Ext-projectives in suspended subcategories

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Abstract

Let $\mathcal{H}$ be a hereditary category with tilting object and $\mathbf{D^b}(\mathcal{H})$ denote the category of bounded complexes over $\mathcal{H}$. This paper is devoted to a study of suspended subcategories of $\mathbf{D^b}(\mathcal{H})$ by means of their Ext-projectives.

Introduction. The concept of a t-structure in a triangulated category $\mathcal{T}$ was introduced in the early eighties in [BBD]. It was meant as a technique to construct various abelian subcategories of $\mathcal{T}$ (the "hearts" of the t-structures) and is helpful for the understanding of the structure of $\mathcal{T}$. Our own motivation for their study comes from the representation theory of artin algebras, which now involves the derived category as an essential tool. In this context, t-structures have been especially useful, in particular due to their relationship with tilting theory (see, for instance, [H2],[KV], [P]). In [KV] Keller and Vossieck exhibited a bijection between the t-structures in a given triangulated category $\mathcal{T}$ and the contravariantly finite suspended subcategories of $\mathcal{T}$, which they called aisles.

Our objective in this paper is to study the t-structures and the aisles, from the point of view of the Ext-projectives. The concept of Ext-projective in a subcategory of a module category was introduced by Auslander and Smalø in

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their study of relative almost split sequences [AS]. Since then, they turned out to be very useful in various contexts (see, for instance [A], [AK], [BR], [Kl]). In our situation it follows from [KV], [BR]) that the Ext-projectives in an aisle $\mathcal{U}$ are the projectives in the heart of the t-structure determined by $\mathcal{U}$. Moreover if $\mathcal{U}$ is generated by the Ext-projectives, then the heart of the t-structure is a module category. This explains our interest in studying them.

We now describe the context of the paper. We start by studying some of the more immediate properties of Ext-projectives. These results lay the ground for the main result of our paper, which we now state. We refer the reader to section (2.3) for the definition of exceptional sequence.

Theorem (A): *Let $\mathcal{H}$ be a hereditary category with tilting object and $\mathcal{U}$ be a suspended subcategory in the bounded derived category $\mathbf{D^b}(\mathcal{H})$. Then*

(a) The indecomposable Ext-projectives in $\mathcal{U}$ can be ordered to form an exceptional sequence.
(b) The number of isomorphism classes of indecomposable Ext-projectives in $\mathcal{U}$ does not exceed the rank of $\mathrm{K}_0(\mathcal{H})$.

We then observe that an Ext-projective in an aisle is necessarily a silting complex, in the sense of [KV], and that the suspended subcategories in $\mathbf{D^b}(\mathrm{mod}\, A)$ (when A has finite global dimension) having a given silting complex as Ext-projective form a partially ordered set having a unique maximal and a unique minimal element (compare with [AK] (1.3)). Returning to the case of the bounded derived category of a hereditary category $\mathcal{H}$ with tilting object, we deduce a procedure which we call "deconstruction of aisles". For the definition of the operation $*$, we refer the reader to [KV] or section 3 below.

Theorem (B): *Let $\mathcal{U}$ be a suspended subcategory, and M be an object in $\mathbf{D^b}(\mathcal{H})$, then:*

(a) M is a silting complex if and only if M is Ext-projective in $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^{\perp})$.
(b) The following conditions are equivalent
(i) M is Ext-projective in $\mathcal{U}$.
(ii) $\mathcal{U} = \mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^{\perp})$ and M is a silting complex.
(iii) $\mathcal{U} = \mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^{\perp})$ and $\mathcal{U}^{\perp} = (\mathcal{U}^{\perp} \cap \mathcal{B}_M^{\perp}) * {}_{\tau M}\mathcal{U}$.

As a consequence, we obtain that the aisles in $\mathbf{D^b}(\mathcal{H})$ which have $rk\mathrm{K}_0(\mathcal{H})$ isomorphism classes of indecomposable Ext-projective objects are in bijection with the silting complexes having the same number of isomorphism classes of indecomposable summands.

Our final section contains applications to the class of supported algebras. In [HRS], Happel, Reiten and Smalø have introduced the class $\mathcal{L}_A$ which consists of the indecomposable A-modules all of whose predeccesors have projective dimension less than or equal than 1. While this definition was meant for the study of quasi-tilted algebras, it leds to generalizations of the latter class. In particular, left supported algebras were introduced in [ACT] as being those algebras A such that the additive closure of $\mathcal{L}_A$ is contravariantly finite in mod A (see also [ALR], [ACPT]). Here we give several characterizations of left supported algebras using aisles.

In a forthcoming paper, we shall apply our techniques to classify classes of aisles.

1 Basic definitions and results

Throughout this paper, we assume that k is a commutative field, and that $\mathcal{T}$ is a triangulated Krull-Schmidt k-category. Given a full subcategory $\mathcal{U}$ of $\mathcal{T}$, we write $X \in \mathcal{U}$ to express that X is an object in $\mathcal{U}$. For $i \in \mathbb{Z}$, the $i^{th}-$translate of $X \in \mathcal{T}$ is denoted by $X[i]$. The *right* and the *left orthogonal* of $\mathcal{U}$ are respectively defined by

$$\mathcal{U}^{\perp} = \{Y \in \mathcal{T}, \operatorname{Hom}_{\mathcal{T}}(X,Y) = 0 \text{ for every } X \in \mathcal{U}\}$$

$$^{\perp}\mathcal{U} = \{Y \in \mathcal{T}, \operatorname{Hom}_{\mathcal{T}}(Y,X) = 0 \text{ for every } X \in \mathcal{U}\}.$$

A full, additive subcategory $\mathcal{U}$ of $\mathcal{T}$, closed under direct summands, is called *suspended* if it is closed under positive translations and under extensions, that is, if

(1) if $X \in \mathcal{U}$, then $X[i] \in \mathcal{U}$, for every $i > 0$, and
(2) if $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{T}$ with $X, Z \in \mathcal{U}$ then $Y \in \mathcal{U}$.

The dual notion is that of a *cosuspended* subcategory.

We recall the following definition from [AS]. Let $\mathcal{U}$ be a full additive subcategory of $\mathcal{T}$ closed under extensions, and X be a non-zero object in $\mathcal{U}$

(1) X is called *Ext-projective* in $\mathcal{U}$ if and only if $\operatorname{Hom}_{\mathcal{T}}(X, Y[1]) = 0$ for all $Y \in \mathcal{U}$.
(2) X is called *Ext-injective* in $\mathcal{U}$ if and only if $\operatorname{Hom}_{\mathcal{T}}(Y, X[1]) = 0$, for all $Y \in \mathcal{U}$.

Remark 1.1 *(a) Let $\mathcal{U}$ be a full additive subcategory of $\mathcal{T}$. A non-zero objet X in $\mathcal{T}$ is Ext-projective in $\mathcal{U}$ if and only if $X \in \mathcal{U} \cap {}^{\perp}\mathcal{U}[1]$. Dually, $X \in \mathcal{T}$ is Ext-injective in $\mathcal{U}$ if and only if $X \in \mathcal{U} \cap \mathcal{U}^{\perp}[-1]$.*
(b) If X is Ext-projective in a suspended subcategory $\mathcal{U}$ of $\mathcal{T}$, then clearly $\operatorname{Hom}_{\mathcal{T}}(X, Y[i]) = 0$, for all $Y \in \mathcal{U}$ and all $i > 0$. Dually, if X is Ext-injective in a cosuspended subcategory $\mathcal{U}$ of $\mathcal{T}$ then $\operatorname{Hom}_{\mathcal{T}}(Y[i], X[1]) = 0$, for all $Y \in \mathcal{U}$ and all $i \leq 0$.
(c) If $\mathcal{U}$ is a suspended subcategory, then it has no Ext-injectives. Indeed, if $X \in \mathcal{U}$ were Ext-injective, then $X[1] \in \mathcal{U}$ implies $\operatorname{Hom}_{\mathcal{T}}(X[1], X[1]) = 0$ and so $X = 0$. Dually, a cosuspended subcategory has no Ext-projectives.

The following lemma will be used essentially in the proof of our theorem (2.3).

Lemma 1.2 *Let $\mathcal{U}$ be a suspended subcategory in $\mathcal{T}$ and X be Ext-projective in $\mathcal{U}$ then, for every $j \neq 0$, $X[j]$ is not Ext-projective in $\mathcal{U}$.*

Proof. Suppose that $j > 0$ is such that $X[j]$ is Ext-projective. Then, for every $Y \in \mathcal{U}$, $\mathrm{Hom}_{\mathcal{T}}(X[j], Y[1]) = 0$. Taking $Y = X[j-1]$ (and observing that $Y \in \mathcal{U}$) yields the contradiction $X = 0$. Thus, let $j < 0$ be such that $X[j]$ is Ext-projective in $\mathcal{U}$. However, $X[j] \in \mathcal{U}$ implies $X[-1] \in \mathcal{U}$ because $\mathcal{U}$ is suspended, and then the Ext-projectivity of X yields $X[-1] \in \mathcal{U} \cap {}^{\perp}\mathcal{U} = \{0\}$. This contradiction shows that $X[j]$ is not in $\mathcal{U}$. Hence $X[j]$ cannot be Ext-projective in $\mathcal{U}$. □

A suspended subcategory $\mathcal{U}$ (or a triangulated subcategory $\mathcal{B}$) of $\mathcal{T}$ is called an *aisle* (or a *Bousfield localization*) in $\mathcal{T}$ if the inclusion functor $\mathcal{U} \to \mathcal{T}$ (or $\mathcal{B} \to \mathcal{T}$, respectively) admits a right adjoint $\tau_{\leq 0} : \mathcal{T} \to \mathcal{U}$.

