functors

2.1. **Notation.** Throughout this paper, all algebras are connected finite dimensional algebras over a fixed commutative field k (and, unless otherwise specified, basic). We sometimes consider an algebra A as a k-category, of which the object class is a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents, and the set of morphisms from e_i to e_j is e_iAe_j. An algebra B is a *full subcategory* of A if there exists an idempotent $e \in A$, sum of (some of) the distinguished idempotents $\{e_i\}$ such that$B = eAe$. It is *convex* in A, if whenever there is a subset $\{e_{i_0}, e_{i_1}, \cdots, e_{i_t}\}$ of $\{e_i\}$ such that $e_{il+1}Ae_{il} \neq 0$ for $0 \leq l < t$ and e_{i_0}, e_{i_t} belong to B, then all the e_{i_l} belong to B.

For an algebra A we only consider its finitely generated left A-modules, and we denote by mod$A$ their category. For a full subcategory $\mathcal{C}$ of mod$A$, we write $X \in \mathcal{C}$ to express that X is an object in $\mathcal{C}$. We denote by add X the full subcategory having as objects the direct sums of direct summands of X, and by Gen X the full subcategory of mod A having as objects the modules Y which are generated by X (that is, such that there exist $d > 0$ and an epimorphism $X^d \longrightarrow Y$). Given an algebra A, we denote by $\mathrm{K}_0(A)$ the Grothendieck group of A. The projective (or injective) dimension of ${}_AX$ is denoted as pd_AX (or

id_AX, respectively). The standard duality $D : \text{mod}A \longrightarrow \text{mod}A^{op}$ is $D = \text{Hom}_k(-,k)$

For further definitions or facts needed on the module category, we refer to [3], [11],[4].

2.2. **The context.** Let B be a finite dimensional k-algebra, and P_0 be a fixed projective B-module. We denote by $A = B[P_0]$ the one-point extension of B by P_0, that is, the matrix algebra

$$A = \begin{bmatrix} B & P_0 \\ 0 & k \end{bmatrix}$$

with the ordinary matrix addition and the multiplication induced from the module structure of P_0.

Thus, B is a full convex subcategory of A, and there is a unique projective A-module P which is not a projective B-module. Also, the simple top S of P is an injective A-module and $pd_AS \leq 1$.

Since we consider at the same time A-modules and B-modules, and in order to avoid confusion, we denote the A-modules by the letters $X, Y, Z \cdots$ and the B-modules by the letters $L, M, N \cdots$.

Let e_B denote the identity of B, so that $B = e_BAe_B$. Consider the $B - A-$bimodule $U = e_BA$. It is clearly projective as right A-module, but also as a left B-module, since ${}_BU \cong_B B \oplus_B P_0$.

We consider the following two functors, respectively called the restriction and the extension functor

$$\mathcal{R} =_B U_A \otimes - : \text{mod}A \longrightarrow \text{mod}B$$

and

$$\mathcal{E} = \text{Hom}\,({}_BU_A, -) : \text{mod}B \longrightarrow \text{mod}A.$$

Clearly, $(\mathcal{R}, \mathcal{E})$ is an adjoint pair of functors, and the left-right projectivity of U implies that both are exact.

The functor $\mathcal{R}$ may be expressed otherwise:

$$U_A \otimes - \cong \text{Hom}_A(B, -)$$

indeed, this is the usual "restriction by zeros" functor: it associates to an A-module X the B-module $U \otimes_A X \cong e_BX$ (in particular, $\mathcal{R}$ does not preserve indecomposability). If we consider mod$B$ as embedded in mod$A$ under the usual embedding functor (as we shall always do), we see that $\mathcal{R}X$ is a submodule of X. Thus $\mathcal{R}$ is a subfunctor of the identity on mod$A$. We now prove that it is a torsion radical.

Lemma. (a) *The functor $\mathcal{R}$ is the torsion radical of the torsion pair* (mod$B$, add$S$) *in* mod$A$.

(b) *The canonical sequence of an A-module X in this torsion pair*

$$0 \longrightarrow \mathcal{R}X \longrightarrow X \longrightarrow S^{r_x} \longrightarrow 0$$

satisfies $r_x = \dim_k \mathrm{Hom}_A(X, S)$.

Proof. (a) Clearly, an A-module X is a B-module if and only if $X \cong \mathcal{R}X$. Also $\mathcal{R}S = 0$. Letting e_k denote the primitive idempotent of A corresponding to the new projective P, we see that, as k-vector space, X admits a decomposition $X \cong e_B X \oplus e_k X$ and moreover, as A-modules, $e_k X \cong S^m$ for some $m \geq 0$. In particular, $\mathcal{R}X \cong \mathcal{R}(e_B X) \oplus \mathcal{R}(S^m) \cong \mathcal{R}(e_B X)$. This implies that $\mathcal{R}X = \mathcal{R}^2 X$ and moreover $\mathcal{R}X = 0$ if and only if $X \in \mathrm{add}S$. Applying the exact functor $\mathcal{R}$ to the short exact sequence of A-modules

$$0 \longrightarrow \mathcal{R}X \longrightarrow X \longrightarrow X/\mathcal{R}X \longrightarrow 0$$

yields $\mathcal{R}(X/\mathcal{R}X) = 0$. This establishes the statement.

(b) Applying $\mathrm{Hom}_A(-, S)$ to the canonical sequence yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(S^{r_X}, S) \longrightarrow \mathrm{Hom}_A(X, S) \longrightarrow Hom_A(\mathcal{R}X, S) = 0$$

so that $r_X = \dim_k \mathrm{Hom}_A(X, S)$, as required. □

The canonical sequence of (b) will be called the *restriction sequence* for X. We note that the pair $(\mathrm{mod}\, B, \mathrm{add}S)$ is a hereditary torsion pair (but we shall not use this fact).

2.3. As a first consequence of the existence of restriction sequences, we obtain the following corollary.

Corollary. *For any A-module X the B-module $\mathcal{R}X$ is projective (in which case, $pd_A X \leq 1$) or else $pd_B \mathcal{R}X = pd_A X$.*

Proof. We consider the restriction sequence

$$0 \longrightarrow \mathcal{R}X \longrightarrow X \longrightarrow S^{r_x} \longrightarrow 0$$

and recall that projective B-modules are projective in mod A. If ${}_B\mathcal{R}X$ is projective, then $pd_A S \leq 1$ implies $pd_A X \leq 1$. If not, assume $pd_B \mathcal{R}X = d$ Then $pd_A \mathcal{R}X = d$ and the above sequence gives $pd_A X = d$. □

2.4. We shall need the following lemma.

Lemma. *For any B-module M, we have an isomorphism of k-vector spaces*

$$\mathrm{Ext}_A(S, M) \cong \mathrm{Hom}_B(P_0, M)$$

.

Proof. Applying $\mathrm{Hom}_A(-, M)$ to the minimal projective resolution

$$0 \longrightarrow P_0 \longrightarrow P \longrightarrow S \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(S, M) \longrightarrow \mathrm{Hom}_A(P, M) \longrightarrow \mathrm{Hom}_A(P_0, M) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1_A(S, M) \longrightarrow \mathrm{Ext}^1_A(P, M) = 0.$$

Since M is a B-module, $\mathrm{Hom}_A(P, M) = 0$. Hence

$$\mathrm{Ext}^1_A(S, M) \cong \mathrm{Hom}_A(P_0, M).$$

Finally, since B is a full convex subcategory of A, then $\mathrm{Hom}_A(P_0, M) \cong \mathrm{Hom}_B(P_0, M)$. □

2.5. Since $(\mathcal{R}, \mathcal{E})$ is an adjoint pair of functors, there are, associated with it, a co-unit $\epsilon : \mathcal{R}\mathcal{E} \longrightarrow id_{\mathrm{mod}\, B}$ and a unit $\delta : id_{\mathrm{mod}\, A} \longrightarrow \mathcal{E}\mathcal{R}$ defined as follows. Let M be a B-module, then

$$\epsilon_M : U \otimes_A \mathrm{Hom}_B(U, M) \longrightarrow M$$

is given by

$$u \otimes f \mapsto f(u)$$

(for $u \in U$ and $f \in \mathrm{Hom}_B(U, M)$ Let X be an A-module, then

$$\delta_X : X \longrightarrow \mathrm{Hom}_B(U, U \otimes_A X)$$

is given by

$$x \mapsto (u \mapsto u \otimes x)$$

(for $x \in X$ and $u \in U$). The next proposition lists relevant properties of these functorial morphisms.

