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COILS AND MULTICOIL ALGEBRAS

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Dedicated to the memory of Maurice Auslander.

ABSTRACT. The purpose of these notes is first to explain how to construct coils and multicoil algebras, and next, by surveying some recent results, to show that multicoil algebras have a relatively simple representation theory.

Introduction

These notes are a faithful, though somewhat extended, account of the talk given by the first author during the "Tame Day" (19th August, 1994) of the Workshop on Representation Theory and Related Topics, held at the Universidad Nacional Autónoma de México from the 16th to the 20th of August, 1994. The aim of this talk was to present a class of finite dimensional algebras of polynomial growth, namely the class of multicoil algebras, which has attracted a lot of interest recently (see, for instance, [4, 5, 6, 16, 17, 19, 23, 27, 28, 29]). The main motivation for studying multicoil algebras comes from the following. To study the representation theory of an algebra, one strategy consists in using covering techniques to reduce the problem to the case where the algebra is simply connected, then in solving the problem in this latter case. At present, little is known about representation-infinite simply connected algebras (see however [1, 2, 11, 16, 17, 23, 24, 27]) so we are still far from a satisfactory theory. We do however have the following result of the second author (see [23], [27] or [22] (9.4)): a strongly simply connected algebra (in the sense of [24]) is of polynomial growth if and only if it is a multicoil algebra. This shows that, in order to understand the module categories of strongly simply connected algebras of polynomial growth, one needs to understand those of multicoil algebras.

The purpose of these notes is twofold. First, we wish to explain how to construct multicoil algebras. Second, we wish to show that multicoil algebras have a relatively simple representation theory. It will indeed follow at once from the definition that one has a full control of the behaviour of the cycles of the module category, but

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also, we shall see that one can give a precise description of all the indecomposable modules. Throughout, we shall focus on the computation of examples. The results quoted are mainly joint results of the authors, or the authors and B. Tomé.

These notes are organised as follows. After a brief introductory section 1, in which we fix the notation and recall some of the relevant definitions, we define in section 2 the notions of admissible operations and coils. Section 3 is devoted to the study of coil enlargements of algebras, section 4 to multicoil algebras, and section 5 to the description of the indecomposable modules over a multicoil algebra.

1. Notation and preliminary definitions

1.1. Throughout this paper, k denotes a fixed algebraically closed field. By an algebra A is meant a basic, connected, associative finite dimensional k-algebra with an identity. Thus, there exists a connected bound quiver (Q_A, I) and an isomorphism $A \cong kQ_A/I$. Equivalently, $A = kQ_A/I$ may be considered as a k-linear category, of which the object class A_0 is the set of points of Q_A, and the set of morphisms from x to y is the quotient of the k-vector space $kQ_A(x,y)$ of all formal k-linear combinations of paths in Q_A from x to y by the subspace $I(x,y) = I \cap kQ_A(x,y)$, see [10]. A full subcategory C of A is called **convex** (in A) if, for any path $a_0 \to a_1 \to \cdots \to a_t$ in A such that $a_0, a_t \in C_0$, we have $a_i \in C_0$ for all $0 \le i \le t$. It is called **triangular** if Q_C contains no oriented cycle.