The notion dual to that of aisle is that of *coaisle* (where $\tau_{\geq 0} : \mathcal{T} \to \mathcal{U}$ is the corresponding adjoint functor).

A pair of subcategories $(\mathcal{U}, \mathcal{V})$ of $\mathcal{T}$ is called a *t-structure* if $\mathcal{U}$ is a suspended subcategory and, for every $X \in \mathcal{T}$, there exists a triangle

$$X_{\mathcal{U}} \to X \to X_{\mathcal{V}} \to X_{\mathcal{U}}[1]$$

in $\mathcal{T}$ with $X_{\mathcal{U}} \in \mathcal{U}$ and $X_{\mathcal{V}} \in \mathcal{V}[-1]$ where $X_{\mathcal{U}} = \tau_{\leq 0}X$ and $X_{\mathcal{V}} = \tau_{>0}X$.

It is shown in [KV], (1.1) that a suspended subcategory $\mathcal{U}$ is an aisle if and only if $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure. Since $\mathcal{T}$ is a Krull-Schmidt category, $\mathcal{U}$ is an aisle if and only if it is contravariantly finite in $\mathcal{T}$ (see [KV](1.3)).

The proof of the following lemma is implicit in [KV](5.1) and [BR](3.1). We recall that *the heart* $\mathcal{C}_{\mathcal{U}}$ of a t-structure, $(\mathcal{U}, \mathcal{U}^{\perp}[1])$, is the subcategory $\mathcal{C}_{\mathcal{U}} = \mathcal{U} \cap \mathcal{U}^{\perp}[1]$. It is shown in [BBD](1.3.6) that $\mathcal{C}_{\mathcal{U}}$ is an abelian category. We denote by $H^0 = \tau_{\leq 0}\tau_{\geq 0} : \mathcal{T} \to \mathcal{C}_{\mathcal{U}}$ the cohomological functor associated to the t-structure (see [BBD](1.3.6)).

Lemma 1.3 *Let $\mathcal{U}$ be an aisle in $\mathcal{T}$. Then*

(a) If X is Ext-projective in $\mathcal{U}$ then $H^0(X)$ is projective in $\mathcal{C}_{\mathcal{U}}$.
(b) The functor $H^0 |_{\mathcal{U} \cap {}^{\perp}\mathcal{U}[1]} : \mathcal{U} \cap {}^{\perp}\mathcal{U}[1] \to \mathcal{C}_{\mathcal{U}}$ is full and faithful.

Proof. (*a*) By [BR] Ch.III (3.2), $H^0(X)$ is projective in $\mathcal{C}_{\mathcal{U}}$ if and only if

$$\mathrm{Hom}_{\mathcal{T}}(\tau_{\geq 0}X[-1], C) = 0$$

for all $C \in \mathcal{C}_{\mathcal{U}}$, where the truncation $\tau_{\geq 0}X[-1]$ is computed with respect to the coaisle $\mathcal{U}^{\perp}[1]$. Consider the triangle

$$\tau_{\leq -1}X \to X \to H^0(X) \to \tau_{\leq -1}X[1].$$

Applying $\mathrm{Hom}_{\mathcal{T}}(-, C)$, with $C \in \mathcal{C}_{\mathcal{U}}$, yields an exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(\tau_{\leq -1}X, C) \to \mathrm{Hom}_{\mathcal{T}}(H^0(X), C[1]) \to \mathrm{Hom}_{\mathcal{T}}(X, C[1]).$$

Since X is Ext-projective in $\mathcal{U}$ and $C \in \mathcal{C}_{\mathcal{U}}$ then $\mathrm{Hom}_{\mathcal{T}}(X, C[1]) = 0. Since \tau_{\leq -1}X = \tau_{\leq 0}(X[-1])[1]$ we get $\mathrm{Hom}_{\mathcal{T}}(\tau_{\leq -1}X, C) = 0$ because $C \in \mathcal{U}^{\perp}[1]$. Therefore $\mathrm{Hom}_{\mathcal{T}}(H^0(X), C[1]) = 0$ and our statement follows.

(b) follows from [KV]Lemma(5.1)(a). □

We deduce the following easy corollary.

Corollary 1.4 *Let A be a finite dimensional k-algebra, and $\mathbf{D^b}(\mathrm{mod}\, A)$ be the derived category of bounded complexes of finitely generated A-modules. Then the Ext-projectives in the so-called canonical aisle*

$$\mathbf{D^{\leq 0, b}}(\mathrm{mod}\, A) = \{X \in \mathbf{D^b}(\mathrm{mod}\, A) \text{ such that } H^j X = 0, \text{ for all } j > 0\}$$

are just the projective $A-$modules. □.

We recall that a triangulated Krull-Schmidt k-category $\mathcal{T}$ is said to have a *Serre duality* if there exists a triangulated equivalence $\tau : \mathcal{T} \to \mathcal{T}$ and, for all X, Y, there exists an isomorphism

$$\mathbf{D}\mathrm{Hom}_{\mathcal{T}}(X, Y[1]) \to \mathrm{Hom}_{\mathcal{T}}(Y, \tau X),$$

functorial in both variables, and called the Auslander-Reiten formula (see [RV1]). Here $\mathbf{D} = \mathrm{Hom}_k(-, k)$ denotes the usual vector space duality.

An example of a triangulated Krull-Schmidt category with Serre duality is the derived category $\mathbf{D^b}(\mathrm{mod}\, A)$, where A is a finite dimensional k-algebra with finite global dimension (see [H2]).

Lemma 1.5 *[AS](3.4) (3.7) Let $\mathcal{U}$ be a full additive subcategory of a triangulated category $\mathcal{T}$ with Serre duality closed under extensions and let $X \in \mathcal{U}$ be an indecomposable object. Then*

(a) X is Ext-projective in $\mathcal{U}$ if and only if $\tau X \in \mathcal{U}^{\perp}$.
(b) X is Ext-injective in $\mathcal{U}$ if and only if $\tau^{-1} X \in {}^{\perp}\mathcal{U}$.

Proof: (a) By the Auslander-Reiten formula, X is Ext-projective in $\mathcal{U}$ if and only if $\mathbf{D}\mathrm{Hom}_{\mathcal{T}}(Y, \tau X) \simeq \mathrm{Hom}_{\mathcal{T}}(X, Y[1]) = 0$ for all $Y \in \mathcal{U}$,that is, if and only if $\tau X \in \mathcal{U}^{\perp}$. □

We deduce the following.

Corollary 1.6 *Let $\mathcal{U}$ be a full additive subcategory of a triangulated category with Serre duality $\mathcal{T}$ and let $X \in \mathcal{T}$ be indecomposable. Then:*

(a) If X is Ext-projective in $\mathcal{U}$, then τX is Ext-injective in $\mathcal{U}^{\perp}$.
(b) If, moreover, $\mathcal{U}$ is an aisle, then the converse also holds true (thus τ defines a bijection between the isomorphism classes of indecomposable Ext-projectives in $\mathcal{U}$, and Ext-injectives in $\mathcal{U}^{\perp}$).

Proof (*a*) By (1.5)(a) we have $\tau X \in \mathcal{U}^\perp$. By the Auslander-Reiten formula, X is Ext-projective in $\mathcal{U}$ if and only if $\mathrm{Hom}_{\mathcal{T}}(Y, \tau X[1]) \simeq \mathbf{D}\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for all $Y \in \mathcal{U}^\perp$. Our statement follows.

(*b*) By [BBD], ${}^\perp(\mathcal{U}^\perp) = \mathcal{U}$. Hence, by (1.5)(b), τX is Ext-injective in $\mathcal{U}^\perp$ and this implies $X = \tau^{-1}(\tau X) \in \mathcal{U}$. Also, $\mathrm{Hom}_{\mathcal{T}}(X, Y[1]) \simeq \mathbf{D}\mathrm{Hom}_{\mathcal{T}}(Y, \tau X)$ for all $Y \in \mathcal{U}$. □

Let A be a finite dimensional k-algebra. A complex $X \in \mathbf{D^b}(\mathrm{mod}\, A)$ is called a *partial tilting complex* if X belongs to the homotopy category $\mathbf{K^b}(\mathrm{proj}\, A)$ of bounded complexes of projectives, and moreover $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(X, X[j]) = 0$ for all $j \neq 0$.