Proposition. *The adjoint pair of functors $(\mathcal{R}, \mathcal{E})$ satisfies the following properties:*

(a) *The co-unit ϵ is a functorial isomorphism.*
(b) *For every A-module X, the kernel and the cokernel of δ_X belong to* add$S$.
(c) *Let X be an A-module. The following conditions are equivalent:*
  i) *δ_X is a monomorphism.*
  ii) *S is not a direct summand of X*

iii) $\operatorname{Hom}_A(S, X) = 0$

Proof. (a) Let M be a B-module. Since ${}_BM$ is generated by ${}_BU \cong_B B \oplus_B P_0$, the morphism ϵ_M is surjective. On the other hand, we have isomorphisms of k-vector spaces

$$\mathcal{R}\mathcal{E}M \cong \operatorname{Hom}_A(B, \operatorname{Hom}_B(U, M)) \cong \operatorname{Hom}_B(U \otimes_A B, M)$$

$$\cong \operatorname{Hom}_B(B, M) \cong M$$

because $U \otimes_A B \cong e_B B = B$. Hence ϵ_M is an isomorphism.

(b)By (a), δ_X restricts to an isomorphism $\delta_{\mathcal{R}X} : \mathcal{R}X \longrightarrow \mathcal{R}X \cong \mathcal{R}\mathcal{E}\mathcal{R}X$. So, the restriction sequences for X and $\mathcal{E}\mathcal{R}X$ yield a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}X & \longrightarrow & X & \longrightarrow & S^{r_X} & \longrightarrow & 0 \\
 & & \cong \downarrow \delta_{\mathcal{R}X} & & \downarrow \delta_X & & \downarrow \delta' & & \\
0 & \longrightarrow & \mathcal{R}X & \longrightarrow & \mathcal{E}\mathcal{R}X & \longrightarrow & S^{r_{\mathcal{E}\mathcal{R}X}} & \longrightarrow & 0
\end{array}$$

where δ' is induced by passing to cokernels. By the Snake lemma, Ker $\delta_X \cong$ Ker δ' and Coker $\delta_X \cong$ Coker δ'. The statement follows.

(c) i) implies ii). If S is a direct summand of X, there exist $m \geq 1$ and a decomposition $X \cong X' \oplus S^m$ with $S \notin$ add X'. Since $\mathcal{R}S = 0$, this yields $\mathcal{E}\mathcal{R}X \cong \mathcal{E}\mathcal{R}X'$. Hence Ker $\delta_X \supseteq S^m$ so that δ_X is not a monomorphism.

ii) implies iii) If δ is not a direct summand of X, then $\operatorname{Hom}_A(S, X) = 0$ because S is simple injective.

iii) implies i). This follows from the fact that, by (b), Ker $\delta_X \in$ add S. □

2.6. One important consequence is the following corollary.

Corollary. *The functor $\mathcal{E}$ is full and faithful. In particular, it preserves indecomposability.*

Proof. Let M, N be B-modules and $g : \mathcal{E}M \longrightarrow \mathcal{E}N$ be a morphism in mod A. Since $\mathcal{R}\mathcal{E} \cong id_{\text{ mod } B}$, the morphism $\mathcal{R}g : M \longrightarrow N$ satisfies $\mathcal{E}\mathcal{R}g = g$. Thus, $\mathcal{E}$ is full. Faithfulness is proven similarly. The last statement follows since, for each B-module M, we have $\operatorname{End}_A\mathcal{E}M \cong \operatorname{End}_B M$. □

3. Homological properties of the extension and restriction functors

3.1. The *right perpendicular category* of S is the full subcategory of mod A defined by

$$S^{perp} = \{X \in \text{mod } A \mid \text{Hom}_A(S,X)=0,\ \text{Ext}^1_A(S,X)=0\}.$$

Lemma . *Let $X \in S^{perp}$. Then $\delta_X : X \longrightarrow \mathcal{E}\mathcal{R}X$ is a functorial isomorphism.*

Proof. Since $\text{Hom}_A(S,X)=0$, it follows from (1.5) (c) that δ_X is a monomorphism, so there exist $m \geq 0$ and a short exact sequence

$$0 \longrightarrow X \xrightarrow{\delta_X} \mathcal{E}\mathcal{R}X \longrightarrow S^m \longrightarrow 0$$

Since $\text{Ext}^1_A(S,X)=0$, this sequence splits. On the other hand, by adjunction,

$$\text{Hom}_A(S,\mathcal{E}\mathcal{R}X) = \text{Hom}_B(\mathcal{R}S,\mathcal{R}X) = 0.$$

So δ_X is an isomorphism. □

3.2. We now construct a short exact sequence relating a B-module M to the extended module $\mathcal{E}M$. We first note that, by (1.5), the unit δ_M is a monomorphism and $\text{Coker}\delta_M \in \text{add}S$, so that there exist $e_M \geq 0$ and a short exact sequence

$$0 \longrightarrow M \xrightarrow{\delta_M} \mathcal{E}\mathcal{R}M \longrightarrow S^{e_M} \longrightarrow 0$$

which clearly coincides with the restriction sequence for $\mathcal{E}\mathcal{R}M \cong \mathcal{E}M$. In particular, $e_M = r_{\mathcal{E}\mathcal{R}M}$. We call this sequence the *extension sequence* for M.

Proposition. *Let M be a B-module. The extension sequence*

$$0 \longrightarrow M \xrightarrow{\delta_M} \mathcal{E}\mathcal{R}M \longrightarrow S^{e_M} \longrightarrow 0$$

satisfies the following properties:

(a) $e_M = \dim_k \text{Ext}^1_A(S,M)$.
(b) *The connecting morphism* $\text{Hom}_A(S,S^{e_M}) \longrightarrow \text{Ext}^1_A(S,M)$ *is an isomorphism.*
(c) $\mathcal{E}M \in S^{perp}$.