By an A-module is meant a finitely generated right A-module. We denote by mod A the category of A-modules and by ind A a full subcategory consisting of a complete set of representatives of the isomorphism classes of indecomposable A-modules. For a full subcategory $\mathcal{C}$ of mod A, we denote by add $\mathcal{C}$ the additive full subcategory of mod A consisting of the direct sums of indecomposable direct summands of the objects in $\mathcal{C}$. If $\mathcal{C}$ consists of a single module M, we write add $\mathcal{C}$ = add M. For two full subcategories $\mathcal{C}, \mathcal{C}'$ of mod A, the notation $\mathrm{Hom}_A(\mathcal{C}, \mathcal{C}') = 0$ means that $\mathrm{Hom}_A(M, M') = 0$ for all M in $\mathcal{C}$ and M' in $\mathcal{C}'$. A **cycle** in mod A is a sequence of non-zero non-isomorphisms $M_0 \to M_1 \to \cdots \to M_t = M_0$, where the M_i are indecomposable modules. An indecomposable module M is called **directing** if it lies on no cycle in mod A. For a point i in Q_A, we denote by $S(i)$ the corresponding simple A-module and by $P(i)$ (or $I(i)$) the projective cover (or injective envelope, respectively) of $S(i)$. The **dimension-vector** of a module M is the vector $\underline{\dim} M = (\dim_k \mathrm{Hom}_A(P(i), M))_{i \in A_0}$. The **support** of an A-module M is the full subcategory Supp M of A with object class $\{i \in A_0 | \mathrm{Hom}_A(P(i), M) \neq 0\}$

1.2. We use freely properties of the Auslander-Reiten translations $\tau_A = DTr$ and $\tau_A^{-1} = TrD$, and the Auslander-Reiten quiver $\Gamma(\text{mod } A)$ of A, for which we refer to [9, 20]. We agree to identify points in $\Gamma(\text{mod } A)$ with the corresponding indecomposable A-modules, and components with the corresponding full subcategories of ind A. A component Γ of $\Gamma(\text{mod } A)$ is called **standard** if Γ is equivalent to its mesh category $k(\Gamma)$, see [10, 20].

Given a standard component Γ of $\Gamma(\text{mod } A)$, and an indecomposable module X in Γ, the **support** $\mathcal{S}(X)$ of the functor $\mathrm{Hom}_A(X, -)|_\Gamma$ is the k-linear category defined as follows. Let $\mathcal{H}_X$ denote the full subcategory of Γ consisting of the indecomposable modules M in Γ such that $\mathrm{Hom}_A(X, M) \neq 0$, and $\mathfrak{I}_X$ denote the ideal of $\mathcal{H}_X$ consisting of the morphisms $f : M \to N$ (with M, N in $\mathcal{H}_X$) such that $\mathrm{Hom}_A(X, f) = 0$. We define $\mathcal{S}(X)$ to be the quotient category $\mathcal{H}_X/\mathfrak{I}_X$. Following the above convention, we usually identify the k-linear category $\mathcal{S}(X)$ with its quiver.

A translation quiver Γ is called a **tube** [12, 20] if it contains a cyclical path and its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$ (where S^1 is the unit circle, and $\mathbb{R}^+$ the non-negative real line). A tube has only two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. This also applies to sectional paths, that is, paths $x_0 \to x_1 \to \cdots \to x_m$ in Γ such that $x_{i-1} \neq \tau x_{i+1}$ for all i, $0 < i < m$. A maximal sectional path consisting of arrows pointing to infinity (or to the mouth) is called a **ray** (or a **coray**, respectively). Tubes containing neither projectives nor injectives are called **stable**. The **rank** of a stable tube Γ is the least positive integer r such that $\tau^r x = x$ for all x in Γ. A tube of rank $r = 1$ is called **homogeneous**.

1.3. The **one−point extension** of an algebra A by an A-module X is the matrix algebra

$$A[X] = \begin{bmatrix} A & 0 \\ X & k \end{bmatrix}$$

with the usual addition and multiplication of matrices. The quiver of $A[X]$ contains Q_A as a full subquiver and there is an additional (extension) point which is a source. The $A[X]$-modules are usually identified with the triples (V, M, φ), where V is a k-vector space, M an A-module and $\varphi : V \to \operatorname{Hom}_A(X, M)$ is a k-linear map. An $A[X]$-linear map $(V, M, \varphi) \to (V', M', \varphi')$ is then identified with a pair (f, g), where $f : V \to V'$ is k-linear, $g : M \to M'$ is A-linear and $\varphi' f = \operatorname{Hom}_A(X, g)\varphi$. One defines dually the **one−point coextension** $[X]A$ of A by X.