Let X be a partial tilting complex and $B = \mathrm{End}_{\mathbf{D^b}(\mathrm{mod}\, A)}(X)$. The functor $F = -\otimes^{\mathbb{L}}_B X : \mathbf{D^b}(\mathrm{mod}\, B) \to \mathbf{D^b}(\mathrm{mod}\, A)$ is well-known to be a full embedding [K].

Corollary 1.7 *Let* $X = \oplus_{i=1}^s X_i$ *be a partial tilting complex with the* X_i *indecomposable and non-isomorphic, and let* $\mathcal{U}$ *denote the essential image under* $-\otimes^{\mathbb{L}}_B X$ *in* $\mathbf{D^b}(\mathrm{mod}\, A)$ *of the canonical aisle* $\mathbf{D}^{\leq 0, \mathbf{b}}(\mathrm{mod} B)$ *(where* $B = \mathrm{End}_{\mathbf{D^b}(\mathrm{mod}\, A)}(X)$*). Then*

- *(a) The objects* $X_1, \cdots, X_s$ *are a full set of representatives of the isomorphism classes of indecomposable Ext-projectives in* $\mathcal{U}$.
- *(b) The functor* $\mathbb{R}\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(X, -)$ *induces an equivalence between the Ext-projectives in* $\mathcal{U}$ *and the projective* $B-$*modules.*
- *(c)* $s \leq rk\mathrm{K}_0\mathbf{D^b}(\mathrm{mod}\, A)$.

Proof: Since $F(B_B) = X$, the functor F induces an equivalence between $\mathbf{D}^{\leq 0, \mathbf{b}}(\mathrm{mod} B)$ and $\mathcal{U}$, whose quasi-inverse is the functor $\mathbb{R}\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(X, -)$. Under this equivalence, the isomorphism classes of indecomposables Ext-projectives in $\mathcal{U}$ correspond to the isomorphism classes of indecomposable Ext-projectives in $\mathbf{D}^{\leq 0, \mathbf{b}}(\mathrm{mod} B)$, which, by (1.4) coincide with the isomorphism classes of indecomposable projectives B-modules. Hence s equals the number of isomorphism classes of indecomposable Ext-projectives in $\mathcal{U}$, and also of projective B-modules. Thus $s = rk\mathrm{K}_0(B) = rk\mathrm{K}_0(\mathbf{D^b}(\mathrm{mod}\, B)) \leq rk\mathrm{K}_0(\mathbf{D^b}(\mathrm{mod}\, A)) = rk\mathrm{K}_0(A)$. □.

2 Aisles in derived hereditary categories

From now on, we let $\mathcal{H}$ be a hereditary category with a tilting object. A connected abelian $k-$category $\mathcal{H}$ is hereditary if the bifunctor $\mathrm{Ext}^2_{\mathcal{H}}(-, ?)$ vanishes, and moreover the sets $\mathrm{Hom}_{\mathcal{H}}(X, Y)$ and $\mathrm{Ext}^1_{\mathcal{H}}(X, Y)$ are finite dimensional $k-$ vector spaces for all $X, Y \in \mathcal{H}$. An object $M \in \mathcal{H}$ is called a *partial tilting object* if $\mathrm{Ext}^1_{\mathcal{H}}(M, M) = 0$. It is a *generator* if $\mathrm{Hom}_{\mathcal{H}}(M, X) = 0$

and $\mathrm{Ext}^1_{\mathcal{H}}(M,X) = 0$ imply $X = 0$, and a *tilting object* if it is both a partial tilting object and a generator. Hereditary categories with tilting object were classified in [H2]. It is shown there that there exists a finite dimensional $k-$algebra A which is hereditary or canonical, such that $\mathbf{D^b}(\mathcal{H}) \simeq \mathbf{D^b}(\mathrm{mod}\, A)$.

We start with an easy and well-known observation. We recall that an object M in a triangulated category $\mathcal{T}$ is called *a generator* if $\mathrm{Hom}_{\mathcal{T}}(M[j], X) = 0$ for all $j \in \mathbb{Z}$ implies $X = 0$. The dual notion is that of *cogenerator*. Clearly, this notion of generator generalizes the above one.

Lemma 2.1 *Let $\mathcal{T}$ be a triangulated category with Serre duality. An object M in $\mathcal{T}$ is a generator if and only if it is a cogenerator.*

Proof. Let M be a generator, and X be such that $\mathrm{Hom}_{\mathcal{T}}(X, M[j]) = 0$ for all $j \in \mathbb{Z}$. We claim that $X = 0$. Since the Serre duality holds in $\mathcal{T}$, we have $\mathrm{Hom}_{\mathcal{T}}(M[j], \tau X) = 0$, for all $j \in \mathbb{Z}$. Since M is a generator, $\tau X = 0$. Hence $X = 0$ and so M is a cogenerator. The converse is shown in the same way. □.

Before stating and proving the next lemma, we recall that, by [HR1] (3.5), we may assume that $\mathcal{H}$ satisfies the additional condition that there is no non-zero morphism from an object of finite length to one of infinite length. For quasi-tilted algebras, we refer the reader to [HRS].

Lemma 2.2 *Let M be a partial tilting object in $\mathcal{H}$ then* $\mathrm{End}_{\mathcal{H}}(M)$ *is a quasi-tilted algebra.*

Proof. We may assume that $\mathcal{H}$ is not a module category (for, otherwise, the result follows from [H2] p. 145). If M is a tilting object in $\mathcal{H}$ there is nothing to show. So assume it is not. In particular, the number of isomorphism classes of indecomposable summands of M is strictly smaller than the rank of $\mathrm{K}_0(\mathcal{H})$ (see [HR2] (1.2)). We then proceed by lowering this rank.

Since M is not tilting, then by [HR2](1.4), it is not a tilting complex in $\mathbf{D^b}(\mathcal{H})$. Hence by (2.1) it is not a cogenerator of $\mathbf{D^b}(\mathcal{H})$ and there exists an indecomposable non-zero object $X \in \mathcal{H}$ such that $\mathrm{Ext}^1_{\mathcal{H}}(X,X) = 0$, $\mathrm{Ext}^1_{\mathcal{H}}(X,M) = 0$ and $\mathrm{Hom}_{\mathcal{H}}(X,M) = 0$. By [HR1](2.1)(2.8), we have $\mathrm{End}_{\mathcal{H}}(X) = k$ and moreover

$$X^{\perp} = \{Y \in \mathcal{H}, \mathrm{Ext}^1_{\mathcal{H}}(X,Y) = 0, \mathrm{Hom}_{\mathcal{H}}(X,Y) = 0\}$$

is a hereditary abelian category with tilting object. We claim that $rk\mathrm{K}_0(X^{\perp}) < rk\mathrm{K}_0(\mathcal{H})$. Since $\mathrm{Ext}^1_{\mathcal{H}}(X,X) = 0$, and $\mathrm{End}_{\mathcal{H}}(X) = k$, it follows from [RV2](3.5)(3.4) that the smallest triangulated subcategory $\mathcal{B}_X$ of $\mathbf{D^b}(\mathcal{H})$ containing X is a Bousfield localization and , moreover $\mathrm{K}_0(\mathcal{H}) \simeq \mathrm{K}_0(\mathcal{B}_X) \oplus \mathrm{K}_0(\mathcal{B}_X^{\perp})$. Since $\mathrm{End}_{\mathcal{H}}(X) = k$, we have $\mathrm{K}_0(\mathcal{H}) \simeq \mathbb{Z} \oplus \mathrm{K}_0(\mathcal{B}_X^{\perp})$. By [BL], proof of Theorem 1, $\mathcal{B}_X^{\perp} \simeq \mathbf{D^b}(X^{\perp})$. Therefore,

$$rk\mathrm{K}_0(X^{\perp}) = rk\mathrm{K}_0(\mathbf{D^b}(X^{\perp})) = rk\mathrm{K}_0(\mathcal{B}^{\perp}) = rk\mathrm{K}_0(\mathcal{H}) - 1.$$

Since M is clearly a partial tilting object in $X^{\perp}$, the statement follows by induction. □

We recall that a sequence $(M_i)_{i\geq 1}$ of indecomposable objects in $\mathbf{D^b}(\mathcal{H})$ is called an *exceptional sequence* if

(1) $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M_i, M_i[j]) = 0$ for all $i \geq 1$ and all $j > 0$, and
(2) $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M_i, M_l[j]) = 0$ for all $i < l$ and all $j \in \mathbb{Z}$.

If A is a triangular algebra, then the indecomposable Ext-projectives in the canonical aisle, which are the indecomposable projective $A-$modules (see (1.4)) can be ordered to form an exceptional sequence. This is clearly not the case if A is not triangular.

We are now able to prove our first theorem.