Proof. (a) We have isomorphisms of k-vector spaces

$$\mathcal{E}M = \text{Hom}_B(U,M) \cong \text{Hom}_B(B\oplus P_0,M) \cong M \oplus \text{Hom}_B(P_0,M)$$

so that, by (1.4),

$$e_M = \dim_k\mathcal{E}M - \dim_k M = \dim_k \text{Hom}_B(P_0,M) = \dim_k \text{Ext}^1_A(S,M).$$

(b) Since S is simple injective, applying $\operatorname{Hom}_A(S,-)$ to the extension sequence yields a long exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(S,M) \longrightarrow \operatorname{Hom}_A(S,\mathcal{E}M) \longrightarrow \operatorname{Hom}_A(S,S^{e_M}) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^1_A(S,M) \longrightarrow \operatorname{Ext}^1_A(S,\mathcal{E}M) \longrightarrow \operatorname{Ext}^1_A(S,S^{e_M}) = 0.$$

Moreover, $\operatorname{Hom}_A(S,\mathcal{E}M) \cong \operatorname{Hom}_B(\mathcal{R}S,M) = 0$ so that the connecting morphism is injective. It follows from (a) that it is an isomorphism. □

3.3. We deduce that mod B and S^{perp} are equivalent categories.

Corollary. *The functors $\mathcal{E}$ and $\mathcal{R}$ induce an equivalence between* mod B *and S^{perp}.*

Proof. This follows from (2.2)(c) and (1.5) (a). □

3.4. The next corollary follows immediately from the equivalence.

Corollary. *Let M be a B-module and $X \in S^{perp}$. Then, for each $j \geq 0$, we have $\operatorname{Ext}^j_A(\mathcal{E}M,X) \cong \operatorname{Ext}^j_B(M,\mathcal{R}X)$.*

Proof. Since $\mathcal{R}\mathcal{E}M \cong M$, this follows from (2.3). □

3.5. The following corollary generalises the adjunction property.

Corollary. *Let X be an A-module, and M be a B-module, then, for all $j \geq 0$, we have $\operatorname{Ext}^j_A(X,\mathcal{E}M) \cong \operatorname{Ext}^j_B(\mathcal{R}X,M)$.*

Proof. Applying $\operatorname{Hom}_A(\mathcal{R}X,-)$ to the extension sequence

$$0 \longrightarrow M \longrightarrow \mathcal{E}M \longrightarrow S^{e_M} \longrightarrow 0$$

corresponding to M yields isomorphisms $\operatorname{Ext}^j_A(\mathcal{R}X,M) \cong \operatorname{Ext}^j_A(\mathcal{R}X,\mathcal{E}M)$ for each $j \geq 0$. On the other hand, since $\mathcal{E}M \in S^{perp}$ applying $\operatorname{Hom}_A(-,\mathcal{E}M)$ to the restriction sequence

$$0 \longrightarrow \mathcal{R}X \longrightarrow X \longrightarrow S^{r_X} \longrightarrow 0$$

yields isomorphisms $\operatorname{Ext}^j_A(X,\mathcal{E}M) \cong \operatorname{Ext}^j_A(\mathcal{R}X,\mathcal{E}M)$ for each $j \geq 0$ (because $pd_A S \leq 1$). The statement now follows from the convexity of B in A. □

3.6. We now compare the extension groups of two modules and their respective restrictions.

Proposition. *Let X, Y be A-modules Then:*
(a) *There is an epimorphism* $\mathrm{Ext}\,^1_A(X,Y) \longrightarrow \mathrm{Ext}\,^1_B(\mathcal{R}X, \mathcal{R}Y)$.
(b) *There is an isomorphism* $\mathrm{Ext}\,^j_A(X,Y) \cong \mathrm{Ext}\,^j_B(\mathcal{R}X, \mathcal{R}Y)$, *for each* $j \geq 2$.
(c) *If* $Y \in S^{perp}$, *then the epimorphism of* (a) *is an isomorphism.*

Proof. Applying $\mathrm{Hom}\,_A(\mathcal{R}X, -)$ to the restriction sequence

$$0 \longrightarrow \mathcal{R}Y \longrightarrow Y \longrightarrow S^{r_Y} \longrightarrow 0$$

yields (because S is injective and because $\mathrm{Hom}\,_A(\mathcal{R}X, S) = 0$) an isomorphism

$$\mathrm{Ext}\,^j_A(\mathcal{R}X, \mathcal{R}Y) \cong \mathrm{Ext}\,^j_A(\mathcal{R}X, Y)$$

for each $j \geq 1$. Applying now $\mathrm{Hom}\,_A(-, Y)$ to the restriction sequence

$$0 \longrightarrow \mathcal{R}X \longrightarrow X \longrightarrow S^{r_X} \longrightarrow 0$$

yields, because $pd_A S \leq 1$, a right exact sequence

$$\mathrm{Ext}\,^1_A(S^{r_X}, Y) \longrightarrow \mathrm{Ext}\,^1_A(X, Y) \longrightarrow \mathrm{Ext}\,^1_A(\mathcal{R}X, Y) \longrightarrow 0$$

and an isomorphism $\mathrm{Ext}\,^j_A(X,Y) \cong \mathrm{Ext}\,^j_A(\mathcal{R}X, Y)$ for each $j \geq 2$. This, together with the convexity of B in A, gives (a) and (b). Finally, (c) follows from the above right exact sequence because $\mathrm{Ext}\,^1_A(S, Y) = 0$. □

3.7. We call an A-module X *self-orthogonal* if $\mathrm{Ext}\,^j_A(X,X) = 0$ for all $j \geq 1$ and *exceptional* if, in addition, $pd_A X < \infty$.

Corollary. *The functors $\mathcal{R}$ and $\mathcal{E}$ preserve self-orthogonality and exceptionality.*

Proof. Let M be a self-orthogonal B-module. By (2.2)(c), $\mathcal{E}M \in S^{perp}$ so that, by (2.3),

$$\mathrm{Ext}\,^j_A(\mathcal{E}M, \mathcal{E}M) \cong \mathrm{Ext}\,^j_B(\mathcal{R}\mathcal{E}M, \mathcal{R}\mathcal{E}M) \cong \mathrm{Ext}\,^j_B(X, X) = 0$$

for each $j \geq 1$. Thus $\mathcal{E}M$ is self-orthogonal.

Let X be a self-orthogonal A-module, then (2.6)(b) yields, for each $j \geq 2$,

$$\mathrm{Ext}\,^j_B(\mathcal{R}X, \mathcal{R}X) \cong \mathrm{Ext}\,^j_A(X, X) = 0$$

and, by (2.6)(a), $\mathrm{Ext}\,^1_A(X,X) = 0$ implies $\mathrm{Ext}\,^1_B(\mathcal{R}X, \mathcal{R}X) = 0$. Thus $\mathcal{R}X$ is self-orthogonal. This shows that $\mathcal{R}$ and $\mathcal{E}$ preserve self-orthogonality. The statement about exceptionality follows from (1.3). □

4. Extension and restriction maps

4.1. We recall a few definitions. Let X be an A-module. The *right orthogonal* $X^\perp$ of X is a full subcategory of mod A defined by

$$X^\perp = \{Y \in \text{mod } A \mid \text{Ext}_A^j(X,Y) = 0 \text{ for each } j \geq 1\}$$

An exceptional A-module T is a *tilting module* if there exists an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^r \longrightarrow 0$$

with $T^i \in \text{add}T$ for all i. It is shown in [6] that an exceptional module T is tilting if and only if $T^\perp \subsetneqq \text{Gen}T$. A tilting module T is *multiplicity-free* if, for an indecomposable direct sum decomposition $T = \oplus_i T_i$ of T, we have $T_i \ncong T_j$ for $i \neq j$.

Proposition. (a) *Let T be a multiplicity-free tilting A-module, then $T' = \mathcal{R}T$ is a tilting B-module.*
(b) *Let M be a multiplicity-free tilting B-module, then $S \oplus \mathcal{E}M$ is a tilting A-module.*

Proof. (a) By (2.7), T' is exceptional. Let $M \in T'^\perp$. By (2.4), $\mathcal{E}M \in T^\perp$. Since T is tilting, $T^\perp \subsetneqq \text{Gen } T$ so that $\mathcal{E}M \in \text{Gen } T$, that is, there exist $\overline{T} \in \text{add}T$ and an epimorphism $p : \overline{T} \longrightarrow \mathcal{E}M$. We deduce a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{R}\overline{T} & \longrightarrow & \overline{T} & \longrightarrow & S^{r_{\overline{T}}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\
0 & \longrightarrow & M & \longrightarrow & \mathcal{E}M & \longrightarrow & S^{e_M} & \longrightarrow & 0
\end{array}$$

where $p' = \mathcal{R}p : \mathcal{R}\overline{T} \longrightarrow \mathcal{R}\mathcal{E}M \cong M$ and p'' is induced by passing to cokernels. The Snake lemma yields an epimorphism $f : \text{Ker } p'' \longrightarrow \text{Coker } p'$. Since $\text{Ker } p'' \in \text{add}S$ and $\text{Coker } p' \in \text{mod } B$, we have $f = 0$ so $\text{Coker } p' = 0$. Therefore, $M \in \text{Gen } \mathcal{R}\overline{T} \subsetneqq \text{Gen } T'$ (because $\mathcal{R}T \in \text{add } T'$).

