ENDOMORPHISM RINGS OF PROJECTIVES OVER LAURA ALGEBRAS

Ibrahim Assem and Flávio Ulhoa Coelho

Introduction.

Since its introduction by Happel and Ringel in the early eighties [HR], the class of tilted algebras has been extensively studied in the representation theory of artin algebras. It is now considered to be one of the classes whose representation theory is best understood and most useful for the general theory. It was therefore natural to consider various generalisations of this notion. Thus, over the years, the following classes of algebras were defined and investigated: the quasi-tilted algebras (which generalise the tilted and the canonical algebras of [R]) [HRS], the shod algebras (which generalise the quasi-tilted) [CL1], the weakly shod algebras (which generalise the shod and the representation-directed algebras) [CL2, CL3], the left and the right glued algebras (which generalise the tilted and the representation-finite algebras) [AC1] and, finally, the laura algebras (which generalise all the previous classes) [AC2]. We recall that an artin algebra A is said to be a laura algebra if all but at most finitely many non-isomorphic indecomposable A-modules belong to the class $\mathcal{L}_A$ (that is, are such that all its predecessors have projective dimension at most one) or to the class $\mathcal{R}_A$ (that is, are such that all its successors have injective dimension at most one).

It was reasonable to ask the following question: if A is an artin algebra belonging to one of the classes above, and e is an idempotent in A, then does the endomorphism algebra eAe of the projective module eA belong also to the same class? The answer has already been shown to be positive for tilted algebras by Happel [H] (III.6.5), for quasi-tilted algebras by Happel, Reiten and Smalø [HRS] (II.1.15) and for shod algebras by Kleiner, Skowroński and Zacharia [KSZ] (1.2). The objective of this note is to show that it is positive as well for the remaining classes. Moreover, our proof yields also the quasi-tilted and the shod cases. Thus, we prove the following theorem.

THEOREM. *Let A be a connected artin algebra, and e be an idempotent in A such that $B = eAe$ is connected.*

(a) *If A is a laura algebra, then so is B.*
(b) *If A is a left (or right) glued algebra, then so is B.*
(c) *If A is a weakly shod algebra, then so is B.*
(d) *If A is a shod algebra, then so is B.*
(e) *If A is a quasi-tilted algebra, then so is B.*

The paper is organized as follows. The first section is devoted to preliminary results, including a new characterisation of weakly shod algebras. In our second we study how properties of B-modules are related to those of A-modules. The proof of our theorem occupies the third section, together with remarks and examples.

1. A characterisation of weakly shod algebras.

1.1. Notation. Throughout this note, our algebras are connected artin algebras. For an algebra A, we denote by $\operatorname{mod} A$ its category of finitely generated right modules, and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of one representative from each isomorphism class of indecomposable modules. We also denote by $\operatorname{rk} K_0(A)$ the rank of the Grothendick group $K_0(A)$ of A. For an A-module M, we denote by $\operatorname{pd} M$ (or $\operatorname{id} M$) its projective dimension (or injective dimension, respectively), and by $\operatorname{add} M$ the full subcategory of $\operatorname{mod} A$ consisting of the finite direct sums of the direct summands of M.

We use freely and without further reference facts about $\operatorname{mod} A$ and the Auslander-Reiten translations τ_A and τ_A^{-1} of $\operatorname{mod} A$, as can be found, for instance in [ARS, R].

We are particularly interested in paths. Given two modules M, N in $\operatorname{ind} A$, a *path* from M to N is a sequence

$$(*) \qquad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_t} M_t = N$$

where all the M_i are in $\operatorname{ind} A$, and all the f_i are non-zero morphisms. In this case, we say that M is a predecessor of N, and N is a successor of M. It will sometimes be necessary to assume that the f_i are non-isomorphisms, in which case we shall explicitely say so. The path $(*)$ is called a *path of irreducible morphisms* if each of the f_i is irreducible. A path of irreducible morphisms $(*)$ is called *sectional* if $\tau_A M_{i+1} \not\cong M_{i-1}$ for each i such that $1 \leq i \leq t$. A *refinement* of the path $(*)$ is a path in $\operatorname{mod} A$

$$M = M'_0 \xrightarrow{f'_1} M'_1 \xrightarrow{f'_2} M'_2 \longrightarrow \cdots \xrightarrow{f'_s} M'_s = N$$

with $s \geq t$, together with an order-prederving function $\sigma : \{1, 2, \ldots, t-1\} \to \{1, 2, \ldots, s-1\}$ such that, for each i with $1 \leq i \leq t$, we have $M_i \cong M'_{\sigma(i)}$.