Theorem 2.3 *Let $\mathcal{U}$ be a suspended subcategory in $\mathbf{D^b}(\mathcal{H})$. Then*

(a) The indecomposable Ext-projectives in $\mathcal{U}$ can be ordered to form an exceptional sequence.
(b) The number of isomorphism classes of indecomposable Ext-projectives in $\mathcal{U}$ does not exceed the rank of $\mathrm{K}_0(\mathcal{H})$.

Proof. (a) If M_i and M_j are indecomposable Ext-projectives in $\mathcal{U}$, we know that there exist indecomposable objects X_i, X_j in $\mathcal{H}$, and integers $k_i, k_j \in \mathbb{Z}$ such that $M_i = X_i[k_i]$ and $M_j = X_j[k_j]$. By (1.2) $M_i \neq M_j$ implies $X_i \neq X_j$.

Let us fix $k \in \mathbb{Z}$, and consider the set $(X_i[k])_{i\in I}$ of all Ext-projectives which are concentrated in degree k. We note that, for all i, j we have $\mathrm{Ext}^1_{\mathcal{H}}(X_i, X_j) = \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X_i[k], X_j[k+1]) = 0$ by definition of Ext-projectives. Consequently, for each finite subset $F \subset I$ the direct sum $\oplus_F X_i \in \mathcal{H}$ is a partial tilting complex. By [HR2](1.2), we know that $|F| \leq rk\mathrm{K}_0(\mathcal{H})$ and hence $|I| \leq rk\mathrm{K}_0(\mathcal{H})$.

By (2.2), the algebra $B = \mathrm{End}_{\mathcal{H}}(\oplus_{i\in I} X_i)$ is quasi-tilted, and hence triangular. This allows us to order the projective $B-$modules P_i (which are in one-to-one correspondence with the X_i) so that $\mathrm{Hom}_B(P_i, P_j) = 0$ for all $j > i$. Consequently, the objects X_i can be ordered in $\mathcal{H}$ so that $\mathrm{Hom}_{\mathcal{H}}(X_i, X_j) = 0$, for all $j > i$.

This shows that, for each $k \in \mathbb{Z}$, the set $(X_i[k])_{i\in I}$ can be ordered to form an exceptional sequence.

We now consider two Ext-projectives in $\mathcal{U}$ of the form $X_i[k_i]$ and $X_j[k_j]$ with $k_i > k_j$. The definition of Ext-projectives yields

$$\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X_i[k_i], X_j[k_j + l]) = 0$$

for all $l > 0$. Setting $l = k_i - k_j + 1$, this yields $\mathrm{Ext}^1_{\mathcal{H}}(X_i, X_j) = 0$. On the other hand, since $\mathcal{H}$ is hereditary, we have

$$\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X_i[k_i], X_j[k_j]) = 0.$$

We proceed to order all the Ext-projectives in $\mathcal{U}$ in such a way as to form an exceptional sequence. Let

$$\mathcal{X}^+ = (X_i[k_i])_{k_i\geq 0}, \text{ and } \mathcal{X}^- = (X_i[k_i])_{k_i<0}.$$

We order each of the sets $\mathcal{X}^+$ and $\mathcal{X}^-$ in the following way. For each fixed $k \geq 0$, the above argument shows that the Ext-projectives concentrated in k can be ordered so as to form an exceptional sequence

$$S_k = \{X_{k_1}[k], \cdots, X_{k_{s_k}}[k]\}.$$

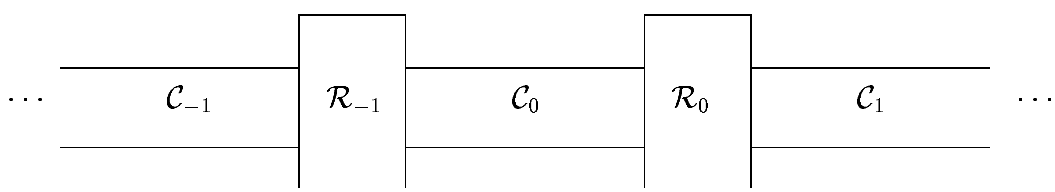
Given $0 \leq k < l$, we set $S_k < S_l$. Then, clearly,

$$S_k \cup S_l = \{X_{k_1}[k], \cdots, X_{k_{s_k}}[k], X_{l_1}[l], \cdots, X_{l_{s_l}}[l]\}$$

is an exceptional sequence. This induces an order on $\mathcal{X}^+$ so that the Ext-projectives in $\mathcal{X}^+$ form an exceptional sequence. Finally, we order the elements of $\mathcal{X}^+ \cup \mathcal{X}^-$ as an exceptional sequence by setting $\mathcal{X}^- < \mathcal{X}^+$. This completes the proof of (a).

(b) By [HR1](2.1), for each indecomposable Ext-projective M_i in $\mathcal{U}$, the algebra $\mathrm{End}_{\mathbf{D^b}(\mathcal{H})}(M_i)$ is a field. It follows then from (a) and [RV2](3.5) that, if $M = \oplus_{i=1}^{s} M_i$ is the direct sum of all indecomposable Ext-projectives in $\mathcal{U}$ then $\mathrm{K}_0(\mathbf{D^b}(\mathcal{H})) \simeq \mathbb{Z}^s \oplus \mathrm{K}_0(M^\perp)$. Consequently $s \leq rk\mathrm{K}_0(\mathbf{D^b}(\mathcal{H})) = rk\mathrm{K}_0(\mathcal{H})$. □

Example 2.4 *We end this section with an example of a suspended subcategory without Ext-projectives in $\mathbf{D^b}(\mathcal{H})$, when $\mathcal{H} = \mathrm{mod}\, H$ and H is a representation-infinite hereditary algebra. It follows from [H2] that the Auslander-Reiten quiver of $\mathbf{D^b}(\mathcal{H})$ is of the form*



where the $\mathcal{C}_i$ $(i \in \mathbb{Z})$ are transjective components, while the $\mathcal{R}_i$ $(i \in \mathbb{Z})$ are families of regular components (each being a family of stable tubes, or of components of the form $\mathbb{Z}A_\infty$, according as H is tame or wild, respectively). We use the following notation from [R]: let $\mathcal{C}, \mathcal{C}'$ denote classes of objects in $\mathbf{D^b}(\mathcal{H})$ such that $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(C', C) = 0$ for all $C' \in \mathcal{C}'$ and $C \in \mathcal{C}$. We let $\mathcal{C} \cup \mathcal{C}'$ stand for the class of objects whose indecomposables belong to $\mathcal{C}$ or $\mathcal{C}'$. Let now $\mathcal{U}$ be the full additive subcategory of $\mathbf{D^b}(\mathcal{H})$ generated by $\mathcal{R}_0 \cup \mathcal{C}_1 \cup \mathcal{R}_1 \cup \mathcal{C}_2 \cup \cdots$ Then $\mathcal{U}$ has no Ext-projective: in fact, if X is an indecomposable Ext-projective object in $\mathcal{U}$ then $\tau X \in \mathcal{U}^\perp$ (by (1.5)) and thus $\mathcal{U}$ and $\mathcal{U}^\perp$ intersect the same component of $\mathbf{D^b}(\mathcal{H})$ and this is impossible, because $\mathcal{U}^\perp$ is the full additive subcategory generated by $\cdots \mathcal{R}_{-2} \cup \mathcal{C}_{-1} \cup \mathcal{R}_{-1} \cup \mathcal{C}_0$. □

3 Ext-projectives and silting complexes

The concept of a silting complex was introduced in [KV] in order to study t-structures in the derived category of a hereditary algebra. The same idea was used by other authors in more general contexts, see, for instance [BR,S]

Let $\mathcal{T}$ be a triangulated category. A complex $M \in \mathcal{T}$ is called a *silting complex* if $\mathrm{Hom}_{\mathcal{T}}(M, M[l]) = 0$ for all $l > 0$.

Thus for instance, any partial tilting complex is a silting complex.

We recall that, if $\mathcal{U}, \mathcal{V}$ are suspended subcategories, then $\mathcal{W} = \mathcal{U} * \mathcal{V}$ denotes the full additive subcategory consisting of all the objects Y' such that there exists an object $Y = Y' \oplus Y''$ and a triangle

$$X \to Y \to Z \to X[1]$$

with $X \in \mathcal{U}$ and $Z \in \mathcal{V}$.

We also need to recall a well-known construction for the smallest suspended subcategory $\mathcal{U}_M$ containing a given object M. Denote by

$$\mathcal{E}_0^+ = \mathrm{add}(\oplus_{j \geq 0} M\,[j])$$

the full subcategory of $\mathcal{T}$ having as objects the direct summands of finite direct sums of copies of non-negative translates of M. If $j > 0$ and $\mathcal{E}_0^+, \mathcal{E}_{j-1}^+$ are known we set $\mathcal{E}_j^+ = \mathcal{E}_0^+ * \mathcal{E}_{j-1}^+$. It is then seen that $\mathcal{U}_M$ is equal to $\mathcal{U}_M = \cup_{j \geq 0} \mathcal{E}_j^+$.