(b) By (2.7), $S \oplus \mathcal{E}M$ is exceptional. Since $\mathcal{E}$ preserves indecomposability (by (1.6)),$S \oplus \mathcal{E}M$ has exactly $1 + \text{rkK}_0(B) = \text{rkK}_0(A)$ isomorphism classes of indecomposable summands.Therefore, by [1](5.12), it suffices to prove that $(S \oplus \mathcal{E}M)^\perp$ is covariantly finite.

Consider the restriction sequence

$$0 \longrightarrow X' \stackrel{f}{\longrightarrow} X \stackrel{g}{\longrightarrow} S^{r_X} \longrightarrow 0$$

(where $X' = \mathcal{R}X$) and an approximation $u : X' \longrightarrow F_{X'}$ with respect to $M^{\perp}$ (which exists because of [1] (5.5)). Next consider the extension sequence

$$0 \longrightarrow F_{X'} \xrightarrow{f'} \tilde{F}_X \xrightarrow{g'} S^{e_{FX'}} \longrightarrow 0$$

(where $\tilde{F}_{X'} = \mathcal{E}F_{X'}$). Since, by (2.2), $\tilde{F}_{X'} \in S^{perp}$, then $\mathrm{Ext}\,^1_A(S, \tilde{F}_{X'}) = 0$ so that, applying (2.4) and using the fact that $pd_A S \leq 1$ yield

$$\mathrm{Ext}\,^j_A(S \oplus \mathcal{E}M, \tilde{F}_{X'}) \cong \mathrm{Ext}\,^j_A(\mathcal{E}M, \tilde{F}_{X'}) \cong \mathrm{Ext}\,^j_B(M, \mathcal{R}\tilde{F}_{X'}) \cong$$

$$\cong \mathrm{Ext}\,^j_B(M, F_{X'}) = 0$$

for each $j \geq 1$, because $F_{X'} \in M^{\perp}$. This shows that $\tilde{F}_{X'} \in (S \oplus \mathcal{E}M)^{\perp}$.

Now, applying $\mathrm{Hom}\,_A(-, \tilde{F}_{X'})$ to the restriction sequence for X yields an exact sequence

$$\mathrm{Hom}\,_A(X, \tilde{F}_{X'}) \longrightarrow \mathrm{Hom}\,_A(X', \tilde{F}_{X'}) \longrightarrow \mathrm{Ext}\,^1_A(S, \tilde{F}_{X'}) = 0$$

therefore there exists $v : X \longrightarrow \tilde{F}_{X'}$ such that $vf = f'u$. We claim that $v' = \begin{bmatrix} v \\ g \end{bmatrix} : X \longrightarrow \tilde{F}_{X'} \oplus S^r X$ is the required $(S \oplus \mathcal{E}M)^{\perp}$-approximation. Let thus $h : X \longrightarrow Y$ with $Y \in (S \oplus \mathcal{E}M)^{\perp}$. We can assume without loss of generality that Y is indecomposable.

Assume first $Y \cong S$, then $hf = 0$ and there exists $h' : S^{r_X} \longrightarrow Y$ such that $h = h'g = \begin{bmatrix} 0 & h' \end{bmatrix} \begin{bmatrix} v \\ g \end{bmatrix}$.

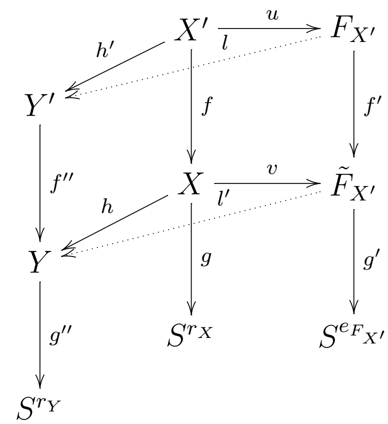
We may thus suppose $Y \ncong S$. In particular, $\mathrm{Hom}\,_A(S, Y) = 0$. We claim that h factors through v. Indeed, consider the restriction sequence for Y

$$0 \longrightarrow Y' \xrightarrow{f} Y \xrightarrow{g} S^{r_Y} \longrightarrow 0$$

$(Y' = \mathcal{R}Y)$. Then $h' = \mathcal{R}h$ satisfies $hf = f''h'$. Since $Y \in (\mathcal{E}M)^{\perp}$, we have, for each $j \geq 0$, $\mathrm{Ext}\,^j_A(M, Y') \cong \mathrm{Ext}\,^j_A(\mathcal{E}M, Y) = 0$ by (2.4). Therefore, $Y' \in M^{\perp}$ and, since u is an approximation, there exists $l : F_{X'} \longrightarrow Y'$ such that $lu = h'$. Applying $\mathrm{Hom}\,_A(-, Y)$ to the extension sequence for $F_{X'}$ yields an exact sequence

$$\mathrm{Hom}\,_A(\tilde{F}_{X'}, Y) \longrightarrow \mathrm{Hom}\,_A(F_{X'}, Y) \longrightarrow \mathrm{Ext}\,^1_A(S^{e_{FX'}}, Y) = 0$$

so there exists $l' : \tilde{F}_{X'} \longrightarrow Y$ such that $l'f' = f''l$



We claim that $h = l'v$. Now

$$(h-l'v)f = hf-l'vf = f'h'-l'f'u = f''h'-f''lu = f''(h'-lu) = f''0 = 0$$

Therefore $h-l'v$ factors through g, that is, there exists $w : S^{r_X} \longrightarrow Y$ such that $wg = h - l'v$. However, $\mathrm{Hom}_A(S, Y) = 0$. Therefore $w = 0$ and so $h = l'v$, as required. □

4.2. Let C be a finite dimensional algebra and T be a tilting C-module. For each $i \geq 0$, denote by $\mathfrak{X}^i_C(T)$ the full subcategory of mod C defined by

$$\mathfrak{X}^i_C(T) = \{X \in \mathrm{mod}\ C \mid \mathrm{Ext}^j_C(T, X) = 0 \text{ for all } j \neq i\}$$

(see [6] p.114).

Proposition. *Let M be a tilting B-module. The functors $\mathcal{E}$ and $\mathcal{R}$ induce, for each $i \geq 0$, quasi-inverse equivalences between $\mathfrak{X}^i_B(M)$ and $\mathfrak{X}^i_A(S \oplus \mathcal{E}M) \cap S^{perp}$.*

Proof. Let $N \in \mathfrak{X}^i_B(M)$. By (2.2)(i), $\mathcal{E}N \in S^{perp}$. By (2.4), for each $j \geq 0$,

$$\mathrm{Ext}^j_A(M, N) \cong \mathrm{Ext}^j_A(\mathcal{E}M, \mathcal{E}N).$$

This, and the fact that $pd_A S \leq 1$, imply that $\mathcal{E}N \in \mathfrak{X}^i_A(S \oplus \mathcal{E}M)$. Hence $\mathcal{E}N \in \mathfrak{X}^i_A(S \oplus \mathcal{E}M) \cap S^{perp}$.