Following [HRS], for an artin algebra A, we denote by $\mathcal{L}_A$ the full subcategory of $\operatorname{ind} A$ consisting of the modules M such that, if L is a predecessor of M, then $\operatorname{pd} L \leq 1$. Dually, we denote by $\mathcal{R}_A$ the full subcategory of $\operatorname{ind} A$ consisting of the modules M such that, if N is a successor of M, then $\operatorname{id} N \leq 1$. Clearly, $\mathcal{L}_A$ is closed under predecessors, while $\mathcal{R}_A$ is closed under successors.

1.2. We need the following lemma [AC2](1.4).

LEMMA. *Let A be an artin algebra.*

(a) *Let P be an indecomposable projective A-module. Then there exist at most finitely many modules M in $\mathcal{R}_A$ such that there exists a path from M to P. Moreover, any such path can be refined to a path of irreducible morphisms, and any path of irreducible morphisms from M to P is sectional.*

(b) *Let I be an indecomposable injective A-module. Then there exist at most finitely many modules M in $\mathcal{L}_A$ such that there exists a path from I to M. Moreover, any such path can be refined to a path of irreducible morphisms, and any path of irreducible morphisms from I to M is sectional.* □

1.3 The following corollary [AC2](1.5) will also be useful.

COROLLARY. *Let A be an artin algebra.*

(a) *$\mathcal{R}_A$ consists of the modules M in $\operatorname{ind} A$ such that, if there exists a path from M to an indecomposable projective module, then this path can be refined to a path of irreducible morphisms, and the latter is sectional.*

(b) *$\mathcal{L}_A$ consists of the modules M in $\operatorname{ind} A$ such that, if there exists a path from an indecomposable injective module to M, then this path can be refined to a path of irreducible morphisms, and the latter is sectional.* □

1.4. We recall that an artin algebra is called *weakly shod* whenever the length of any path from an indecomposable injective to an indecomposable projective is bounded [CL3]. We now give a characterisation of weakly shod algebras.

PROPOSITION. *An artin algebra A is weakly shod if and only if there exists an $\ell \geq 0$ such that any path from an indecomposable module not lying in $\mathcal{L}_A$ to an indecomposable module not lying in $\mathcal{R}_A$ has length at most ℓ.*

Proof. We first prove the necessity. Assume that there exists an indecomposable module M not in $\mathcal{L}_A$ and an indecomposable module N not in $\mathcal{R}_A$ as well as a path $M \to \cdots \to N$ of arbitrary length. Since M is not in $\mathcal{L}_A$, it has a predecessor M' such that pd $M' \geq 2$. By [R] p.74, there exist an indecomposable injective A-module I and a path $I \to \tau_A M' \to * \to M' \to \cdots \to M$. Similarly, there exists a successor N' of N, an indecomposable projective A-module P and a path $N \to \cdots \to N' \to * \to \tau_A^{-1} N' \to P$. Combining these with the given path from M to N yields a path from I to P of arbitrary length, a contradiction to A being weakly shod.

We now prove the sufficiency. Suppose that A is not weakly shod. Then, for each $t \geq 0$, there exists an indecomposable injective A-module I, an indecomposable projective A-module P as well as a path in ind A:

$$I = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t \longrightarrow \overset{(\theta_t)}{\cdots} \longrightarrow N_t \xrightarrow{g_t} \cdots \longrightarrow N_1 \xrightarrow{g_1} N_0 = P$$

with the f_i, g_i irreducible, and (θ_t) a path of length greater than t. We choose $t > 1 + 2 \operatorname{rk} K_0(A)$ and claim that M_t does not belong to $\mathcal{L}_A$.

