Coverings of Laura Algebras: the Standard Case

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Abstract

In this paper, we study the covering theory of laura algebras. We prove that if a connected laura algebra is standard (that is, it is not quasi-tilted of canonical type and its connecting components are standard), then this algebra has nice Galois coverings associated to the coverings of the connecting component. As a consequence, we show that the first Hochschild cohomology group of a standard laura algebra vanishes if and only if it has no proper Galois coverings.

Introduction

The Hochschild cohomology groups are interesting invariants of associative algebras. In this paper, we are interested in the first group, which can be thought of as the group of classes of outer derivations of the algebra. In [42, §3, Pb. 1], Skowroński has related the vanishing of the first Hochschild cohomology group $\mathsf{HH}^1(A)$ of an algebra (with coefficients in the bimodule ${}_AA_A$) to the simple connectedness of A. A basic connected finite dimensional algebra over an algebraically closed field k is called *simply connected* if has no proper Galois covering or, equivalently, if the fundamental group (in the sense of [36]) of any presentation is trivial. In particular, Skowroński has conjectured that a tame triangular algebra A is simply connected if and only if $\mathsf{HH}^1(A) = 0$. Since then, this equivalence has been shown for several classes of algebras not necessarily tame or triangular. Thus, for instance, it has been proved for standard representation-finite algebras [15], for piecewise hereditary algebras [34] and for weakly shod algebras [32] (see also [6]).

Here, we study this conjecture for laura algebras. These are defined as follows. Let $\mathsf{mod}\,A$ be the category of finitely generated right A-modules, and $\mathsf{ind}\,A$ be a full subcategory consisting of exactly one representative of each isomorphism class of indecomposable A-modules. We define the *left part* $\mathcal{L}_A$ *of* $\mathsf{mod}\,A$ to be the full subcategory of $\mathsf{ind}\,A$ consisting of those modules whose predecessors have projective dimension at most one, and we define its *right part* $\mathcal{R}_A$ dually. These classes were introduced in [28] in order to study the module categories of quasi-tilted algebras. Following [3, 44], we say that A is *laura* provided $\mathsf{ind}\,A\setminus(\mathcal{L}_A\cup\mathcal{R}_A)$ has only finitely many objects. Thus, any representation-finite algebra, any quasi-tilted algebra or any weakly shod algebra is laura. The class of laura algebras has been studied in [3, 4, 5, 7, 8, 23, 44]. Here, we concentrate on the conjecture that a laura algebra A is simply connected if and only if $\mathsf{HH}^1(A) = 0$.

One approach to this conjecture, already used in [33, 34], is to use the covering theory of laura algebras. Covering theory was introduced by Gabriel and his school (see [14, 25, 38], for instance) and consists in replacing an algebra by a locally bounded category, called its covering, which is sometimes easier to study. In the case of laura algebras, a non-quasi-tilted laura algebra A has a unique faithful non-semiregular Auslander-Reiten component which is *quasi-directed* (that is, it is generalised standard and almost all its modules are directed). This component is called the *connecting component* (see [3]) of the laura algebra, in analogy to the connecting components of tilted algebras. Thus, any laura algebra which is not quasi-tilted of canonical type has at least one, and at most two, connecting components (actually, it has two if and only if the algebra is concealed). If, in particular, A is representation-finite, then its connecting component coincides with the whole Auslander-Reiten quiver. We say that a laura algebra A which is not quasi-tilted of canonical type is *standard* provided its connecting components are all standard (it is known from [40] that the connecting components of a concealed algebra are standard). This generalises the notion of standard representation-finite algebra (see [14]). Several classes of laura algebras are standard, notably all tilted algebras or all weakly shod algebras. Our first main theorem says that the universal covering of the connecting component induces a covering of the algebra.

Theorem A. *Let A be laura and not quasi-tilted of canonical type. There exists a connected locally bounded k-category $\widetilde{A}$ and a covering functor $F\colon \widetilde{A}\to A$ whose fibres are in bijection with the fundamental group $\pi_1(\Gamma)$ of the connecting component Γ. If moreover A is standard, then F is a Galois covering with group $\pi_1(\Gamma)$.*

Note that if A is concealed, then the fundamental groups of its two connecting components, which are postprojective and preinjective respectively, are isomorphic.

In order to prove Theorem A, we need to consider a more general situation. Namely, we first consider an Auslander-Reiten component of an algebra, which contains a left section (in the sense of [1]). We then show that to the universal cover of that component corresponds a covering of its support with nice properties. Applying this result to the connecting component of a laura algebra yields the required covering.

By the theorem, if A is standard, then we are able to work with the Galois coverings which are notably easier to handle than covering functors. We prove that if A is a standard laura algebra, then any Galois covering of the connecting component of A induces a Galois covering of A itself, with the same group. This allows us to prove our second main theorem, which settles the conjecture for the case of standard laura algebras.

Theorem B. *Let A be a standard laura algebra, and Γ its connecting component(s). The following are equivalent:*

(a) A is simply connected.

(b) $\mathsf{HH}^1(A) = 0$.

(c) Γ is simply connected.

(d) The orbit graph $\mathcal{O}(\Gamma)$ is a tree.

Moreover, if these conditions are verified, then A is weakly shod.

If one drops the standard condition, then the above theorem may fail. Indeed, there are examples of non-standard representation-finite algebras which are simply connected and with non-zero first Hochschild cohomology group (see [14, 15], or below). However, some implications are still true in Theorem B without assuming standardness. Indeed, we always have: (c) and (d) are equivalent and (c) implies (a) and (b).

In our final section, we consider the case of special biserial laura algebras which have been characterised in [23], and we prove that the conjecture holds in this case also, although we do not know whether they are standard or not.

Our paper is organised as follows. After a short preliminary section, we prove a few preparatory lemmata on covering functors in Section 2. In Section 3, we give examples of standard laura algebras. Section 4 is devoted to properties of tilting modules which are in the image of the push-down functor associated to a covering functor. In Section 5 we study the coverings of Auslander-Reiten components having left sections. The proof of Theorem A occupies Section 6. We then concentrate on the case of Galois coverings in Section 7, which leads to the proof of Theorem B in Section 8. Finally, Section 9 deals with the special biserial case.

1 Preliminaries

Categories and modules

Throughout this paper, k denotes a fixed algebraically closed field. All our categories are locally bounded k-categories, in the sense of [14, 2.1]. We assume that all locally bounded k-categories are small and that all functors are k-linear (the categories of finite dimensional modules and their bounded derived categories are skeletally small).

Let $F\colon \mathcal{E} \to \mathcal{B}$ be a k-linear functor and let G be a group acting on $\mathcal{E}$ and $\mathcal{B}$ by automorphisms. Then F is called *G-equivariant* if $F \circ g = g \circ F$ for every $g \in G$.

A basic finite dimensional algebra A can be considered equivalently as a locally bounded k-category as follows: we fix a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents, then the object set of A is the set $\{e_1, \dots, e_n\}$ and the morphisms space from e_i to e_j is $e_j A e_i$. The composition of morphisms is induced by the multiplication in A.

Let $\mathcal{C}$ be a locally bounded k-category. We denote by $\mathcal{C}_o$ its object class. A *right $\mathcal{C}$-module* M is a k-linear functor $M\colon \mathcal{C}^{op} \to \mathsf{MOD}\, k$, where $\mathsf{MOD}\, k$ is the category of k-vector spaces. We write $\mathsf{MOD}\, \mathcal{C}$ for the category of $\mathcal{C}$-modules and $\mathsf{mod}\, \mathcal{C}$ for the full subcategory of the *finite dimensional* $\mathcal{C}$-modules, that is, those modules M such that $\sum_{x \in \mathcal{C}_o} \dim\ M(x) < \infty$. If $\mathcal{A}$ is a subcategory of $\mathsf{MOD}\, \mathcal{C}$, we use the notation $X \in \mathcal{A}$ to express that X is an object in $\mathcal{A}$. For every $x \in \mathcal{C}_o$, the indecomposable projective $\mathcal{C}$-module associated to x is $\mathcal{C}(-, x)$. The standard duality $\mathsf{Hom}_k(-, k)$ is denoted by D. Let M be a $\mathcal{C}$-module. If $\mathcal{B}$ is a full subcategory of $\mathcal{C}$, then $M_{|\mathcal{B}}$ denotes the induced $\mathcal{B}$-module. If $\mathcal{X}$ is a subcategory of $\mathsf{mod}\, \mathcal{C}$, then the $\mathcal{X}$-module $\mathsf{Hom}_{\mathcal{C}}(-, M)_{|\mathcal{X}}$ is denoted by $\mathsf{Hom}_{\mathcal{C}}(\mathcal{X}, M)$. Also, $\mathsf{Hom}_{\mathcal{C}}(M, \mathcal{C})$ denotes the $\mathcal{C}^{op}$-module $\mathsf{Hom}_{\mathcal{C}}(M, \bigoplus_{x \in \mathcal{C}_o} \mathcal{C}(-, x))$ (if $A = \mathcal{C}$ is a finite dimensional algebra, this is just the left A-module $\mathsf{Hom}_A(M, A)$).

We let $\mathsf{ind}\, \mathcal{C}$ be a full subcategory of $\mathsf{mod}\, \mathcal{C}$ consisting of a complete set of representatives of the isomorphism classes of the indecomposable $\mathcal{C}$-modules. We write $\mathsf{proj}\, \mathcal{C}$ and $\mathsf{inj}\, \mathcal{C}$ for the full subcategories of $\mathsf{ind}\, \mathcal{C}$ of projective and injective modules, respectively. Whenever we speak about an indecomposable $\mathcal{C}$-module, we always mean that it belongs to $\mathsf{ind}\, \mathcal{C}$.

For a full subcategory $\mathcal{A}$ of $\mathsf{mod}\, \mathcal{C}$, we denote by $\mathsf{add}\, \mathcal{A}$ the full subcategory of $\mathsf{mod}\, \mathcal{C}$ with objects the direct sums of summands of modules in $\mathcal{A}$. If M is a module, then $\mathsf{add}\, M$ denotes $\mathsf{add}\, \{M\}$.

The Auslander-Reiten translations of $\mathsf{mod}\, \mathcal{C}$ are denoted by $\tau_{\mathcal{C}} = D\, \mathsf{Tr}$ and $\tau_{\mathcal{C}}^{-1} = \mathsf{Tr}\, D$. The Auslander-Reiten quiver of $\mathcal{C}$ is denoted by $\Gamma(\mathsf{mod}\, \mathcal{C})$. For a component Γ of $\Gamma(\mathsf{mod}\, \mathcal{C})$, we denote by $\mathcal{O}(\Gamma)$ its orbit graph (see [14, 4.2], its definition is recalled in Section 8 below). It is *non-semiregular* if it contains both an injective and a projective module. It is *faithful* if its *annihilator* $\mathsf{Ann}\,\Gamma = \bigcap_{X\in\Gamma} \mathsf{Ann}\,X$ is zero. Following [43], a component of Γ is *generalised standard* if $\mathsf{rad}^{\infty}(X,Y)=0$ for every $X,Y\in\Gamma$. It is *standard* if there exists an isomorphism of k-categories $k(\Gamma)\xrightarrow{\sim}\mathsf{ind}\,\Gamma$ which extends the identity on vertices, and which maps meshes to almost split sequences. Let $\pi\colon\widetilde{\Gamma}\to\Gamma$ be a morphism of translation quivers. Let $\mathcal{X}$ be a full convex subquiver of $\widetilde{\Gamma}$. We let $k(\mathcal{X})$ be the full subcategory of $k(\widetilde{\Gamma})$ with objects the vertices in $\mathcal{X}$. Following [14, 3.1], a functor $p\colon k(\mathcal{X})\to\mathsf{ind}\,\Gamma$ is called *well-behaved* (with respect to π) if the following conditions are satisfied:

1. $p(X)=\pi(X)$ for every $X\in\mathcal{X}$.
2. Let $X\in\mathcal{X}$. Let $(u_i\colon Z_i\to X)_{i=1,\dots,t}$ be all the arrows in $\mathcal{X}$ ending at X (or $(v_j\colon X\to Y_j)_{j=1,\dots,s}$ be all the arrows in $\mathcal{X}$ starting from X), then the morphism $\begin{bmatrix}p(u_1) & \dots & p(u_t)\end{bmatrix}\colon \bigoplus_{i=1}^{t} p(Z_i)\to p(X)$ (or $\begin{bmatrix}p(v_1) & \dots & p(v_s)\end{bmatrix}^t\colon p(X)\to\bigoplus_{j=1}^{s} Y_j$, respectively) is irreducible.
3. Let $X\in\widetilde{\Gamma}$ be non projective. Let $(u_i\colon\tau X\to E_i)_{i=1,\dots,s}$ (or $(v_i\colon E_i\to X)_{i=1,\dots,s}$) be all the arrows in $\widetilde{\Gamma}$ starting from τX (or ending at X, respectively). If the mesh in $\widetilde{\Gamma}$ ending at X is contained in $\mathcal{X}$, then $\sum_{i=1}^{s} p(g_i)p(f_i)=0$.

Conditions 2 and 3 above imply that if a mesh in $\widetilde{\Gamma}$ is contained in $\mathcal{X}$, then p maps this mesh to an almost split sequence.

For more notions and results on modules, we refer the reader to [9, 10]. For a reminder on coverings and fundamental groups of translation quivers, we refer the reader to [14, §1].

Paths

Let $\mathcal{C}$ be a locally bounded k-category. Given $X,Y\in\mathsf{ind}\,\mathcal{C}$, a *path* $X\rightsquigarrow Y$ from X to Y in $\mathsf{ind}\,\mathcal{C}$ is a sequence of non-zero morphisms:

$$(\star)\quad X=X_0\xrightarrow{f_1}X_1\to\dots\to X_{t-1}\xrightarrow{f_t}X_t=Y\ (t\geqslant 0)$$

where $X_i\in\mathsf{ind}\,\mathcal{C}$ for all i. We then say that X is a *predecessor* of Y and that Y is a *successor* of X. A path from X to X involving at least one non-isomorphism is a *cycle*. A module $X\in\mathsf{ind}\,\mathcal{C}$ which lies on no cycle is *directed*. If each f_i in $(\star)$ is irreducible, we say that $(\star)$ is a *path of irreducible morphisms* or a *path in* $\Gamma(\mathsf{mod}\,\mathcal{C})$. A path $(\star)$ of irreducible morphisms is *sectional* if $\tau_{\mathcal{C}}X_{i+1}\neq X_{i-1}$ for all i with $0<i<t$.

An indecomposable module $M\in\mathcal{L}_A$ is Ext-*injective* in $\mathsf{add}\,\mathcal{L}_A$ if $\mathsf{Ext}^1_A(-,M)_{|\mathcal{L}_A}=0$ (see [11]). This is the case if and only if $\tau_A^{-1}M\notin\mathcal{L}_A$.

The endomorphism algebra of the direct sum of the indecomposable projective modules lying in $\mathcal{L}_A$ is called the *left support of* A. If A is laura, then its left support is a product of tilted algebras (see [5, 4.4, 5.1]).

An algebra A is *weakly shod* if the length of any path in $\mathsf{ind}\,A$ from an injective to a projective is bounded [20]. Also, A is *quasi-tilted* if its global dimension $\mathsf{gl.dim}\,A$ is at most two and $\mathsf{ind}\,A=\mathcal{L}_A\cup\mathcal{R}_A$, see [28].

2 Preliminaries on covering functors

We recall some notions on covering functors. A k-linear functor $F\colon\mathcal{E}\to\mathcal{B}$ is a *covering functor* if (see [14, 3.1]):

1. $F^{-1}(x)\neq\emptyset$ for every $x\in\mathcal{B}_o$.
2. For every $x,y\in\mathcal{E}_o$, the two following k-linear maps are bijective:

$$\bigoplus_{F(y')=F(y)}\mathcal{E}(x,y')\to\ \mathcal{B}(F(x),F(y)),\ \text{and}\qquad\bigoplus_{F(x')=F(x)}\mathcal{E}(x',y)\to\ \mathcal{B}(F(x),F(y))\ .$$

Following [25, § 3], F is a *Galois covering with group* G if there exists a group morphism $G\to\mathsf{Aut}(\mathcal{C})$ such that G acts freely on $\mathcal{E}_o$, $F\circ g=F$ for every $g\in G$ and the functor $\mathcal{E}/G\to\mathcal{B}$ induced by F is an isomorphism. We refer the reader to [25, 3.1] for the definition of $\mathcal{E}/G$. Galois coverings are covering functors.

If $F\colon\mathcal{E}\to\mathcal{B}$ is a covering functor, then F defines an adjoint pair $(F_\lambda,F_.)$ of functors $F_\lambda\colon\mathsf{MOD}\,\mathcal{E}\to\mathsf{MOD}\,\mathcal{B}$ and $F_.\colon\mathsf{MOD}\,\mathcal{B}\to\mathsf{MOD}\,\mathcal{E}$ (see [14, 3.2]). The functor $F_.$ is called the *pull-up functor* and F_λ is called the *push-down functor*. We recall briefly their construction: If $M\in\mathsf{MOD}\,\mathcal{B}$, then $F_.M=M\circ F^{op}$; if $M\in\mathsf{MOD}\,\mathcal{E}$, then $F_\lambda M$ is the $\mathcal{B}$-module such that $F_\lambda M(x)=\bigoplus_{F(\widetilde{x})=x}M(\widetilde{x})$, for every $x\in\mathcal{B}_o$. Both F_λ and $F_.$ are exact.

Let $F\colon\mathcal{E}\to\mathcal{B}$ be a covering functor between locally bounded k-categories. We prove a few facts relative to F. Some of these are easy to prove in case F is a Galois covering. However, in general, the proofs are more complicated. This can be explained simply by the following fact: $F^{op}\colon \mathcal{E}^{op} \to \mathcal{B}^{op}$ is also a covering functor, and $D\,F_\lambda^{op} \simeq F_\lambda\, D$ if F is Galois. However, this isomorphism no longer exists in the general case of covering functors (see [14, 3.4], for instance).

As a motivation for the results in this section, we start with the following proposition which is essential in our paper. It is mainly due to Riedtmann (see [38, 2.2]). In the sequel, we use it without further reference.

Proposition 2.1. *Let A be a basic finite dimensional algebra. Let Γ be a component of $\Gamma(\mathsf{mod}\,A)$. Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be the universal Galois covering of translation quivers. Then there exists a well-behaved functor $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$. If, moreover, Γ is generalised standard then p is a covering functor.*

Proof: The functor p was first constructed in [38, 2.2] for the stable part of the Auslander-Reiten quiver of a self-injective representation-finite algebra. The covering property was proved in [38, 2.3] under the same setting. The construction of p was generalised to any Auslander-Reiten component in [14, 3.1]. Finally, it is easy to check that the arguments given in [38, 2.3] to prove that p is a covering functor apply to the case of generalised standard components. ■

The results of this section will be applied to covering functors as in 2.1. We now turn to the general situation where $F\colon \mathcal{E} \to \mathcal{B}$ is a covering functor between locally bounded k-categories.

Since F_λ and $F_.$ are exact, we still have an adjunction at the level of derived categories. Here and in the sequel, $\mathcal{D}(\mathsf{MOD}\,\mathcal{E})$ and $\mathcal{D}^b(\mathsf{mod}\,\mathcal{E})$ denote the derived category of $\mathcal{E}$-modules and the bounded derived category of finite dimensional $\mathcal{E}$-modules, respectively. The following lemma is immediate. For a background on derived categories, we refer the reader to [26, Chap. III].

Lemma 2.2. *F_λ and $F_.$ induce an adjoint pair $(F_\lambda, F_.)$ of exact functors:*

$$\mathcal{D}(\mathsf{MOD}\,\mathcal{E}) \underset{F_.}{\overset{F_\lambda}{\rightleftarrows}} \mathcal{D}(\mathsf{MOD}\,\mathcal{B}) \quad .$$

Moreover $F_\lambda(\mathcal{D}^b(\mathsf{mod}\,\mathcal{E})) \subseteq \mathcal{D}^b(\mathsf{mod}\,\mathcal{B})$. ■

Let $x \in \mathcal{E}_o$. By condition 2 in the definition of a covering functor, F induces a canonical isomorphism $F_\lambda(\mathcal{E}(-,x)) \xrightarrow{\sim} \mathcal{B}(-,F(x))$ of $\mathcal{B}$-modules (see [14, 3.2]). In the sequel, we always identify these two modules by means of this isomorphism. Using this identification we get the following result.

Lemma 2.3. *Let $M \in \mathcal{D}(\mathsf{MOD}\,\mathcal{E})$. Then F_λ induces two linear maps for every $x \in \mathcal{B}_o$:*

$$\bigoplus_{F(\widetilde{x})=x} \mathcal{D}(\mathsf{MOD}\,\mathcal{E})(M,\mathcal{E}(-,\widetilde{x})) \to \mathcal{D}(\mathsf{MOD}\,\mathcal{B})(F_\lambda M, \mathcal{B}(-,x))\ ,$$

$$\textit{and} \quad \bigoplus_{F(\widetilde{x})=x} \mathcal{D}(\mathsf{MOD}\,\mathcal{E})(\mathcal{E}(-,\widetilde{x}),M) \to \mathcal{D}(\mathsf{MOD}\,B)(\mathcal{B}(-,x), F_\lambda M)\ .$$

These maps are functorial in M, and they are bijective if M is quasi-isomorphic to a bounded complex of finite dimensional projective modules (for example, if $\mathsf{gl.dim}\,\mathcal{E} < \infty$ and $M \in \mathsf{mod}\,\mathcal{E}$).

Proof: Let φ_M be the first map. If $M = P[l]$ where $l \neq 0$ and P is a projective $\mathcal{E}$-module, then φ_M is bijective (because $\mathsf{Ext}_{\mathcal{E}}^{-l}(P,\mathcal{E}(-,\widetilde{x})) = 0$). Also, if M is an indecomposable projective $\mathcal{E}$-module, then φ_M is bijective (because F is a covering functor). Finally, if $M \to M' \to M'' \to M[1]$ is a triangle in $\mathcal{D}(\mathsf{MOD}\,\mathcal{E})$, then φ_M, $\varphi_{M'}$ and $\varphi_{M''}$ are bijective as soon as two of them are so. Consequently, φ_M is bijective if M is quasi-isomorphic to a bounded complex of finite dimensional projective $\mathcal{E}$-modules. The second map is handled similarly. ■

It is not always true that F_λ commutes with the Auslander-Reiten translation. However, we have the following.

Lemma 2.4. *Let $X \in \mathsf{ind}\,\mathcal{E}$ be such that $F_\lambda X \in \mathsf{ind}\,\mathcal{B}$ and $\mathsf{pd}\,X < \infty$. Then $\dim \tau_{\mathcal{E}} X = \dim \tau_{\mathcal{B}} F_\lambda X$.*

Proof: Let $X \in \mathsf{mod}\,\mathcal{E}$ be any module. Let $P_1 \to P_0 \to X \to 0$ be a minimal projective presentation in $\mathsf{mod}\,\mathcal{E}$. By [14, 3.2], we deduce that $F_\lambda P_1 \to F_\lambda P_0 \to F_\lambda X \to 0$ is a minimal projective presentation in $\mathsf{mod}\,\mathcal{B}$. So we have exact sequences in $\mathsf{mod}\,\mathcal{E}^{op}$ and $\mathsf{mod}\,\mathcal{B}^{op}$, respectively:

$$0 \to \mathsf{Hom}_{\mathcal{E}}(X,\mathcal{E}) \to \mathsf{Hom}_{\mathcal{E}}(P_0,\mathcal{E}) \to \mathsf{Hom}_{\mathcal{E}}(P_1,\mathcal{E}) \to \mathsf{Tr}_{\mathcal{E}} X \to 0\ ,$$

$$\text{and } 0 \to \mathsf{Hom}_{\mathcal{B}}(F_\lambda X,\mathcal{B}) \to \mathsf{Hom}_{\mathcal{B}}(F_\lambda P_0,\mathcal{B}) \to \mathsf{Hom}_{\mathcal{B}}(F_\lambda P_1,\mathcal{B}) \to \mathsf{Tr}_{\mathcal{B}} F_\lambda X \to 0\ ,$$

Let $X \in \mathsf{mod}\,\mathcal{E}$ be of finite projective dimension, thus quasi-isomorphic to a bounded complex of finite dimensional projective $\mathcal{E}$-modules. The bijections of 2.3 imply that $\dim \mathsf{Hom}_{\mathcal{E}}(X,\mathcal{E}) = \sum_{x\in\mathcal{E}_o} \dim \mathsf{Hom}_{\mathcal{E}}(X,\mathcal{E}(-,x)) = \sum_{x\in\mathcal{B}_o} \dim \mathsf{Hom}_{\mathcal{B}}(X,\mathcal{B}(-,x)) = \dim \mathsf{Hom}_{\mathcal{B}}(F_\lambda X,\mathcal{B})$. Using the above exact sequences, we deduce that $\dim \mathsf{Tr}_{\mathcal{E}} X = \dim \mathsf{Tr}_{\mathcal{B}} F_\lambda X$. Whence the lemma in case X and $F_\lambda X$ are indecomposable. ■

3 Standard laura algebras

In this section, our objective is to derive sufficient conditions for a laura algebra to be standard. Weakly shod algebras are particular cases of laura algebras. It is proved in [20, § 4] that if A is weakly shod and not quasi-tilted, then A can be written as a one-point extension $A = B[M]$ such that the connecting component of A can be recovered from M and from the connecting components of B. This motivates the following definition.