Conversely, let $X \in \mathfrak{X}^i_A(S \oplus \mathcal{E}M) \cap S^{perp}$. Since $X \in S^{perp}$ then by (2.1), the B-module $N = \mathcal{R}X$ satisfies $X \cong \mathcal{E}N$. Hence, by (2.4)

$$\mathrm{Ext}^j_A(M, N) \cong \mathrm{Ext}^j_A(\mathcal{E}M, X)$$

for each $j \geq 0$. Therefore $N \in \mathfrak{X}^i_B(M)$. □

Remark. By (1.3), tilting B-modules of projective dimension (at most) one correspond under the maps defined in (3.1), to tilting A-modules of projective dimension (at most) one. In this case, $(\mathfrak{X}_A^0, \mathfrak{X}_A^1)$ and $(\mathfrak{X}_B^0, \mathfrak{X}_B^1)$ and torsion pairs in mod A and mod B respectively (see [6]). Clearly, for $i \geq 2$ we have $\mathfrak{X}_A^i = 0$ and $\mathfrak{X}_B^i = 0$.

4.3. It follows from the proof of (3.1), that, if M is a multiplicity-free tilting B-module, then the tilting A-module $S \oplus \mathcal{E}M$ is always multiplicity-free. On the other hand, it is generally not true that, if T is a multiplicity-free tilting A-module, then the tilting B-module $\mathcal{R}T$ is multiplicity-free.

Lemma. *Let T be a multiplicity-free tilting A-module. The following conditions are equivalent:*

(a) *$\mathcal{R}T$ is multiplicity-free.*
(b) *S is a direct summand of T.*
(c) *There exists a B-module M such that $T = S \oplus \mathcal{E}M$.*

Proof. (a) implies (b). Assume S is not a summand of T, and let $T = \oplus_{i=1}^r T_i$ be an indecomposable decomposition. Then $r = rk\mathrm{K}_0(A)$ (see, for instance, [5](1.1)). Since $\mathcal{R}T = \oplus_{i=1}^r \mathcal{R}T_i$ and $\mathcal{R}T_i \neq 0$ for each i, then $\mathcal{R}T$ has at least r isomorphism classes of indecomposables summands. Since $r = 1 + rk\mathrm{K}_0(B) > rk\mathrm{K}_0(B)$, then $\mathcal{R}T$ cannot be multiplicity-free.

(b) implies (c). Assume that $T = S \oplus X$ is a multiplicity-free tilting A-module. In particular S is not a summand of X, so that $\mathrm{Hom}_A(S, X) = 0$. Since also $\mathrm{Ext}_A^1(S, X) = 0$, we have $X \in S^{perp}$ and the B-module $M = \mathcal{R}X$ satisfies $T = S \oplus X \cong S \oplus \mathcal{E}M$.

(c)implies (a). Since T is multiplicity-free, so is $\mathcal{E}M$, hence so is $M \cong \mathcal{R}\mathcal{E}M \cong \mathcal{R}T$. □

4.4. Let C be a finite dimensional algebra and $\mathcal{T}_C$ be a complete set of representatives of the isomorphism classes of multiplicity-free tilting C-modules. For $T, T' \in \mathcal{T}_C$, we define $T \leq T'$ to mean that $T^{\perp} \subseteqq T'^{\perp}$. Clearly, this defines a partial order on $\mathcal{T}_C$.

Corollary. *The functors $\mathcal{R}$ and $\mathcal{E}$ induce two maps*

$$r : \mathcal{T}_A \longrightarrow \mathcal{T}_B$$

$$e : \mathcal{T}_B \longrightarrow \mathcal{T}_A$$

such that $re = id_{\mathcal{T}_B}$. These maps are defined as follows: if $M \in \mathcal{T}_B$, then $eM = S \oplus \mathcal{E}M$ and, if $T \in \mathcal{T}_A$, then $rT = T^*$, where T^* is *a (unique up to isomorphism) multiplicity-free tilting B-module such that $T^* = \mathrm{add}\mathcal{R}T$.*

Proof. By (3.1), r and e are maps, and the relation $re = id_{\mathcal{T}_B}$ follows from $\mathcal{R}\mathcal{E} \cong id_{\mathrm{mod}B}$. □

The maps r and e are respectively called *restriction* and *extension maps.*

Example. If one extends (even a hereditary algebra) by a non-projective module, then neither the restriction nor the extension define maps between the corresponding posets of tilting modules.

Let indeed B be the path algebra of the quiver

$$1\circ \longleftarrow \circ 2$$

and let $A = B[S_2]$. then A is given by the quiver

$$1\circ \xleftarrow{\ \beta\ } \circ 2 \xleftarrow{\ \alpha\ } \circ 3$$

bound by $\beta\alpha = 0$. Here, and in the sequel, we denote by P_x, S_x respectively the indecomposable projective and the simple module corresponding to the point x of the quiver.

(a) Extending the tilting B-module $M = P_1 \oplus P_2$ yields the A-module $eM = P_1 \oplus P_2 \oplus S_3$ which is not tilting, because $\mathrm{Ext}\,^2_A(S_3, P_1) \neq 0$.

(b) Restricting the tilting A-module $T = P_1 \oplus P_2 \oplus P_3$ yields the B-module $\mathcal{R}T = P_1 \oplus P_2 \oplus S_2$ which is not tilting, because $\mathrm{Ext}\,^1_B(S_2, P_1) \neq 0$.

4.5. If $\mathcal{C}$ is an additive full subcategory of mod A, closed under extensions, then a non-zero module $X \in \mathcal{C}$ is called *Ext-projective* in $\mathcal{C}$ if $\mathrm{Ext}\,^1_A(X, -)|_{\mathcal{C}} = 0$, see [2].

Lemma. (a) *Let P_1 be a non-zero projective B-module, then $\mathcal{E}P_1$ is an Ext-projective in S^{perp}.*

(b) *Let I_1 be a non-zero injective B-module, then $\mathcal{E}I_1$ is an injective A-module.*

Proof. (a) By (2.2), $\mathcal{E}P_1 \in S^{perp}$. Let $X \in S^{perp}$, then, by (2.1) $X \cong \mathcal{E}\mathcal{R}X$ hence

$$\mathrm{Ext}\,^1_A(\mathcal{E}P_1, X) \cong \mathrm{Ext}\,^1_A(\mathcal{E}P_1, \mathcal{E}\mathcal{R}X) \cong \mathrm{Ext}\,^1_B(P_1, \mathcal{R}X) = 0$$

(b)Let S' be a simple A-module. Suppose first $S' \ncong S$. Applying $\mathrm{Hom}\,_A(S', -)$ to the extension sequence

$$0 \longrightarrow I_1 \longrightarrow \mathcal{E}I_1 \longrightarrow S^{e_{I_1}} \longrightarrow 0$$

yields an isomorphism $\mathrm{Ext}\,^1_A(S', \mathcal{E}I_1) \cong \mathrm{Ext}\,^1_A(S', I_1)$. Since $S' \not\cong S$, then S' is a B-module and the injectivity of I_1 in mod B yields $\mathrm{Ext}\,^1_A(S', \mathcal{E}I_1) \cong \mathrm{Ext}\,^1_B(S', I_1) = 0$. On the other hand, $\mathrm{Ext}\,^1_A(S, \mathcal{E}I_1) = 0$ because $\mathcal{E}I_1 \in S^{perp}$ by (2.2). Therefore $\mathcal{E}I_1$ is an injective A-module. □

4.6. An algebra C is called a *Gorenstein algebra* if $pdDC < \infty$ and $idC < \infty$ (see [1]).

Corollary. *If B is a Gorenstein algebra, then $e(DB) = DA$.*

Proof. Since B is Gorenstein, then DB is a tilting B-module. Moreover, $DA = S \oplus DB$. □

5. Comparing the quivers of tilting modules

5.1. We now prove our key lemma.

Lemma. (a) *The maps $e : \mathcal{T}_B \longrightarrow \mathcal{T}_A$ and $r : \mathcal{T}_A \longrightarrow \mathcal{T}_B$ are morphisms of posets.*