We denote by (ξ_t) the subpath $I = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_t} M_t$ of the above path. We have one of two cases. If (ξ_t) is not sectional, then there exists j such that $M_{j-1} \cong \tau_A M_{j+1}$ and the path $I = M_0 \xrightarrow{f_1} M_1 \longrightarrow \cdots \xrightarrow{f_j} M_j$ is sectional. By [B, IT], $\operatorname{Hom}_A(I, \tau_A M_{j+1}) \neq 0$ and therefore pd $M_{j+1} \geq 2$. Thus, M_{j+1} does not belong to $\mathcal{L}_A$, and neither does M_t. If, on the other hand, (ξ_t) is sectional, then it contains at least $1 + \operatorname{rk} K_0(A)$ indecomposable modules which are not injective. By [S], there exist p, q such that $1 \leq p, q \leq t$ and $\operatorname{Hom}_A\left(\tau_A^{-1} M_p, M_q\right) \neq 0$. Since $\operatorname{Hom}_A(I, M_p) \neq 0$, we have pd $\tau_A^{-1} M_p \geq 2$, so $\tau_A^{-1} M_p$ does not lie in $\mathcal{L}_A$. Therefore, M_q does not belong to $\mathcal{L}_A$, and neither does M_t. This establishes our claim.

Similarly, N_t does not lie in $\mathcal{R}_A$. Then, for each $t > 1 + 2 \operatorname{rk} K_0(A)$, we have a path (θ_t) from an indecomposable not in $\mathcal{L}_A$ to an indecomposable not in $\mathcal{R}_A$, and of length greather than t, a contradiction to our hypothesis. □

2. Passing from mod A to mod B.

2.1. From now on, and until the end of this paper, we assume that A is a connected artin algebra, that e is an idempotent in A chosen (without loss of generality) so that the algebra $B = eAe$ is connected, and that $P = eA$ is the corresponding projective A-module. We denote by pres P the full subcategory of mod A consisting of the modules M which are P-presented, that is, which admit a presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with P_0, P_1 in add P. We recall that the functor $\operatorname{Hom}_A(P, -) : \operatorname{mod} A \to \operatorname{mod} B$ induces an equivalence pres $P \cong \operatorname{mod} B$, under which the summands of P correspond to the projective B-modules ([ARS] (II.2.1) p.33 and (II.2.5) p.35). We need the left inverse of this functor.

LEMMA. *Let X be a B-module, then the A-module $X \otimes_B P$ is P-presented, and we have $\mathrm{Hom}_A(P, X \otimes_B P) \cong X$, functorially.*

Proof. There exist $m, n > 0$ and an exact sequence in $\mathrm{mod}\, B$

$$B_B^m \longrightarrow B_B^n \longrightarrow X_B \longrightarrow 0\,.$$

Applying the right exct functor $- \otimes_B P_A : \mathrm{mod}\, B \to \mathrm{mod}\, A$ yields an exact sequence

$$P_A^m \longrightarrow P_A^n \longrightarrow X \otimes_B P \longrightarrow 0\,.$$

Thus, $X \otimes_B P$ is P-presented. Applying now the exact functor $\mathrm{Hom}_A(P, -)$ to the previous exact sequence yields a commutative diagram with exact rows.

$$\begin{array}{ccccccc} \mathrm{Hom}_A(P, P^m) & \longrightarrow & \mathrm{Hom}_A(P, P^n) & \longrightarrow & \mathrm{Hom}_A(P, X \otimes_B P) & \longrightarrow & 0 \\ \| \wr & & \| \wr & & & & \\ B_B^m & \longrightarrow & B_B^n & \longrightarrow & X_B & \longrightarrow & 0\,. \end{array}$$ □

PROPOSITION. *Let M be a P-presented A-module.*

(a) *If* $\mathrm{pd}\ M_A \le 1$, *then* $\mathrm{pd}\ \mathrm{Hom}_A(P, M) \le 1$.
(b) *If M lies in $\mathcal{L}_A$, then* $\mathrm{Hom}_A(P, M)$ *lies in $\mathcal{L}_B$.*
(c) *If M lies in $\mathcal{R}_A$, but* $\mathrm{Hom}_A(P, M)$ *does not lie in $\mathcal{R}_B$, then there exist a projective A-module P' in $\mathcal{R}_A$ and a path from M to P'.*
(d) *If M lies in $\mathcal{R}_A \setminus \mathcal{L}_A$, then* $\mathrm{Hom}_A(P, M)$ *lies in $\mathcal{R}_B$.*