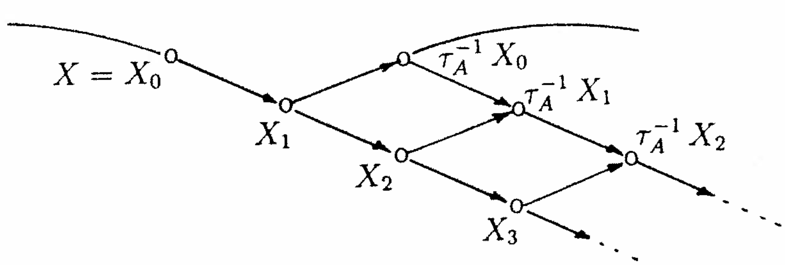
2. Admissible operations and coils

2.1. A coil is a translation quiver constructed inductively from a stable tube by a sequence of operations called admissible. Our first task is thus to define the latter. Throughout this section, let A be an algebra, and Γ be a standard component of $\Gamma(\operatorname{mod} A)$. For an indecomposable module X in Γ, called the **pivot**, the admissible operation to apply to Γ depends on the shape of the support $\mathcal{S}(X)$ of $\operatorname{Hom}_A(X, -)|_\Gamma$.

(ad 1) Assume $\mathcal{S}(X)$ consists of an infinite sectional path starting at X:

$$X = X_0 \to X_1 \to X_2 \to \ldots$$

Thus the component Γ may look as follows:



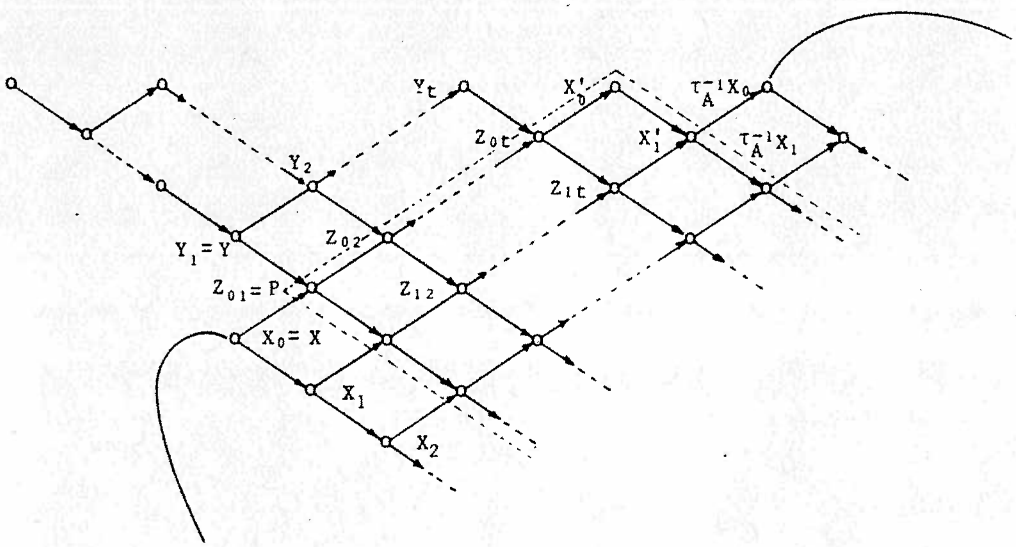
In this case, we let $t \geq 1$ be a positive integer,

$$D = T_t(k) = \begin{bmatrix} k & 0 & \dots & 0 \\ k & k & \dots & 0 \\ \vdots & & \ddots & \vdots \\ k & \dots & \dots & k \end{bmatrix}$$

denote the full $t \times t$ - lower triangular matrix algebra and $Y_1, \dots, Y_t$ denote the indecomposable injective D-modules with $Y = Y_1$ the unique indecomposable projective-injective. We define the **modified algebra** A' of A to be the one-point extension

$$A' = (A \times D)[X \oplus Y]$$

and the **modified component** Γ' of Γ to be



where $Z_{ij} = (k, X_i \oplus Y_j, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ for $i \geq 0$ and $1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$. The morphisms are defined in the obvious way. The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01} = P$ is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau_A^{-1} X_i) = X'_i$ provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining points of Γ (or $\Gamma(\text{mod } D)$), the translation τ' coincides with τ_A (or τ_D, respectively).