Definition 3.1. Let A be a connected laura algebra which is not quasi-tilted of canonical type. An indecomposable projective A-module P lying in the connecting component Γ is a *maximal projective* if it has an injective predecessor and no projective proper successor in $\mathsf{ind}\,A$. Furthermore, A is a *maximal extension* of B if there exists a maximal projective $P = eA$ such that $B = (1-e)A(1-e)$ and $A = B[M]$, where $M = \mathsf{rad}P$.

By definition, a maximal projective belongs to $\mathcal{R}_A$. In particular, by [6, 2.2], it is directed. The notions of *minimal injective* or *maximal coextension* are defined dually. If A is a tilted algebra which is the endomorphism algebra of a regular tilting module, then it has neither maximal projective, nor minimal injective (see [41]).

Proposition 3.2. *Let $A = B[M]$ be a maximal extension. Then B is a product of laura algebras not quasi-tilted of canonical type. Moreover, if every connecting component of B is standard, then so is A.*

Proof: Let $P_m \in \mathsf{ind}\,A$ be the maximal projective such that $\mathsf{rad}P_m = M$ and denote by Γ the component of $\Gamma(\mathsf{mod}\,A)$ in which P_m lies. So $P_m \in \mathcal{R}_A \cap \Gamma$. In particular, P_m is directed. Note that every proper predecessor of P_m is an indecomposable B-module.

Let us prove the first assertion. If it is false, then a connected component B' of B is quasi-tilted of canonical type. Since A is connected, at least one indecomposable summand M' of M lies in $\mathsf{ind}\,B'$. Assume first that M' is not directed. In particular, $M' \in \Gamma$ implies that $M' \notin \mathcal{L}_A \cup \mathcal{R}_A$. Therefore there is a non-sectional path $M' \rightsquigarrow P$ in $\mathsf{ind}\,A$ with P projective. If $P = P_m$, then there exists a non-sectional path $M' \rightsquigarrow M''$ with M'' an indecomposable summand of $M = \mathsf{rad}P_m$. This is impossible because P_m is directed (see [29, Thm. 1 of § 2]). So $P \neq P_m$. By maximality of P_m, the path $M' \rightsquigarrow P$ is a non-sectional path in $\mathsf{ind}\,B'$ ending at a projective. So $M' \notin \mathcal{R}_{B'}$. On the other hand, $M' \notin \mathcal{L}_A$ means that there exists a non-sectional path $I \rightsquigarrow M'$ in $\mathsf{ind}\,A$, where I is injective. By maximality of P_m, this is a non-sectional path in $\mathsf{ind}\,B'$. For the same reason, we have $\mathsf{Hom}_A(P_m, I) = 0$, so that I is injective as a B'-module. So $M' \notin \mathcal{L}_{B'} \cup \mathcal{R}_{B'}$. This is impossible because B' is quasi-tilted. Therefore M' is directed. Since B' is quasi-tilted of canonical type, the component Γ' of $\Gamma(\mathsf{mod}\,B')$ containing M' is either the unique postprojective or the unique preinjective component (see [35, Prop. 4.3]). Assume that Γ' is the unique postprojective component of $\Gamma(\mathsf{mod}\,B')$. Then $\Gamma' \subseteq \mathcal{L}_{B'} \backslash \mathcal{R}_{B'}$ (see [21, 5.2]). In particular, there exists a non-sectional path $M' \rightsquigarrow P$ in $\mathsf{ind}\,B'$ with P projective. Since P_m is maximal, this is also a non-sectional path in $\mathsf{ind}\,A$. Since P is projective and since $M' \in \Gamma$, we deduce that $P \in \Gamma$ and that the path is refinable to a non-sectional path in $\Gamma(\mathsf{mod}\,A)$ and therefore in $\Gamma(\mathsf{mod}\,B')$ because P_m is maximal. Consequently, M' lies in the postprojective component Γ' of $\Gamma(\mathsf{mod}\,B')$ and is the starting point of a non-sectional path in $\Gamma(\mathsf{mod}\,B')$ ending at a projective. This is absurd. If Γ' is the unique preinjective component of $\Gamma(\mathsf{mod}\,B')$, then, using dual arguments, we also get a contradiction. Thus, B' cannot be quasi-tilted of canonical type.

Now, we assume that every connected component of B is standard, and prove that A is standard. Later, in 5.7, we shall see that if A is tilted then all its connecting components are standard. So we assume that A is not tilted. Let Γ be the connecting component of $\Gamma(\mathsf{mod}\,A)$ and Γ' be the disjoint union of the connecting components of the Auslander-Reiten quivers of the components of B. We compare Γ and Γ'. More precisely, let $\mathcal{X}$ be the full subquiver of Γ with vertices those modules which are not successors of P_m. So $\mathcal{X}$ is a full subquiver of $\Gamma(\mathsf{mod}\,B)$ stable under predecessors in $\Gamma(\mathsf{mod}\,B)$, and it contains $\Gamma \backslash \mathcal{R}_A$. We claim that $\mathcal{X}$ is contained in Γ'. For this purpose, we prove a series of assertions.

(a) The left supports of A and B coincide. Indeed, we have $\mathcal{L}_A \cap \mathsf{ind}\,B \subseteq \mathcal{L}_B$ (see [4, 2.1]). On the other hand, if $P \in \mathsf{ind}\,B$ is a projective not lying in $\mathcal{L}_A$, then there is a non-sectional path $I \rightsquigarrow P$ in $\mathsf{ind}\,A$ with I injective. Since P_m is maximal, this is a non-sectional path in $\mathsf{ind}\,B$. For the same reason, $\mathsf{Hom}_A(P_m, I) = 0$, so that I is injective as a B-module. So $P \notin \mathcal{L}_B$. Thus A and B have the same left support.

(b) Let $P \neq P_m$ be a projective lying in Γ. Then $P \in \Gamma'$. Indeed, if there exists a path $I \rightsquigarrow P$ in Γ with I injective, then the maximality of P_m implies that this path lies entirely in $\mathsf{ind}\,B$ and starts from an injective B-module. So $P \in \Gamma'$. If there is no such path, then $P \in \mathcal{L}_A \cap \Gamma$. So P lies in a connecting component of one of the components of the left support of A, which is also the left support of B. From [3, 5.4], we deduce that P lies in Γ'.

(c) Let $X \in \mathcal{X}$. There exists $m \geqslant 0$ such that $\tau_A^m X \in \Gamma'$. By assumption on X, we have $\tau_B X = \tau_A X$. Assume first that $\tau_A^m X = P$ for some $m \geqslant 0$ and some projective P. So $P \neq P_m$. From (b), we get that $P \in \Gamma'$. Now assume that X is left stable and non-periodic. If $X \in \mathcal{R}_A$, there exists $l \geqslant 0$ such that $\tau_A^l X$ is Ext-projective in $\mathcal{R}_A$. Since X is left stable, we deduce that $\tau_A^{l+1} X \in \Gamma \backslash \mathcal{R}_A$. So we can assume that $X \in \Gamma \backslash \mathcal{R}_A$. Since A is laura, there exists m such that $\tau_A^m X \in \Gamma \cap \mathcal{L}_A$. So $\tau_A^m X$ lies in one of the connecting components of the left support of A. So $\tau_A^m X \in \Gamma'$ because the left supports of A and B are equal. Finally, assume that X is periodic.

Then there exists a projective module $P \in \Gamma$, a periodic direct summand Y of $\mathsf{rad}P$, and a path $Y \rightsquigarrow X$ in $\Gamma \backslash \mathcal{R}_A$, and therefore in $\Gamma(\mathsf{mod}\, B)$. Since Y is periodic, then $P \neq P_m$ (otherwise P_m would be a proper successor of itself). Since $P \in \Gamma'$, we have $Y \in \Gamma'$ and therefore $X \in \Gamma'$.

(d) $\mathcal{X}$ is contained in Γ'. Indeed, we already know that $\mathcal{X}$ is a full subquiver of $\Gamma(\mathsf{mod}\, B)$. Also, we proved that for every $X \in \mathcal{X}$, there exists $m \geqslant 0$ such that $\tau_A^m X = \tau_B^m X \in \Gamma'$. So $\mathcal{X}$ is contained in Γ'.

We now show that Γ is standard. By hypothesis, there exists a well-behaved functor $\varphi\colon k(\Gamma') \to \mathsf{ind}\,\Gamma'$. Since $\mathcal{X}$ is a full subquiver of Γ' stable under predecessors in $\Gamma(\mathsf{mod}\, B)$, there exists a well-behaved functor $\psi\colon k(\mathcal{Y}) \to \mathsf{ind}\,\Gamma$ where $\mathcal{Y}$ is a full subquiver of Γ such that:

1. $\mathcal{Y}$ contains $\mathcal{X}$.
2. $\mathcal{Y}$ is stable under predecessors in $\Gamma(\mathsf{mod}\, A)$.
3. ψ and φ coincide on $\mathcal{X}$.
4. $\mathcal{Y}$ is maximal for these properties.

We wish to show that $\mathcal{Y} = \Gamma$. Assume that $\mathcal{Y} \neq \Gamma$. Since $\mathcal{Y}$ contains $\mathcal{X}$, it contains $\Gamma \backslash \mathcal{R}_A$, so there exists a source X in $\Gamma \backslash \mathcal{Y}$. If X is projective, then $X = P_m$. So ψ is defined on every indecomposable summand Y of $\mathsf{rad}P_m$. We set $\psi(X) = P_m$. Let $\alpha_1 : X_1 \to P_m, \ldots, \alpha_t : X_t \to P_m$ be the arrows ending at X. Then $X_1 \oplus \ldots \oplus X_t = \mathsf{rad}P_m$, and we let $\psi(\alpha_i)$ be the inclusion $X_i \hookrightarrow P_m$. If X is not projective, then the mesh ending at X has the following shape:

$$\begin{array}{ccccc} & \overset{u_1}{\nearrow} & X_1 & \overset{v_1}{\searrow} & \\ \tau_A X & & \vdots & & X \\ & \underset{u_n}{\searrow} & X_n & \underset{v_n}{\nearrow} & \end{array} \quad . \qquad (\star)$$

Since X is a source of $\Gamma \backslash \mathcal{Y}$, then ψ is already defined on the full subquiver of the mesh consisting of all vertices except X. In particular, the following map is right minimal almost split:

$$\begin{bmatrix}\psi(u_1) & \ldots & \psi(u_n)\end{bmatrix}^t : \tau_A X \to X_1 \oplus \ldots \oplus X_n \ .$$

We let $\psi(X) = X$, and $\begin{bmatrix}\psi(v_1) & \ldots & \psi(v_n)\end{bmatrix} : X_1 \oplus \ldots \oplus X_n \to X$ be the cokernel of the above map. This construction is inspired from [14, 3.1, Ex. b]. Clearly, this construction contradicts the maximality of $\mathcal{Y}$. So $\mathcal{Y} = \Gamma$ and there exists a well-behaved functor $\psi\colon k(\Gamma) \to \mathsf{ind}\,\Gamma$ which is the identity on objects. The arguments in the proof of [14, 5.1] show that this is an isomorphism. So Γ is standard. ■

Corollary 3.3. *Let A be a weakly shod algebra not quasi-tilted of canonical type, then A is standard.*

Proof: By [6, 3.3], there exists a sequence of full convex subcategories

$$C = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_m = A$$

with C tilted and, for each $i \geqslant 0$, the algebra A_{i+1} is a maximal extension of A_i. The result follows from 3.2 and induction because C is standard (see 5.7 below). ■

The preceding result motivates the following definition, inspired from [6, 2.3].

Definition 3.4. Let A be a laura algebra. We say that A admits a *maximal filtration* if there exists a sequence

$$C = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_m = A \qquad (f)$$

of full convex subcategories with C a product of representation-finite algebras and, for each $i \geqslant 0$, the algebra A_{i+1} is a maximal extension, or a maximal coextension, of A_i.

Corollary 3.5. *Let A be a laura algebra admitting a maximal filtration (f):*

(a) If C is a product of standard representation-finite algebras, then A is standard.

(b) If the Auslander-Reiten quiver of every connected component of C is simply connected, then A is standard.

(c) If $\mathsf{HH}^1(A) = 0$, then A is standard.

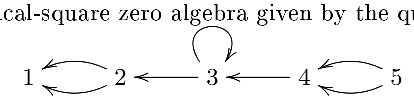
Proof: Statement (a) follows directly from 3.2.
(b) This follows from 3.2 and the fact that if a representation-finite connected algebra C has vanishing first Hochschild cohomology group $\mathsf{HH}^1(C)$, or equivalently, if its Auslander-Reiten quiver is simply connected, then C is standard [15, 4.2].
(c) We use induction on the length m of a maximal filtration. If $m = 0$, then A is representation-finite and the result follows from [15, 4.2]. Now assume that $m \geqslant 1$ and that the statement holds for algebras admitting maximal filtrations of length less than m. Without loss of generality, we may assume that $A = A_{m-1}[M]$ is a maximal extension. We claim that $\mathsf{Ext}^1_{A_{m-1}}(M, M) = 0$. Indeed, if this is not the case, then there exists

an indecomposable summand N of M such that $\mathsf{Ext}^1_{A_{m-1}}(M,N) \neq 0$. Write $M \simeq N \oplus N'$ and let P be the indecomposable projective such that $M = \mathsf{rad}P$. Then N' is a submodule of P and $L = P/N'$ is indecomposable. By [28, III.2.2, (a)] we have $\mathsf{id}\, L \geqslant 2$. But this contradicts the fact that $L \in \mathcal{R}_A$ because it is a successor of the maximal projective P. So $\mathsf{Ext}^1_{A_{m-1}}(M,M) = 0$. Applying [27, 5.3], the exact sequence

$$\mathsf{HH}^1(A) \to \mathsf{HH}^1(A_{m-1}) \to \mathsf{Ext}^1_{A_{m-1}}(M,M)\ .$$

yields $\mathsf{HH}^1(A_{m-1}) = 0$. By the induction hypothesis, A_{m-1} is standard. By 3.2, so is A. ■

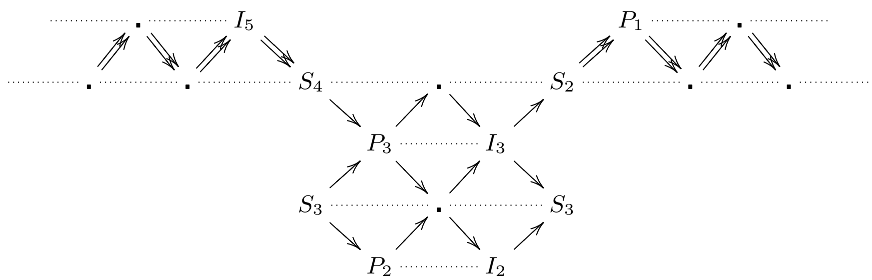
Examples 3.6. (*a*) Let A be the radical-square zero algebra given by the quiver



This is a laura algebra (see [3, 2.3]). Here and in the sequel, we denote by P_x, I_x and S_x the indecomposable projective, the indecomposable injective, and the simple corresponding to the vertex x, respectively. It is easily seen that P_5 is maximal projective and I_1 is minimal injective. Letting C be the representation-finite full convex subcategory with objects $\{2,3,4\}$ we see that

$$C \subsetneq [S_2 \oplus S_2]C \subsetneq A$$

is a maximal filtration. Since C is standard, so is A. The connecting component is drawn below:



where the two copies of S_3 are identified.

(*b*) Let B, C be products of standard laura algebras, and A an articulation of B, C (in the sense of [24]). Then A is laura not quasi-tilted of canonical type (see *loc. cit.*). Using the description of the connecting component of A (see [24, 3.9]) it is easy to check that A is standard.

The section motivates the following questions.

Problem 1. Which laura algebras admit maximal filtrations?

Problem 2. Assume that A is a laura algebra which does not admit a maximal filtration. If $\mathsf{HH}^1(A) = 0$, do we have that A is standard?

4 Tilting modules of the first kind with respect to covering functors

For tilting theory, we refer to [9]. Let B be a product of tilted algebras and n be the rank of its Grothendieck group. In [34, Cor. 4.5], it is proved that tilting modules are of the first kind with respect to any Galois covering of B. More precisely, let $F\colon \widetilde{B} \to B$ be a Galois covering with group G, where $\widetilde{B}$ is locally bounded. Denote by $\mathcal{T}$ the class of complexes $T \in \mathcal{D}^b(\mathsf{mod}\, B)$ such that:

1. T is multiplicity-free and has n indecomposable summands.
2. $\mathcal{D}^b(\mathsf{mod}\, B)(T, T[i]) = 0$ for every $i \geqslant 1$ (so T is a *silting complex* in the sense of [30]).
3. T generates the triangulated category $\mathcal{D}^b(\mathsf{mod}\, B)$.

Any multiplicity-free tilting module lies in $\mathcal{T}$. It was proved in [34, § 4] that for any $T \in \mathcal{T}$ and for any indecomposable summand X of T, there exists $\widetilde{X} \in \mathcal{D}^b(\mathsf{mod}\, \widetilde{B})$ such that:

1. $F_\lambda \widetilde{X} \simeq X$.
2. ${}^g\widetilde{X} \not\simeq {}^h\widetilde{X}$ for $g \neq h$.
3. If $Y \in \mathcal{D}^b(\mathsf{mod}\, \widetilde{B})$ is such that $F_\lambda Y \simeq X$, then $Y \simeq {}^g\widetilde{X}$ for some $g \in G$.

Given $T \in \mathcal{T}$ and an indecomposable summand X of T, we fix $\widetilde{X} \in \mathcal{D}^b(\mathsf{mod}\, \widetilde{B})$ arbitrarily such that $F_\lambda \widetilde{X} \simeq X$.

For later reference, we recall some facts on the objects $\widetilde{X}$ when T is a tilting module. The following result was proved in [34, Lem. 3.7].

Lemma 4.1. *Let $F\colon \widetilde{B} \to B$ be a Galois covering with group G. Let $T \in \mathsf{mod}\, B$ be a multiplicity-free tilting module. Let $T = T_1 \oplus \ldots \oplus T_n$ be an indecomposable decomposition. For every i, there exists $\widetilde{T}_i \in \mathsf{ind}\, \widetilde{B}$ such that $F_\lambda \widetilde{T}_i = T_i$. Moreover:*

(a) ${}^g\widetilde{T}_i \not\simeq {}^h\widetilde{T}_j$ *for* $(g,i) \neq (h,j)$.

(b) $\mathsf{pd}\, \widetilde{T}_i \leqslant 1$ *for every* i.

(c) $\mathsf{Ext}^1_{\widetilde{B}}({}^g\widetilde{T}_i,\ {}^h\widetilde{T}_j) = 0$ *for every* g,h,i,j.

(d) For every indecomposable projective $\widetilde{B}$-module P, there exists an exact sequence $0 \to P \to T^{(1)} \to T^{(2)} \to 0$ with $T^{(1)}, T^{(2)}$ in $\mathsf{add}\,\{\,{}^g\widetilde{T}_i \mid g \in G,\ i \in \{1,\ldots,n\}\}$.

■

We shall use similar facts about covering functors which need not be Galois. Therefore we prove the following result.

Proposition 4.2. *Let $F\colon \widetilde{B} \to B$ be a Galois covering with group G, where $\widetilde{B}$ is locally bounded. With the above setting, let $p\colon \widetilde{B} \to B$ be a covering functor such that $F(x) = p(x)$ for every $x \in \widetilde{B}_o$. Let $T \in \mathcal{T}$ and X be an indecomposable summand of T. Then:*

(a) There exists an isomorphism $p_\lambda({}^g\widetilde{X}) \xrightarrow{\sim} X$, for every $g \in G$.

(b) If $M \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$ is such that $p_\lambda M \simeq X$, then $M \simeq {}^g\widetilde{X}$ for some $g \in G$.

(c) For every $M \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$, the following maps induced by p_λ and by the isomorphisms of (a) are linear bijections:

$$\varphi_{X,M}\colon \bigoplus_{g\in G} \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})({}^g\widetilde{X}, M) \xrightarrow{\sim} \mathcal{D}^b(\mathsf{mod}\,B)(X, p_\lambda M)\,,$$
$$\text{and } \psi_{X,M}\colon \bigoplus_{g\in G} \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})(M, {}^g\widetilde{X}) \xrightarrow{\sim} \mathcal{D}^b(\mathsf{mod}\,B)(p_\lambda M, X)\,.$$

In order to prove the proposition, we need the following lemma. In case p a Galois covering, the lemma was proved in [34, Lems. 4.2, 4.3] (see also [33, Lems. 3.2, 3.3]). For simplicity, we write $\mathsf{Hom}(X,Y)$ for the space of morphisms in the derived category.

Lemma 4.3. *Let $T, T' \in \mathcal{T}$ be such that 4.2 holds true for T and for T'. Consider a triangle in $\mathcal{D}^b(\mathsf{mod}\,B)$:*

$$X \to X'_1 \oplus \ldots \oplus X'_t \to Y \to X[1]\ , \qquad (\Delta)$$

where $X \in \mathsf{add}\,T$ and $X'_1,\ldots,X'_t$ are indecomposable summands of T'. Assume that $\mathsf{Hom}(Y, X'_i[1]) = 0$ for all i (we do not assume that $Y \in \mathsf{add}\,T$ or $Y \in \mathsf{add}\,T'$). Then for every $g \in G$, there exist $\widetilde{Y} \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$ and $g_1,\ldots,g_t \in G$ such that the triangle Δ is isomorphic to the image under p_λ of a triangle in $\mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$ as follows:

$${}^g\widetilde{X} \to {}^{g_1}\widetilde{X}'_1 \oplus \ldots \oplus {}^{g_t}\widetilde{X}'_t \to \widetilde{Y} \to {}^g\widetilde{X}[1]\ .$$

Dually, consider a triangle in $\mathcal{D}^b(\mathsf{mod}\,B)$:

$$Y \to X'_1 \oplus \ldots \oplus X'_t \to X \to Y[1]\ , \qquad (\Delta')$$

where $X \in \mathsf{add}\,T$ and $X'_1,\ldots,X'_t$ are indecomposable summands of T'. Assume that $\mathsf{Hom}(X'_i, Y[1]) = 0$ for all i. Then for every $g \in G$, there exist $\widetilde{Y} \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$ and $g_1,\ldots,g_t \in G$ such that the triangle Δ' is isomorphic to the image under p_λ of a triangle in $\mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$ as follows:

$$\widetilde{Y} \to {}^{g_1}\widetilde{X}'_1 \oplus \ldots \oplus {}^{g_t}X'_t \to {}^g\widetilde{X} \to \widetilde{Y}[1]\ .$$

Proof: The proofs of [34, Lems. 4.2, 4.3] use the following key property of a Galois covering $F\colon \widetilde{B} \to B$ with group G. Given $L, M \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$, the two following linear maps induced by F_λ are bijective:

$$\bigoplus_{g\in G} \mathsf{Hom}({}^gL, M) \xrightarrow{\sim} \mathsf{Hom}(F_\lambda L, F_\lambda M) \text{ and } \bigoplus_{g\in G} \mathsf{Hom}(L, {}^gM) \xrightarrow{\sim} \mathsf{Hom}(F_\lambda L, F_\lambda M)\ .$$

Of course, these bijections no longer exist for a covering functor which is not Galois. However, using our hypothesis that 4.2 holds true for T and for T', it is easy to check that the proofs of [34, Lems. 4.2, 4.3] still work in the present case. Whence the lemma. ■

Proof of 4.2: We proceed in several steps.