Proof. (a) Since M lies in $\mathrm{pres}\, P$, there exists a presentation $P_1' \to P_0' \to M \to 0$ with P_0', P_1' in $\mathrm{add}\, P$. Since $\mathrm{pd}\ M_A \le 1$, there exists a minimal projective redolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0\,.$$

It follows from minimality that, for each $i \in \{0, 1\}$, P_i is a summand of P_i' and thus lies in $\mathrm{add}\, P$. Applying the exact functor $\mathrm{Hom}_A(P, -)$ yields a projective resolution

$$0 \longrightarrow \mathrm{Hom}_A(P, P_1) \longrightarrow \mathrm{Hom}_A(P, P_0) \longrightarrow \mathrm{Hom}_A(P, M) \longrightarrow 0\,.$$

(b) Let $X_0 \xrightarrow{u_1} X_1 \xrightarrow{u_2} X_2 \longrightarrow \cdots \xrightarrow{u_t} X_t = \mathrm{Hom}_A(P, M)$ be a path in $\mathrm{ind}\, B$. Setting, for each i, $M_i = X_i \otimes_B P$ and $f_i = u_i \otimes_B P$, we deduce a path in $\mathrm{ind}\, A$

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_t} M_t = M$$

with all M_i in $\mathrm{pres}\, P$. Since M lies in $\mathcal{L}_A$, we have $\mathrm{pd}\ M_0 \le 1$. Applying (a) yields $\mathrm{pd}\ M_0 = \mathrm{pd}\ \mathrm{Hom}_A(P, M_0) \le 1$.

(c) Since $X_B = \mathrm{Hom}_A(P, M)$ is not in $\mathcal{R}_B$, it has a successor Y_1 such that $\mathrm{id}\ Y_1 \ge 2$. By [R] p.74, there exists a projective B-module Q such that $\mathrm{Hom}_B\left(\tau_B^{-1} Y_1, Q\right) \ne 0$. This yields a path in $\mathrm{ind}\, B$

$$(*) \qquad X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_t = Y_1 \xrightarrow{v_1} Y_2 \xrightarrow{v_2} Y_3 \longrightarrow Q$$

with $Y_3 = \tau_B^{-1} Y_1$ and v_1, v_2 irreducible. As before, $(*)$ induces a path in $\mathrm{ind}\, A$

$$(**) \qquad M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots \longrightarrow M_t = N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \longrightarrow P'$$

where all modules lie in $\operatorname{pres} P$, $N_i = Y_i \otimes_B P$ for $i = 1, 2, 3$, $P' = Q \otimes_B P$ and $f_i = v_i \otimes_B P$ for $i = 1, 2$. Moreover, P' is projective (and lies in $\operatorname{add} P$). Since M lies in $\mathcal{R}_A$, so does P'.

(d) We construct as in (c) the paths (∗) and (∗∗). We note that since, by hypothesis, M lies in $\mathcal{R}_A \setminus \mathcal{L}_A$, then so do all the modules on the path (∗∗).

Now, there exists an almost split sequence in $\operatorname{mod} B$

$$0 \longrightarrow Y_1 \xrightarrow{\begin{bmatrix} v_1 \\ v_1' \end{bmatrix}} Y_2 \oplus Y' \xrightarrow{[v_2 \ v_2']} Y_3 \longrightarrow 0\,.$$

Since $\operatorname{mod} B \cong \operatorname{pres} P$, we have a short exact sequence

$$0 \longrightarrow N_1 \xrightarrow{\begin{bmatrix} f_1 \\ f_1' \end{bmatrix}} N_2 \oplus N' \xrightarrow{[f_2 \ f_2']} N_3 \longrightarrow 0$$