If now $t = 0$, we define the modified algebra A' to be the one-point extension

$$A' = A[X]$$

and the modified component Γ' to be the component obtained from Γ by inserting only the sectional path consisting of the X'_i.

It is important to observe that this operation does not affect standardness.

LEMMA[5](2.2). *With the above notation, the component of* $\Gamma(\text{mod } A')$ *containing* X*, considered as an* A'*-module, is equal to* Γ' *and is standard.* □

Intuitively, this operation amounts to "opening" the component Γ along the arrows $X_i \to \tau_A^{-1} X_{i-1}$, then "glueing" Γ with $\Gamma(\text{mod } D)$ by inserting the infinite rectangle (indicated by the dotted lines in the figure above) consisting of the points Z_{ij} and X'_i. This rectangle is equal to the support $\mathcal{S}(P)$ in Γ' of the functor $\text{Hom}_{A'}(P,-)|_{\Gamma'}$, where P is the new projective. We say that Γ' is obtained from Γ and $\Gamma(\text{mod } D)$ by inserting the rectangle determined by P.

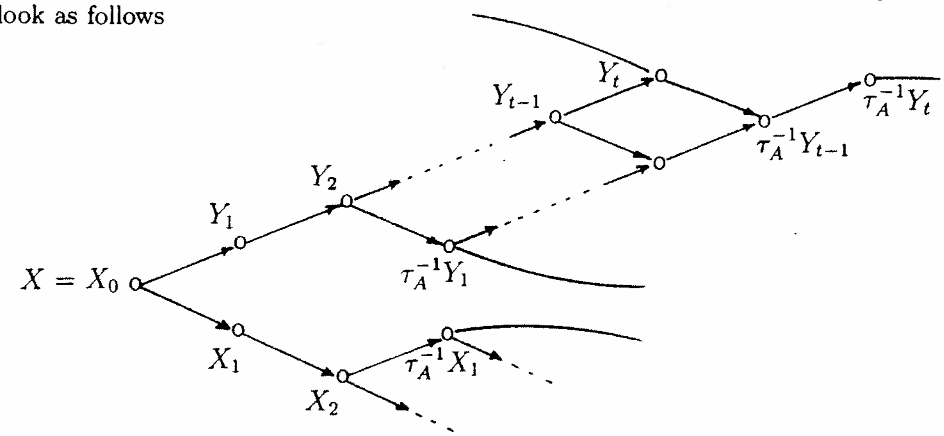
The non-negative integer t is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \dots$ in the inserted rectangle equals $t+1$. We call t the **parameter** of the operation.

In case Γ is a stable tube, it is clear that any module on the mouth of Γ satisfies the condition for being a pivot for the above operation. Actually, the above operation is, in this case, the tube insertion as considered in [12] and the above lemma is just [12](2.3).

(ad2) Assume $\mathcal{S}(X)$ to consist of two sectional paths starting at X, the first infinite and the second finite with at least one arrow

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \dots$$

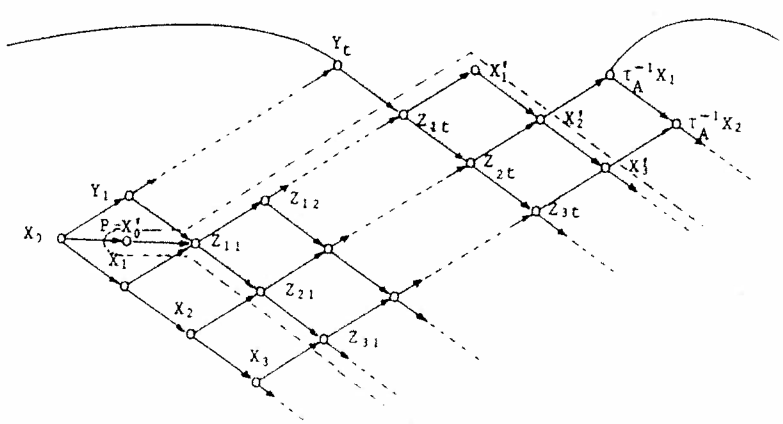
where $t \geq 1$. In particular, X is necessarily injective. The component Γ may then look as follows