Step 1: If $T = B$, then 4.2 holds true. The following facts are direct consequences of the definition of a covering functor (see also [14, 3.2]):

1. $Y \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$ is a projective module if and only if $p_\lambda Y$ is a projective module.
2. $p_\lambda\left(\widetilde{B}(-,x)\right) \simeq F_\lambda\left(\widetilde{B}(-,x)\right) \simeq B(-,F(x)) = B(-,p(x))$ for every $x \in \widetilde{B}_o$.
3. ${}^g\widetilde{B}(-,x) = \widetilde{B}(-,gx)$ for every $x \in \widetilde{B}_o$ and every $g \in G$.

Therefore 4.2 holds true for $T = B$.

Given an object X in a triangulated category, we write $\langle X\rangle$ for the smallest full subcategory containing X which is stable under sums, summands and shifts (in both directions).

Step 2: If $T, T' \in \mathcal{T}$ are such that $T' \in \langle T\rangle$, then 4.2 holds true for T if and only if it does for T'. This follows directly from the compatibility of p_λ with the shift.

For the next step, we consider the following situation. Assume that $T, T' \in \mathcal{T}$ are such that:

1. $T = M \oplus \overline{T}$, where M is indecomposable.
2. $T' = M' \oplus \overline{T}$, where M' is indecomoposable.
3. There exists a non-split triangle $\Delta:\ M \xrightarrow{u} E \xrightarrow{v} M' \to M[1]$ where u is a left minimal $\mathsf{add}\,\overline{T}$-approximation and v is a right minimal $\mathsf{add}\,\overline{T}$-approximation.

Step 3: If $T, T' \in \mathcal{T}$ are as above, then 4.2 holds true for T if and only if it does for T'. We prove that the condition is necessary. Clearly it suffices to prove that the assertions (a), (b), and (c) of 4.2 are true for M'. For simplicity, we identify $p_\lambda({}^g\widetilde{X})$ and X via the isomorphism used to define $\varphi_{X,-}$ and $\psi_{X,-}$ for every indecomposable summand X of T and $g \in G$.

Fix an indecomposable decomposition $E = \bigoplus_{i=1}^{t} E_i$. Recall from [34, Lem. 4.4] that Δ is isomorphic to the image under F_λ of a triangle $\widetilde{\Delta}$ in $\mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$:

$$\widetilde{M} \xrightarrow{\widetilde{u}} \bigoplus_{i=1}^{t} {}^{g_i}\widetilde{E}_i \xrightarrow{\widetilde{v}} {}^{g_0}\widetilde{M}' \to \widetilde{M}[1]\ , \qquad (\widetilde{\Delta})$$

for some $g_0, g_1, \ldots, g_t \in G$. Moreover, $\widetilde{u}$ is a left minimal $\mathsf{add}\,\mathcal{X}$-approximation and $\widetilde{v}$ is a right minimal $\mathsf{add}\,\mathcal{X}'$-approximation, where $\mathcal{X}$ and $\mathcal{X}'$ are the following full subcategories of $\mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$:

- $\mathcal{X} = \{\,{}^g\widetilde{X} \mid g \in G,\ X$ an indecomposable summand of T and ${}^g\widetilde{X} \not\simeq \widetilde{M}\}$.
- $\mathcal{X}' = \{\,{}^g\widetilde{X} \mid g \in G,\ X$ an indecomposable summand of T' and ${}^g\widetilde{X} \not\simeq \widetilde{M}'\}$.

Fix $g \in G$. Since 4.2 holds for T, we may apply 4.3 to construct a triangle $\widetilde{\Delta}':\ {}^g\widetilde{M} \xrightarrow{\widetilde{u}'} \bigoplus_{i=1}^{t} {}^{g_i'}\widetilde{E}_i \xrightarrow{\widetilde{v}'} Z_g \to {}^g\widetilde{M}[1]$ whose image under p_λ is isomorphic to Δ. In particular, $p_\lambda(Z_g) \simeq M'$. For simplicity, we assume that Δ is equal to the image of $\widetilde{\Delta}$ under p_λ, and we set $\widetilde{E}' = \bigoplus_{i=1}^{t} {}^{g_i'}\widetilde{E}_i$. Let us prove that $Z_g \simeq {}^{gg_0}\widetilde{M}'$. It suffices to prove that $\widetilde{\Delta}'$ and ${}^g\widetilde{\Delta}$ are isomorphic. For this purpose, we only need to prove that $\widetilde{u}'$ is a left minimal $\mathsf{add}\ {}^g\mathcal{X}$-approximation. Let $f\colon {}^g\widetilde{M} \to {}^{g'}\widetilde{Y}$, where Y is an indecomposable summand of T such that ${}^{g'}\widetilde{Y} \in {}^g\mathcal{X}$. Since $\varphi_{M,\widetilde{M}}$ is bijective and since $\mathsf{End}(M) = k$, we have $Y \in \mathsf{add}\,\overline{T}$. So we have a factorisation by $u = p_\lambda(\widetilde{u}')$:

$$\begin{array}{ccc} M & \xrightarrow{u} & E \\ & {\scriptstyle p_\lambda(f)}\searrow & \downarrow{\scriptstyle f'} \\ & & Y \end{array} .$$

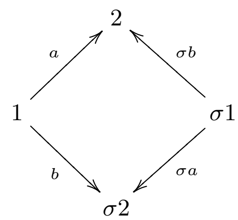
Since $\psi_{Y,\widetilde{E}_i}$ is bijective for every i, we have $f' = \sum_{h\in G} p_\lambda(f_h')$, where $(f_h')_h \in \bigoplus_{h\in G} \mathsf{Hom}(\widetilde{E}, {}^h\widetilde{Y})$. So $p_\lambda(f - f_{g'}'\widetilde{u}') - \sum_{h\neq g'} p_\lambda(f_h'\widetilde{u}') = 0$. Using 4.2, we get $f = f_{g'}'\widetilde{u}'$. Hence $\widetilde{u}'$ is a left $\mathsf{add}\ {}^g\mathcal{X}$-approximation. On the other hand, $\widetilde{u}'$ is left minimal because $u = p_\lambda(\widetilde{u}')$ is left minimal and p_λ is exact. As explained above, these facts imply that $Z_g \simeq {}^{gg_0}\widetilde{M}'$. So $p_\lambda({}^g\widetilde{M}') \simeq M'$, for every $g \in G$.

Let $Y \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$. Using the triangles ${}^g\widetilde{\Delta}$ ($g \in G$) and using that 4.2 holds true for T, the maps $\varphi_{M',Y}$ and $\psi_{M',Y}$ are bijective (recall that Hom-functors are cohomological).

Finally, if $Y \in \mathcal{D}^b(\mathsf{mod}\,\widetilde{B})$, and if $f\colon p_\lambda Y \to M'$ is an isomorphism, then $f = \sum_{g\in G} p_\lambda(f_g)$ with $(f_g)_g \in \bigoplus_{g\in G} \mathsf{Hom}(Y, {}^g\widetilde{M}')$. Since $p_\lambda Y$ and M' are indecomposable, there exists $g_1 \in G$ such that $p_\lambda(f_{g_1})$ is an isomorphism. Since p_λ is exact, we deduce that $f_{g_1}\colon Y \to {}^{g_1}\widetilde{M}'$ is an isomorphism. This finishes the proof of the assertion: 4.2 holds true for T' if it holds true for T. The converse implication is proved using similar arguments.

Step 4: If $T \in \mathcal{T}$, then 4.2 holds true. This follows directly from the three preceding steps, and from [34, Prop. 3.6]. ■

Example 4.4. Let $B = kQ$ be the path algebra of the Kronecker quiver $1 \overset{a}{\underset{b}{\rightrightarrows}} 2$. There is a Galois covering $F\colon \widetilde{B} \to B$ with group $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$, where $\widetilde{B} = k\widetilde{Q}$ is the path algebra of the following quiver:



and where F is the functor such that $F(\sigma^i \alpha) = \alpha$ for every arrow α and every $i \in \{0,1\}$. On the other hand, there is a covering functor $p\colon B' \to B$ such that $p(a) = p(\sigma b) = b$, $p(a) = a$ and $p(\sigma a) = a + b$. The module $T = e_2 B \oplus \tau_B^{-1}(e_1 B)$ is tilting. One checks easily that $F_\lambda(e_2 \widetilde{B}) = e_2 B$, $F_\lambda(\tau_{\widetilde{B}}^{-1}(e_1 \widetilde{B})) = \tau_B^{-1}(e_1 B)$ and that $p_\lambda(e_2 \widetilde{B}) \simeq e_2 B$, $p_\lambda(\tau_{\widetilde{B}}^{-1}(e_1 \widetilde{B})) \simeq \tau_B^{-1}(e_1 B)$.

5 Coverings of left sections

Let A be a basic finite dimensional k-algebra, Γ a component of $\Gamma(\mathsf{mod}\, A)$, $\pi\colon \widetilde{\Gamma} \to \Gamma$ the universal cover of translation quivers and $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\, \Gamma$ a well-behaved covering functor. A *left section* (see [1, 2.1]) in Γ is a full subquiver Σ such that: Σ is acyclic; it is convex in Γ; and, for any $x \in \Gamma$, predecessor in Γ of some $y \in \Sigma$, there exists a unique $n \geqslant 0$ such that $\tau^{-n} x \in \Sigma$. Assume that Σ is a left section in Γ and let $B = A/\mathsf{Ann}\,\Sigma$. In this section, we construct a covering functor $F\colon \widetilde{B} \to B$ associated to p and we construct a functor $\varphi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\, \widetilde{B}$. Both F and φ are essential in the proofs of Theorems A and B.

By [1, Thm. A], the algebra B is a full convex subcategory of A and a product of tilted algebras and the components of Σ form complete slices of the components of B. So we assume that Q is a finite quiver with no oriented cycle and $T \in \mathsf{mod}\, kQ$ is a tilting module such that $B = \mathsf{End}_{kQ}(T)$. Any module $X \in \mathsf{mod}\, B$ defines the Σ-module $\mathsf{Hom}_B(\Sigma, X)$ which, as a functor, assigns the vector space $\mathsf{Hom}_B(E, X)$ to the object E of Σ. By the above properties of B, the map $x \mapsto \mathsf{Hom}_{kQ}(T, D(kQe_x))$ defines an isomorphism of k-categories $kQ \xrightarrow{\sim} \Sigma$. We denote by $\Gamma_{\leqslant \Sigma}$ the full subquiver of Γ generated by all the predecessors of Σ in Γ.

The covering of the left section Σ

Let $\widetilde{\Sigma}$ be the full subcategory of $k(\widetilde{\Gamma})$ whose objects are the points $x \in k(\widetilde{\Gamma})$ such that $p(x) \in \Sigma$. Therefore $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\, \Gamma$ induces a covering functor $p\colon \widetilde{\Sigma} \to \Sigma$. Note that $\widetilde{\Sigma}$ and $\widetilde{\Gamma}_{\leqslant \widetilde{\Sigma}}$ are stable under $\pi_1(\Gamma)$, as subquivers of $\widetilde{\Gamma}$. Since Σ is hereditary, so is $\widetilde{\Sigma}$. Therefore we have $\widetilde{\Sigma} = k\widetilde{Q}$ for some quiver $\widetilde{Q}$. In particular, the isomorphism $kQ \xrightarrow{\sim} \Sigma$ and the covering functor $p\colon \widetilde{\Sigma} \to \Sigma$ defines a covering functor $q\colon k\widetilde{Q} \to kQ$.

The covering functor of B

Since π and p coincide on vertices, π induces a Galois covering of quivers $\pi\colon \widetilde{Q} \to Q$ with group $\pi_1(\Gamma)$. We write $\pi\colon k\widetilde{Q} \to kQ$ for the induced Galois covering with group $\pi_1(\Gamma)$. Note that $\widetilde{Q}$ is a disjoint union of copies of the universal cover of Q because $\widetilde{\Gamma}$ is simply connected. Also, remark that thanks to the Galois covering $\pi\colon \widetilde{Q} \to Q$ there is an action of $\pi_1(\Gamma)$ on $\mathsf{mod}\, k\widetilde{Q}$. Let $T = T_1 \oplus \ldots \oplus T_n$ be an indecomposable decomposition and $\widetilde{B}$ be the full subcategory of $\mathsf{mod}\, k\widetilde{Q}$ with objects the modules ${}^g\widetilde{T}_i$ (with $i \in \{1, \ldots, n\}$, $g \in \pi_1(\Gamma)$, see 4.3).

Lemma 5.1. *The k-category $\widetilde{B}$ is locally bounded. The push-down functor $p_\lambda\colon \mathsf{mod}\, k\widetilde{Q} \to \mathsf{mod}\, kQ$ induces a covering functor:*

$$\begin{array}{cccc} F\colon & \widetilde{B} & \to & B \\ & {}^g\widetilde{T}_i & \mapsto & T_i = p_\lambda({}^g\widetilde{T}_i) \end{array} .$$

Moreover, if $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\, \Gamma$ is a Galois covering with group $\pi_1(\Gamma)$, then so is F.

Proof: We apply the results of the preceding section to the covering functor $q\colon k\widetilde{Q} \to kQ$ and the Galois covering $\pi\colon k\widetilde{Q} \to kQ$. The first assertion follows from 4.1 and 4.2, and the second follows from 4.2. Finally, the last assertion was proved in [33, Lem. 2.2]. ∎

We also have a Galois covering $\widetilde{B} \to B$ induced by the push-down $\pi_\lambda \colon \mathsf{mod}\, k\widetilde{Q} \to \mathsf{mod}\, kQ$ (see [33, Lem. 2.2]). In particular, the covering functor $F\colon \widetilde{B} \to B$ and the Galois covering $\widetilde{B} \to B$ coincide on objects. Therefore we may apply the results of the preceding section to F.

In the sequel, we write $\widetilde{T}$ for the $k\widetilde{Q}$-module $\bigoplus\{{}^g\widetilde{T}_i \mid i \in \{1,\ldots,n\}, g \in \pi_1(\Gamma)\}$. Although $\widetilde{T}$ is not necessarily finite dimensional, it follows from 4.2 that it induces a well-defined functor:

$$\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T},-)\colon \mathsf{mod}\, k\widetilde{Q} \to \mathsf{mod}\, \widetilde{B}\ .$$

More precisely, if $X \in \mathsf{mod}\, k\widetilde{Q}$, then $\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T}, X)$ is the $\widetilde{B}$-module defined by ${}^g\widetilde{T}_i \mapsto \mathsf{Hom}_{k\widetilde{Q}}({}^g\widetilde{T}_i, X)$. In particular, an object x in $\widetilde{\Sigma} = k\widetilde{Q}$ defines the injective $k\widetilde{Q}$-module $D(k\widetilde{Q}(x,-))$ which gives rise to the $\widetilde{B}$-module $\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T}, D(k\widetilde{Q}(x,-)))$. Therefore every $\widetilde{B}$-module X defines a $\widetilde{\Sigma}$-module:

$$\begin{array}{rcl} \widetilde{\Sigma}^{op} & \to & \mathsf{mod}\, k \\ x & \mapsto & \mathsf{Hom}_{\widetilde{B}}(\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T}, D(k\widetilde{Q}(x,-))), X) \end{array} .$$

For reasons that will become clear later, this module is denoted by $\mathsf{Hom}_{\widetilde{B}}(\widetilde{\Sigma}, X)$. This way, we get a functor $\mathsf{Hom}_{\widetilde{B}}(\widetilde{\Sigma},-)\colon \mathsf{mod}\, \widetilde{B} \to \mathsf{mod}\, \widetilde{\Sigma}$. We need the following result for later reference.

Lemma 5.2. *The following diagram commutes up to isomorphism of functors:*

$$\begin{array}{ccccc} \mathsf{mod}\, k\widetilde{Q} & \xrightarrow{\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T},-)} & \mathsf{mod}\, \widetilde{B} & \xrightarrow{\mathsf{Hom}_{\widetilde{B}}(\widetilde{\Sigma},-)} & \mathsf{mod}\, \widetilde{\Sigma} \\ \downarrow{\scriptstyle q_\lambda} & & \downarrow{\scriptstyle F_\lambda} & & \downarrow{\scriptstyle p_\lambda} \\ \mathsf{mod}\, kQ & \xrightarrow[\mathsf{Hom}_{kQ}(T,-)]{} & \mathsf{mod}\, B & \xrightarrow[\mathsf{Hom}_{B}(\Sigma,-)]{} & \mathsf{mod}\, \Sigma \end{array} .$$

Moreover:

(a) The two top horizontal arrows are $\pi_1(\Gamma)$-equivariant.

(b) If $\theta\colon \mathsf{mod}\, kQ \to \mathsf{mod}\, \Sigma$ (or $\widetilde{\theta}\colon \mathsf{mod}\, k\widetilde{Q} \to \mathsf{mod}\, \widetilde{\Sigma}$) denotes the composition of the two bottom (or top) horizontal arrows, then it induces an equivalence from the full subcategory of injective kQ-modules (or injective $k\widetilde{Q}$-modules) to the full subcategory of projective Σ-modules (or projective $\widetilde{\Sigma}$-modules, respectively).

(c) Let $\widetilde{\alpha}\colon \widetilde{I} \to \widetilde{J}$ be a surjective morphism between injective $k\widetilde{Q}$-modules. Let $\alpha\colon I \to J$ be equal to $q_\lambda(\widetilde{\alpha})$. Then F_λ maps the connecting morphism $\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T}, \widetilde{J}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\ \widetilde{\alpha})$ to the connecting morphism $\mathsf{Hom}_{kQ}(T, J) \to \mathsf{Ext}^1_{kQ}(T, \mathsf{Ker}\ \alpha)$.

Proof: The commutativity is an easy exercise on covering functors, and left to the reader.

(a) This assertion follows from a direct computation.

(b) We know from tilting theory that θ induces an equivalence (see [9, Chap. VIII Thm. 3.5]):

$$\begin{array}{rrcl} \Phi\colon & \mathsf{inj}\, kQ & \to & \mathsf{proj}\, \Sigma \\ & I & \mapsto & \mathsf{Hom}_B(\Sigma, \mathsf{Hom}_{kQ}(T, I)) \end{array} .$$

Let $I \in \mathsf{inj}\, k\widetilde{Q}$. Then $p_\lambda\widetilde{\theta}(I) = \theta q_\lambda(I)$. Moreover, q_λ maps indecomposable injective $k\widetilde{Q}$-modules to indecomposable injective kQ-modules (see 4.2). So $p_\lambda\widetilde{\theta}(I)$ is indecomposable projective, and therefore so is $\widetilde{\theta}(I)$ (see [14, 3.2]). Consequently, $\widetilde{\theta}$ induces the following functor:

$$\begin{array}{rrcl} \Psi\colon & \mathsf{inj}\, k\widetilde{Q} & \to & \mathsf{proj}\, \widetilde{\Sigma} \\ & I & \mapsto & \mathsf{Hom}_{\widetilde{B}}(\widetilde{\Sigma}, \mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T}, I)) \end{array} .$$

So we have a commutative diagram:

$$\begin{array}{ccc} \mathsf{inj}\, k\widetilde{Q} & \xrightarrow{\Psi} & \mathsf{proj}\, \widetilde{\Sigma} \\ \downarrow{\scriptstyle q_\lambda} & & \downarrow{\scriptstyle p_\lambda} \\ \mathsf{inj}\, kQ & \xrightarrow[\Phi]{\sim} & \mathsf{proj}\, \Sigma \end{array} .$$

In this diagram, p_λ, q_λ and Φ are faithful. Hence, so is Ψ. Let $I, J \in \mathsf{inj}\, k\widetilde{Q}$ and let $f\colon \Psi(I) \to \Psi(J)$. Let $h\colon q_\lambda I \to q_\lambda J$ be such that $\Phi(h) = p_\lambda(f)$. Using 4.2, we have $h = \sum_{g\in\pi_1(\Gamma)} q_\lambda(h_g)$, where $(h_g)_g \in \bigoplus_{g\in\pi_1(\Gamma)} \mathsf{Hom}_{k\widetilde{Q}}(\widetilde{I}, {}^g\widetilde{J})$.

So $p_\lambda(f) = \sum\limits_{g\in G} p_\lambda \Psi(h_g)$. Using 4.2 again, we deduce that $f = \Psi(h_1)$. So Ψ is full. Finally, we know from the preceding section that $q_\lambda\colon \mathsf{inj}\, k\widetilde{Q} \to \mathsf{inj}\, kQ$ is dense. Also, so is $p_\lambda\colon \mathsf{proj}\,\widetilde{\Sigma} \to \mathsf{proj}\,\Sigma$ (see [14, 3.2], for instance). Since Φ is an equivalence, we deduce that Ψ is dense. Therefore Ψ is an equivalence.

(c) The push-down functors q_λ and F_λ are exact. So we have a commutative diagram up to isomorphism:

$$\begin{array}{ccc}
\mathcal{D}^b(\mathsf{mod}\, k\widetilde{Q}) & \xrightarrow{\mathbf{RHom}_{k\widetilde{Q}}(\widetilde{T},-)} & \mathcal{D}^b(\mathsf{mod}\, \widetilde{B}) \\
\downarrow{\scriptstyle q_\lambda} & & \downarrow{\scriptstyle F_\lambda} \\
\mathcal{D}^b(\mathsf{mod}\, kQ) & \xrightarrow[\mathbf{RHom}_{kQ}(T,-)]{} & \mathcal{D}^b(\mathsf{mod}\, B)
\end{array} .$$

The statement follows from this diagram. ■

We now wish to construct a functor $\varphi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\,\widetilde{B}$. We proceed in several steps:

1. Define a functor $\varphi_0\colon k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}) \to \mathsf{mod}\,\widetilde{B}$ where $k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}})$ denotes the full subcategory of $k(\widetilde{\Gamma})$ with objects the vertices in $\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$.
2. Define φ on objects, so that it coincides with φ_0 on predecessors of $\widetilde{\Sigma}$.
3. Define φ on morphisms, so that it extends φ_0.

The functor $\varphi_0\colon k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}) \to \mathsf{mod}\,\widetilde{B}$

Before constructing φ_0, we prove the following lemma. In the case of a Galois covering whose group acts freely on indecomposables, a corresponding result was proved in [25, 3.6]. We know that $p\colon \widetilde{\Sigma} \to \Sigma$ and $F\colon \widetilde{B} \to B$ are covering functors, and that the latter coincides on objects with a Galois covering $\widetilde{B} \to B$ with group $\pi_1(\Gamma)$. Finally, if $X \in \mathsf{ind}\, B$ is a summand of a tilting B-module, then $\widetilde{X} \in \mathsf{ind}\,\widetilde{B}$ is a module such that $p_\lambda(\widetilde{X}) \simeq X$ (see 4.2).