The dual construction is also useful. Let $\mathcal{E}_0^- = \mathrm{add}(\oplus_{j \leq 0} M\,[j])$ be the full subcategory of $\mathcal{T}$ having as objects the direct summands of finite direct sums of copies of non-positive translates of M. If $j < 0$ and that $\mathcal{E}_0^-, \mathcal{E}_{j+1}^-$ are known we set $\mathcal{E}_j^- = \mathcal{E}_0^- * \mathcal{E}_{j+1}^-$. The smallest cosuspended subcategory ${}_M\mathcal{U}$ containing M is equal to ${}_M\mathcal{U} = \cup_{j \leq 0} \mathcal{E}_j^-$.

The smallest triangulated subcategory generated by M, $\mathcal{B}_M$, can be constructed in a similar way that $\mathcal{U}_M$ considering all possible translations of M instead of the positive ones. The dual construction gives the smallest triangulated subcategory cogenerated by M, ${}_M\mathcal{B}$.

Note that the orthogonal have the following easy characterizations:

- $X \in \mathcal{U}_M^{\perp}$ (respectively $X \in \mathcal{B}_M^{\perp}$) if and if $\mathrm{Hom}_{\mathcal{T}}(M[j], X) = 0$ for all $j \geq 0$ (respectively for all $j \in \mathbb{Z}$).
- $X \in {}^{\perp}{}_M\mathcal{U}$ (respectively $X \in {}^{\perp}{}_M\mathcal{B}$) if and only if $\mathrm{Hom}_{\mathcal{T}}(X, M[j]) = 0$ for all $j \leq 0$ (respectively for all $j \in \mathbb{Z}$).

Lemma 3.1 *Let $M \in \mathbf{D}^{\mathbf{b}}(\mathrm{mod}\, A)$ be a silting complex. Then:*

(a) M is Ext-projective in $\mathcal{U}_M$.
(b) M is Ext-projective in ${}^{\perp}_{\tau M}\mathcal{U}$.

Proof.(*a*) It follows from the construction above that, for every $X \in \mathcal{E}_0^+$, we have $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, X[1]) = 0$. Suppose that $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, X[1]) = 0$ for all $X \in \mathcal{E}_i^+$ with $i < j$ and let $Y' \in \mathcal{E}_j^+$. Then there exist $Y = Y' \oplus Y''$ and a triangle $X \to Y \to Z \to X[1]$ where $X \in \mathcal{E}_0^+$ and $Y \in \mathcal{E}_{j-1}^+$. Applying $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, -)$ to this triangle yields $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, Y[1]) = 0$ and hence $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, Y'[1]) = 0$. This completes the proof.

(*b*) By an induction similar to the one used in (*a*), we get $V \in {}^{\perp}_{\tau M}\mathcal{U}$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(V, \tau M[-i]) = 0$ for all $i \geq 0$. That is, if and only if $\mathbf{D}\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, V[i+1]) \simeq \mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(V[i], \tau M) = 0$ for all $i \geq 0$, or equivalently, if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, V[j]) = 0$ for all $j > 0$. Setting $V = M$ yields $M \in {}^{\perp}_{\tau M}\mathcal{U}$. Moreover, M is Ext-projective in ${}^{\perp}_{\tau M}\mathcal{U}$ since the same equivalence shows that $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M, V[1]) = 0$ for all $V \in {}^{\perp}_{\tau M}\mathcal{U}$. □

In the situation of (3.1) the Ext-projectives in $\mathcal{U}_M$ correspond to the projectives in the heart $\mathcal{C}_{\mathcal{U}_M} = \mathcal{U}_M \cap \mathcal{U}_M^{\perp}[1]$, which is a module category by [BR](3.3)(3.4). Another consequence is the following corollary:

Corollary 3.2 *Let $\mathcal{H}$ be a hereditary category. Let $M = \oplus_{i=1}^{s} M_i$ be a silting complex in $\mathbf{D^b}(\mathcal{H})$ with the M_i indecomposable and pairwise non isomorphic. Then*

(a) The M_i can be ordered to form an exceptional sequence.
(b) The smallest suspended subcategory $\mathcal{U}_M$ (or triangulated subcategory $\mathcal{B}_M$) containing M is an aisle (or a Bousfield localization, respectively) in $\mathbf{D^b}(\mathcal{H})$.

Proof. (*a*) By (3.1) the M_i are indecomposable Ext-projectives in the suspended subcategory $\mathcal{U}_M$. We then apply Theorem (2.3).

(*b*) this follows from (*a*) and [RV2], [B]. □

The following statement should be compared with [AK](1.3).

Proposition 1 *Let M be a silting complex in $\mathbf{D^b}(\mathrm{mod}\, A)$, where A is a finite dimension k-algebra of finite global dimension. The suspended subcategories having M as Ext-projective form a partially ordered set under inclusion. This set has $\mathcal{U}_M$ as unique minimal element and ${}^{\perp}_{\tau M}\mathcal{U}$ as unique maximal element. Moreover, $\mathcal{U}_M = {}^{\perp}_{\tau M}\mathcal{U}$ if and only if M is a generator of $\mathbf{D^b}(\mathrm{mod}\, A)$.*

Proof. By (3.1), since M is a silting complex, then M is Ext-projective in $\mathcal{U}_M$ and in ${}^{\perp}_{\tau M}\mathcal{U}$. Now, let $\mathcal{U}$ be a suspended subcategory of $\mathbf{D^b}(\mathrm{mod}\, A)$ in which M is Ext-projective. Then $\mathcal{U}_M \subset \mathcal{U}$. On the other hand, by (1.5), $\tau M \in \mathcal{U}^{\perp}$, hence ${}_{\tau M}\mathcal{U} \subset \mathcal{U}^{\perp}$. Therefore,

$$\mathcal{U}_M \subset \mathcal{U} \subset {}^{\perp}(\mathcal{U}^{\perp}) \subset {}^{\perp}_{\tau M}\mathcal{U}.$$

This shows the first statement.

Now, suppose that $\mathcal{U}_M = {}^{\perp}_{\tau M}\mathcal{U}$. In order to show that M is a generator, assume that $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(M[j], X) = 0$, for all $j \in \mathbb{Z}$. In particular, $X \in$

$\mathcal{U}_M^{\perp}$. On the other hand, $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(M[j], X) = 0$ for all $j < 0$ implies $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(X, \tau M[j+1]) \simeq \mathbf{D}\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(M[j], X) = 0$ for all $j < 0$, and then $X \in {}^{\perp}_{\tau M}\mathcal{U} = \mathcal{U}_M$. Therefore, $X \in \mathcal{U}_M^{\perp} \cap \mathcal{U}_M = 0$ and we are done.

Conversely, suppose that M is a generator in $\mathbf{D^b}(\mathrm{mod}\,A)$. By the proof of (3.1) above, $X \in \mathcal{U}_M$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(M[j], X) = 0$ for all $j < 0$. Using the Auslander-Reiten formula, $X \in \mathcal{U}_M$ if and only if $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}X, \tau M[j]) = 0$ for all $j \leq 0$, that is, if and only if, $X \in {}^{\perp}_{\tau M}\mathcal{U}$. □

Corollary 3.3 *Let M be a partial tilting complex in $\mathbf{D^b}(\mathrm{mod}\,A)$. Then M is a tilting complex if and only if $\mathcal{U}_M = {}^{\perp}_{\tau M}\mathcal{U}$.*

Proof. This follows from prop.1 and the definition. □.

Corollary 3.4 *If $M = \oplus_{i=1}^{s} M_i$ is a silting complex in $\mathbf{D^b}(\mathrm{mod}\,A)$ then*

(1) $\mathcal{U}_M \subset \cap_{i=1}^{s} {}^{\perp}_{\tau M_i}\mathcal{U}$.
(2) If, moreover, M is a generator, then $\mathcal{U}_M = \cap_{i=1}^{s} {}^{\perp}_{\tau M_i}\mathcal{U}$.

Proof. (a) Since M is a silting complex then, for each i, we have $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(M_i, M[j]) = 0$ for all $j > 0$. Hence by the proof of (3.1) $M \in {}^{\perp}_{\tau M_i}\mathcal{U}$ for each i, so $M \in \cap_{i=1}^{s} {}^{\perp}_{\tau M_i}\mathcal{U}$. The statement follows.