(b) *An arrow $\alpha : M_1 \longrightarrow M_2$ in $\overrightarrow{\mathcal{K}}_B$ induces an arrow $e(M_1) \longrightarrow e(M_2)$ in $\overrightarrow{\mathcal{K}}_A$ (which we denote by $e(\alpha)$).*

(c) *If $\beta : T_1 \longrightarrow T_2$ is an arrow in $\overrightarrow{\mathcal{K}}_A$. then either $r(T_1) = r(T_2)$, or else there exists an arrow $r(T_1) \longrightarrow r(T_2)$ (which we denote by $r(\beta)$).*

Proof. (a) Assume first that $M_1, M_2 \in \mathcal{T}_B$ are such that $M_1 \leq M_2$. We claim that $S \oplus \mathcal{E}M_1 \leq S \oplus \mathcal{E}M_2$ in $\mathcal{T}_A$, that is, $S \oplus \mathcal{E}M_1 \in (S \oplus \mathcal{E}M_2)^{\perp}$ or, equivalently, $\mathrm{Ext}\,^j_A(S \oplus \mathcal{E}M_2, S \oplus \mathcal{E}M_1) = 0$ for each $j \geq 1$. Now, by (2.3), we have $\mathrm{Ext}\,^j_A(\mathcal{E}M_2, \mathcal{E}M_1) \cong \mathrm{Ext}\,^j_B(M_2, M_1) = 0$ for each $j \geq 1$ because $M_2 \in M_1^{\perp}$. The required statement follows.

Assume next that $T_1, T_2 \in \mathcal{T}_A$ are such that $T_1 \leq T_2$. We claim that $\mathrm{Ext}\,^j_B(\mathcal{R}T_2, \mathcal{R}T_1) = 0$ for each $j \geq 1$. Now, by (2.6)(b), we have

$$\mathrm{Ext}\,^j_B(\mathcal{R}T_2, \mathcal{R}T_1) \cong \mathrm{Ext}\,^j_A(T_2, T_1) = 0$$

for each $j \geq 2$ while, for $j = 1$, the epimorphism

$$\mathrm{Ext}\,^1_A(T_2, T_1) \longrightarrow \mathrm{Ext}\,^1_B(\mathcal{R}T_2, \mathcal{R}T_1)$$

of (2.6)(a) gives $\mathrm{Ext}\,^1_B(\mathcal{R}T_2, \mathcal{R}T_1) = 0$. Therefore $\mathcal{R}T_1 \in (\mathcal{R}T_2)^{\perp}$.

(b) Let $\alpha : M_1 \longrightarrow M_2$ be an arrow in $\overrightarrow{\mathcal{K}}_B$. There exist indecomposable B-modules N_1, N_2 such that $M_1 = L \oplus N_1$, $M_2 = L \oplus N_2$ and a non-split short exact sequence

$$0 \longrightarrow N_1 \longrightarrow \overline{L} \longrightarrow N_2 \longrightarrow 0$$

with $\overline{L} \in \text{add}L$. The exact functor $\mathcal{E}$ yields a non-split short exact sequence

$$0 \longrightarrow \mathcal{E}N_1 \longrightarrow \mathcal{E}\overline{L} \longrightarrow \mathcal{E}N_2 \longrightarrow 0$$

Since $eM_i = S \oplus \mathcal{E}M_i = S \oplus \mathcal{E}L \oplus \mathcal{E}N_i$ for $i = 1, 2$, there exists an arrow $eM_1 \longrightarrow eM_2$.

(c) Let $\beta : T_1 \longrightarrow T_2$ be an arrow in $\overrightarrow{\mathcal{K}}_A$. There exist indecomposable A-modules V_1, V_2 such that $T_1 = W \oplus V_1$, $T_2 = W \oplus V_2$ and a non split exact sequence

$$0 \longrightarrow V_1 \longrightarrow \overline{W} \longrightarrow V_2 \longrightarrow 0$$

with $\overline{W} \in \text{add}W$. The exact functor $\mathcal{R}$ yields an exact sequence

$$0 \longrightarrow \mathcal{R}V_1 \longrightarrow \mathcal{R}\overline{W} \longrightarrow \mathcal{R}V_2 \longrightarrow 0$$

If it splits, then $\text{add}\mathcal{R}T_1 = \text{add}\mathcal{R}T_2$ so that $r(T_1) = r(T_2)$. If it does not, then there exists an arrow $rT_1 \longrightarrow rT_2$. □

5.2. We now complete the proof of our main result. In view of (3.4) and (4.1), it suffices to prove the following theorem.

Theorem. (a) *The map $e : \mathcal{T}_B \longrightarrow \mathcal{T}_A$ induces a full embedding of quivers $e : \overrightarrow{\mathcal{K}}_B \longrightarrow \overrightarrow{\mathcal{K}}_A$.*

(b) *The image of e in $\overrightarrow{\mathcal{K}}_A$ is closed under successors.*

(c) *If a point of $\overrightarrow{\mathcal{K}}_A$ lies in the image of e, then all but exactly one of it immediate predecessors lies in the image.*

(d) *Distinct connected components of $\overrightarrow{\mathcal{K}}_B$ map to distinct connected components of $\overrightarrow{\mathcal{K}}_A$.*

Proof. (a) and (b). Since, by (4.1), e is an embedding of quivers, we only have to show that, for any arrow $e(M) \longrightarrow T$ in $\overrightarrow{\mathcal{K}}_A$, there exists a point M' in $\overrightarrow{\mathcal{K}}_B$ such that $T = e(M')$ and moreover, there exists an arrow $M \longrightarrow M'$ in $\overrightarrow{\mathcal{K}}_B$.

We have $eM = S \oplus \mathcal{E}M \cong X \oplus W$, $T = Y \oplus W$ (with X, Y indecomposable) and a non-split short exact sequence

$$0 \longrightarrow X \longrightarrow \overline{W} \longrightarrow Y \longrightarrow 0$$

with $\overline{W} \in \text{add}W$.

Notice first that S is necessarily a summand of W. Indeed, if not, then $S \cong X$ and the injectivity of S would force the above sequence to split.

On the other hand, S is not a summand of $\overline{W}$. Indeed, if it is, then S would map non-trivially to Y and, since S is simple injective, we would get $Y \cong S$ and the sequence would again split.

Since S is a summand of W, we can write $T = S \oplus V$ for some A-module V. Since T is a tilting module and S is simple injective, then $V \in S^{perp}$ so that $V \cong \mathcal{ER}(V)$ by (2.4). Therefore $T = e(\mathcal{R}V)$. There remains to show the existence of an arrow $M \longrightarrow \mathcal{R}V$. Consider the exact sequence

$$0 \longrightarrow \mathcal{R}X \longrightarrow \mathcal{R}\overline{W} \longrightarrow \mathcal{R}Y \longrightarrow 0.$$

If it splits, $\mathcal{R}X$ is a summand of $\mathcal{R}W$, so $\mathcal{ER}X$ is a summand of $\mathcal{ER}W$. Now $X \in S^{perp}$, hence $\mathcal{ER}X \cong X$. On the other hand, S is not a summand of $\overline{W}$ hence $\overline{W} \in S^{perp}$ and thus $\mathcal{ER}\overline{W} \cong \overline{W}$. This implies that X is a summand of $\overline{W}$, a contradiction. So the sequence does not split and the required arrow exists.