(where $N' = Y' \otimes_B P$) in $\operatorname{pres} P$, hence in $\operatorname{mod} A$. Since the former sequence does not split, neither does the latter. So $\operatorname{Ext}^1_A(N_3, N_1) \neq 0$. However, since $\operatorname{Hom}_A(N_3, P') \neq 0$, we have $\operatorname{id} \tau_A N_3 \geq 2$, so $\tau_A N_3$ does not lie in $\mathcal{R}_A$. On the other hand, $\mathcal{R}_A \setminus \mathcal{L}_A$ is closed under successors, hence the full subcategory $\operatorname{add}(\mathcal{R}_A \setminus \mathcal{L}_A)$ of $\operatorname{mod} A$ consisting of the direct sums of modules in $\mathcal{R}_A \setminus \mathcal{L}_A$ is a torsion class. By [AS], N_3 is an Ext-projective in $\operatorname{add}(\mathcal{R}_A \setminus \mathcal{L}_A)$. Since N_1 itself lie in $\mathcal{R}_A \setminus \mathcal{L}_A$, we get a contradiction to $\operatorname{Ext}^1_A(N_3, N_1) \neq 0$. □

2.3 COROLLARY. *If M is a P-presented A-module in $\mathcal{L}_A \cup \mathcal{R}_A$, then* $\operatorname{Hom}_A(P, M)$ *belongs to $\mathcal{L}_B \cup \mathcal{R}_B$.* □

2.4 COROLLARY. *There exist only finitely many non-isomorphic indecomposable P-presented A-modules M which lie in $\mathcal{R}_A$, but are such that* $\operatorname{Hom}_A(P, M)$ *does not lie in $\mathcal{R}_B$.*

Proof. This follows from (2.2)(c) and (1.2). □

2.5. We recall a few definitions. Let A be an artin algebra. An A-module T is called a *tilting module* of $\operatorname{pd} T \leq 1$, $\operatorname{Ext}^1_A(T, T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals $\operatorname{rk} K_0(A)$. The algebra A is called *tilted* [HR] if there exists a tilting A-module T such that $\operatorname{End} T_A$ is hereditary. The algebra A is called *shod* [CL1] if, for any indecomposable A-module M, we have $\operatorname{pd} M \leq 1$ or $\operatorname{id} M \leq 1$. Finally, it is *quasi-tilted* [HRS] if it is shod and of global dimension at most two. Tilted algebras are quasi-tilted, quasi-tilted algebras are shod and shod algebras are weakly shod.

COROLLARY. *Let A be a quasi-tilted algebra which is not tilted, and M be a P-presented A-module. If M lies in $\mathcal{L}_A \cap \mathcal{R}_A$, then* $\operatorname{Hom}_A(P, M)$ *lies in $\mathcal{L}_B \cap \mathcal{R}_B$.*

Proof. By (2.2)(b), $\operatorname{Hom}_A(P, M)$ lies in $\mathcal{L}_B$. Assume that $\operatorname{Hom}_A(P, M)$ does not lie in $\mathcal{R}_B$. By (2.2)(c), there exists a projective A-module P' in $\mathcal{R}_A$ which is a successor of M. By [CS](2.5), this implies that A is tilted, a contradiction. □

3. Proof of the theorem.

3.1. We need two more definitions. An artin algebra A is called a *laura* algebra if the class $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$, that is, if all but at most finitely many modules in $\operatorname{ind} A$ lie in $\mathcal{L}_A \cup \mathcal{R}_A$ (see [AC2]). It is called *right glued* (or *left glued*) if the class of all modules M in $\operatorname{ind} A$ such that $\operatorname{pd} M \leq 1$ (or $\operatorname{id} M \leq 1$, respectively) is cofinite in $\operatorname{mod} A$ (see [AC1]). Weakly shod algebras are laura. We need the following lemma [AC2](2.2).

LEMMA. *An artin algebra is right (or left) glued if and only if $\mathcal{L}_A$ (or $\mathcal{R}_A$, respectively) is cofinite in* $\operatorname{ind} A$. □

3.2. Proof of the theorem (a) Assume that A is a laura algebra, and that X is an indecomposable B-module. If X does not lie in $\mathcal{L}_B \cup \mathcal{R}_B$, then, by (2.3), the P-presented A-module $X \otimes_B P$ does not lie in $\mathcal{L}_A \cup \mathcal{R}_A$. Since $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$, we are done.