We define the **modified algebra** A' of A to be the one-point extension

$$A' = A[X]$$

and the **modified component** Γ' of Γ to be



where $Z_{ij} = (k,\ X_i \oplus Y_j, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ for $i \geq 1, 1 \leq j \leq t$ and $X'_i = (k, X_i, 1)$ for $i \geq 0$. The morphisms are the obvious ones. The translation τ' of Γ' is defined as follows: $P = X'_0$ is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X'_i = Z_{i-1,t}$ if $i \geq 2, \tau' X'_1 = Y_t, \tau'(\tau_A^{-1} X_i) = X'_i$ if $i \geq 1$, provided X_i is not an injective A-module, otherwise X'_i is injective in Γ'. For the remaining points of Γ', the translation τ' coincides with the translation τ_A. Under a reasonable condition (that will always be satisfied in the sequel), the above operation does not affect standardness.

LEMMA[5](2.3). *With the above notation, the component of* $\Gamma(\text{mod } A')$ *containing* X*, considered as an* A'*-module, is equal to* Γ'*. Further, if any walk in* Γ *from* X *to some* $\tau_A^{-1} Y_{i-1}$ *factors through one of the arrows* $Y_j \to \tau_A^{-1} Y_{j-1}$ *(where* $1 < j \leq t$*), then* Γ' *is standard.* □

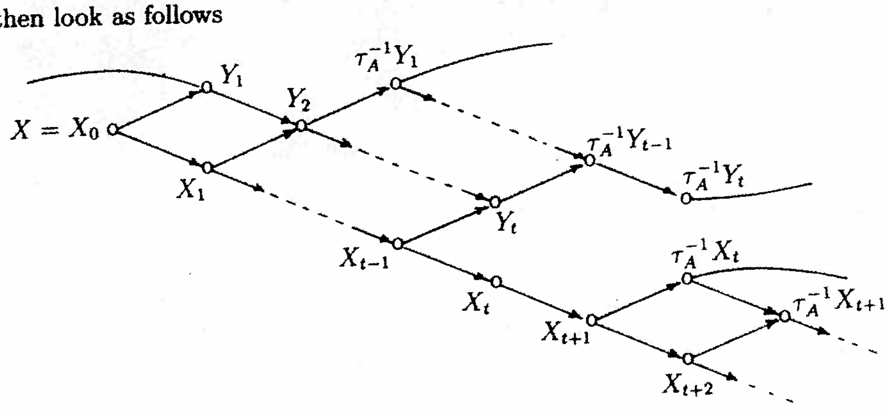
Intuitively, the above operation amounts to "opening" the component Γ along the arrows $X_i \to \tau_A^{-1} X_{i-1}$, "plugging" a new projective-injective P and inserting the infinite rectangle (indicated by the dotted lines in the figure above) consisting of the points Z_{ij} and X'_i. On the other hand, those modules M such that there is a walk from M to $\tau_A^{-1} Y_{j-1}$ for some j, $2 < j \leq t$, not factoring through one of the arrows $Y_j \to \tau_A^{-1} Y_{j-1}$ are "removed" from the component. The inserted rectangle is equal to the support $\mathcal{S}(P)$ in Γ' of the functor $\text{Hom}_{A'}(P, -)|_{\Gamma'}$, where P is the new projective-injective. We say that Γ' is obtained from Γ by inserting the rectangle determined by P.