(c) Let $T_1, T_2, \cdots, T_r$ be the immediate predecessors of $e(M) = S \oplus \mathcal{E}M$ in $\overrightarrow{\mathcal{K}}_A$. Assume S is a summand of T_i. By (3.3), T_i lies in the image of e. We claim that there is exactly one i_0 such that T_{i_0} is not in the image of e or, equivalently, S is not a summand of T_{i_0}. By construction of $\overrightarrow{\mathcal{K}}_A$, there is at most one such T_{i_0}. We prove that there is at least one such T_{i_0}. The extension sequence for M gives $S \in \mathrm{Gen}\mathcal{E}M$. Now by [5](1.3) (see also [8]) there exists an exact sequence

$$0 \longrightarrow X \longrightarrow \overline{\mathcal{E}W} \longrightarrow S \longrightarrow 0$$

such that $\overline{\mathcal{E}W} \in \mathrm{add}\mathcal{E}W$ and $X \oplus \mathcal{E}M$ is a tilting A-module (of which S is not a summand). This implies the claim.

Since the image of e only contains modules having S as a summand, we are done.

(d) Assume that two points M, M' in $\overrightarrow{\mathcal{K}}_B$ lie in distinct connected components, but are such that their images eM, eM' lie in the same component of $\overrightarrow{\mathcal{K}}_A$. Then there exists a walk in the latter component.

$$eM_1 - T_2 - \cdots - T_r - eM_2$$

Applying the restriction maps, we get, by (4.1), a walk from $M = re(M)$ to $M' = re(M')$, a contradiction. □

5.3. To any poset E, one can associate a simplicial complex $|E|$, called its *chain complex* as follows: an i-simplex is a set of $i+1$ distinct elements $\{x_0, x_1, \cdots, x_i\}$ of E such that $x_0 \le x_1 \le \cdots \le x_i$.

Corollary. *The simplicial complex $|\mathcal{T}_B|$ is (homeomorphic to) a retract of $|\mathcal{T}_A|$.*

Proof. Since $re = id_{\mathcal{T}_B}$, it suffices to observe that, by (4.1)(b)(c), the maps e and r induce simplicial maps between $|\mathcal{T}_A|$ and $|\mathcal{T}_B|$. □

5.4. For an algebra C, we denote by $\mathcal{P}^{<\infty}(C)$ the full subcategory of mod C consisting of all modules of finite projective dimension.

Corollary. *$\mathcal{P}^{<\infty}(A)$ is contravariantly finite in* mod A *if and only if $\mathcal{P}^{<\infty}(B)$ is contravariantly finite in* mod B.

Proof. By [9] (3.3), it suffices to prove that $\mathcal{T}_A$ has a minimal element if and only if so does B. If $\mathcal{T}_B$ has a minimal element then, since the image of e is closed under successors, so does $\mathcal{T}_A$. By [9](3.2), a minimal element T in $\mathcal{T}_A$ must admit S as a summand. By (3.3), there exists a tilting B-module M such that $T = S \oplus \mathcal{E}M = eM$. By (4.2)(b), M is a minimal element in $\overrightarrow{\mathcal{K}}_B$. □

5.5. Let C be an algebra. A point in $\overrightarrow{\mathcal{K}}_C$ is called *saturated* if the number of its neighbours equals $rk\mathrm{K}_0(C)$.

Corollary. *Assume B is hereditary, and a point M in $\overrightarrow{\mathcal{K}}_B$ is saturated. Then its image eM in $\overrightarrow{\mathcal{K}}_A$ is saturated.*

Proof. Since M is saturated, it has $rk\mathrm{K}_0(B)$ neighbours in $\overrightarrow{\mathcal{K}}_B$. By (4.2)(b) and (c), eM has exactly $1+rk\mathrm{K}_0(B) = rk\mathrm{K}_0(A)$ neighbours. □

5.6. We deduce a sufficient condition to $\overrightarrow{\mathcal{K}}_A$ to have infinitely many components.

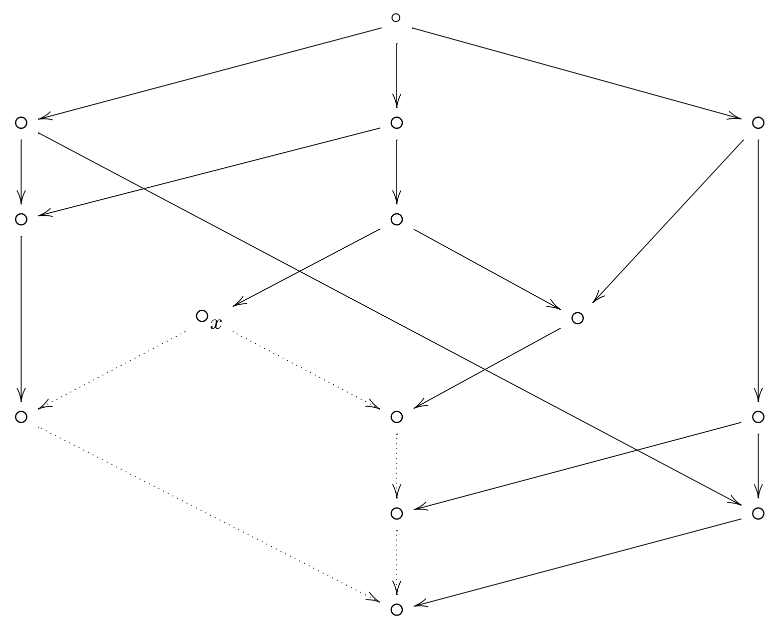
Corollary. *If A is hereditary, and contains a wild full convex subcategory B with 3 simple modules, then $\overrightarrow{\mathcal{K}}_A$ has infinitely many connected components.*

Proof. By [12], $\overrightarrow{\mathcal{K}}_B$ has infinitely many connected components. Our statement follows from (4.2)(d), its dual and an obvious induction. □

5.7. Examples. (a) Let A be the path algebra of the quiver

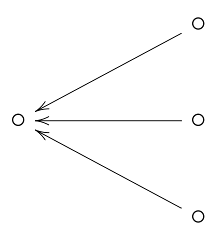
$$1\circ \longleftarrow \circ 2 \longleftarrow \circ 3 \longleftarrow \circ 4$$

and B be the full convex subcategory generated by all points except 4. The quiver $\overrightarrow{\mathcal{K}}_A$ is:



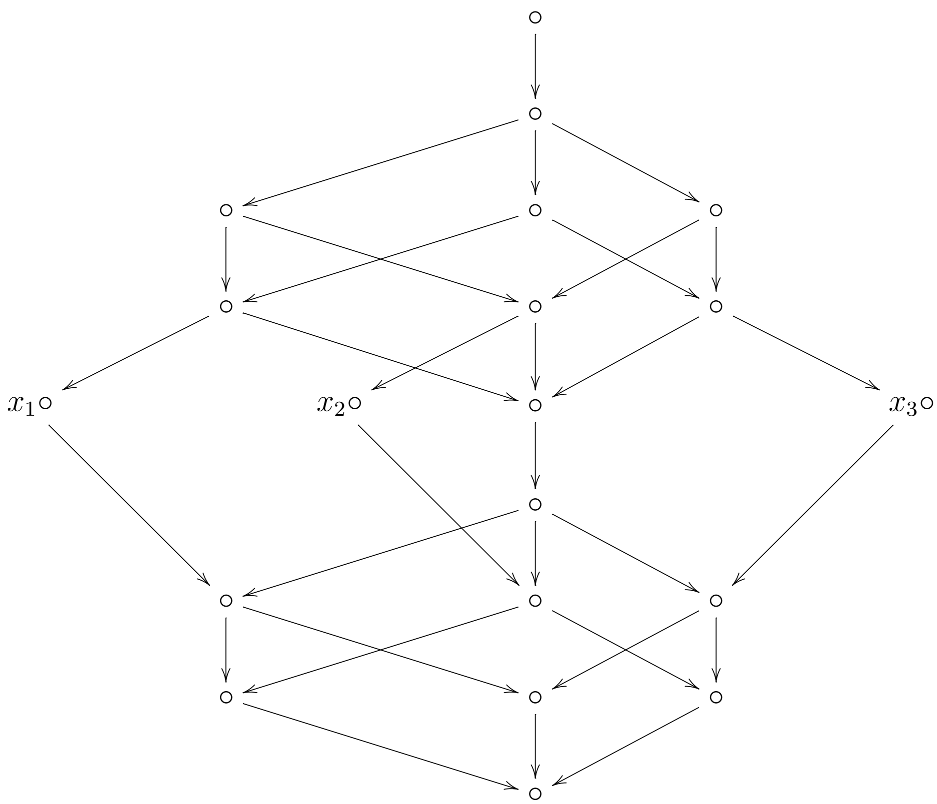
Then $\overrightarrow{\mathcal{K}}_B$ is the subquiver indicated by dotted lines, consisting of all succesors of the point x.