Lemma 5.3. *Let $X \in \Gamma_{\leqslant\Sigma}$ and $g_0 \in \pi_1(\Gamma)$. If $\widetilde{u}\colon \widetilde{E} \to {}^{g_0}\widetilde{X}$ is right minimal almost split, then so is $p_\lambda\widetilde{u}\colon p_\lambda\widetilde{E} \to X$. Consequently, $p_\lambda\tau_{\widetilde{B}}({}^{g_0}\widetilde{X}) \simeq \tau_B X$ if X is not projective.*

Proof: Notice that X is an indecomposable summand of some tilting B-module. So we may apply the results of 4.2. If X is projective, the assertion follows from [14, 3.2]. So we assume that X, and therefore ${}^{g_0}\widetilde{X}$, are not projective. Let $u\colon E \to X$ be right minimal almost split and $E = E_1 \oplus \ldots \oplus E_t$ an indecomposable decomposition. Since $\Gamma_{\leqslant\Sigma}$ has no oriented cycle (see [1, 2.2]), we have $\mathsf{Ext}^1_B(E, \tau_B X) = 0$. Also, the linear map $\bigoplus\limits_{g\in\pi_1(\Gamma)} \mathsf{Hom}_{\widetilde{B}}({}^{g}\widetilde{E}_i, {}^{g_0}\widetilde{X}) \to \mathsf{Hom}_B(E_i, X)$ is bijective, for every i (see 4.2). Therefore we apply 4.3 to the exact sequence $0 \to \tau_B X \to E \xrightarrow{u} X \to 0$: There exist $g_1, \ldots, g_t \in G$ and morphisms $\widetilde{u}_i\colon {}^{g_i}\widetilde{E}_i \to {}^{g_0}\widetilde{X}$ ($i \in \{1, \ldots, t\}$) fitting into a commutative diagram whose vertical arrow on the left is an isomorphism:

$$\begin{array}{ccc}
E & \xrightarrow{p_\lambda[\widetilde{u}_1,\ldots,\widetilde{u}_t]} & X \\
{\scriptstyle\sim}\downarrow & & \| \\
E & \xrightarrow[u]{} & X
\end{array} .$$

We identify u and $p_\lambda[\widetilde{u}_1, \ldots, \widetilde{u}_t]$ via this diagram. Let $i \in \{1, \ldots, n\}$. Then $\widetilde{u}_i\colon {}^{g_i}\widetilde{E}_i \to {}^{g_0}\widetilde{X}$ is not a retraction because ${}^{g_i}\widetilde{E}_i$ and ${}^{g_0}\widetilde{X}$ are non-isomorphic indecomposable modules. So $\widetilde{u}_i$ factors through $\widetilde{u}$, for every i. Applying p_λ to each factorisation shows that u factors through $p_\lambda(\widetilde{u})$. On the other hand, $p_\lambda\widetilde{u}$ is not a retraction because X is not a direct summand of E. So $p_\lambda(\widetilde{u})$ factors through u. The right minimality of u implies that the morphism u is a direct summand of $p_\lambda(\widetilde{u})$. Finally, the following equality follows from 2.4:

$$\dim \mathsf{Ker}\, u = \dim \tau_B X = \dim \tau_B p_\lambda\, {}^{g_0}\widetilde{X} = \dim p_\lambda \tau_{\widetilde{B}}\, {}^{g_0}\widetilde{X} = \dim \mathsf{Ker}\, p_\lambda(\widetilde{u}) \ .$$

So $p_\lambda(\widetilde{u})$ and u are isomorphic, and $p_\lambda(\widetilde{u})$ is right minimal almost split. ■

Thanks to the preceding lemma, we can construct a functor $\varphi_0\colon k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}) \to \mathsf{ind}\,\widetilde{B}$.

Lemma 5.4. *There exists a full and faithful functor, $\pi_1(\Gamma)$-equivariant on vertices $\varphi_0\colon k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}) \to \mathsf{ind}\,\widetilde{B}$. This functor maps arrows in $\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$ to irreducible maps, and meshes to almost split sequences. Moreover, it* *commutes with the translations and it extends the canonical functor* $\widetilde{\Sigma} \to \operatorname{ind} \widetilde{B}$ *defined on the objects by* $x \mapsto \operatorname{Hom}_{k\widetilde{Q}}(\widetilde{T}, D(k\widetilde{Q}(x,-))$. *Finally, the following diagram is commutative up to isomorphism of functors:*

$$\begin{array}{ccc} k(\widetilde{\Gamma}_{\leqslant \widetilde{\Sigma}}) & \xrightarrow{\varphi_0} & \operatorname{ind} \widetilde{B} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle F_\lambda} \\ \operatorname{ind}(\Gamma_{\leqslant \Sigma}) & \hookrightarrow & \operatorname{ind} B \end{array}$$

Proof: Step 1: Clearly there is a functor $\varphi_0\colon \widetilde{\Sigma} \to \operatorname{mod} \widetilde{B}$ given by $\widetilde{x} \mapsto \operatorname{Hom}_{k\widetilde{Q}}(\widetilde{T}, D(k\widetilde{Q}(\widetilde{x},-)))$. Note that $\widetilde{\Sigma}$ (or Σ) is naturally equivalent to the full subcategory of $\operatorname{mod} k\widetilde{Q}$ (or $\operatorname{mod} kQ$, respectively), consisting of the indecomposable injective modules. Therefore 5.2 shows that this functor is full and faithful, and that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\varphi_0} & \operatorname{ind} \widetilde{B} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle F_\lambda} \\ \Sigma & \hookrightarrow & \operatorname{ind} B \end{array} .$$

Note that $\varphi_0(M)$ is indecomposable for every M because so is $F_\lambda \varphi_0(M) = p(M)$. The functor $\varphi_0\colon \widetilde{\Sigma} \to \operatorname{ind} \widetilde{B}$ is $\pi_1(\Gamma)$-equivariant: Indeed, for every $g \in \pi_1(\Gamma)$, and every $\widetilde{x} \in \widetilde{Q}_0$, we have the following equalities:

$$\varphi_0(g\widetilde{x}) = \operatorname{Hom}_{k\widetilde{Q}}(\widetilde{T}, D(k\widetilde{Q}(g\widetilde{x},-))) = \operatorname{Hom}_{k\widetilde{Q}}(\widetilde{T}, {}^{g}D(k\widetilde{Q}(\widetilde{x},-))) = {}^{g}\operatorname{Hom}_{k\widetilde{Q}}(\widetilde{T}, D(k\widetilde{Q}(\widetilde{x},-))) = {}^{g}\varphi_0(\widetilde{x}) \ .$$

Step 2: If $M \in k(\widetilde{\Gamma}_{\leqslant \widetilde{\Sigma}})$, there exists a unique $n \in \mathbb{N}$ such that $\tau^{-n}M \in \widetilde{\Sigma}$. Let $\varphi_0(M)$ be the $\widetilde{B}$-module:

$$\varphi_0(M) = \tau^{n}_{\widetilde{B}} \varphi_0(\tau^{-n}M) \ .$$

It follows from 5.3 that $F_\lambda \varphi_0(M) = p(M)$. Also $\varphi_0({}^{g}M) = {}^{g}\varphi_0(M)$ for every $g \in \pi_1(\Gamma)$ and for every vertex M because τ commutes with the action of $\pi_1(\Gamma)$.

Step 3: In order to define φ_0 on morphisms, we construct inductively a sequence of $\pi_1(\Gamma)$-invariant left sections $\widetilde{\Sigma}_i$ of $\widetilde{\Gamma}$ such that $\widetilde{\Sigma}_0 = \widetilde{\Sigma}$, such that $\widetilde{\Sigma}_{i+1} \backslash \widetilde{\Sigma}_i$ consists of the $\pi_1(\Gamma)$-orbit of a vertex, and such that, if $\bigcup\limits_{t=1}^{i} \widetilde{\Sigma}_t$ denotes the full subcategory of the path category $k\widetilde{\Gamma}$ whose vertices are given by those of $\widetilde{\Sigma}_0, \dots, \widetilde{\Sigma}_i$, then $k\widetilde{\Gamma}_{\leqslant \widetilde{\Sigma}} = \bigcup\limits_{i \geqslant 0} \widetilde{\Sigma}_i$. Each inductive step defines a functor $\varphi_0\colon \bigcup\limits_{t=1}^{i} \widetilde{\Sigma}_t \to \operatorname{ind} \widetilde{B}$ which maps arrows to irreducible maps and extends the constructions of the two preceding steps. This functor makes the following diagram commute:

$$\begin{array}{ccc} \bigcup\limits_{t=1}^{i} \widetilde{\Sigma}_t & \xrightarrow{\varphi_0} & \operatorname{ind} \widetilde{B} \\ \downarrow & & \downarrow{\scriptstyle F_\lambda} \\ \operatorname{ind}(\Gamma_{\leqslant \Sigma}) & \hookrightarrow & \operatorname{ind} B \end{array} ,$$

where the vertical arrow on the left is naturally induced by p. Assume that $\varphi_0\colon \bigcup\limits_{t=1}^{i} \widetilde{\Sigma}_t \to \operatorname{ind} \widetilde{B}$ has been defined for some $i \geqslant 0$. Since $\widetilde{\Sigma}_i$ is acyclic, it has a sink. Assume first that all sinks are projective. Let P be a projective sink, and let $\widetilde{\Sigma}_{i+1}$ be equal to $\widetilde{\Sigma}_i \backslash \{ {}^{g}P \mid g \in \pi_1(\Gamma)\}$; then, $\widetilde{\Sigma}_{i+1}$ is a left section, and there is a unique $\varphi_0\colon \bigcup\limits_{t=0}^{i+1} \widetilde{\Sigma}_t \to \operatorname{ind} \widetilde{B}$ satisfying the required conditions. Assume that there is a non projective sink M in $\widetilde{\Sigma}_i$. Then there exists a mesh in $\widetilde{\Gamma}$:

$$\begin{array}{ccccc} & & N_1 & & \\ & {\scriptstyle v_1}\nearrow & & \searrow{\scriptstyle u_1} & \\ \tau M & & \vdots & & M \\ & {\scriptstyle v_s}\searrow & & \nearrow{\scriptstyle u_s} & \\ & & N_s & & \end{array} ,$$

and $M, N_1, \dots, N_s \in \widetilde{\Sigma}_i$ because $M \in \widetilde{\Sigma}_i$ is a sink. In particular, $\varphi_0(u_j)$ is defined, and $F_\lambda\, \varphi_0(u_j) = p(u_j)_{|B}$ for every i. For simplicity, we write $\varphi_0(u)$ for $\begin{bmatrix}\varphi_0(u_1) & \dots & \varphi_0(u_s)\end{bmatrix}$ and $p(u)$ for $\begin{bmatrix}p(u_1) & \dots & p(u_s)\end{bmatrix}$. Then $\varphi_0(u)$ is right minimal almost split in $\operatorname{mod} \widetilde{B}$: Indeed, there exists a right minimal almost split morphism $L \xrightarrow{w} \varphi_0(M)$. Since $\varphi_0(u)\colon \bigoplus\limits_{j} \varphi_0(N_j) \to \varphi_0(M)$ is not a retraction (because each $\varphi_0(u_j)$ is an irreducible morphism, by the induction hypothesis), there exists a morphism $w'\colon \bigoplus_j \varphi_0(N_j) \to L$ such that $\varphi_0(u) = ww'$; applying F_λ, we have $F_\lambda\varphi_0(u) = F_\lambda(w)F_\lambda(w')$; but now $F_\lambda\varphi_0(u) = p(u)_{|B}$ is right minimal almost split by construction, and so is $F_\lambda(w)$ (see 5.3); hence, $F_\lambda(w')$ is an isomorphism and therefore so is w' because F_λ is exact. We let $\begin{bmatrix}\varphi_0(v_1) & \dots & \varphi_0(v_s)\end{bmatrix}^t : \varphi_0(\tau M) \to \bigoplus_{j=1}^{s} \varphi_0(N_j)$ be the kernel of $\varphi_0(u)$. For simplicity, we write $\varphi_0(v)$ for $\begin{bmatrix}\varphi_0(v_1) & \dots & \varphi_0(v_s)\end{bmatrix}^t$ and $p(v)$ for $\begin{bmatrix}p(v_1) & \dots & p(v_s)\end{bmatrix}^t$. We let $\widetilde{\Sigma}_{i+1} = \left(\widetilde{\Sigma}_i \backslash \{ {}^gM \mid g \in \pi_1(\Gamma)\}\right) \bigcup \{ {}^g\tau M \mid g \in \pi_1(\Gamma)\}$. Clearly, $\widetilde{\Sigma}_{i+1}$ is a left section. We now show that we may assume $\varphi_0(v)$ to be taken such that $F_\lambda\varphi_0(v) = p(u)_{|B}$. Indeed, the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_\lambda\varphi_0(\tau M) & \xrightarrow{F_\lambda\varphi_0(v)} & F_\lambda\varphi_0\left(\bigoplus_{j=1}^{s} N_j\right) & \xrightarrow{F_\lambda\varphi_0(u)} & F_\lambda\varphi_0(M) & \longrightarrow & 0 \\
 & & & & \| & & \| & & \\
0 & \longrightarrow & p(\tau M)_{|B} & \xrightarrow[p(v)_{|B}]{} & p\left(\bigoplus_{j=1}^{s} N_j\right)_{|B} & \xrightarrow[p(u)_{|B}]{} & p(M)_{|B} & \longrightarrow & 0
\end{array}$$

shows that there exists an isomorphism $F_\lambda\varphi_0(\tau M) \to p(\tau M)_{|B}$ which makes the left square commute. Since $F_\lambda\varphi_0(\tau M) = p(\tau M)_{|B}$ is a brick (because it belongs to $\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$), this isomorphism is the multiplication by a non-zero constant c. Hence, $p(v)_{|B} = c\,F_\lambda\varphi_0(v)$. Replacing $\varphi_0(v)$ by $c\,\varphi_0(v)$ does the trick. Thus, we have defined $\varphi_0\colon \bigcup_{t=1}^{i+1} \widetilde{\Sigma}_t \to \mathsf{ind}\,\widetilde{B}$. It is clear that the required conditions are satisfied. This induction shows that there is a functor $\varphi_0\colon k\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}} \to \mathsf{ind}\,\widetilde{B}$ mapping arrows to irreducible maps and meshes to almost split sequences, and such that the following diagram commutes:

$$\begin{array}{ccc}
k\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}} & \xrightarrow{\varphi_0} & \mathsf{ind}\,\widetilde{B} \\
\downarrow & & \downarrow F_\lambda \\
\mathsf{ind}\,\Gamma_{\leqslant\Sigma} & \hookrightarrow & \mathsf{ind}\,B
\end{array},$$

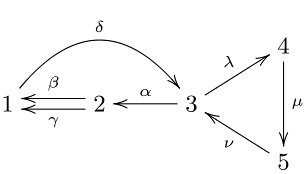
where the vertical arrow on the left is induced by p. Since F_λ is faithful, φ_0 induces a functor $\varphi_0\colon k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}) \to \mathsf{ind}\,\widetilde{B}$. It is now clear that this functor satisfies the conditions of the lemma. ■

It was shown in [1, 3.2] that the existence of a left section Σ in an Auslander-Reiten component Γ implies that $\Gamma_{\leqslant\Sigma}$ is generalised standard. We now prove that it is standard.

Corollary 5.5. *Let A be a finite dimensional k-algebra and Γ be a component of $\Gamma(\mathsf{mod}\,A)$ having a left section Σ. Then $\Gamma_{\leqslant\Sigma}$ is standard.*

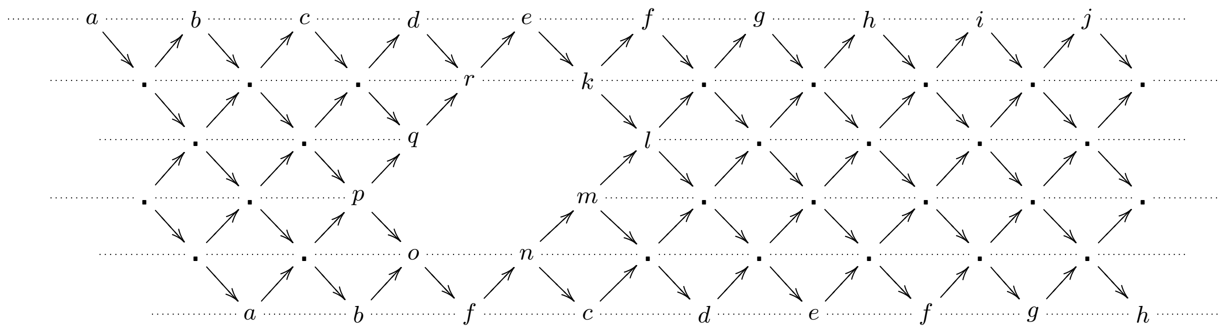
Proof: Let $B = A/\mathsf{Ann}\,\Sigma$. Then B is a product of tilted algebras and the components of Σ form complete slices for the components of B. Let Γ' be the union of the components of $\Gamma(\mathsf{mod}\,B)$ intersecting Σ. The arguments of the proof of 5.4 show that there exists a full and faithful functor $k(\Gamma'_{\leqslant\Sigma}) \to \mathsf{ind}\,\Gamma'_{\leqslant\Sigma}$ extending the identity on vertices. So $\Gamma_{\leqslant\Sigma} = \Gamma'_{\leqslant\Sigma}$ is standard. ■

Example 5.6. Let A be the cluster-tilted algebra of type $\widetilde{\mathbb{A}}$ given by the quiver

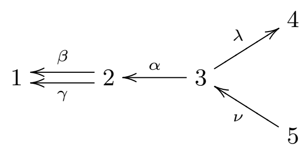


and the potential $w = \delta\beta\alpha + \nu\mu\lambda$ (or, equivalently, by the relations $\beta\alpha = 0$, $\delta\beta = 0$, $\alpha\delta = 0$, $\mu\lambda = 0$, $\nu\mu = 0$

and $\lambda\nu = 0$). Then the transjective component Γ of $\Gamma(\mathsf{mod}\,A)$ is of the form



where the vertices with the same label are identified. Then Γ admits a left section $\Sigma = \{e, r, q, p, o, f\}$ and $B = A/\mathsf{Ann}\,\Sigma$ is the algebra given by the quiver:



with the inherited relations. As we have seen, $\Gamma_{\leqslant\Sigma}$ is standard (and generalised standard) while, clearly, Γ itself is not.

The following corollary seems to be well-known. However we have been unable to find a reference.

Corollary 5.7. *Let B be a tilted algebra and Γ be a connecting component of B. Then Γ is standard.*

Proof: If B is concealed, this follows from [40, 2.4 (11) p. 80]. Assume that B is not concealed. So Γ is the unique connecting component of B. Let Σ be a complete slice in Γ. As observed in 5.5, we have a full and faithful functor $k(\Gamma_{\leqslant\Sigma}) \to \mathsf{ind}\,\Gamma$ extending the identity on vertices. A dual construction extends this functor to a full and faithful functor $k(\Gamma) \to \mathsf{ind}\,\Gamma$ extending the identity on vertices. So Γ is standard. ■

From now on, we identify $\widetilde{\Sigma}$ to a full subcategory of $\mathsf{mod}\,\widetilde{B}$ by means of φ_0.

Construction of φ on objects

We prove that for any $M \in \widetilde{\Gamma}$, there exists $\varphi(M) \in \mathsf{mod}\,\widetilde{B}$ whose image under $F_\lambda \colon \mathsf{mod}\,\widetilde{B} \to \mathsf{mod}\,B$ coincides with $p(M)_{|B}$, in such a way that $\varphi({}^gM) = {}^g\varphi(M)$, for every $g \in \pi_1(\Gamma)$. We define $\mathcal{L}_\Sigma$ to be the full subcategory of $\mathsf{ind}\,B$ which consists of the predecessors of the complete slice Σ. Also a *minimal* $\mathsf{add}\,\mathcal{L}_\Sigma$*-presentation* of a module R is a sequence of morphisms:

$$E_1 \to E_2 \to R$$

where the morphism on the right is a minimal $\mathsf{add}\,\mathcal{L}_\Sigma$-approximation and the one on the left is a minimal $\mathsf{add}\,\mathcal{L}_\Sigma$-approximation of its kernel. Before constructing $\varphi(M)$, we prove two lemmata.

Lemma 5.8. *Let $R \in \mathsf{mod}\,B$ be a module with no direct summand in $\mathcal{L}_\Sigma$. There exists an exact sequence in $\mathsf{mod}\,B$, which is a minimal $\mathsf{add}\,\mathcal{L}_\Sigma$-presentation:*

$$0 \to E_1 \to E_2 \to R \to 0 \qquad (\star)$$

with $E_1, E_2 \in \mathsf{add}\,\Sigma$. Moreover, the functor $\mathsf{Hom}_{kQ}(T, -)$ induces a bijection between the class of all such exact sequences, and the class of minimal injective copresentations:

$$0 \to \mathsf{Tor}_1^B(R,T) \to I_1 \to I_2 \to 0 \quad .$$

Finally, there is an isomorphism in $\mathsf{mod}\,B$:

$$R \simeq \mathsf{Ext}^1_{kQ}(T, \mathsf{Tor}_1^B(R,T)) \quad .$$

Proof: Let $\mathcal{X}(T)$ be the torsion class induced by T in $\mathsf{mod}\,B$. So R lies in $\mathcal{X}(T)$ and has no direct summand in Σ. Therefore R is the epimorphic image of a module in $\mathsf{add}\,\Sigma$. The first assertion then follows from [8, 2.2, (d)].

Let $f\colon I_1 \to I_2$ be the morphism between injective kQ-modules such that $\mathsf{Hom}_{kQ}(T, f)$ is equal to the morphism $E_1 \to E_2$ in $(\star)$. Because of the Brenner-Butler Theorem (see [9, Chap. VI, Thm. 3.8, p.207]), the functor $- \underset{B}{\otimes} T$ applied to $(\star)$ yields an injective copresentation in $\mathsf{mod}\,kQ$:

$$0 \to \mathsf{Tor}_1^B(R,T) \to I_1 \to I_2 \to 0 \quad .$$

The minimality of this copresentation follows from the minimality of $E_2 \to R$. With these arguments, it is straightforward to check that there is a well-defined bijection which carries the equivalence class of the exact sequence $0 \to E_1 \to E_2 \to R \to 0$ to the equivalence class of the exact sequence $0 \to \mathsf{Tor}_1^B(R,T) \to I_1 \to I_2 \to 0$.

The last assertion follows from the Brenner-Butler Theorem and from the fact that $R \in \mathcal{X}(T)$. ■

Lemma 5.9. *$\mathsf{add}\,\mathcal{L}_\Sigma$ is contravariantly finite in $\mathsf{mod}\,A$. Therefore if $X \in \Gamma\backslash\mathcal{L}_\Sigma$, then $X_{|B}$ lies in the torsion class induced by T in $\mathsf{mod}\,B$.*

Proof: By [1, Thm. B], the algebra B is the endomorphism algebra of the indecomposable projective A-modules lying in $\mathcal{L}_\Sigma$. In particular, a projective B-module is projective as an A-module so that the projective dimensions in $\mathsf{mod}\,A$ and in $\mathsf{mod}\,B$ coincide on $\mathcal{L}_\Sigma$. Also, by [1, Thm. B], all modules in $\mathcal{L}_\Sigma$ have projective dimension at most one as B-modules. Therefore $\mathcal{L}_\Sigma \subseteq \mathcal{L}_A$. Moreover, $\bigoplus \Sigma$ is sincere as a B-module. Hence, [1, 8.2] implies that $\mathsf{add}\,\mathcal{L}_\Sigma$ is contravariantly finite in $\mathsf{mod}\,A$. Let $X \in \Gamma\backslash\mathcal{L}_\Sigma$. Let $P \twoheadrightarrow M_{|B}$ be the projective cover in $\mathsf{mod}\,B$. As noticed above, we have $P \in \mathsf{add}\,\mathcal{L}_\Sigma$. Therefore $P \twoheadrightarrow M_{|B}$ factors through $\mathsf{add}\,\Sigma$. Thus, X lies in the torsion class. ■

Lemma 5.10. *There exists a map $\varphi\colon \widetilde{\Gamma}_o \to \mathsf{mod}\,\widetilde{B}$ extending φ_0, and such that $F_\lambda(\varphi(M)) = p(M)_{|B}$, for every $M \in \widetilde{\Gamma}$. Moreover, $\varphi({}^gM) = {}^g\varphi(M)$ for every $g \in \pi_1(\Gamma)$ and $M \in \widetilde{\Gamma}$.*

Proof: Note that φ is already defined on $\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$ because of 5.4. Let $M \in \widetilde{\Gamma}\backslash\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$. Then $p(M) \in \Gamma\backslash\Gamma_{\leqslant\Sigma} = \Gamma\backslash\mathcal{L}_\Sigma$. By 5.9, the module $p(M)_{|B}$ lies in the torsion class induced by T in $\mathsf{mod}\,B$. So there is a decomposition in $\mathsf{mod}\,B$:

$$p(M)_{|B} = R \oplus E \quad ,$$

where $E \in \mathsf{add}\,\Sigma$ and R has no indecomposable summand in $\mathcal{L}_\Sigma$. Also, fix a decomposition in $\mathsf{mod}\,\widetilde{\Sigma}$:

$$k(\widetilde{\Gamma})(\widetilde{\Sigma}, M) = \widetilde{R} \oplus \widetilde{P} \quad ,$$

where $\widetilde{P}$ is projective and maximal for this property. Let $\widetilde{E} \in \mathsf{add}\,\widetilde{\Sigma}$ be such that $\widetilde{P} = k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E})$.