(b) Assume now that M is a generator. By (3.3), $\mathcal{U}_M = {}^{\perp}_{\tau M}\mathcal{U}$. Now, $X \in {}^{\perp}_{\tau M}\mathcal{U}$ if and only if, for each $j < 0$, we have

$$\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(X, (\tau M)[j]) = \oplus_{i=1}^{s}\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(X, \tau M_i)[j]) = 0$$

that is, if and only if, for all i with $0 \leq i \leq s$ and all $j < 0$, we have

$$\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(X, (\tau M_i)[j]) = 0$$

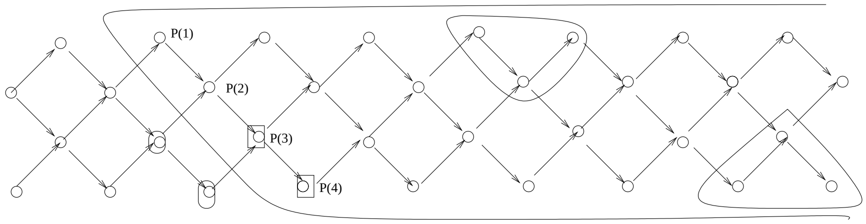
This is equivalent to saying that $X \in {}^{\perp}_{\tau M_i}\mathcal{U}$ for all i. □

Example 3.5 *Let A be the path algebra of the Dynkin quiver*

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet} \longleftarrow \overset{4}{\bullet}$$

and let $M = P(3) \oplus P(4)$ be the direct sum of the indecomposable projective modules corresponding to the points 3 and 4. We draw the Auslander-Reiten quiver of $\mathbf{D^b}(\mathrm{mod}\,A)$ in which we show the subcategories $\mathcal{U}_M$ and ${}^{\perp}_{\tau M}\mathcal{U}$.

Here, $\mathcal{U}_M$ consist of $P(3), P(4)$ and all the complexes inside the small triangles. We notice that the heart $\mathcal{U}_M \cap \mathcal{U}_M^{\perp}[1]$ of the corresponding t-structure is the module category of the path algebra of the Dynkin quiver $\mathbb{A}_2$. Also, while $P(3)$ and $P(4)$ are the only Ext-projectives in $\mathcal{U}_M$, the aisle ${}^{\perp}_{\tau M}\mathcal{U}$ (which coincides with the canonical aisle) has two more Ext-projectives, namely $P(1)$ and $P(2)$.



4 Deconstructing aisles

We first see a procedure for restricting aisles.

Lemma 4.1 *Let $\mathcal{U}$ be a suspended subcategory in $\mathbf{D^b}(\mathcal{H})$ and let $(M_i)_{i=1}^m$ be a set of indecomposable, pairwise non-isomorphic, Ext-projectives in $\mathcal{U}$. Let $s < m$ and $M = \oplus_{i=1}^s M_i$.*

(a) If $j > s$, then M_j is Ext-projective in $\mathcal{U} \cap \mathcal{B}_M^\perp$.
(b) If $\mathcal{U}$ is an aisle in $\mathbf{D^b}(\mathcal{H})$ then $\mathcal{U} \cap \mathcal{B}_M^\perp$ is an aisle in $\mathcal{B}_M^\perp$.

Proof. (*a*) By (2.3), the M_i can be ordered to form an exceptional sequence. Then, by definition, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M_i[l], M_j) = 0$ for all $i \leq s$ and all $l \in \mathbb{Z}$. Consequently, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[l], M_j) = 0$ for all $l \in \mathbb{Z}$. This implies $M_j \in \mathcal{B}_M^\perp$, so $M_j \in \mathcal{U} \cap \mathcal{B}_M^\perp$. Since M_j is Ext-projective in $\mathcal{U}$, it is clearly so in $\mathcal{U} \cap \mathcal{B}_M^\perp$.

(*b*) Let $Y \in \mathcal{B}_M^\perp$. Since $\mathcal{U}$ is an aisle in $\mathbf{D^b}(\mathcal{H})$ there exists a triangle $X \to Y \to Z \to X[1]$ with $X \in \mathcal{U}$ and $Z \in \mathcal{U}^\perp$. Applying the functor $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, -)$ to this triangle yields, since $Y \in \mathcal{B}_M^\perp$, an isomorphism

$$\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j], X) \simeq \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j+1], Z)$$

for each $j \in \mathbb{Z}$. Since M is Ext-projective in $\mathcal{U}$, we have $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j], X) = 0$ for all $j < 0$, hence $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j+1], Z) = 0$ for all $j < 0$. Since, on the other hand, $Z \in \mathcal{U}^\perp$ we have $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j+1], Z) = 0$ for all $j \geq -1$. This shows that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j], Z) = 0$ for all $j \in \mathbb{Z}$ and thus $Z \in \mathcal{U}^\perp \cap \mathcal{B}_M^\perp$. Now, this is equivalent to saying that $Z \in \mathcal{B}_M^\perp$ and $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(U, Z) = 0$ for all $U \in \mathcal{U}$. In particular, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(U, Z) = 0$ for all $U \in \mathcal{U} \cap \mathcal{B}_M^\perp$ and therefore Z belongs to the right orthogonal, inside $\mathcal{B}_M^\perp$ (!) of $\mathcal{U} \cap \mathcal{B}_M^\perp$. This completes the proof. □

We are able now to prove our second theorem inspired from [AK], Theorem (A).

Theorem 4.2 *Let $\mathcal{U}$ be a suspended subcategory and M be an object in $\mathbf{D^b}(\mathcal{H})$. Then*

(a) M is a silting complex if and only if M is Ext-projective in $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$.
(b) The following conditions are equivalent:
(i) M is Ext-projective in $\mathcal{U}$.
(ii) $\mathcal{U} = \mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$ and M is a silting complex.
(iii) $\mathcal{U} = \mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$ and $\mathcal{U}^\perp = (\mathcal{U}^\perp \cap \mathcal{B}_M^\perp) * {}_{\tau M}\mathcal{U}$.

Proof. (*a*) We first assume that M is a silting complex in $\mathbf{D^b}(\mathcal{H})$ and prove that M is Ext-projective in $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$. Let $Y' \in \mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$. Then there exist $Y = Y' \oplus Y''$ and a triangle

$$X \to Y \to Z \to X[1]$$

with $X \in \mathcal{U}_M$ and $Z \in \mathcal{B}_M^\perp \cap \mathcal{U}$. Applying the functor $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, -)$ yields an exact sequence

$$\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, X[1]) \to \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, Y[1]) \to \mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, Z[1]).$$

Since, by (3.1) M is Ext-projective in $\mathcal{U}_M$ we have $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, X[1]) = 0$. Since $Z \in \mathcal{B}_M^\perp$ we also have $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, Z[1]) = 0$. Therefore, $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, Y[1]) = 0$ and so $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M, Y'[1]) = 0$. This shows that M is Ext-projective in $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$.

Conversely, if M is Ext-projective in $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp)$, it is clear that M is a silting complex.

(*b*) (*i*) implies (*ii*). Assume M is Ext-projective in $\mathcal{U}$. Since $M \in \mathcal{U}$, then $\mathcal{U}_M \subset \mathcal{U}$ and also $\mathcal{U} \cap \mathcal{B}_M^\perp \subset \mathcal{U}$ so that $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp) \subset \mathcal{U}$. The Ext-projectivity of M shows that it is a silting complex, then $\mathcal{U}_M$ is an aisle (by (3.2)). Now, let $Y \in \mathcal{U}$, there is a triangle $X \to Y \to Z \to X[1]$ such that $X \in \mathcal{U}_M$ and $Z \in \mathcal{U}_M^\perp$. It suffices to show that, actually, $Z \in \mathcal{U} \cap \mathcal{B}_M^\perp$. Since $X, Y \in \mathcal{U}$, then $Z \in \mathcal{U}$ because $\mathcal{U}$ is closed under extensions. In order to prove that $Z \in \mathcal{B}_M^\perp$, we must prove that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j], Z) = 0$ for all $j \in \mathbb{Z}$. This equality certainly holds for $j \geq 0$, because $Z \in \mathcal{U}_M^\perp$. On the other hand, the Ext-projectivity of M in $\mathcal{U}$ shows that it holds as well as for $j < 0$. This establishes that $\mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp) = \mathcal{U}$.

(*ii*) implies (*iii*) By (*a*) and (1.5) we have $\tau M \in \mathcal{U}^\perp$, hence ${}_{\tau M}\mathcal{U} \subset \mathcal{U}^\perp$. Since $\mathcal{U}^\perp \cap \mathcal{B}^\perp \subset \mathcal{U}^\perp$, we have $(\mathcal{U}^\perp \cap \mathcal{B}_M^\perp) * {}_{\tau M}\mathcal{U} \subset \mathcal{U}^\perp$.

Conversely, let $Y \in \mathcal{U}^\perp$. Since ${}_{\tau M}\mathcal{U}$ is a coaisle (by the dual of (3.2)) there exists a triangle

$$X \to Y \to Z \to X[1]$$

such that $X \in {}^\perp{}_{\tau M}\mathcal{U}$ and $Z \in {}_{\tau M}\mathcal{U}$. We claim that, actually, $X \in \mathcal{U}^\perp \cap \mathcal{B}_M^\perp$. Since $Z, Y \in \mathcal{U}^\perp$, then $X \in \mathcal{U}^\perp$. In order to prove that $X \in \mathcal{B}_M^\perp$, we must prove that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j], X) = 0$ for all $j \in \mathbb{Z}$. Since $X \in {}^\perp{}_{\tau M}\mathcal{U}$, we have

$$\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j-1], X) \simeq \mathbf{D}\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(X, \tau M[j]) = 0,$$

for all $j \leq 0$. On the other hand, for all $j \geq 0$ we have $M[j] \in \mathcal{U}$ while $X \in \mathcal{U}^\perp$, so that $\mathrm{Hom}_{\mathbf{D^b}(\mathcal{H})}(M[j], X) = 0$. This shows that $\mathcal{U}^\perp = (\mathcal{U}^\perp \cap \mathcal{B}_M^\perp) * {}_{\tau M}\mathcal{U}$.