(b) Let A be the path algebra of the quiver Q



In this case, we have three possible embeddings of the quiver Q'

inside Q. Letting B be the path algebra of Q', we see that there are 3 different embeddings of $\overrightarrow{\mathcal{K}}_B$ inside $\overrightarrow{\mathcal{K}}_A$, obtained by identifying $\overrightarrow{\mathcal{K}}_B$ with the subquivers of $\overrightarrow{\mathcal{K}}_A$ consisting of the successors of x_1, x_2 and x_3 respectively.



6. Comparing endomorphism algebras

6.1. In this section, we assume that B (or, equivalently, $A = B[P_0]$) is hereditary. For an algebra C, we denote by $\nu_C = DC \otimes_C -$ the Nakayama functor, and by $\tau_C = DTr$ the Auslander-Reiten translation in mod C (for details, we refer to [3], Chapters (IV) and (V), [4], Chapter (IV)).

Proposition. *Let M be a tilting B-module. Then $End_A eM$ is the one-point extension of* $\operatorname{End}_B M$ *by the module* $\operatorname{Hom}_B(M, \nu_B P_0)$.

Proof. Consider the almost split sequence

$$0 \longrightarrow \tau_A S \longrightarrow E \longrightarrow S \longrightarrow 0$$

in mod A. Then E is injective and in fact, is the direct sum of all indecomposable injectives I_x such that S is a summand of I_x such that S is a summand of I_x/S_x (see [3] p.154). Hence $\mathcal{R}E = \nu_B P_0$. Applying $\operatorname{Hom}_A(\mathcal{E}M, -)$ to the sequence above yields an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\mathcal{E}M, \tau_A S) \longrightarrow \operatorname{Hom}_A(\mathcal{E}M, E) \longrightarrow$$

$\longrightarrow \mathrm{Hom}_A(\mathcal{E}M, S) \longrightarrow \mathrm{Ext}^1_A(\mathcal{E}M, \tau_A S)$.

Since $pdS \leq 1$, $id\tau_A S \leq 1$, the Auslander-Reiten formulae (see [11], p.75) yield $\mathrm{Hom}_A(\mathcal{E}M, S) = D\mathrm{Ext}^1_A(S, \mathcal{E}M) = 0$ and $\mathrm{Ext}^1_A(\mathcal{E}M, \tau_A S) \cong D\mathrm{Hom}_A(S, \mathcal{E}M) = 0$ because $\in S^{perp}$ (by (2.2). Thus, $\mathrm{Hom}_A(\mathcal{E}M, E) \cong \mathrm{Hom}_A(\mathcal{E}M, S)$. We infer that $\mathrm{Hom}_A(\mathcal{E}M, S) \cong \mathrm{Hom}_B(M, \mathcal{R}E) \cong \mathrm{Hom}_B(M, \nu_B P_0)$. The statement follows. □

6.2. We deduce that $\mathrm{End}_B rT$ is representation finite whenever $\mathrm{End}_A T$ is.

Proposition. *Let T be a tilting A-module such that* $\mathrm{End}_A T$ *is representation finite. Then* $\mathrm{End}_B rT$ *is representation finite.*

Proof. By tilting theory (see, for instance, [6] p.144) there is a one-to-one correspondence between the isomorphism classes of the indecomposable $\mathrm{End}_B rT$-modules and of the indecomposable A-modules lying in one of the classes $\mathfrak{X}^0_A(T)$ and $\mathfrak{X}^1_A(T)$ of mod A. The statement then follows from (3.2). □

6.3. If S is a summand of a tilting A-module then by (3.3), there exists a tilting B-module M such that $T = S \oplus \mathcal{E}M$. In particular $\mathrm{End}_A T$ is a one-point extension of $\mathrm{End}_B \mathcal{R}T \cong \mathrm{End}_B M$, so $\mathrm{End}_B M$ is a quotient algebra of $\mathrm{End}_A T$. We now consider the case where S is not a summand of T.

Proposition. *If X is an A-module such that S is not a direct summand of X, then* $\mathrm{End}_A X$ *is (isomorphic to) a subalgebra of* $\mathrm{End}_B \mathcal{R}X$.

Proof. Applying the functor $\mathrm{Hom}_A(\mathcal{R}X, -)$ to the restriction sequence

$$0 \longrightarrow \mathcal{R}X \xrightarrow{f} X \xrightarrow{g} S^{r_X} \longrightarrow 0$$

yields an isomorphism

$$\mathrm{Hom}_A(\mathcal{R}X, \mathcal{R}X) \cong \mathrm{Hom}_A(\mathcal{R}X, X)$$

given as follows: if $u \in \mathrm{Hom}_A(\mathcal{R}X, X)$ then $gu = 0$ implies the existence of a unique $v \in \mathrm{Hom}_A(\mathcal{R}X, \mathcal{R}X)$ such that $u = fv$.

On the other hand, since S is not a summand of X, the map

$$\mathrm{Hom}_A(X, X) \longrightarrow \mathrm{Hom}_A(\mathcal{R}X, X)$$

given by $w \mapsto wf$ (and obtained by applying $\mathrm{Hom}_A(-, X)$ to the above sequence) is a monomorphism.

Composing yields an injection $\mathrm{Hom}_A(X, X) \longrightarrow \mathrm{Hom}_B(\mathcal{R}X, \mathcal{R}X)$ defined as follows: $w \mapsto w'$ where $w' : \mathcal{R}X \longrightarrow \mathcal{R}X$ satisfies $wf = fw'$.

But this implies $w' = \mathcal{R}w$ (since the latter is the unique morphism verifying this equality). In particular, $w \mapsto w'$ is a morphism of algebras. □

6.4. The above proposition applies in particular to tilting modules. Considering the trivial one yields the following (surprising) corollary.

Corollary . *The algebra A is (isomorphic to) a subalgebra of $B' = \operatorname{End}_B U$ and B' is Morita-equivalent to B.*

Proof. By (4.3), A is isomorphic to a subalgebra of $\operatorname{End}\mathcal{R}A$. Now $\mathcal{R}A \cong \mathcal{R}B \oplus \mathcal{R}P \cong_B B \oplus_B P_0 =_B U$. Thus, $\operatorname{End}_B\mathcal{R}A \cong \operatorname{End}_B U$ is Morita equivalent to B. □

Example. Let A be the path algebra of the quiver

$$1\circ \longleftarrow 2\circ \longleftarrow \circ 3$$

then $\mathcal{R}A = P_1 \oplus P_2^2$ and $\operatorname{End}_B\mathcal{R}A$ is the 3×3-matrix algebra

$$\operatorname{End}_B\mathcal{R}A = \{ \begin{bmatrix} a & b & c \\ 0 & d & d' \\ 0 & e & e' \end{bmatrix} | a, b, c, d, d', e, e' \in k \}$$

Clearly, A is a subalgebra of the latter.

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