We claim that $p_\lambda\colon \mathsf{mod}\,\widetilde{\Sigma} \to \mathsf{mod}\,\Sigma$ maps $\widetilde{R}$ and $\widetilde{P}$ to $\mathsf{Hom}_B(\Sigma, R)$ and $\mathsf{Hom}_B(\Sigma, E)$ respectively. Indeed, since $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ is a covering functor inducing $p\colon \widetilde{\Sigma} \to \Sigma$, the image of $k(\widetilde{\Gamma})(\widetilde{\Sigma}, M)$ under $p_\lambda\colon \mathsf{mod}\,\widetilde{\Sigma} \to \mathsf{mod}\,\Sigma$ is $\mathsf{Hom}_B(\Sigma, p(M))$. Moreover, the decomposition $p(M)_{|B} = R \oplus E$ in $\mathsf{mod}\,B$ gives a decomposition $\mathsf{Hom}_B(\Sigma, p(M)) = \mathsf{Hom}_B(\Sigma, R) \oplus \mathsf{Hom}_B(\Sigma, E)$ in $\mathsf{mod}\,\Sigma$ where $\mathsf{Hom}_B(\Sigma, E)$ is projective and $\mathsf{Hom}_B(\Sigma, R)$ has no non-zero projective direct summand. The claim then follows from [14, 3.2].

In order to prove that R is the image of a $\widetilde{B}$-module under F_λ, we consider a minimal projective presentation in $\mathsf{mod}\,\widetilde{\Sigma}$:

$$0 \to \widetilde{P}_1 \to \widetilde{P}_2 \to \widetilde{R} \to 0 \; .$$

Then there exists a morphism $\widetilde{f}\colon \widetilde{I}_1 \to \widetilde{I}_2$ between injective $k\widetilde{Q}$-modules such that the morphism $\widetilde{P}_1 \to \widetilde{P}_2$ equals $\widetilde{\theta}(\widetilde{f})$ (here $\widetilde{\theta}$ is as in 5.2). Let $f\colon I_1 \to I_2$ be the image of $\widetilde{f}$ under $q_\lambda\colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$. Hence, the image of $\mathsf{Ker}\,\widetilde{f}$ under $q_\lambda\colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$ is $\mathsf{Ker}\,f$. Let $P_1 \to P_2$ be the image of $\widetilde{\theta}(\widetilde{f})$ under the same functor. Therefore the commutativity of the diagram in 5.2 gives a minimal projective presentation in $\mathsf{mod}\,\Sigma$:

$$0 \to P_1 \to P_2 \to \mathsf{Hom}_B(\Sigma, R) \to 0 \quad .$$

On the other hand, 5.2 shows that $P_1 \to P_2$ is equal to the following morphism in $\mathsf{mod}\,\Sigma$:

$$\mathsf{Hom}_B(\Sigma, \mathsf{Hom}_{kQ}(T, I_1)) \xrightarrow{\mathsf{Hom}_B(\Sigma, \mathsf{Hom}_{kQ}(T,f))} \mathsf{Hom}_B(\Sigma, \mathsf{Hom}_{kQ}(T, I_2)) \quad .$$

Therefore we have a minimal $\mathsf{add}\,\mathcal{L}_\Sigma$-presentation:

$$\mathsf{Hom}_{kQ}(T, I_1) \xrightarrow{\mathsf{Hom}_{kQ}(T,f)} \mathsf{Hom}_{kQ}(T, I_2) \to R \quad ,$$

Because of 5.8, the sequence $0 \to \mathsf{Hom}_{kQ}(T, I_1) \to \mathsf{Hom}_{kQ}(T, I_2) \to R \to 0$ is exact and $\mathsf{Ker}\,f = \mathsf{Tor}_1^B(R,T)$. In other words, $q_\lambda\colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$ maps $\mathsf{Ker}\,\widetilde{f}$ to $\mathsf{Tor}_1^B(R,T)$. Using 5.8 and the last diagram in the proof of 5.2, we get $F_\lambda(\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f})) = R$.

We give an explicit construction of φ. Let $M \in k(\widetilde{\Gamma})$. We fix a minimal projective presentation in $\mathsf{mod}\,\widetilde{\Sigma}$:

$$0 \to \widetilde{P}_1 \xrightarrow{\widetilde{u}} \widetilde{P}_2 \to \widetilde{R} \to 0 \quad ,$$

and fix injective $k\widetilde{Q}$-modules $\widetilde{I}_1$ and $\widetilde{I}_2$, together with a morphism $\widetilde{f}\colon \widetilde{I}_1 \to \widetilde{I}_2$ such that $\widetilde{u} = \widetilde{\theta}(\widetilde{f})$. Then we let $\varphi(M)$ be the following $\widetilde{B}$-module:

$$\varphi(M) = \varphi_0(\widetilde{E}) \oplus \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}) \quad .$$

Where $\varphi_0(\widetilde{E}) = \varphi_0(\widetilde{E}_1)\oplus\ldots\oplus\varphi_0(\widetilde{E}_s)$ if $\widetilde{E} = \widetilde{E}_1\oplus\ldots\oplus\widetilde{E}_s$ with $\widetilde{E}_1,\ldots,\widetilde{E}_s \in \widetilde{\Sigma}$. This finishes the construction of the map $\varphi\colon \widetilde{\Gamma}_o \to \mathsf{mod}\,\widetilde{B}$. We now prove the $\pi_1(\Gamma)$-equivariance property. Let $M \in k(\widetilde{\Gamma})$ be a vertex and let $g \in \pi_1(\Gamma)$. We keep the above notation $\widetilde{R}$, $\widetilde{E}$, *etc.* introduced for M, and we adopt the dashed notation $\widetilde{R}'$, $\widetilde{E}'$, *etc.* for the corresponding objects associated to ${}^gM$. We have $k(\widetilde{\Gamma})(\widetilde{\Sigma}, {}^gM) = {}^gk(\widetilde{\Gamma})(\widetilde{\Sigma}, M)$. Indeed, the $\widetilde{\Sigma}$-modules $k(\widetilde{\Gamma})(\widetilde{\Sigma}, {}^gM)$ and ${}^gk(\widetilde{\Gamma})(\widetilde{\Sigma}, M)$ are given by the functors $X \mapsto k(\widetilde{\Gamma})(X, {}^gM)$ and $X \mapsto k(\widetilde{\Gamma})({}^{g^{-1}}X, M)$ from $\widetilde{\Sigma}^{op}$ to $\mathsf{mod}\,k$, respectively. These two functors coincide because $\pi_1(\Gamma)$ acts on $k(\widetilde{\Gamma})$. Hence, $\widetilde{E}' = {}^g\widetilde{E}$ and $\widetilde{R}' = {}^g\widetilde{R}$. Therefore any minimal projective presentation of $\widetilde{R}'$ in $\mathsf{mod}\,\widetilde{\Sigma}$ is obtained from a minimal projective presentation of $\widetilde{R}$ by applying g. Since, moreover, $\widetilde{\theta}$ is $\pi_1(\Gamma)$-equivariant (see 5.2), we deduce that $\widetilde{f}' = {}^g\widetilde{f}$. Finally, the $\pi_1(\Gamma)$-action on $\mathsf{mod}\,k\widetilde{Q}$ implies, as above, that $\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,{}^g\widetilde{f}) = \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, {}^g\mathsf{Ker}\,\widetilde{f}) = {}^g\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f})$. From the construction of φ, we get $\varphi({}^gM) = {}^g\varphi(M)$. ■

Construction of φ on morphisms

We complete the construction of φ by proving the following lemma.

Lemma 5.11. *Let $u\colon M \to N$ be a morphism in $k(\widetilde{\Gamma})$. Then there exists a unique morphism $\varphi(u)\colon \varphi(M) \to \varphi(N)$ in $\mathsf{mod}\,\widetilde{B}$, such that $F_\lambda(\varphi(u)) = p(u)_{|B}$.*

Proof: Since F_λ is exact, it is faithful so the morphism $\varphi(u)$ is unique. We prove the existence. By 5.4, we have constructed $\varphi(u) = \varphi_0(u)$ in case $N \in \widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$. So we may assume that $N \in \widetilde{\Gamma}\backslash\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$. Since any path in $\widetilde{\Gamma}$ from a vertex in $\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}$ to N has a vertex in $\widetilde{\Sigma}$, we may also assume that $M \in \left(\widetilde{\Gamma}\backslash\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}\right)\bigcup\widetilde{\Sigma}$. Notice that the functor $\varphi_0\colon k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}}) \to \mathsf{mod}\,\widetilde{B}$ naturally extends to a unique functor $\varphi_0\colon \mathsf{add}\,(k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}})) \to \mathsf{mod}\,\widetilde{B}$, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathsf{add}\,(k(\widetilde{\Gamma}_{\leqslant\widetilde{\Sigma}})) & \xrightarrow{\varphi_0} & \mathsf{mod}\,\widetilde{B} \\ {\scriptstyle \mathsf{add}\,p}\downarrow & & \downarrow{\scriptstyle F_\lambda} \\ \mathsf{add}\,(\mathsf{ind}\,\Gamma_{\leqslant\Sigma}) & \hookrightarrow & \mathsf{mod}\,B \end{array} .$$

Let us fix decompositions in $\mathsf{mod}\,\widetilde{\Sigma}$ as in the proof of 5.10:

$$k(\widetilde{\Gamma})(\widetilde{\Sigma}, M) = \widetilde{P} \oplus \widetilde{R}, \quad\text{and}\quad k(\widetilde{\Gamma})(\widetilde{\Sigma}, N) = \widetilde{P}' \oplus \widetilde{R}' \quad ,$$

where $\widetilde{P}, \widetilde{P}'$ are projective and $\widetilde{R}, \widetilde{R}'$ have no non-zero projective direct summand. We let $\widetilde{E}, \widetilde{E}' \in \mathsf{add}\,\widetilde{\Sigma}$ be such that $\widetilde{P} = k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E})$ and $\widetilde{P}' = k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E}')$, respectively. Therefore the morphism $k(\widetilde{\Gamma})(\widetilde{\Sigma}, u)$ can be written as:

$$k(\widetilde{\Gamma})(\widetilde{\Sigma}, u) = \begin{bmatrix} \widetilde{u}_1 & 0 \\ \widetilde{u}_2 & \widetilde{u}_3 \end{bmatrix} : k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E}) \oplus \widetilde{R} \to k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E}') \oplus \widetilde{R}' \quad .$$

Similarly, we fix decompositions in $\mathsf{mod}\,B$:

$$p(M)_{|B} = E \oplus R, \quad p(N)_{|B} = E' \oplus R' \quad ,$$

where $E, E' \in \mathsf{add}\,\Sigma$, and R, R' have no direct summand in Σ. As above, the morphism $p(u)_{|B}$ decomposes as:

$$p(u)_{|B} = \begin{bmatrix} u_1 & 0 \\ u_2 & u_3 \end{bmatrix} : E \oplus R \to E' \oplus R' \quad .$$

Recall from the proof of 5.10 that $p_\lambda\colon \mathsf{mod}\,\widetilde{\Sigma} \to \mathsf{mod}\,\Sigma$ maps $k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E})$, $\widetilde{R}$, $k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E}')$ and $\widetilde{R}'$ to $\mathsf{Hom}_B(\Sigma, E)$, $\mathsf{Hom}_B(\Sigma, R)$, $\mathsf{Hom}_B(\Sigma, E')$ and $\mathsf{Hom}_B(\Sigma, R')$, respectively. As a consequence, it maps $\widetilde{u}_i$ to $\mathsf{Hom}_B(\Sigma, u_i)$, for every i. As in the proof of 5.10, we have morphisms $\widetilde{f}\colon \widetilde{I}_1 \to \widetilde{I}_2$ and $\widetilde{f}'\colon \widetilde{I}_1' \to \widetilde{I}_2'$ between injective $k\widetilde{Q}$-modules such that there exist minimal projective presentations in $\mathsf{mod}\,\widetilde{\Sigma}$:

$$\widetilde{\theta}(\widetilde{I}_1) \xrightarrow{\widetilde{\theta}(\widetilde{f})} \widetilde{\theta}(\widetilde{I}_2) \to \widetilde{R} \to 0 \quad\text{and}\quad \widetilde{\theta}(\widetilde{I}_1') \xrightarrow{\widetilde{\theta}(\widetilde{f}')} \widetilde{\theta}(\widetilde{I}_2') \xrightarrow{\widetilde{v}} \widetilde{R}' \to 0 \quad .$$

With these notations, we have:

$$\varphi(M) = \varphi_0(\widetilde{E}) \oplus \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}) \quad\text{and}\quad \varphi(N) = \varphi_0(\widetilde{E}') \oplus \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}') \quad .$$

Also, if $M \in \widetilde{\Sigma}$, then $\widetilde{f} = 0$, so $\varphi(M) = \varphi_0(\widetilde{E})$.

It suffices to prove that each of u_1, u_2, u_3 is the image under F_λ of some morphism $\varphi_0(\widetilde{E}) \to \varphi_0(\widetilde{E}')$, $\varphi_0(\widetilde{E}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}')$ and $\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}')$ respectively. Clearly $u_1\colon k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E}) \to k(\widetilde{\Gamma})(\widetilde{\Sigma}, \widetilde{E}')$ is induced by a morphism $\widetilde{E} \to \widetilde{E}'$ in $\mathsf{add}\,\widetilde{\Sigma}$. This and 5.4 imply that u_1 is the image under F_λ of a morphism

$\varphi_0(\widetilde{E}) \to \varphi_0(\widetilde{E}')$. We now prove that u_2 is the image under F_λ of a morphism $\varphi_0(\widetilde{E}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}')$. Let $f' \colon I'_1 \to I'_2$ be the image of $\widetilde{f}' \colon \widetilde{I}'_1 \to \widetilde{I}'_2$ under $q_\lambda \colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$. Therefore we have a minimal projective presentation in $\mathsf{mod}\,\Sigma$ (see 5.2):

$$0 \to \theta(I'_1) \xrightarrow{\theta(f')} \theta(I'_2) \to \mathsf{Hom}_B(\Sigma, R') \to 0 \ ,$$

together with a minimal injective copresentation in $\mathsf{mod}\,kQ$:

$$0 \to \mathsf{Tor}_1^B(R', T) \to I'_1 \xrightarrow{f'} I'_2 \to 0 \ \ .$$

Recall that $\mathsf{Tor}_1^B(R', T)$ is equal to the image of $\mathsf{Ker}\,\widetilde{f}'$ under $q_\lambda \colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$. Therefore we have an exact sequence in $\mathsf{mod}\,B$, which is also a minimal $\mathsf{add}\,\mathcal{L}_\Sigma$-presentation:

$$0 \to \mathsf{Hom}_{kQ}(T, I'_1) \to \mathsf{Hom}_{kQ}(T, I'_2) \xrightarrow{v} R' \to 0 \ \ ,$$

where the morphism v is such that $\mathsf{Hom}_B(\Sigma, v)$ is the image of $\widetilde{v} \colon \widetilde{\theta}(\widetilde{I}'_2) \to \widetilde{R}'$ under $p_\lambda \colon \mathsf{mod}\,\widetilde{\Sigma} \to \mathsf{mod}\,\Sigma$ (see the diagram in 5.2). The projective cover $\widetilde{v}$ of $\widetilde{R}'$ in $\mathsf{mod}\,\widetilde{\Sigma}$ yields a morphism $\widetilde{\delta} \colon \widetilde{I} \to \widetilde{I}'_2$ in $\mathsf{mod}\,k\widetilde{Q}$, where $\widetilde{I}$ is the injective $k\widetilde{Q}$-module such that $\widetilde{P} = \widetilde{\theta}(\widetilde{I})$, and such that the following diagram of $\mathsf{mod}\,\widetilde{\Sigma}$ is commutative:

$$\begin{array}{ccc} & & \widetilde{P} \\ & \swarrow^{\widetilde{\theta}(\widetilde{\delta})} & \downarrow{\scriptstyle \widetilde{u}_2} \\ \widetilde{\theta}(\widetilde{I}'_2) & \xrightarrow[\widetilde{v}]{} & \widetilde{R}' \end{array} \quad .$$

Therefore if $\delta \colon I \to I'_2$ denotes the image of $\widetilde{\delta} \colon \widetilde{I} \to \widetilde{I}'_2$ under $q_\lambda \colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$, then $\mathsf{Hom}_B(\Sigma, u_2)$ equals the composition $\theta(I) \xrightarrow{\theta(\delta)} \theta(I'_2) \xrightarrow{\mathsf{Hom}_B(\Sigma,v)} \mathsf{Hom}_B(\Sigma, R')$. This is an equality of morphisms in $\mathsf{mod}\,\Sigma$, hence, of morphisms between contravariant functors from $\mathsf{add}\,\Sigma$ to $\mathsf{mod}\,k$. Applying this equality to E yields that u_2 equals the composition $E \xrightarrow{\mathsf{Hom}_{kQ}(T,\delta)} \mathsf{Hom}_{kQ}(T, I'_2) \xrightarrow{v} R'$. On the other hand, the morphism:

$$\mathsf{Hom}_{kQ}(T, I'_2) \xrightarrow{v} R' = \mathsf{Ext}^1_{kQ}(T, \mathsf{Ker}\,f')$$

is the connecting morphism of the sequence resulting from the application of $\mathsf{Hom}_{kQ}(T, -)$ to the exact sequence $0 \to \mathsf{Ker}\,f' \to I'_1 \xrightarrow{f'} I'_2 \to 0$. Therefore 5.2 implies that v equals the image under F_λ of the connecting morphism of the sequence resulting from the application of $\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T}, -)$ to the exact sequence $0 \to \mathsf{Ker}\,\widetilde{f}' \to \widetilde{I}'_1 \to \widetilde{I}'_2 \to 0$. Consequently, u_2 equals the image under F_λ of the composition $\varphi_0(\widetilde{E}) \xrightarrow{\mathsf{Hom}_{k\widetilde{Q}}(\widetilde{T},\delta)} \varphi_0(\mathsf{Hom}_{kQ}(\widetilde{T}, \widetilde{I}'_2)) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}')$.

It only remains to prove that $u_3 \colon R \to R'$ equals the image under F_λ of a morphism $\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}')$ in $\mathsf{mod}\,\widetilde{B}$. Using the projective presentations of $\widetilde{R}$ and $\widetilde{R}'$, we find morphisms $\widetilde{\alpha} \colon \widetilde{I}_2 \to \widetilde{I}'_2$ and $\widetilde{\beta} \colon \widetilde{I}_1 \to \widetilde{I}'_1$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\theta}(\widetilde{I}_1) & \xrightarrow{\widetilde{\theta}(\widetilde{f})} & \widetilde{\theta}(\widetilde{I}_2) & \longrightarrow & \widetilde{R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\theta}(\widetilde{\beta})} & & \downarrow{\scriptstyle \widetilde{\theta}(\widetilde{\alpha})} & & \downarrow{\scriptstyle \widetilde{u}_3} & & \\ 0 & \longrightarrow & \widetilde{\theta}(\widetilde{I}'_1) & \xrightarrow[\widetilde{\theta}(\widetilde{f}')]{} & \widetilde{\theta}(\widetilde{I}'_2) & \longrightarrow & \widetilde{R}' & \longrightarrow & 0 \end{array} \quad .$$

Therefore there exists a morphism $\widetilde{\gamma} \colon \mathsf{Ker}\,\widetilde{f} \to \mathsf{Ker}\,\widetilde{f}'$ making the following diagram in $\mathsf{mod}\,k\widetilde{Q}$ commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ker}\,\widetilde{f} & \longrightarrow & \widetilde{I}_1 & \xrightarrow{\widetilde{f}} & \widetilde{I}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\gamma}} & & \downarrow{\scriptstyle \widetilde{\beta}} & & \downarrow{\scriptstyle \widetilde{\alpha}} & & \\ 0 & \longrightarrow & \mathsf{Ker}\,\widetilde{f}' & \longrightarrow & \widetilde{I}'_1 & \xrightarrow[\widetilde{f}']{} & \widetilde{I}'_2 & \longrightarrow & 0 \end{array} \quad .$$

We claim that the image of $\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \widetilde{\gamma}) \colon \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \mathsf{Ker}\,\widetilde{f}')$ under F_λ is equal to u_3. Indeed, let α, β, γ be the respective images of $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$ under $q_\lambda \colon \mathsf{mod}\,k\widetilde{Q} \to \mathsf{mod}\,kQ$. Then the image of $\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T}, \widetilde{\gamma})$ under F_λ is equal to (see 5.2):

$$\mathsf{Ext}^1_{kQ}(T, \gamma) \colon \mathsf{Ext}^1_{kQ}(T, \mathsf{Ker}\,f) \to \mathsf{Ext}^1_{kQ}(T, \mathsf{Ker}\,f') \ \ .$$

On the other hand, we have two commutative diagrams in $\mathsf{mod}\, kQ$ and $\mathsf{mod}\, \widetilde{B}$ respectively:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Ker}\, f = \mathsf{Tor}_1^B(R,T) & \longrightarrow & I_1 & \longrightarrow & I_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \mathsf{Ker}\, f' = \mathsf{Tor}_1^B(R',T) & \longrightarrow & I_1' & \longrightarrow & I_2' & \longrightarrow & 0 \end{array} \quad , \text{ and}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{Hom}_{kQ}(T,I_1) & \longrightarrow & \mathsf{Hom}_{kQ}(T,I_2) & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathsf{Hom}_{kQ}(T,\beta)} & & \downarrow{\scriptstyle \mathsf{Hom}_{kQ}(T,\alpha)} & & \downarrow{\scriptstyle u_3} & & \\ 0 & \longrightarrow & \mathsf{Hom}_{kQ}(T,I_1') & \longrightarrow & \mathsf{Hom}_{kQ}(T,I_2') & \longrightarrow & R' & \longrightarrow & 0 \end{array} \quad ,$$

from which it is straightforward to check that $u_3\colon R \to R'$ coincides with $\mathsf{Ext}^1_{kQ}(T,\gamma)$. Thus, u_3 is equal to the image under F_λ of the morphism $\mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T},\widetilde{\gamma})\colon \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T},\mathsf{Ker}\,\widetilde{f}) \to \mathsf{Ext}^1_{k\widetilde{Q}}(\widetilde{T},\mathsf{Ker}\,\widetilde{f}')$. This completes the proof. ■

We summarise our results in the following theorem.

Theorem 5.12. *Let A be a finite dimensional k-algebra and Γ be a component of $\Gamma(\mathsf{mod}\, A)$ containing a left section Σ. Let $B = A/\mathsf{Ann}\,\Sigma$ and $\pi\colon \widetilde{\Gamma} \to \Gamma$ be the universal cover of translation quivers. To a well-behaved covering functor $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ corresponds a covering $F\colon \widetilde{B} \to B$ with $\widetilde{B}$ locally bounded and a functor $\varphi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\,\widetilde{B}$ which is $\pi_1(\Gamma)$-equivariant on vertices and makes the following diagram commute:*

$$\begin{array}{ccc} k(\widetilde{\Gamma}) & \xrightarrow{\ \varphi\ } & \mathsf{mod}\,\widetilde{B} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle F_\lambda} \\ \mathsf{ind}\,\Gamma & \xrightarrow[\mathsf{Hom}_A(B,-)]{} & \mathsf{mod}\, B \end{array} \quad .$$

Proof: The functor φ is constructed as above. The $\pi_1(\Gamma)$-equivariance follows from 5.10 and 5.11. ■

Corollary 5.13. *If $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ is a Galois covering (with respect to the action of $\pi_1(\Gamma)$ on $k(\widetilde{\Gamma})$), then the functor $\varphi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\,\widetilde{B}$ of the preceding theorem is $\pi_1(\Gamma)$-equivariant.*

Proof: We already know that φ is $\pi_1(\Gamma)$-equivariant on objects. Note also that $F\colon \widetilde{B} \to B$ is a Galois covering with group $\pi_1(\Gamma)$ (see 5.1). Let $f\colon M \to N$ be a morphism in $k(\widetilde{\Gamma})$, and $g \in \pi_1(\Gamma)$. Then $\varphi({}^g f)\colon \varphi({}^g M) \to \varphi({}^g N)$ and ${}^g\varphi(f)\colon {}^g\varphi(M) \to {}^g\varphi(N)$ are two morphisms in $\mathsf{mod}\,\widetilde{B}$ such that $F_\lambda(\varphi({}^g f)) = p({}^g f)_{|B} = p(f)_{|B} = F_\lambda({}^g\varphi(f))$ (recall that $F_\lambda = F_\lambda \circ g$ for every $g \in \pi_1(\Gamma)$ because it is the push-down functor of a Galois covering with group $\pi_1(\Gamma)$). So we deduce that $\varphi({}^g f) = {}^g\varphi(f)$. ■

6 The main theorem

The aim of this section is to prove Theorem A. Thus we assume that A is a connected laura algebra, not quasi-tilted of canonical type. For simplicity, we assume that A is not concealed. We use the following notation:

- Γ is the connecting component of $\Gamma(\mathsf{mod}\, A)$, and $\pi\colon \widetilde{\Gamma} \to \Gamma$ is the universal cover of translation quivers (if A is concealed, we may equivalently choose Γ to be the unique postprojective component or the unique preinjective component).
- $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ is a well-behaved covering functor. If Γ is standard, we assume that p equals the composition of $k(\pi)\colon k(\widetilde{\Gamma}) \to k(\Gamma)$ with some isomorphism $k(\Gamma) \xrightarrow{\sim} \mathsf{ind}\,\Gamma$, so that p is a Galois covering with group $\pi_1(\Gamma)$.
- Σ is the full subcategory of $\mathsf{ind}\,\Gamma$ whose objects are the Ext-injective objects in $\mathcal{L}_A$.
- B is the left support of A.