(b) Assume that A is a right glued algebra, and that X is an indecomposable B-module. If X does not lie in $\mathcal{L}_B$, then the P-presented A-module $X \otimes_B P$ does not lie in $\mathcal{L}_A$ (by (2.2)(b)). Since, by (3.1), $\mathcal{L}_A$ is cofinite in $\operatorname{ind} A$, this shows the statement for right glued algebras. The one for left glued algebras follows by duality.

(d) Assume that A is a shod algebra, and that X is an indecomposable B-module. Since A is shod, then, by [CL1], the P-presented A-module $X \otimes_B P$ either belongs to $\mathcal{L}_A$ or to $\mathcal{R}_A \setminus \mathcal{L}_A$. In the first case, (2.2)(b) implies that X lies in $\mathcal{L}_B$ and, in the second case, (2.2)(c) implies that it lies in $\mathcal{R}_B$. Then, B is shod.

(e) Assume that A is a quasi-tilted algebra. Then A is shod and, therefore, so is B (by (d) above). Assume that B is not quasi-tilted. Then there exists an indecomposable projective B-module Q in $\mathcal{R}_B \setminus \mathcal{L}_B$. Since A is quasi-tilted, the projective A-module $Q \otimes_B P$ lies in $\mathcal{L}_A$. By (2.2)(b), we infer that Q lies in $\mathcal{L}_B$, a contradiction.

(c) Assume that A is a weakly shod algebra. We may, by (e), assume that A is not quasi-tilted. Let s_1 be the number of P-presented A-modules M in $\mathcal{R}_A$ such that $\operatorname{Hom}_A(P, M)$ is not in $\mathcal{R}_B$. By (2.4), s_1 is finite. Let also s_2 be the number of modules in $\operatorname{ind} A$ which do not lie in $\mathcal{L}_A \cup \mathcal{R}_A$. Since A is weakly shod, s_2 is also finite. In view of (1.4), it suffices to show that $\ell = s_1 + s_2 - 1$ is a bound on the length of paths from an indecomposable not in $\mathcal{L}_B$ to one not in $\mathcal{R}_B$.

Assume that we have a path in $\operatorname{ind} B$

$$X = X_0 \xrightarrow{u_1} X_1 \xrightarrow{u_2} X_2 \longrightarrow \cdots \xrightarrow{u_t} X_t = Y$$

with X not in $\mathcal{L}_B$ and Y not in $\mathcal{R}_B$. Suppose $t > s_1 + s_2 - 1$. Setting $M_i = X_i \otimes_B P$ and $f_i = u_i \otimes_B P$ for each i, we get a path in $\operatorname{ind} A$

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_t} M_t = N$$

where all the M_i are P-presented. By (2.2)(b), M does not lie in $\mathcal{L}_A$. Moreover, by definition of s_1, the module M_{t-s_1} does not lie in $\mathcal{R}_A$. Thus, we have a subpath

$$(\xi) \qquad M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow \cdots \xrightarrow{f_{t-s_1}} M_{t-s_1}$$

with M not in $\mathcal{L}_A$ and M_{t-s_1} not in $\mathcal{R}_A$. This implies that none of the modules on (ξ) lies in $\mathcal{L}_A \cup \mathcal{R}_A$. Consequently, by [CL3](2.5), (ξ) lies in the unique pip-bounded component of the Auslander-Reiten quiver of A. Since this component is generalised standard and contains no oriented cycles, the path (ξ) has no repetitions. Since (ξ) contains $t - s_1 + 1 > s_2$ indecomposables, none of which lies in $\mathcal{L}_A \cup \mathcal{R}_A$, we have reached a contradiction which completes the proof. □