(*iii*) implies (*i*). From the existence of the triangle $M \to M \to 0 \to M[1]$ follows that $M \in \mathcal{U}_M * (\mathcal{U} \cap \mathcal{B}_M^\perp) = \mathcal{U}$, and from the existence of the triangle $0 \to \tau M \to \tau M \to 0$ follows that $\tau M \in (\mathcal{U}^\perp \cap \mathcal{B}_M^\perp) * {}_{\tau M}\mathcal{U} = \mathcal{U}^\perp$. Invoking (1.5) concludes the proof. □

Corollary 4.3 *Let $\mathcal{U}$ be a suspended subcategory in $\mathbf{D^b}(\mathcal{H})$ and let $M = \oplus_{i=1}^{s} M_i$ be Ext-projective in $\mathcal{U}$, with the M_i indecomposable and pairwise non-isomorphic. Then*

$$\mathcal{U} = \mathcal{U}_{M_1} * \cdots * \mathcal{U}_{M_s} * (\mathcal{U} \cap \mathcal{B}_M^{\perp}).$$

Proof. By induction on s. If $s = 1$, this is Theorem 4.2. Assume that the statement holds for $s-1$, then $\mathcal{U} = \mathcal{U}_{M_1} * \cdots * \mathcal{U}_{M_{s-1}} * (\mathcal{U} \cap \mathcal{B}_N^{\perp})$, with $N = \oplus_{i=1}^{s-1} M_i$.

By definition of an exceptional sequence, we have $M_s \in \mathcal{B}_N^{\perp}$. Thus applying (4.2) yields $\mathcal{U} \cap \mathcal{B}_N^{\perp} = U_{M_s} * (\mathcal{U} \cap \mathcal{B}_N^{\perp} \cap \mathcal{B}_{M_s}^{\perp}) = U_{M_s} * (\mathcal{U} \cap \mathcal{B}_M^{\perp})$. The statement follows at once. □

Corollary 4.4 *Let $\mathcal{U}$ be an aisle in $\mathbf{D^b}(\mathcal{H})$ and let $M = \oplus_{i=1}^{s} M_i$ be Ext-projective in $\mathcal{U}$, with the M_i indecomposable and pairwise non-isomorphic. Then $s = rk\mathrm{K}_0(\mathcal{H})$ if and only if M is a generator of $\mathbf{D^b}(\mathcal{H})$. Moreover, if this is the case, then $\mathcal{U} = \mathcal{U}_M = \mathcal{U}_{M_1} * \cdots * \mathcal{U}_{M_s}$.*

Proof. By (2.3) the objects $(M_i)_{i=1}^{s}$ can be assumed to form an exceptional sequence. Hence from [RV2](3.4), (3.5),

$$\mathrm{K}_0(\mathcal{H}) \simeq \mathrm{K}_0(\mathbf{D^b}(\mathcal{H})) \simeq \mathrm{K}_0(\mathcal{B}_M) \oplus \mathrm{K}_0(\mathcal{B}_M^{\perp}) \simeq \mathbb{Z}^s \oplus \mathrm{K}_0(\mathcal{B}_M^{\perp}).$$

On the other hand, by [BL], proof of Theorem 1, there exists a hereditary category $\mathcal{H}'$ such that $\mathcal{B}_M^{\perp} \simeq \mathbf{D^b}(\mathcal{H}')$. Hence

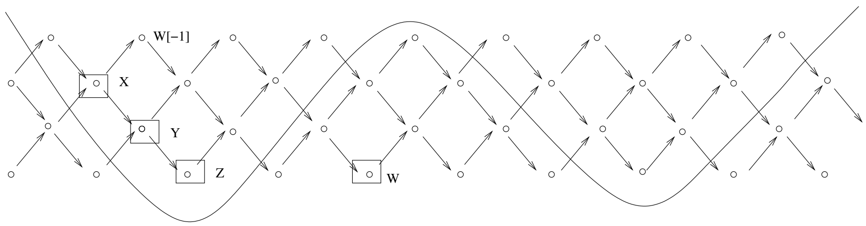
$$\mathrm{K}_0(\mathcal{B}_M^{\perp}) \simeq \mathrm{K}_0(\mathbf{D^b}(\mathcal{H}')) \simeq \mathrm{K}_0(\mathcal{H}')$$

Thus by [HRS], $\mathrm{K}_0(\mathcal{B}_M^{\perp}) = 0$ if and only if $\mathcal{B}_M^{\perp} = 0$. Now, $s = rk\mathrm{K}_0(\mathcal{H})$ if and only if $\mathrm{K}_0(\mathcal{B}_M^{\perp}) = 0$, thus, if and only if $\mathcal{B}_M^{\perp} = 0$ or equivalently, if and only if, $\mathcal{B}_M = \mathbf{D^b}(\mathcal{H})$, that is, if and only if M is a generator of $\mathbf{D^b}(\mathcal{H})$. The second statement follows form (4.3) and the fact that $\mathcal{B}_M^{\perp} = 0$.

Corollary 4.5 *The aisles in $\mathbf{D^b}(\mathcal{H})$ which have exactly $rk\mathrm{K}_0(\mathcal{H})$ isomorphism classes of indecomposable Ext-projectives are in bijective correspondance with the silting complexes having $rk\mathrm{K}_0(\mathcal{H})$ isomorphism classes of indecomposable summands in $\mathbf{D^b}(\mathcal{H})$.*

Proof. Let $\mathcal{U}$ be an aisle in $\mathbf{D^b}(\mathcal{H})$ which has exactly $rk\mathrm{K}_0(\mathcal{H})$ isomorphism classes of Ext-projective objects $M_1, \cdots, M_s$. Then, setting $M = \oplus_{i=1}^{s} M_i$ we have $\mathcal{U} = \mathcal{U}_M$ by (4.4). Conversely, if $M = \oplus_{i=1}^{s} M_i$ is a silting complex with $s = rk\mathrm{K}_0(\mathcal{H})$ then by (3.2) $\mathcal{U}_M$ is an aisle and, by (3.1) the M_i are indecomposable Ext-projectives in $\mathcal{U}_M$. Moreover, it follows from (2.3)(b) that the number of isomorphism classes of indecomposable Ext-projectives in $\mathcal{U}_M$ is bounded by $rk\mathrm{K}_0(\mathcal{H}) = s$ □

Example 4.6 *Let A be as in the example (3.5) and let X, Y, Z, W be the indicated objects in $\mathbf{D^b}(\mathrm{mod}\, A)$*



Then, it is easily seen that $M = X \oplus Y \oplus Z \oplus W$ is a silting complex, which is not tilting, because $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\,A)}(X, W[-1]) \neq 0$. *Observe that $End_A M$ is the product of two connected components one equal to k, and the other the path algebra of the quiver*

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet}$$

In particular, $End_A M$ and A are not derived equivalent.

5 Applications to left supported algebras

We illustrate our techniques by giving several characterizations of left supported algebras, introduced in [ACT] and studied in [ACPT]. Let A be a finite dimensional $k-$algebra, and ind$A$ denote a full subcategory of mod$A$ consisting of exactly one representative from each isomorphism class of indecomposable $A-$modules. Given $L, M \in \mathrm{ind}A$ we say that L is a *predecessor* of M, or that M is a *successor* of L, if there exists a sequence of non-zero morphisms

$$L = L_0 \to L_0 \to L_1 \to \cdots \to L_t = M$$

with $L_i \in \mathrm{ind}A$ for all i. Following [HRS], we let $\mathcal{L}_A$ (or $\mathcal{R}_A$) be the class of all $M \in \mathrm{ind}A$ all of whose predecessors have projective dimension at most one (or all of whose successors have injective dimension at most one, respectively).

An algebra A is *left supported* if add$\mathcal{L}_A$ is contravariantly finite in mod$A$. Let F denote the direct sum of all indecomposable projective $A-$modules not lying in $\mathcal{L}_A$, E_1 denote the direct sum of all $L \in \mathcal{L}_A$ such that $\mathrm{Hom}_A(\mathbf{D}A, L) \neq 0$, and E_2 denote the direct sum of all $L' \in indA - \mathrm{add}E_1$ such that $\mathrm{Hom}_A(F, \tau_A^{-1}L') \neq 0$. Then $\mathcal{E} = \{X \in A,\ X \in \mathrm{add}(E_1 \oplus E_2)\}$ is the class of all indecomposable Ext-injectives in add$\mathcal{L}_A$ (see [ACT](3.1), [ALR](3.2)). The module $T = E_1 \oplus E_2 \oplus F$ is always a partial tilting module (see [ACT](3.3)) and it is a tilting module if and only if A is left supported ([ACT], Theorem A). If this is the case, then the torsion-free class corresponding to the module T is $\mathcal{F}(T) = \mathrm{add}(\mathcal{L}_A - \mathcal{E})$, see [ACT](4.1).