We know from [5, 4.4, 5.1] that the algebra B is a product of tilted algebras. Without loss of generality, we assume that:

- $B = \mathsf{End}_{kQ}(T)$, where $T = T_1 \oplus \ldots \oplus T_n$ is a multiplicity-free tilting kQ-module ($T_i \in \mathsf{ind}\, kQ$).

- Σ is the full subcategory of $\mathsf{mod}\, B$ with objects the modules of the form $\mathsf{Hom}_{kQ}(T, D(kQe_x))$, $x \in Q_0$.

It follows from [1, 2.1 Ex. b] that Σ is a left section of Γ. So we may apply 5.12. The proof of Theorem A is done in the following steps: We first construct a locally bounded k-category $\widetilde{A}$ endowed with a free $\pi_1(\Gamma)$-action in case A is standard; then construct a covering functor $F\colon \widetilde{A} \to A$ extending the functor $F\colon \widetilde{B} \to B$ of 5.12 and satisfying the conditions of the theorem; we also construct a functor $\Phi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\, \widetilde{A}$ which extends the functor $\varphi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\, \widetilde{B}$ of 5.12 and is essential in the study of the Galois coverings of standard laura algebras; and finally we prove Theorem A.

The category $\widetilde{A}$

We need some notation. Let C be the full subcategory of $\mathsf{ind}\, A$ with objects the indecomposable projective A-modules not in $\mathcal{L}_A$. So C is a full subcategory of $\mathsf{ind}\,\Gamma$. Let $\widetilde{C}$ be the full subcategory $p^{-1}(C)$, so that p induces a covering functor $\widetilde{C} \to C$. Note that if A is standard and p is Galois with group $\pi_1(\Gamma)$, then $p\colon \widetilde{C} \to C$ is a Galois covering with group $\pi_1(\Gamma)$. For every $x \in \widetilde{B}_o$, let $\widetilde{P}_x$ be the corresponding indecomposable projective $\widetilde{B}$-module. Also, $P_x \in \mathsf{mod}\, B$ denotes the indecomposable projective B-module associated to an object $x \in B_o$. We define the $\widetilde{C} - \widetilde{B}$-bimodule $\widetilde{M}$ to be the functor $\widetilde{C} \times \widetilde{B}^{op} \to \mathsf{mod}\, k$ such that for every $\widetilde{P} \in \widetilde{C}_o$ and $x \in \widetilde{B}_o$

$$ {}_{\widetilde{P}}\widetilde{M}_x = \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_x, \varphi(\widetilde{P})) \quad , $$

with obvious actions of $\widetilde{C}$ (using φ) and $\widetilde{B}$. We need the following technical lemma.

Lemma 6.1. *Let $X \in \widetilde{\Gamma}$. Let $p(X)_{|B} = X_1^{l_1} \oplus \ldots \oplus X_t^{l_t}$ be an indecomposable decomposition in $\mathsf{mod}\, B$ with $X_1, \ldots, X_t$ pairwise non isomorphic. Then $X_1, \ldots, X_t \in \mathsf{ind}\, B \cap \Gamma$, and they are direct summands of tilting B-modules. If $N \in \mathsf{ind}\, \widetilde{B}$ is a summand of $\varphi(X)$, then there exist $i \in \{1, \ldots, t\}$ and $g \in \pi_1(\Gamma)$ such that $N \simeq {}^g\widetilde{X}_i$.*

Proof: By the proof of 5.10, the two first assertions are true. Therefore we apply 4.2 to the modules $X_1, \ldots, X_t$. Up to a renumbering, we may assume that $\mathsf{Hom}_B(X_i, X_j) = 0$ if $i < j$. Now $F_\lambda N$ is a direct summand of $p(X)_{|B}$. So there exist indices $1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant t$ such that $F_\lambda N \simeq X_{i_1} \oplus \ldots \oplus X_{i_l}$. In particular, there is a section $s\colon X_{i_1} \to F_\lambda N$ which defines a split triangle in $\mathcal{D}^b(\mathsf{mod}\, B)$:

$$ M \to X_{i_1} \xrightarrow{s} F_\lambda N \to M[1] \ , $$

where $M = \bigoplus_{j \geqslant 2} X_{i_j}[-1]$. In particular, we have $\mathcal{D}^b(\mathsf{mod}\, B)(X_{i_1}, M[1]) = \mathsf{Hom}_B(X_{i_1}, \bigoplus_{j \geqslant 2} X_{i_j}) = 0$. This and the isomorphisms of 4.2 (applied to X_{i_1} which an indecomposable summand of a tilting B-module) show that 4.3 applies here. So, there exist $g \in \pi_1(\Gamma)$, a morphism $\widetilde{s}\colon {}^g\widetilde{X}_{i_1} \to N$ (in $\mathcal{D}^b(\mathsf{mod}\, \widetilde{B})$, and actually in $\mathsf{mod}\, \widetilde{B}$), and a commutative diagram:

$$ \begin{array}{ccc} X_{i_1} & \xrightarrow{F_\lambda \widetilde{s}} & F_\lambda N \\ \sim\downarrow & & \downarrow\sim \\ X_{i_1} & \xrightarrow[s]{} & F_\lambda N \end{array} \quad , $$

where the vertical arrows are isomorphisms. This allows the identification $s = F_\lambda \widetilde{s}$. We claim that $\widetilde{s}$ is a section. For this purpose, we complete $\widetilde{s}$ into a triangle ${}^g\widetilde{X}_{i_1} \xrightarrow{\widetilde{s}} N \to N' \xrightarrow{f} {}^g\widetilde{X}_{i_1}$ in $\mathcal{D}^b(\widetilde{B})$. Since $F_\lambda\colon \mathcal{D}^b(\mathsf{mod}\, \widetilde{B}) \to \mathcal{D}^b(\mathsf{mod}\, B)$ is exact, we deduce a triangle $X_{i_1} \xrightarrow{s} F_\lambda N \to F_\lambda N' \xrightarrow{F_\lambda f} X_{i_1}[1]$ in $\mathcal{D}^b(\mathsf{mod}\, B)$. Since s is a section, this triangle splits, that is $F_\lambda f = 0$. The bijections of 4.2 imply that $f = 0$, that is, $\widetilde{s}$ is a section. Since N is indecomposable, we get $N \simeq {}^g\widetilde{X}_{i_1}$. ■

The following lemma defines $\widetilde{A}$ and its $\pi_1(\Gamma)$-action in case A is standard.

Lemma 6.2. *Let $\widetilde{A} = \begin{bmatrix} \widetilde{B} & 0 \\ \widetilde{M} & \widetilde{C} \end{bmatrix}$. Then $\widetilde{A}$ is locally bounded and $\pi_1(\Gamma)$ acts freely on $\widetilde{A}$ if A is standard.*

Proof: We know that $\widetilde{B}$ and $\widetilde{C}$ are locally bounded. Let $P \in \widetilde{C}_o$. We have the bijection of 4.2:

$$ \bigoplus_{\widetilde{x} \in \widetilde{B}_o} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, \varphi(\widetilde{P})) \xrightarrow{\sim} \bigoplus_{x \in B_o} \mathsf{Hom}_B(P_x, p(\widetilde{P})_{|B}) \quad . \qquad \text{(i)} $$

Since the right hand-side is finite dimensional, then so is $\bigoplus_{\widetilde{x} \in \widetilde{B}_o} {}_{\widetilde{P}}\widetilde{M}_{\widetilde{x}}$, for every $\widetilde{P} \in \widetilde{C}_o$.

Now let $P \in \widetilde{C}_o$, let $\widetilde{x} \in \widetilde{B}_o$, and let us prove that $\bigoplus_{p(P') = p(P)} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, \varphi(P'))$ is finite dimensional. By definition of p, we have $p^{-1}(p(P)) = \{\, {}^gP \mid g \in \pi_1(\Gamma)\}$. Also, we know from 5.12 that φ is $\pi_1(\Gamma)$-equivariant on objects. Therefore:

$$ \bigoplus_{p(P') = p(P)} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, \varphi(P')) = \bigoplus_{g \in \pi_1(\Gamma)} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, {}^g\varphi(P)) \ . \qquad \text{(ii)} $$

Moreover, 6.1 shows that there exist modules $X_1, \ldots, X_t \in \mathsf{ind}\, B$ which are direct summands of tilting B-modules and there exist $g_1, \ldots, g_t$ such that $\varphi(P) = \bigoplus_{i=1}^{t} {}^{g_i}\widetilde{X}_i$. By applying 4.2 to each X_i, we get a bijection of vector spaces:

$$\bigoplus_{g\in\pi_1(\Gamma)} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}},\, {}^{g}\varphi(P)) \simeq \mathsf{Hom}_B(P_{F(\widetilde{x})}, F_\lambda\varphi(P)) \ . \tag{iii}$$

From (ii) and (iii) we infer that $\bigoplus_{p(P')=p(P)} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, \varphi(P'))$ is finite dimensional for every $\widetilde{x} \in \widetilde{B}_o$ and $P \in \widetilde{C}_o$. This shows that $\widetilde{A}$ is locally bounded.

Assume now that A is standard and that $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ is a Galois covering with group $\pi_1(\Gamma)$. Let us define a free $\pi_1(\Gamma)$-action on $\widetilde{A}$. We already have a free $\pi_1(\Gamma)$-action on $\widetilde{B}$ and on $\widetilde{C}$. Also, for every $\widetilde{x} \in \widetilde{B}_o$, $\widetilde{P} \in \widetilde{C}_o$ and $g \in \pi_1(\Gamma)$, we have an isomorphism of vector spaces:

$$_{\widetilde{P}}\widetilde{M}_{\widetilde{x}} = \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, \varphi(P)) \xrightarrow{\sim} {}_{{}^g\widetilde{P}}\widetilde{M}_{g\widetilde{x}} = \mathsf{Hom}_{\widetilde{B}}({}^{g}\widetilde{P}_{\widetilde{x}}, \varphi({}^{g}\widetilde{P})) \tag{$\star$}$$

given by the $\pi_1(\Gamma)$-action on $\mathsf{mod}\,\widetilde{B}$ (recall that φ is $\pi_1(\Gamma)$-equivariant on objects, and that $\widetilde{P}_{g\widetilde{x}} = {}^{g}\widetilde{P}_{\widetilde{x}}$). We define the action of g on morphisms of $\widetilde{A}$ lying in $\widetilde{M}$ using this isomorphism. Since $\pi_1(\Gamma)$ acts on $\mathsf{mod}\,\widetilde{B}$, this defines a $\pi_1(\Gamma)$-action on $\widetilde{A}$, that is, $g(vu) = g(v)g(u)$ whenever u and v are composable morphisms in $\widetilde{A}$. Moreover, $\pi_1(\Gamma)$ acts freely on objects in $\widetilde{B}$ and in $\widetilde{C}$. So we have a free $\pi_1(\Gamma)$-action on $\widetilde{A}$. ■

The functor $F\colon \widetilde{A} \to A$

Lemma 6.3. *There exists a covering functor $F\colon \widetilde{A} \to A$ extending $F\colon \widetilde{B} \to B$. If moreover A is standard, then F can be taken to be Galois with group $\pi_1(\Gamma)$.*

Proof: Note that $A = \begin{bmatrix} B & 0 \\ M & C \end{bmatrix}$ where M is the $C-B$-bimodule such that ${}_PM_x = \mathsf{Hom}_B(P_x, P_{|B})$ for every $P \in C_o$ and $x \in B_o$. Let us define $F\colon \widetilde{A} \to A$ as follows:

- $F_{|\widetilde{B}}$ coincides with the functor $F\colon \widetilde{B} \to B$.
- $F_{|\widetilde{C}}$ coincides with $p\colon \widetilde{C} \to C$.
- Let $x \in \widetilde{B}_o$ and $\widetilde{P} \in \widetilde{C}_o$, then $F\colon\ {}_{\widetilde{P}}\widetilde{M}_x \to\ {}_{F(\widetilde{P})}M_{F(x)}$ is the following map induced by F_λ:

$$\mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_x, \varphi(\widetilde{P})) \to \mathsf{Hom}_B(P_{F(x)}, p(\widetilde{P})_{|B}) \ \ .$$

Since $F_\lambda\colon \mathsf{mod}\,\widetilde{B} \to \mathsf{mod}\,B$ is a functor and $F_\lambda\varphi = p(-)_{|B}$ (see 5.12), we have defined a functor $F\colon \widetilde{A} \to A$. Let us prove that $F\colon \widetilde{A} \to A$ is a covering functor. Since $F\colon \widetilde{B} \to B$ and $p\colon \widetilde{C} \to C$ are covering functors, the bijections (i), (iii) in the proof of 6.2 show that for any $\widetilde{a} \in \widetilde{B}_o$ and any $\widetilde{P} \in \widetilde{C}_o$, the two following maps induced by F_λ are isomorphisms:

$$\bigoplus_{F(\widetilde{x})=F(\widetilde{a})} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{x}}, \varphi(\widetilde{P})) \to \mathsf{Hom}_B(P_{F(\widetilde{a})}, p(\widetilde{P})_{|B}) \ \ ,$$

$$\bigoplus_{p(\widetilde{Q})=p(\widetilde{P})} \mathsf{Hom}_{\widetilde{B}}(\widetilde{P}_{\widetilde{a}}, \varphi(\widetilde{Q})) \to \mathsf{Hom}_B(P_{F(\widetilde{a})}, p(\widetilde{Q})_{|B}) \ \ .$$

So F is a covering functor. Assume now that A is standard. We may suppose that p is a Galois covering with group $\pi_1(\Gamma)$. By 6.2, there is a free $\pi_1(\Gamma)$-action on $\widetilde{A}$. Moreover, $F\colon \widetilde{B} \to B$, and therefore $F_\lambda\colon \mathsf{mod}\,\widetilde{B} \to \mathsf{mod}\,B$, are $\pi_1(\Gamma)$-equivariant, and so is $p\colon \widetilde{C} \to C$, because it restricts the Galois covering $p\colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$. Therefore $F\colon \widetilde{A} \to A$ is $\pi_1(\Gamma)$-equivariant. Finally, the fibres of $F\colon \widetilde{A} \to A$ on objects are the $\pi_1(\Gamma)$-orbits in $\widetilde{A}_o$ because $F\colon \widetilde{B} \to B$ and $p\colon \widetilde{C} \to C$ are Galois coverings. Since $F\colon \widetilde{A} \to A$ is a covering functor, this implies that it is also a Galois covering with group $\pi_1(\Gamma)$ (see for instance the proof of [31, Prop. 6.1.37]). ■

The functor $\Phi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\,\widetilde{A}$

Recall that an $\widetilde{A}$-module is a triple (K, L, f) where $K \in \mathsf{mod}\,\widetilde{B}$, $L \in \mathsf{mod}\,\widetilde{C}$ and $f\colon L \underset{\widetilde{C}}{\otimes} \widetilde{M} \to K$ is a morphism of $\widetilde{B}$-modules. Let $\psi\colon k(\widetilde{\Gamma}) \to \mathsf{mod}\,\widetilde{C}$ be the functor:

$$\psi\colon X \mapsto k(\widetilde{\Gamma})(\widetilde{C}, X) \ \ .$$

Clearly, it is $\pi_1(\Gamma)$-equivariant. Let $L \in k(\widetilde{\Gamma})$. Then $\psi(L) \underset{\widetilde{C}}{\otimes} \widetilde{M}$ is the $\widetilde{B}$-module whose value at $x \in \widetilde{B}_o$ equals:

$$\left(\psi(L) \underset{\widetilde{C}}{\otimes} \widetilde{M}\right)(x) = \left(\bigoplus_{\widetilde{P} \in \widetilde{C}_o} k(\widetilde{\Gamma})(\widetilde{P}, L) \underset{k}{\otimes} \mathsf{Hom}_{\widetilde{B}}\left(\widetilde{P}_x, \varphi(\widetilde{P})\right)\right) /N \quad ,$$

where N is the following subspace:

$$N = \left\langle ff' \otimes u \; - \; f \otimes \varphi(f')u \;\middle|\; f \in k(\widetilde{\Gamma})(\widetilde{P}, L), f' \in k(\widetilde{\Gamma})(\widetilde{P}', \widetilde{P}), u \in \mathsf{Hom}_{\widetilde{B}}\left(\widetilde{P}_x, \varphi(\widetilde{P})\right), \text{ for every } \widetilde{P}, \widetilde{P}' \in \widetilde{C}_o \right\rangle \; .$$

For every $x \in \widetilde{B}_o$ and $\widetilde{P} \in \widetilde{C}_o$, we have a k-linear map:

$$\begin{array}{rrcl} \eta_{L,x,P} \colon & k(\widetilde{\Gamma})(\widetilde{P}, L) \underset{k}{\otimes} \mathsf{Hom}_{\widetilde{B}}\left(\widetilde{P}_x, \varphi(\widetilde{P})\right) & \to & \mathsf{Hom}_{\widetilde{B}}\left(\widetilde{P}_x, \varphi(L)\right) = \varphi(L)(x) \\ & f \otimes u & \mapsto & \varphi(f)\, u \end{array} \quad .$$

It is not difficult to check that the family of maps $\left(\eta_{L,x,\widetilde{P}}\right)_{L,x,\widetilde{P}}$ defines a functorial morphism:

$$\eta \colon \psi(-) \underset{\widetilde{C}}{\otimes} \widetilde{M} \to \varphi \quad .$$

Moreover, if φ is $\pi_1(\Gamma)$-equivariant, then η is $\pi_1(\Gamma)$-equivariant. We let $\Phi \colon k(\widetilde{\Gamma}) \to \mathsf{mod}\,\widetilde{A}$ be the following functor:

$$\Phi \colon L \longmapsto (\varphi(L), \psi(L), \eta_L) \quad .$$

The main theorem

Our Theorem A follows directly from the following result.

Theorem 6.4. *Let A be laura and not quasi-tilted of canonical type. Let $\pi \colon \widetilde{\Gamma} \to \Gamma$ be the universal cover of the connecting component Γ. There exists a covering functor $F \colon \widetilde{A} \to A$, and a commutative diagram:*

$$\begin{array}{ccc} k(\widetilde{\Gamma}) & \xrightarrow{\Phi} & \mathsf{mod}\,\widetilde{A} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle F_\lambda} \\ \mathsf{ind}\,\Gamma & \hookrightarrow & \mathsf{mod}\,A \end{array} \quad ,$$

where p is a well-behaved covering functor, and Φ is faithful. If, moreover, A is standard, then F and p may be assumed to be Galois coverings with group $\pi_1(\Gamma)$, and Φ is then $\pi_1(\Gamma)$-equivariant and full.

Proof: Let $p \colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ be a well-behaved covering functor. The commutativity of the shown diagram follows from the one of 5.12 and from the one of the diagram:

$$\begin{array}{ccc} k(\widetilde{\Gamma}) & \xrightarrow{\quad\psi\quad} & \mathsf{mod}\,\widetilde{C} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_\lambda} \\ \mathsf{ind}\,\Gamma & \xrightarrow[X \mapsto \mathsf{Hom}_A(C,X)]{} & \mathsf{mod}\,C \end{array} \quad ,$$

Since $F_\lambda \Phi = p$ and p is faithful, then Φ is faithful. Therefore $\Phi(k(\widetilde{\Gamma}))$ is contained in a connected component Ω of $\mathsf{mod}\,\widetilde{A}$.

We now prove that $\widetilde{A}$ is connected. Let $x \in \widetilde{A}_o$ and Q_x be the corresponding indecomposable projective $\widetilde{A}$-module. If $\widetilde{F}_\lambda Q_x \in C_o$, then, by construction, Q_x lies in the image of Φ, so that $Q_x \in \Omega$. If $F_\lambda Q_x \notin C_o$, then $F(x) \in B_o$ and $x \in \widetilde{B}_o$. In this case, there is a non-zero morphism $u \colon P_{F(x)} = \widetilde{F}_\lambda Q_x \to E$ in $\mathsf{mod}\,B$, where $E \in \Sigma$. Fix $\widetilde{E} \in p^{-1}(E)$ so that $F_\lambda \Phi(\widetilde{E}) = E$. Since u is non-zero, 4.2 implies that there is a non-zero morphism $Q_x \to {}^g\varphi(\widetilde{E}) = \Phi({}^g\widetilde{E})$ in $\mathsf{mod}\,\widetilde{B}$ (recall that φ is $\pi_1(\Gamma)$-equivariant on vertices). So $Q_x \in \Omega$, and Ω contains all the indecomposable projective $\widetilde{A}$-modules. This proves that $\widetilde{A}$ is connected.

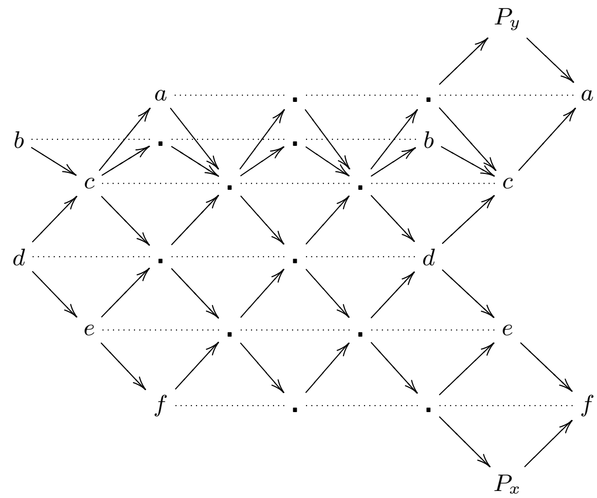
It remains to prove that if A is standard, then Φ is full and $\pi_1(\Gamma)$-equivariant, and F is Galois with group $\pi_1(\Gamma)$. In case A is standard, we suppose that $p \colon k(\widetilde{\Gamma}) \to \mathsf{ind}\,\Gamma$ is Galois with group $\pi_1(\Gamma)$. Therefore φ is $\pi_1(\Gamma)$-equivariant (see 5.13) and so is η. Hence, Φ is $\pi_1(\Gamma)$-equivariant. Also, F is a Galois covering because of 6.3. We prove that Φ is full. Given a morphism $f \colon \Phi(L) \to \Phi(N)$, there exists $(f_g)_g \in \bigoplus_{g \in \pi_1(\Gamma)} \mathsf{Hom}_{k(\Gamma)}(L, {}^gN)$ such that $F_\lambda(f) = \sum_g p(f_g)$ (because p is Galois). So $F_\lambda(f - \Phi(f_1)) - \sum_{g \neq 1} F_\lambda(\Phi(f_g)) = 0$. Since F is Galois with group $\pi_1(\Gamma)$ and since Φ is $\pi_1(\Gamma)$-equivariant, we get $f = \Phi(f_1)$. So Φ is full and the theorem is proved.■

The following example due to Riedtmann shows that F needs not be a Galois covering.