3.3. REMARK In what precedes, we have not proved that, if A is tilted, then so is B (which is shown in [H](III.6.5) with the help of perpendicular categories). In one particular case, however, we do have an immediate answer. Indeed, it follows easily from [R] that, if an algebra has a sincere directing indecomposable module, then it is tilted. Now, it is easy to see that, with the above notation, if A has a sincere indecomposable P-presented module M, then $\operatorname{Hom}_A(P, M)$ is a sincere directing indecomposable B-module. Indeed, it is clear that, if M is sincere and indecomposable in $\operatorname{mod} A$ then $X = \operatorname{Hom}_A(P, M)$ is sincere and indecomposable in $\operatorname{mod} B$. We now show that, if M is P-presented and directing in $\operatorname{mod} A$, then X is directing in $\operatorname{mod} B$. Indeed, if this is not the case, then there exists a path in $\operatorname{ind} B$ of the form

$$X = X_0 \xrightarrow{u_1} X_1 \xrightarrow{u_2} X_2 \longrightarrow \cdots \xrightarrow{u_t} X_t = X$$

with the u_i non-isomorphisms. But then

$$M = X \otimes_B P \xrightarrow{u_1 \otimes_B P} X_1 \otimes_B P \xrightarrow{u_2 \otimes_B P} X_2 \otimes_B P \longrightarrow \cdots \xrightarrow{u_t \otimes_B P} X_t \otimes_B P = M$$

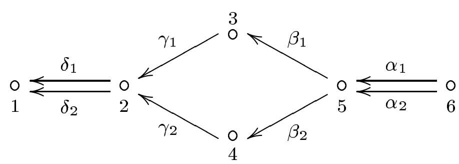
is a path of non-isomorphisms in $\operatorname{ind} A$, a contradiction. □

3.4. The following corollary shows that, for instance, a tubular algebra [R] cannot occur as a full subcategory of a laura algebra which is not quasi-tilted.

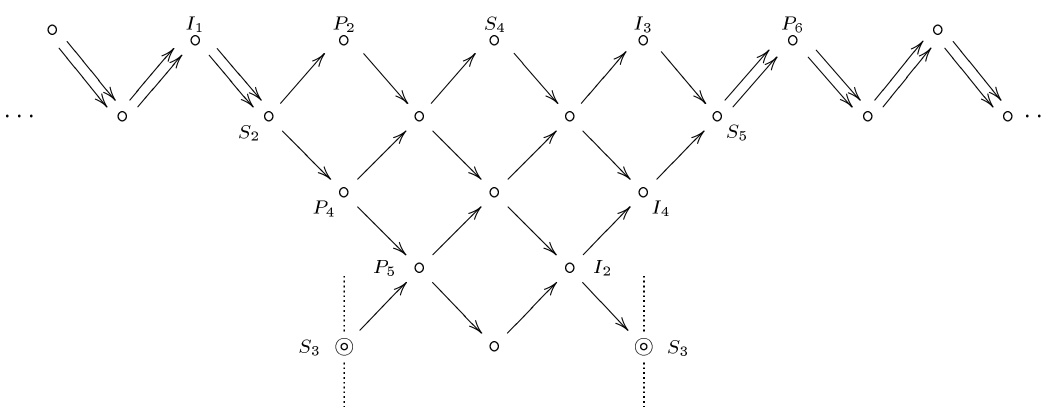
COROLLARY. *Assume that A is a laura algebra which is not quasi-tilted. If B is quasi-tilted, then it is tilted.*

Proof. If all the indecomposable summands of P lie in $\mathcal{L}_A$, then, by [AC2](4.10), there exists a tilted algebra A' such that P is a projective A'-module. By [H](III.6.5), we infer that $B = \operatorname{End} P_A = \operatorname{End} P_{A'}$ is tilted. If this is not the case, then $P = P' \oplus P''$, where P'_A is an indecomposable projective lying in $\mathcal{R}_A \setminus \mathcal{L}_A$. By (2.2)(d), $\operatorname{Hom}_A(P, P')$ is an indecomposable projective B-module lying in $\mathcal{R}_B$. Thus, if B is quasi-tilted, it follows from [HRS](II.3.4) that B is tilted. □

3.5. EXAMPLE The following example shows that any of the remaining cases may occur. Let k be a commutative field, and A be the finite dimensional k-algebra given by the quiver