Further, following [ACPT], we let $\mathcal{R}_0$ be the class of all $M \in \mathrm{ind}A$ which are successors of injectives. Then the module $R = E_1 \oplus \tau_A^{-1}E_2 \oplus F$ is the direct sum of all indecomposable Ext-projectives in $\mathcal{R}_0$, by [ACPT](5.3). It is always a partial tilting module [ACPT](5.4) and it is a tilting module if and only if A is left supported, or equivalently, if and only if $\mathcal{R}_0 = \{X \in \text{ind}A, \text{Hom}_A(R, X) \neq 0\}$, see [ACPT], Theorem (B).

Lemma 5.1 *With the above notation, we have*

$$\mathcal{U}_T^{\perp} \cap \text{mod}\, A = \mathcal{F}(T).$$

Proof: By the definition of right orthogonal, $X \in \mathcal{U}_T^{\perp} \cap \text{mod}\, A$ if and only if, for all $j \geq 0$, $\text{Hom}_{\mathbf{D^b}(\text{mod}\, A)}(T[j], M) = 0$. That is, if and only if $\text{H}om_{\text{mod}\, A}(T, M) = 0$, if and only if, $X \in \mathcal{F}(T)$. □

We are now able to prove the main result of this section.

Theorem 5.2 *The following conditions are equivalent for an artin algebra A,*

(a) A is left supported
(b) $\mathcal{U}_T^{\perp} \cap \text{mod}\, A = \text{add}(\mathcal{L}_A - \mathcal{E})$.
(c) $\mathcal{U}_R \cap \text{mod}\, A = \text{add}\mathcal{R}_0$.

Proof: (a) implies (b) follows from the above lemma and [ACT](4.1).

(a) implies (b). We wish to prove that $T = E_1 \oplus E_2 \oplus F$ is a tilting module. Since T is a partial tilting module, and by hypothesis (and the above lemma) $\mathcal{F}(T) = \text{add}(\mathcal{L}_A - \mathcal{E})$, then the corresponding torsion class (which consists of all modules generated by T) is then $\mathcal{T}(T) = \text{ind}A-(\mathcal{L}_A - \mathcal{E})$. Let I be an indecomposable injective $A-$module. If $I \in \mathcal{L}_A$, then $I \in \mathcal{E}$. Therefore $I \in \mathcal{T}(T)$. Since $\mathcal{T}(T)$ contains all indecomposable injectives, and is generated by a module, then it is induced by a multiplicity-free tilting module X (see[A]). To complete the proof, it suffices to show that $X = T$. It is known that X is Ext-projective in $\mathcal{T}(T)(= \mathcal{T}(X))$. Let now Y be an indecomposable Ext-projective in $\mathcal{T}(T)$. We have two cases:

(1) If Y is not projective, then $\tau_A Y \in \mathcal{L}_A - \mathcal{E}$ (by [AS](3.7)) and $Y \notin \mathcal{L}_A - \mathcal{E}$. Assume $Y \notin \mathcal{L}_A$ then $\tau_A Y \in \mathcal{L}_A$ and $Y = \tau_A^{-1}(\tau_A Y) \notin \mathcal{L}_A$ implies $\tau_A Y \in \mathcal{E}$ (because $\mathcal{E}$ consists of the Ext-injectives in $\text{add}\mathcal{L}_A$), and this is a contradiction. Hence $Y \in \mathcal{L}_A$. Since $Y \notin \mathcal{L}_A - \mathcal{E}$, then $Y \in \mathcal{E}$.
(2) If Y is projective, there are again two cases. If $Y \notin \mathcal{L}_A$ then $Y \in \text{add}F$. If $Y \in \mathcal{L}_A$ then $Y \in \mathcal{E}$ because $Y \in \mathcal{T}(T)$.

This shows that $Y \in \text{add}T$. Then $\text{add}X \subset \text{add}T$. Since T itself is Ext-projective in $\mathcal{T}(T)$, see [A], then $\text{add}T \subset \text{add}X$. Hence $X = T$, and T is a tilting module.

(a) implies (c) We first claim that $\mathcal{U}_R \cap \text{mod}A = \{M \in \text{mod}A, \text{Ext}_A^1(R, M) = 0\}$. Let $X \in \mathcal{U}_R \cap \text{mod}A$, then $\text{Hom}_{\mathbf{D^b}(\text{mod}\, A)}(R[j], X) = 0$ for all $j < 0$. In particular, $\text{Hom}_{\mathbf{D^b}(\text{mod}\, A)}(R[-1], X) = \text{Ext}^1(R, X) = 0$. Conversely, let X be such that $\text{Ext}_A^1(R, X) = 0$. Since the projective dimension of R is at most 1, we have $\text{Hom}_{\mathbf{D^b}(\text{mod}\, A)}(R[j], X) = 0$ for all $j < 0$. Since R is a tilting module (because A is left supported), it is a generator of $\mathbf{D^b}(\text{mod}\, A)$ and hence $\mathcal{U}_R = \{X \in \mathbf{D^b}(\mathrm{mod}\, A) | \mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(R, X[j]) = 0 \text{ for all } j > 0\}$, see [ASZ] (2.2). Thus $X \in \mathcal{U}_R \cap \mathrm{mod}\, A$ and we have established our claim.

By [ACPT], A is left supported if and only if $\mathrm{add}\mathcal{R}_0 = \{M \in \mathrm{mod}A, \mathrm{Ext}^1_A(R, M) = 0\}$. Hence $\mathcal{U}_R \cap \mathrm{mod}A = \mathrm{add}\mathcal{R}_0$.

(c) implies (a) We prove that $\mathcal{R}_0 = \{X \in \mathrm{ind}A, \mathrm{Hom}_A(R, X) \neq 0\}$. Let $X \in \mathcal{R}_0$ then, by hypothesis, $X \in \mathcal{U}_R$. In particular, $X \notin \mathcal{U}_R^{\perp}$. Then there exists $j \leq 0$ such that $\mathrm{Hom}_{\mathbf{D^b}(\mathrm{mod}\, A)}(R, X[j]) \neq 0$. Conversely, if $\mathrm{Hom}_A(R, X) \neq 0$ then, since $R \in \mathcal{R}_0$, and $\mathcal{R}_0$ is closed under successors, then $X \in \mathcal{R}_0$. This completes the proof. □

Another way to characterize left supported algebras is via their left supports. The *left support* A_λ of A is the endomorphism algebra of the direct sum of all indecomposable projective $A-$modules which are not in $\mathcal{L}_A$.

We denote by $\mathcal{B}_{\mathcal{L}_A}$ the smallest triangulated subcategory of $\mathbf{D^b}(\mathrm{mod}\, A_\lambda)$ containing $\mathcal{L}_A$. Furthermore, we denote by ${}_E\mathcal{B}$ and $\mathcal{B}_E$, respectively, the smallest triangulated subcategories of $\mathbf{D^b}(\mathrm{mod}\, A_\lambda)$ cogenerated and generated by $E = E_1 \oplus E_2$. It was shown in [T] that $\mathcal{B}_{\mathcal{L}_A} = \mathbf{D^b}(\mathrm{mod}\, A_\lambda)$.

Proposition 2 *The following conditions are equivalent for an artin algebra A*

(a) A is left supported.
(b) $\mathbf{D^b}(\mathrm{mod}\, A_\lambda) = {}_E\mathcal{B} = \mathcal{B}_E$.
(c) $\mathcal{B}_{\mathcal{L}_A} = {}_E\mathcal{B}$.

Proof. (a) is equivalent to (b) Indeed, it is known that A is left supported if and only if E is a tilting $A_\lambda-$module (see [ACT](3.3)) and this is the case if and only if E is a generator, and also a cogenerator of $\mathbf{D^b}(\mathrm{mod}\, A_\lambda)$ that is, if and only if $\mathbf{D^b}(\mathrm{mod}\, A_\lambda) = {}_E\mathcal{B} = \mathcal{B}_E$.

(b) implies (c). It follows from [T] since $\mathcal{B}_{\mathcal{L}_A} = \mathbf{D^b}(\mathrm{mod}\, A_\lambda)$.

(c) implies (a) Assume ${}_E\mathcal{B} = \mathcal{B}_{\mathcal{L}_A} = \mathbf{D^b}(\mathrm{mod}\, A_\lambda)$. In particular, E is a cogenerator of $\mathbf{D^b}(\mathrm{mod}\, A_\lambda)$. By (2.1) it follows that E is also a generator of $\mathbf{D^b}(\mathrm{mod}\, A_\lambda)$ and hence $\mathbf{D^b}(\mathrm{mod}\, A_\lambda) = \mathcal{B}_E$. Since E is a partial tilting module, then E is a tilting A_λ-module and A is a left supported algebra. □

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