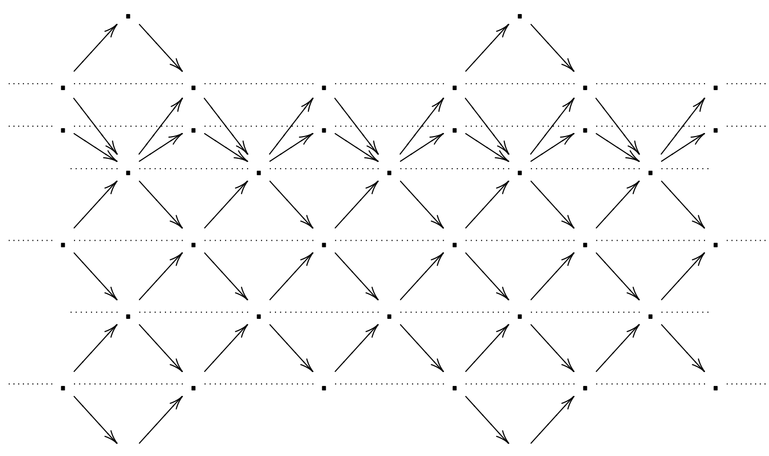
Example 6.5. Assume that $\mathsf{char}(k) = 2$ and let A be given by the bound quiver (see [14, §7, Ex. 14 bis] and [39]):

$$x \underset{\delta}{\overset{\sigma}{\leftrightarrows}} y \circlearrowright \rho\,, \quad \rho^4 = 0,\ \rho^2 = \delta\sigma,\ \sigma\delta = \sigma\rho\delta\ .$$

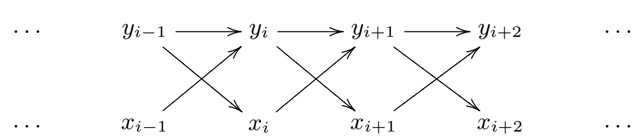
Then A is representation-finite and not standard, with the following Auslander-Reiten quiver:



where the two copies of a, b, c, d, e and f, respectively, are identified. Here, $\pi_1(\Gamma) \simeq \mathbb{Z}$ and $\widetilde{\Gamma}$ is as follows:



Therefore $\widetilde{A}$ is the locally bounded k-category, given by the following bound quiver:



$$\delta_{i+1}\sigma_i = \rho_{i+1}\rho_i,\ \ \sigma_{i+1}\delta_i = 0\ \text{, for all } i,$$

where σ_i, δ_i and ρ_i denote the arrows $y_i \to x_{i+1}$, $x_i \to y_{i+1}$, and $y_i \to y_{i+1}$, respectively. Now the covering functor $F\colon \widetilde{A} \to A$ is as follows:

1. $F(\rho_i) = \rho$ for every i,
2. $F(\sigma_i) = \sigma$ for every $i \equiv 0, 1 \bmod 4$,
3. $F(\sigma_i) = \sigma + \sigma\rho$ for every $i \equiv 2, 3 \bmod 4$,
4. $F(\delta_i) = \delta$ for every $i \equiv 1, 3 \bmod 4$,
5. $F(\delta_i) = \delta + \rho\delta$, for every $i \equiv 0, 2 \bmod 4$.

Obviously, F is a covering functor which is not Galois. Actually, one can easily check that A is simply connected, that is, the fundamental group (in the sense of [36]) of any presentation of A is trivial. Hence, A has no proper Galois covering by a locally bounded and connected k-category.

The following corollary is a particular case of our main theorem. We state it for later purposes.

Corollary 6.6. *Let A be a standard laura algebra and let Γ be its connecting component. There exists a Galois covering $F\colon \widetilde{A} \to A$ with group $\pi_1(\Gamma)$ together with a commutative diagram:*

$$\begin{array}{ccc} k(\widetilde{\Gamma}) & \xrightarrow{\Phi} & \mathrm{mod}\,\widetilde{A} \\ {\scriptstyle k(\pi)}\downarrow & & \downarrow{\scriptstyle F_\lambda} \\ k(\Gamma) & \hookrightarrow & \mathrm{mod}\,A \end{array},$$

where $\pi\colon \widetilde{\Gamma} \to \Gamma$ is the universal cover and where Φ is full, faithful and $\pi_1(\Gamma)$-equivariant. ■

We pose the following problems.

Problem 3. Does there exists a combinatorial characterisation of standardness for laura algebras (as happens for representation-finite algebras, see [13])?

Problem 4. Let A be left supported algebra. Is it possible to construct a covering $\widetilde{A} \to A$ associated to the universal coverings of the components of $\Gamma(\mathrm{mod}\,A)$ containing the Ext-injective modules of $\mathcal{L}_A$?

7 Galois coverings of the connecting component

Theorem 7.1. *Let A be a standard laura algebra, and $p\colon \Gamma' \to \Gamma$ be a Galois covering with group G of its connecting component. Then there exists a Galois covering $F'\colon A' \to A$ with group G, where A' is connected and locally bounded. Moreover, there exists a commutative diagram:*

$$\begin{array}{ccc} k(\Gamma') & \xrightarrow{\Phi'} & \mathrm{mod}\,A' \\ {\scriptstyle k(p)}\downarrow & & \downarrow{\scriptstyle F'_\lambda} \\ k(\Gamma) & \overset{j}{\hookrightarrow} & \mathrm{mod}\,A \end{array},$$

where Φ' is full, faithful and G-equivariant.

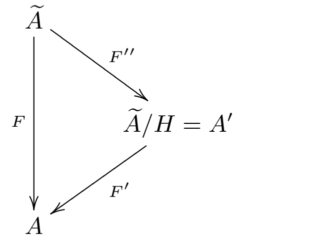
Proof: Since A is standard, there exists a full and faithful functor $j\colon k(\Gamma) \hookrightarrow \mathrm{ind}\,A$ with image $\mathrm{ind}\,\Gamma$, and which maps meshes to almost split sequences. Let $\pi\colon \widetilde{\Gamma} \to \Gamma$ be the universal cover. Then there exists a normal subgroup $H \lhd \pi_1(\Gamma)$ such that $\widetilde{\Gamma}/H \simeq \Gamma'$ and $G \simeq \pi_1(\Gamma)/H$, and such that under these identifications, the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\Gamma} & & \\ & \searrow^{q} & \\ {\scriptstyle \pi}\downarrow & & \widetilde{\Gamma}/H = \Gamma' \\ & \swarrow_{p} & \\ \Gamma & & \end{array},$$

where q is the projection. These identifications imply that $p\colon \Gamma' \to \Gamma$ is induced by $\pi\colon \widetilde{\Gamma} \to \Gamma$ by factoring out by H. By 6.6, there exist a Galois covering $F\colon \widetilde{A} \to A$ with group $\pi_1(\Gamma)$ and a commutative diagram:

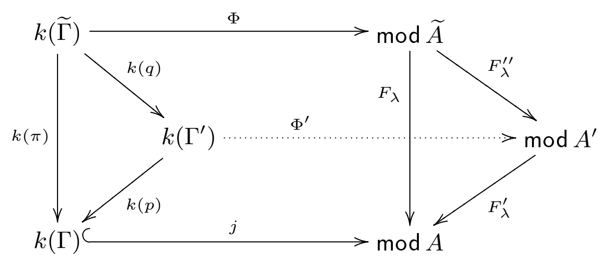
$$\begin{array}{ccc} k(\widetilde{\Gamma}) & \xrightarrow{\Phi} & \mathrm{mod}\,\widetilde{A} \\ {\scriptstyle k(\pi)}\downarrow & & \downarrow{\scriptstyle F_\lambda} \\ k(\Gamma) & \overset{j}{\hookrightarrow} & \mathrm{mod}\,A \end{array},$$

where Φ is full, faithful and $\pi_1(\Gamma)$-equivariant. Setting $A' = \widetilde{A}/H$, we deduce a Galois covering $F' \colon A' \to A$ with group G, making the following diagram commute:



,

where F'' is the natural projection (and F' is deduced from F by factoring out by H). Therefore we have a commutative diagram of solid arrows:



.

We prove the existence of the dotted arrow Φ' such that $\Phi'\ k(q) = F''_\lambda\ \Phi$. For this purpose, recall that $k(q)$ is a Galois covering with group H. Hence, it suffices to prove that $F''_\lambda\ \Phi$ is H-invariant. Indeed, we have $F''_\lambda = \Phi' h = F''_\lambda h \Phi' = F''_\lambda \Phi'$, for every $h \in H$, because Φ is $\pi_1(\Gamma)$-equivariant and F'' is a Galois covering with group H. Now, we prove that the whole diagram commutes. We have:

$$(F'_\lambda\ \Phi')\ k(q) = F'_\lambda F''_\lambda\ \Phi = F_\lambda\ \Phi = j\ k(\pi) = j\ k(p)\ k(q) \quad ,$$

hence, $F'_\lambda\ \Phi' = j\ k(p)$. We prove next that Φ' is full and faithful. Let $f \colon X \to Y$ be a morphism in $k(\Gamma')$ such that $\Phi'(f) = 0$. Fix $\widetilde{X}, \widetilde{Y} \in k(\widetilde{\Gamma})$ such that $q(\widetilde{X}) = X$ and $q(\widetilde{Y}) = Y$. Since $k(q)$ is Galois with group H, there exists $(f_h)_{h\in H} \in \bigoplus_{h\in H} k(\widetilde{\Gamma})(\widetilde{X}, {}^h\widetilde{Y})$ such that $\sum_{h\in H} k(q)(f_h) = f$. The commutativity of the diagram gives:

$$0 = \sum_{h\in H} F''_\lambda(\Phi(f_h)) \quad ,$$

where $(\Phi(f_h))_{h\in H} \in \bigoplus_{h\in H} \mathsf{Hom}_{\widetilde{A}}(\Phi(\widetilde{X}), {}^h\Phi(\widetilde{Y}))$ (recall that Φ is $\pi_1(\Gamma)$-equivariant). Since $F'' \colon \widetilde{A} \to A'$ is Galois with group H, we deduce that $\Phi(f_h) = 0$ for every $h \in H$, so that $f_h = 0$ for every $h \in H$, because Φ is faithful. Thus, $f = \sum_{h\in H} k(q)(f_h) = 0$ and Φ' is faithful. Let $X, Y \in k(\Gamma')$ and $u \colon \Phi'(X) \to \Phi'(Y)$ be a morphism in $\mathsf{mod}\, A'$, and fix $\widetilde{X}, \widetilde{Y} \in k(\widetilde{\Gamma})$ as above. In particular, we have $\Phi'(X) = F''_\lambda(\Phi(\widetilde{X}))$ and $\Phi'(Y) = F''_\lambda(\Phi(\widetilde{Y}))$. Therefore there exists $(\widetilde{u}_h)_{h\in H} \in \bigoplus_{h\in H} \mathsf{Hom}_{\widetilde{A}}(\Phi(\widetilde{X}), {}^h\Phi(\widetilde{Y}))$ such that $u = \sum_{h\in H} F''_\lambda(\widetilde{u}_h)$. Since Φ' is $\pi_1(\Gamma)$-equivariant, we have $\mathsf{Hom}_{\widetilde{A}}(\Phi'(\widetilde{X}), {}^h\Phi'(\widetilde{Y})) = \mathsf{Hom}_{\widetilde{A}}(\Phi'(\widetilde{X}), \Phi'({}^h\widetilde{Y}))$, for every $h \in H$. Using the fullness of Φ', we find $(\widetilde{f}_h)_{h\in H} \in \bigoplus_{h\in H} k(\widetilde{\Gamma})(\widetilde{X}, {}^h\widetilde{Y})$ such that $\widetilde{u}_h = \Phi(\widetilde{f}_h)$ for every $h \in H$. Since $k(q)$ is Galois with group H, we deduce that $\sum_{h\in H} k(q)(\widetilde{f}_h) \in k(\Gamma)(X, Y)$. Moreover, we have:

$$\Phi'\left(\sum_{h\in H} k(q)(\widetilde{f}_h)\right) = \sum_{h\in H} F''_\lambda \Phi(\widetilde{f}_h) = \sum_{h\in H} F''_\lambda \widetilde{u}_h = u \quad ,$$

whence the fullness of Φ'. To finish, it remains to prove that Φ' is G-equivariant. Let $g \in H$ be the residual class of $\sigma \in \pi_1(\Gamma)$ modulo H. We need to prove that $\Phi' \circ g = g \circ \Phi'$. We have $\Phi' \circ g \circ k(q) = \Phi' \circ k(q) \circ \sigma$, because $q \colon \widetilde{\Gamma} \to \Gamma' = \widetilde{\Gamma}/H$ is the canonical projection. Hence, $\Phi' \circ g \circ k(q) = F''_\lambda \circ \sigma \circ \Phi$, because $F''_\lambda \circ \Phi = \Phi' \circ k(q)$, and Φ is $\pi_1(\Gamma)$-equivariant. Since $F''_\lambda \circ \sigma = g \circ F''_\lambda$ (because F'' is deduced from F by factoring out by H), we have $\Phi' \circ g \circ k(q) = g \circ F''_\lambda \circ \Phi = g \circ \Phi' \circ k(q)$, and so $\Phi' \circ g = g \circ \Phi'$. The proof is complete. ■

Corollary 7.2. *In the situation of Theorem 7.1, the full subquiver Ω of $\Gamma(\mathsf{mod}\,A')$ with vertex set equal to $\{X \in \mathsf{ind}\,A' \mid F'_\lambda X \in \Gamma\}$ is a faithful and generalised standard component of $\Gamma(\mathsf{mod}\,A')$, isomorphic, as a translation quiver, to Γ'. Moreover, there exists a Galois covering of translation quivers $\Gamma' \to \Gamma$ with group G extending the map $X \mapsto F'_\lambda X$.*

Proof: Since $F'_\lambda \Phi = j\,k(p)$, the module $\Phi(X)$ is indecomposable and lies in Ω, for every $X \in \Gamma'$. On the other hand if $X \in \Omega$, there exists $X' \in \Gamma'$ such that $F'_\lambda X = k(p)(X')$. Therefore $F'_\lambda X = F'_\lambda \Phi(X')$. Since X and $\Phi(X')$ are indecomposable, there exists $g \in G$ such that $X = {}^g\Phi(X') = \Phi({}^gX') \in \Phi(\Gamma')$. Thus, we have shown that:

(i) Ω coincides with the full subquiver of $\Gamma(\mathsf{mod}\,A')$ with set of vertices $\{\Phi(X) \mid X \in \Gamma'\}$.

Let $X \xrightarrow{u} Y$ be an arrow in Γ'. Since $F'_\lambda \Phi = j\,k(p)$, then $F'_\lambda\ \Phi(u)$ is an irreducible morphism between indecomposable A-modules. Using [32, Lem. 2.1], we deduce that $\Phi(u)$ is irreducible. This proves that:

(ii) The full subquiver of $\Gamma(\mathsf{mod}\,A')$ with set of vertices $\{\Phi(X) \mid X \in \Gamma'\}$ is contained in a connected component of $\Gamma(\mathsf{mod}\,A')$.

Combining (i), (ii) and [32, Lem. 2.3], we deduce that Ω is a component of $\Gamma(\mathsf{mod}\,A')$. The same lemma shows that Ω is faithful and generalised standard because so is Γ.

Let us prove that Φ' induces an isomorphism between Γ' and Ω. Since $q\colon \widetilde{\Gamma} \to \Gamma'$ is surjective on vertices and $F''_\lambda \Phi = \Phi'\,k(q)$, we know that every $X \in \Omega$ lies in the image of F''_λ. Also, $k(q)$ and Φ commute with the translation, and so does F''_λ (see [32, Lem. 2.1]). Hence Φ' commutes with the translation. Finally $k(q)$ maps meshes to meshes, and Φ maps meshes to almost split sequences. So Φ' maps meshes to almost split sequences (see [32, Lem. 2.2]). Therefore there exists a morphism of translation quivers $\Gamma' \to \Omega$ extending the map $X \mapsto \Phi'(X)$ on vertices. Since it is a bijection on vertices, we deduce that it is an isomorphism $\Gamma' \xrightarrow{\sim} \Omega$.

Finally, the stabiliser $G_X = \{g \in G \mid {}^gX \simeq X\}$ of X is trivial, for every $X \in \Omega$ because G acts freely on Γ' and Φ' is G-equivariant. Therefore there exists a Galois covering of translation quivers $\Omega \to \Gamma$ with group G and extending the map $X \mapsto F'_\lambda(X)$ (see [25, 3.6]). ■

Corollary 7.3. *In the situation of Theorem 7.1, if G is finite, then A' is a finite dimensional standard laura algebra.*

Proof: Since G is finite, A' is finite dimensional. By the preceding corollary, Γ' is generalised standard and faithful. Since Γ has only finitely many isomorphism classes of indecomposable modules lying on oriented cycles, the same is true for Γ'. Therefore Γ' is quasi-directed and faithful. Applying [37, 3.1] (or [45, Thm. 2]) shows that A' is a laura algebra with Γ' as a connecting component. Finally, the full and faithful functor $\Phi'\colon k(\Gamma') \to \mathsf{mod}\,A'$ with image equal to $\mathsf{ind}\,\Gamma'$ shows that Γ' is standard, that is, A' is standard. ■

Remark 7.4. The above corollary may be compared with [7, Thm. 1.2] and [32, Thm. 3]. Indeed, if A' is a finite dimensional algebra endowed with the free action of a (necessarily finite) group G, then the category A/G and the skew-group algebra $A[G]$ are Morita equivalent.

We end this section with the following corollary:

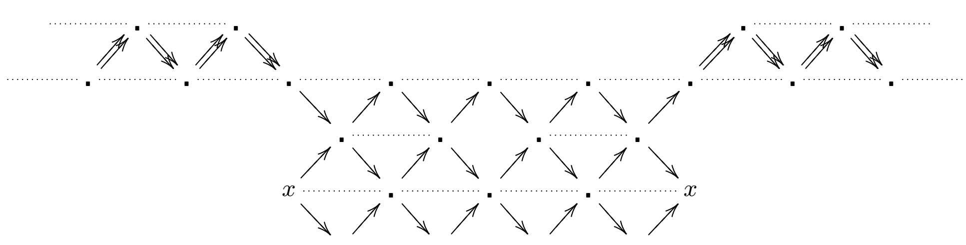
Corollary 7.5. *In the situation of Theorem 7.1, if G is finite, then:*

(a) A is tame if and only if A' is tame.

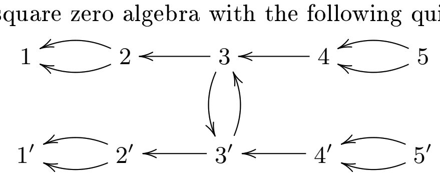
(b) A is wild if and only if A' is wild.

Proof: This follows from Theorem 7.1 and from [3, 5.3, (b)]. ■

Example 7.6. Consider the algebra A of 3.6, (a). The connecting component Γ admits a Galois covering with group $\mathbb{Z}/2\mathbb{Z}$ by the following translation quiver:



where the two copies of x are identified. With our construction, we get a Galois covering $F\colon A' \to A$ with group $\mathbb{Z}/2\mathbb{Z}$, where A' is the radical square zero algebra with the following quiver:



.

Note that both A and A' are tame.

8 Proof of Theorem B

We recall the definition of the orbit graph $\mathcal{O}(\Gamma)$ (see [14, 4.2]). Given a vertex $x \in \Gamma$, its τ orbit x^τ is the set $\{y \in \Gamma \mid y = \tau^l x,\ \text{for some}\ l \in \mathbb{Z}\}$. Also, we fix a polarisation σ in Γ. Recall that the periodic components of Γ are defined as follows. Consider the full translation subquiver of Γ with vertices the periodic vertices in Γ. To this subquiver, add a new arrow $x \to \tau x$ for every vertex x. A *periodic component* of Γ is a connected component of the obtained quiver. Then:

1. The orbit graph $\mathcal{O}(\Gamma)$ has as vertices the periodic components of Γ and the τ-orbits of the non-periodic vertices.
2. For each periodic component, there is a loop attached to the associated vertex in $\mathcal{O}(\Gamma)$.
3. Let u^σ be the σ-orbit of an arrow $u\colon x \to y$. If both x and y are non-periodic, an edge is attached between x^τ and y^τ. If x (or y) is non-periodic and y (or x) is periodic, then an edge is attached between x^τ (or y^τ) and the vertex associated to the periodic component containing y (or x, respectively). Otherwise, there is no arrow associated to u^σ.

By [14, 4.2], the fundamental group of the orbit graph $\mathcal{O}(\Gamma)$ is isomorphic to $\pi_1(\Gamma)$.

Throughout this section, we assume that A is standard laura, having Γ as its connecting component. We use the following two lemmata:

Lemma 8.1. *If $\mathcal{O}(\Gamma)$ is a tree, then A is weakly shod.*

Proof: If $\mathcal{O}(\Gamma)$ is a tree, then Γ is simply connected (see [14, 4.1 and 4.2]). In particular, Γ has no oriented cycle. So A is weakly shod. ■

Lemma 8.2. *Let A be a (non necessarily connected) laura algebra which is not quasi-tilted of canonical type. If the orbit graph of any connecting component is a tree, then A is simply connected and $\mathsf{HH}^1(A) = 0$.*

Proof: This follows from the preceding lemma and from [32, Cor. 2]. ■

We now prove Theorem B whose statement we recall for convenience.

Theorem B. *Let A be a standard laura algebra, and Γ its connecting component(s). The following are equivalent:*

(a) *A is simply connected.*

(b) $\mathsf{HH}^1(A) = 0$.

(c) *Γ is simply connected.*

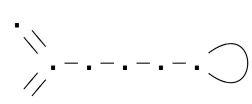
(d) *The orbit graph $\mathcal{O}(\Gamma)$ is a tree.*

Moreover, if these conditions are verified, then A is weakly shod.

Proof: By [14, 4.1, 4.2] and the above lemma, (c) and (d) are equivalent and imply (a) and (b). If A is simply connected, then 6.6 implies that $\pi_1(\Gamma) = 1$. So (a) implies (c). Finally, assume that $\mathsf{HH}^1(A) = 0$. By 6.6, the algebra A admits a Galois covering with group $\pi_1(\Gamma)$. This group is free because of [14, 4.2]. On the other hand, the rank of $\pi_1(\Gamma)$ is less than or equal to $\dim \mathsf{HH}^1(A)$ because of [18, Thm. 4.1]. Therefore $\pi_1(\Gamma) = 1$. So (b) implies (c). So the conditions are equivalent, and they imply that A is weakly shod by 8.1. ■

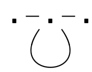
We illustrate Theorem B on the following examples. In particular, note that this theorem does not necessarily hold true if one drops the standard condition.

Example 8.3. (a) Let A be as in 3.6, (a). Then A clearly admits a Galois covering with group a free group of rank 3 by a locally bounded k-category. It is given by the universal cover of the underlying graph of the ordinary quiver. So A is not simply connected. The orbit graph $\mathcal{O}(\Gamma)$ of the connecting component Γ is as follows:

 .

Its fundamental group is free of rank 3. A straightforward computation shows that $\dim \mathsf{HH}^1(A) = 7$ (see also [19, Thm. 1]).

(b) Let A be as in 6.5. As already noticed, A is a simply connected representation-finite algebra. Also, it is not standard. The orbit graph of its Auslander-Reiten quiver is as follows:

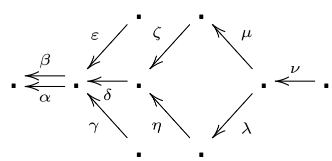


Finally, A admits the following outer derivation, yielding a non-zero element in $\mathsf{HH}^1(A)$ (see [15, 4.2])

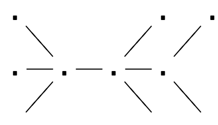
$$\begin{array}{rrcl} d\colon & A & \to & A \\ & \sigma,\delta & \mapsto & 0 \\ & \rho & \mapsto & \rho^3 \end{array} .$$

This example shows that Theorem B may fail if one drops the standard condition. Note that the definition of simple connectedness used in [15, 4.3] differs slightly from ours: In *loc. cit.*, as in [14, § 6], a representation-finite algebra is called simply connected if its Auslander-Reiten quiver is simply connected.

(c) Let A be the algebra given by the quiver:



bound by the relations $\beta\varepsilon = 0$, $\alpha\gamma = 0$, $\beta\delta = \alpha\delta$, $\delta\zeta = 0$, $\delta\eta = 0$, $\zeta\mu = \eta\lambda$, $\zeta\mu\nu = 0$ Then A is laura. Actually, it is right glued in the sense of [2, 3.1]. The orbit graph $\mathcal{O}(\Gamma)$ of its connecting component Γ is as follows:



It is a tree. Also, A is clearly simply connected, and it is not hard to see that $\mathsf{HH}^1(A) = 0$ using, for instance, Happel's long exact sequence (see [27, 5.3]).

We pose the following problem.

Problem 5. Let A be a non-standard laura algebra. How can the vanishing of $\mathsf{HH}^1(A)$ be expressed in terms of topological properties of A?