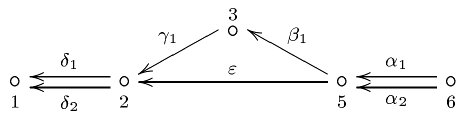


where $\alpha_i\beta_j = 0$, $\gamma_i\delta_j = 0$ (for all $i, j \in \{1, 2\}$) and $\beta_1\gamma_1 = 0$. Then A is a laura algebra whose quasidirected nonsemiregular component [AC2] is of the form



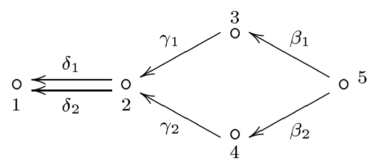
where we identify the two copies of S_3 along the vertical dotted lines. Here, we denote by P_x (or I_x, or S_x, or e_x) the indecomposable projective (or injective, or simple, or idempotent, respectively) corresponding to the point x of the quiver.

(a) Let $e = \sum_{x \neq 4} e_x$. Then $B = eAe$ is the radical square zero algebra given by the quiver



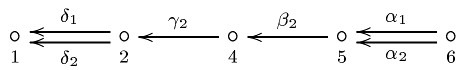
This is also a laura algebra, which is not weakly shod.

(b) Let $e = \sum_{x \neq 6} e_x$. Then $B = eAe$ is given by the quiver



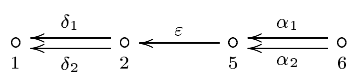
bound by $\gamma_i \delta_j = 0$ (for all $i, j \in \{1, 2\}$) and $\beta_1 \gamma_1 = 0$. Then B is right glued, but is not weakly shod. Dually, if $e' = \sum_{x \neq 1} e_x$, then $B' = e'Ae'$ is left glued, but is not weakly shod.

(c) Let $e = \sum_{x \neq 3} e_x$. Then $B = eAe$ is given by the quiver



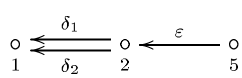
bound by $\alpha_i \beta_2 = 0$ and $\gamma_2 \delta_i = 0$ (for all $i \in \{1, 2\}$). Then B is a weakly shod algebra (of global dimension two), but it is not shod: indeed, the simple B-module S_4 is such that pd $S_4 = 2$, and id $S_4 = 2$.

(d) Let $e = \sum_{x \neq 3,4} e_x$. Then $B = eAe$ is the radical square zero algebra given by the quiver



It is a shod algebra of global dimension three, hence it is not quasi-tilted.

(e) Let $e = e_1 + e_2 + e_5$. Then $B = eAe$ is the radical square zero algebra given by the quiver



It is clearly tilted.

3.6. REMARK Examples (c) and (d) above show that we may increase the global dimension while passing from A to B. However, if e is a convex idempotent (that is, if there exists a sequence $e_i = e_{i_0}, e_{i_1}, \ldots, e_{i_t} = e_j$ of primitive idempotents such that $e_{\ell+1} A e_{i_\ell} \neq 0$ for $o \leq \ell \leq t$ and $ee_i = e_i$, $ee_j = e_j$, then $ee_{i_\ell} = e_{i_\ell}$ for all ℓ), then the global dimension of B does not exceed that of A. Indeed, it is well-known, and easy to see, that, in this case, for any two B-modules X, Y, we have $\mathrm{Ext}^i_B(X, Y) \cong \mathrm{Ext}^i_A(X, Y)$ for all i. Hence, for any simple B-module S, we have pd $S_B \leq$ pd S_A. This yields our statement. Now, if A

(and hence B) are monomial albebras, then the global dimension can be easily computed by looking at the overlaps between the monomial relations [GHZ,G]. In particular, the global dimension equals two if and only if no two monomial relations share (at least) an arrow. Thus, if A is a shod monomial algebra, and e is a convex idempotent such that no two monomials in B share an arrow, then B is quasi-tilted.

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Ibrahim Assem
Mathématiques et d'Informatique
Université de Sherbrooke
Sherbrooke, Qc
Canada, J1K 2R1
ibrahim.assem@dmi.usherb.ca

Flávio Ulhoa Coelho
Departamento de Matemática. IME
Universidade de São Paulo
CP 66281, São Paulo, SP
05315-970, Brasil
fucoelho@ime.usp.br