9 Special biserial laura algebras

Throughout this section, we fix a laura algebra $A = kQ/I$, where (Q, I) is special biserial. This class of algebras is characterised in [23]. As we shall see, Theorem B holds true for A'. In the following proposition, we consider a specific kind of Galois covering of A: Given a Galois covering of bound quivers $(\overline{Q}, \overline{I}) \to (Q, I)$ with group G (in the sense of [16]), we consider the induced Galois covering $\overline{A} = k\overline{Q}/\overline{I} \to A$ with group G.

Proposition 9.1. *Let $A = kQ/I$ be as above. Let $(\overline{Q}, \overline{I}) \to (Q, I)$ be a Galois covering of bound quivers with group G (with $\overline{Q}$ connected). Let $F\colon \overline{A} = k\overline{Q}/\overline{I} \to A$ be the induced Galois covering with group G. Then there exists a Galois covering of translation quivers $\overline{\Gamma} \to \Gamma$ with group G, extending the map $X \mapsto F_\lambda X$, where $\overline{\Gamma}$ is the full translation subquiver of $\Gamma(\mathsf{mod}\,\overline{A})$ with set of objects equal to $\{X \in \mathsf{ind}\,\mathcal{C} \mid F_\lambda X \in \Gamma\} \subseteq \Gamma(\mathsf{mod}\,\mathcal{C})$. Moreover, $\widetilde{\Gamma}$ is a component of $\Gamma(\mathsf{mod}\,\overline{A})$.*

Proof: We only need to prove the two following facts:

1. If $X \in \Gamma$, then there exists $\overline{X} \in \mathsf{ind}\,\overline{A}$ such that $F_\lambda \overline{X} = X$, and $\overline{X} \not\simeq {}^g\overline{X}$ for every $g \in G\backslash\{1\}$.
2. The full subquiver $\overline{\Gamma}$ of $\Gamma(\mathsf{mod}\,\overline{A})$ is a component.

Since Γ has modules which are not τ_A-invariant, all modules in Γ are string modules. Let γ be a string in Q. Since $(\overline{Q}, \overline{I}) \to (Q, I)$ is a covering of bound quivers, there exists a string $\overline{\gamma}$ in $\overline{Q}$ whose image under $Q' \to Q$ is equal to γ. Therefore if we write $M(\gamma)$ for the corresponding string module, then $F_\lambda(M(\overline{\gamma})) = M(\gamma)$, and ${}^gM(\overline{\gamma}) = M({}^g\overline{\gamma})$, so that $G_{M(\overline{\gamma})} = 1$. This proves the first assertion. Also, by [25, 3.6], there exists a Galois covering of translation quivers $\overline{\Gamma} \to \Gamma$ extending the map $X \mapsto F_\lambda X$.

There remains to prove that $\overline{\Gamma}$ is a connected component of $\Gamma(\mathsf{mod}\,\overline{A})$. By [32, Prop. 2.3], $\overline{\Gamma} \subseteq \Gamma(\mathsf{mod}\,\overline{A})$ is a union of components of $\Gamma(\mathsf{mod}\,\overline{A})$. Let $T_1, \ldots, T_n$ be the (pairwise distinct) modules of Γ which are either

Ext-injectives of $\mathcal{L}_A$ or projective modules not in $\mathcal{L}_A$. Because of [5, 4.2, 4.4], the module $T = T_1 \oplus \ldots \oplus T_n$ is a tilting A-module. Let $\mathcal{T}$ be the full subcategory of $\mathsf{ind}\,\overline{A}$ with set of vertices equal to:

$$\{X \in \overline{\Gamma} \mid F_\lambda X \in \{T_1, \ldots, T_n\}\} \ .$$

Let us prove that $\mathcal{T}$ is connected. Let $P \in \mathsf{ind}\,\overline{A}$ be projective. There is an exact sequence $0 \to F_\lambda P \to T^{(1)} \to T^{(2)} \to 0$ in $\mathsf{mod}\,A$, with $T^{(1)}, T^{(2)} \in \mathsf{add}\,T$. Since $\mathsf{Ext}^1_A(T^{(2)}, T^{(1)}) = 0$, 4.3 yields an exact sequence in $\mathsf{mod}\,\overline{A}$:

$$0 \to P \to X_P^{(1)} \to X_P^{(2)} \to 0 \ , \qquad (\Delta_P)$$

with $X^{(1)}, X^{(2)} \in \mathsf{add}\,\mathcal{T}$. Note that $\mathsf{pd}\,X \leqslant 1$ for every $X \in \mathcal{T}$ because F_λ is exact. So:

- $\mathcal{T} \subseteq \mathcal{K}^b(\mathsf{proj}\,\overline{A})$, where $\mathcal{K}^b$ stands for the homotopy category of bounded complexes.
- The exact sequences Δ_P, for $P \in \mathsf{ind}\,\overline{A}$ projective, show that $\mathcal{T}$ generates $\mathcal{K}^b(\mathsf{proj}\,\overline{A})$, as a triangulated category.

Now, $\mathcal{K}^b(\mathsf{proj}\,\overline{A})$ is connected because so is $\overline{A}$. So $\mathcal{T}$ is connected. On the other hand, [32, Prop. 2.3] shows that $\mathsf{rad}^\infty(X, Y) = 0$ for every $X, Y \in \overline{\Gamma}$, because the same property holds for Γ. Therefore there exists a connected component Ω of $\overline{\Gamma}$ containing all the modules in $\mathcal{T}$. Since G acts transitively on the connected components of $\overline{\Gamma}$ (thanks to the Galois covering $\overline{\Gamma} \to \Gamma$) we deduce that $\overline{\Gamma} = \Omega$. So $\overline{\Gamma}$ is a component. ■

Before proving our next lemma, we state the characterisation of [23] of those special biserial algebras which are laura. Given a special biserial quiver (Q, I), consider a walk in Q:

$$\gamma = \rho\omega_1\omega_2\omega_3\rho'$$

such that the following conditions are statisfied:

1. ρ and ρ' are paths lying in I,
2. ω_2 is a cyclic walk, that is, its source and target are equal, and
3. $\omega_1\omega_2\omega_3$ is a string, that is, it contains neither a path lying in I, nor the formal inverse of a path lying in I.

Then γ is called an *intertwined double-zero* if for every $n \geqslant 1$, the walk $\omega_1\omega_2^n\omega_3$ is a string. This combinatorial configuration was defined by Dionne in [23] and called DOZE for short.

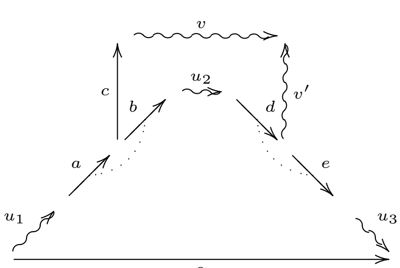
Theorem 9.2. *see [23] Let (Q, I) be a special biserial quiver. Then kQ/I is a laura algebra if and only if there is no DOZE in (Q, I).*

If we let $(\overline{Q}, \overline{I}) \to (Q, I)$ be the universal cover (see [36]), then we get a Galois covering $F\colon \overline{A} \to A$ with group $\pi_1(Q, I)$. Later, we prove that F coincides with the covering functor given by our main result. For this purpose, we need the following lemma. Recall that given an arrow $\alpha\colon x \to y$ in a quiver Q, a path $u = \beta_1 \ldots \beta_s$ from x to y distinct from α is called a *bypass*.

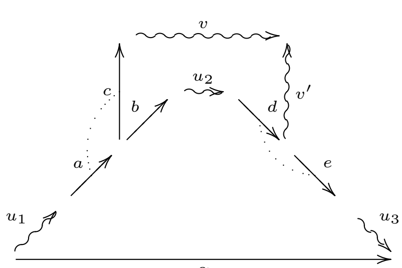
Lemma 9.3. *Let $\sim$ be the homotopy relation defined by (Q, I) (as in [36]). Then $\alpha \not\sim u$ for every bypass u of α in Q.*

Proof of 9.3: We proceed by contradiction and assume that $\alpha \sim u$ for some bypass u of α. Thus, α appears in a path belonging to a binomial relation, and the same holds for at least one arrow of u (with a binomial relation which needs not be the same as the one for α).

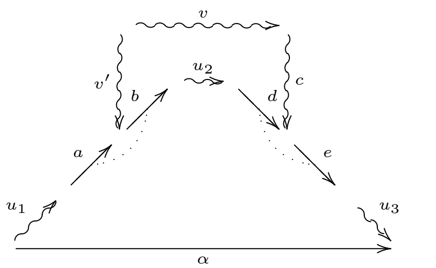
Assume first that u is not an arrow. Let $v_1 - t\,v_2$ be a binomial relation such that v_1, v_2 are paths, and v_1 and u share at least one arrow. Assume that neither v_1 is contained in u, nor u is contained in v_1. Then one can see that (Q, I) contains one of the following five bound quivers, where the monomial relations (denoted by dotted lines) are due to the fact that (Q, I) is special biserial, and where the symbol $\rightsquigarrow$ denotes a path:



$u = u_3edu_2bau_1,\ v_1 = v'du_2b,\ v_2 = vc$
DOZE

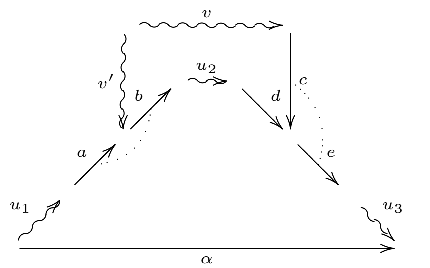


$u = u_3edu_2bau_1,\ v_1 = v'du_2b,\ v_2 = vc$
case 1 .



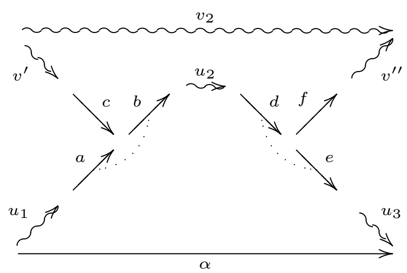
$u = u_3edu_2bau_1,\ v_1 = du_2bv',\ v_2 = ev$

DOZE



$u = u_3edu_2bau_1,\ v_1 = du_2bv',\ v_2 = cv$

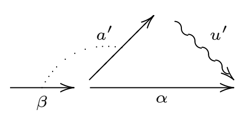
case 2 .



$u = u_3edu_2bau_1,\ v_1 = v''fdu_2bcv'$

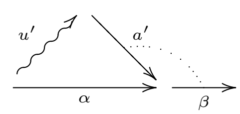
DOZE .

We have specified the cases which obviously contain a DOZE. These cases cannot appear because A is laura. So only cases 1 and 2 are possible. Now, α also appears in a binomial relation: There exists an arrow β such that $\beta\alpha$ or $\alpha\beta$ is a subpath of a path appearing in a binomial relation (and, in particular, we have $\beta\alpha \notin I$ or $\alpha\beta \notin I$, respectively). Thus (Q, I) also contains a bound subquiver of one of the following two shapes:



$u = u'a$

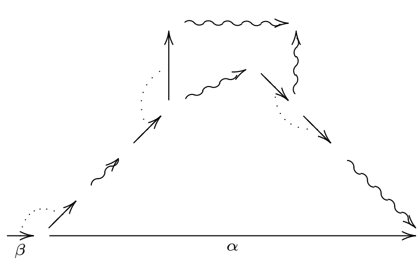
case a



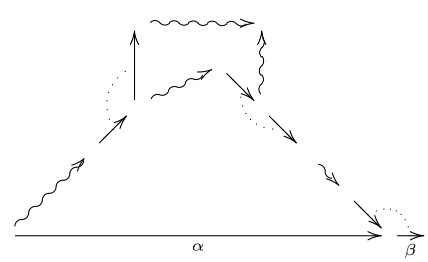
$u = au'$

case b .

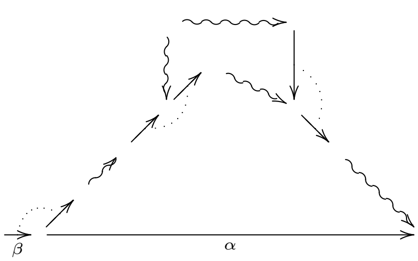
According to the cases 1, 2, a and b, we deduce that (Q, I) contains one of the following bound quivers:



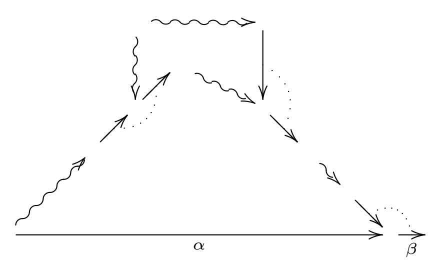
case 1 + case a



case 1 + case b .

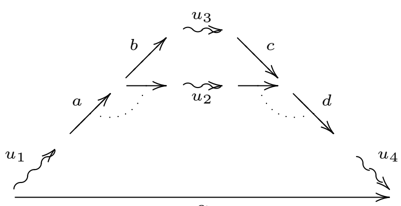


case 2 + case a

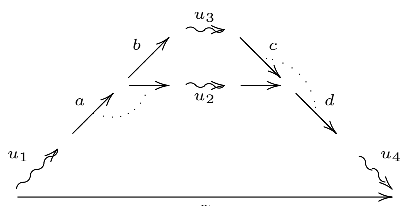


case 2 + case b .

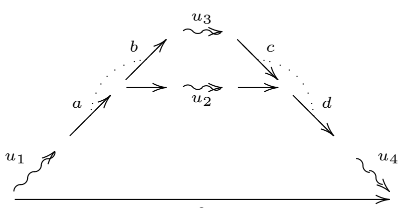
Obviously, (Q, I) contains a DOZE in each of the above four cases, whereas A is laura. This contradiction shows that v_1 is contained in u, or u is contained in v_1. Now, assume that v_1 is contained in u. So (Q, I) contains one of the following four bound quivers:



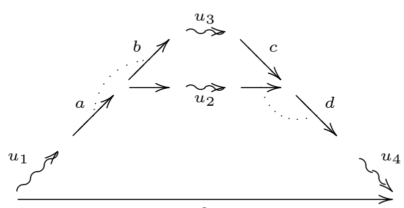
$u = u_4 d u_2 a u_1,\ v_1 = u_2,\ v_2 = c u_3 b$ DOZE



$u = u_4 d u_2 a u_1,\ v_1 = u_2,\ v_2 = c u_3 b$ case 3 .

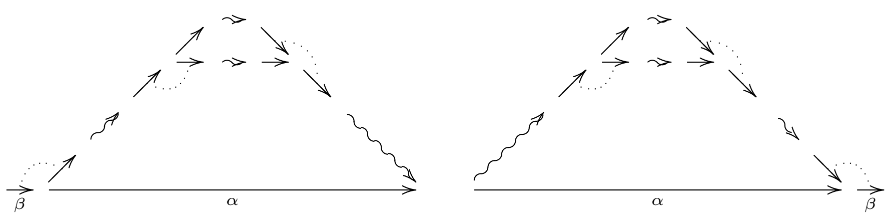


$u = u_4 d u_2 a u_1,\ v_1 = u_2,\ v_2 = c u_3 b$ DOZE

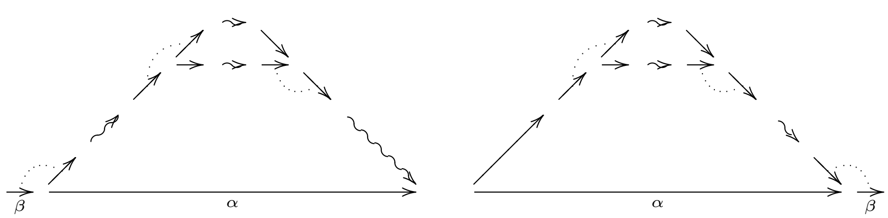


$u = u_4 d u_2 a u_1,\ v_1 = u_2,\ v_2 = c u_3 b$ case 4 .

Again, the cases specified with "DOZE" contain a DOZE. So only cases 3 and 4 are possible. If we combine these two cases with a and b, we find that (Q, I) contains one of the following four bound quivers:

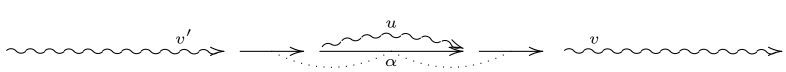


case 3 + case a case 3 + case b .



case 4 + case a case 4 + case b .

Therefore (Q, I) contains a DOZE in each case, whereas A is laura. This contradiction shows that u is necessarily contained in v_1. Remark that since $v_1, v_2 \notin I$, since no vertex of Q is the source (or the target) of three arrows, and since (Q, I) contains no DOZE, we necessarily have $v_1 = v u_1 v'$, with v, v' two non-trivial paths. Consequently, (Q, I) contains a DOZE, because $v_1 \notin I$:

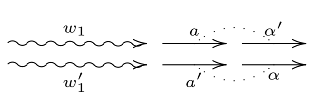


This is absurd. Therefore u is necessarily an arrow.

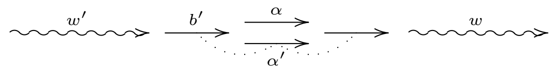
We now assume that $u = \alpha'$ **is an arrow:** $x \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} y$ **.** Let $r = v_1 - t\, v_2$ and $r' = v_1' - t'\, v_2'$ be binomial relations such that α and α' are contained in v_1 and v_1', respectively. We assume first that r ends at y. Since (Q, I) is special biserial, we have:

- $v_1 = \alpha a w_1$, with $a \in Q_1$ and w_1 a path.
- $v_1' = \alpha' a' w_1'$, with $a' \in Q_1$ and w_1' a path.

Therefore (Q, I) contains the following bound quiver:



Thus, (Q, I) contains a DOZE (recall that $v_1, v_1 \notin I$). This is absurd, so r does not end at y. Similarly, r does not start at x. So $v_1 = wb\alpha b'w'$, where $a, a' \in Q_1$ and w, w' are paths. Hence (Q, I) contains the following bound quiver:



.

Again, we find that (Q, I) contains a DOZE. This last contradiction proves the lemma. ■

The preceding lemma implies the following result. A locally bounded k-category $\mathcal{C}$ is called *constricted* if, for any arrow $\alpha\colon x \to y$ in the ordinary quiver of $\mathcal{C}$ we have $\dim_k \mathcal{C}(x, y) = 0$. It is shown in [12] that, if an algebra A is constricted, then the fundamental group does not depend on the particular presentation of A.

Proposition 9.4. *Let (Q, I) be a finite special biserial bound quiver such that kQ/I is laura not quasi-tilted of canonical type. Let $(\overline{Q}, \overline{I}) \to (Q, I)$ be the universal cover with group $\pi_1(Q, I)$. Then $\overline{Q}$ has no bypass, so that $k\overline{Q}/\overline{I}$ is constricted.*

Proof: Following the construction of [36] of the universal cover, we know that: If $\overline{Q}$ has a bypass, then there exists a bypass u of an arrow α in Q such that α and u are homotopic. The preceding lemma shows that this cannot happen. Whence the proposition. ■

In order to prove Theorem B for special biserial algebras, we need to prove that the group $\pi_1(Q, I)$ is free. For this purpose we use the following result of [17]. Note that the setting therein concerned triangular algebras. Actually, one can check that the triangular hypothesis plays no rôle in the following result. We let the reader state the corresponding dual result.

Proposition 9.5. *see [17, Prop. 2.2] Let A be a basic and connected finite dimensional k-algebra. Assume that $A = [M]B = \begin{bmatrix} k & 0 \\ M & B \end{bmatrix}$ where B is an algebra and $M \in \mathsf{mod}\, B$. Let $B_1, \ldots, B_t$ be the connected components of B. Let $M_1, \ldots, M_t$ be the modules over $B_1, \ldots, B_t$ respectively, such that $M = M_1 \oplus \ldots M_t$. Let Q_A and Q_{B_i} be the ordinary quivers of A and B_i respectively, for every i. Let $\nu_A\colon kQ_A \twoheadrightarrow A$ be a bound quiver presentation. For every i, the presentation $\nu_i\colon kQ_{B_i} \twoheadrightarrow B_i$ obtained by restriction defines a Galois covering $F^{(i)}$ with group $\pi_1(Q_{B_i}, \mathsf{Ker}\,\nu_i)$ of B_i. Assume the following:*

(a) For every i, the fundamental group $\pi_1(Q_{B_i}, \mathsf{Ker}\,\nu_i)$ is free.

(b) For every i, every indecomposable summand of M_i lies in the image of the push-down functor $F_\lambda^{(i)}$.

Then the group $\pi_1(Q_A, \mathsf{Ker}\,\nu_A)$ is free. ■

Using the preceding proposition, we can prove the following result.

Proposition 9.6. *Let (Q, I) be a connected special biserial bound quiver. Let $A = kQ/I$ and assume that A is laura. Then $\pi_1(Q, I)$ is a free group.*

Proof: We proceed by induction on $|A_o|$. Assume first that A is representation-finite. Then it is standard. This follows from the fact that (Q, I) is special biserial and from [13, 2.7]. Also, Q has no multiple arrows. It follows from [25, 36] that $\pi_1(Q, I)$ is free. Assume now that A is representation-infinite, and that the proposition holds true for special biserial laura algebras having less objects than A. Then A can be written as one-point extension or coextension (see [3, 5.4]). Assume that $A = [M]B$ (the other case is dealt with similarly). Then B is a product of connected laura algebras (see [4]). Also, the presentation of A by (Q, I) restricts to a presentation of B which is special biserial. Moreover, given a connected component B' of B, the fundamental group associated to the induced presentation of B' is free (by the induction hypothesis). Finally, M is a direct sum of string modules. As observed at the beginning of the proof of 9.1, given a special biserial quiver (Q', I') and given a Galois covering $(Q'', I'') \to (Q, I)$, any string module over kQ'/I' lies in the image of the push-down functor of the induced Galois covering $kQ''/I'' \to kQ'/I'$. Therefore the hypotheses of 9.5 are satisfied. Hence, $\pi_1(Q, I)$ is free. ■

We now prove that Theorem B holds true for special biserial laura algebras.

Theorem 9.7. *Let A be a special biserial laura algebra. The following conditions are equivalent.*

(a) A is simply connected.

(b) $\mathsf{HH}^1(A) = 0$.

If moreover, A is not quasi-tilted of canonical type, then these conditions are equivalent to the following ones:

(c) The connecting component Γ is simply connected.

(d) The orbit graph $\mathcal{O}(\Gamma)$ is a tree.

Proof: The quasi-tilted case was settled in [34, Cor. 2]. So we assume that A is not quasi-tilted. We therefore write $F\colon \widetilde{A} \to A$ for the covering functor given by our main theorem. Recall that (c) and (d) are equivalent (see [14, § 4]). Let (Q,I) be a special biserial quiver such that $A \simeq kQ/I$. According to 9.6, we denote by r the rank of the free group $\pi_1(Q,I)$. We write $F'\colon \overline{A} \to A$ for the Galois covering with group $\pi_1(Q,I)$ induced by the presentation $A \simeq kQ/I$. Recall that F' defines an injective linear map $\mathsf{Hom}(\pi_1(Q,I),k^+) \hookrightarrow \mathsf{HH}^1(A)$ (see [22, Cor. 3]). In particular, we have $\dim \mathsf{HH}^1(A) \geqslant r$.

If $\mathsf{HH}^1(A) = 0$ then $r = 0$. So $\pi_1(Q,I)$ is trivial.

If A is simply connected, then $\pi_1(Q,I)$ is trivial and $F'\colon \overline{A} \to A$ is an isomorphism (see [31, Rem. 6.1.5]). So, A is constricted because of 9.3. Now [31, Prop. 6.2.27] shows that the covering functor $F\colon \widetilde{A} \to A$ is induced by a covering $(Q',I') \to (Q,I)$ of bound quivers. Since $\pi_1(Q,I)$ is trivial, this covering is trivial, that is, $(Q',I') \to (Q,I)$ is an isomorphism (see [36]). In other words, we have $\widetilde{A} \simeq A$ and, therefore, $\widetilde{\Gamma} \simeq \Gamma$. In particular, A is weakly shod (see 8.1).

Thus, any of the hypotheses (a), (b), (c) or (d) implies that A is weakly shod. The equivalence between these conditions is therefore given by [32, Cor. 2]. ■

We end this section with the following problem.

Problem 6. Assume that A is laura special biserial. Is it standard?

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