The integer $t \geq 1$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \ldots$ in the inserted rectangle equals $t + 1$. We call t the **parameter** of the operation.

(ad3) Assume $\mathcal{S}(X)$ to consist of two parallel sectional paths the first infinite and starting at X, the second finite with at least one arrow

$$\begin{array}{ccccccccccccc} & & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_t & & & & \\ & & \uparrow & & \uparrow & & & & \uparrow & & & & \\ X = X_0 & & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & X_{t+1} \longrightarrow & \cdots \end{array}$$

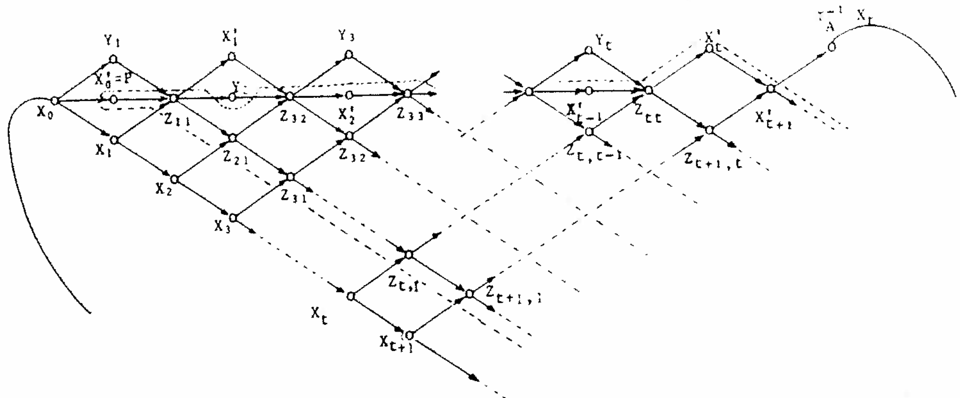
where $t \geq 2$. In particular, X_{t-1} is necessarily injective. The component Γ may then look as follows



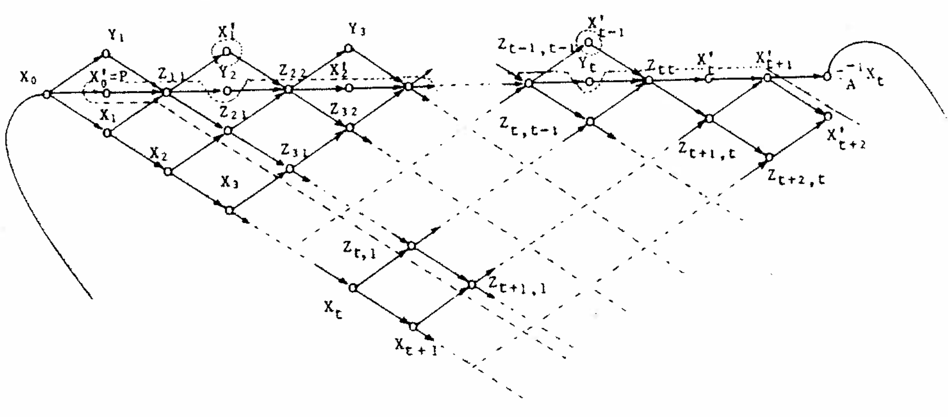
We define the **modified algebra** A' of A to be the one-point extension

$$A' = A[X]$$

and the **modified component** Γ' of Γ to be
- if t is odd:



-if t is even:



where $Z_{ij} = (k, X_i \oplus Y_j, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ for $i \geq 1, 1 \leq j \leq i$ and $X_i' = (k, X_i, 1)$ for $i \geq 1$. The morphisms are the obvious ones. The translation τ' is defined as follows: $P = X_0'$ is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, 2 \leq j \leq i, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' X_i' = Y_i$ if $1 \leq i \leq t, \tau' X_i' = Z_{i-1,t}$ if $i > t, \tau' Y_j = X_{j-2}'$ if $2 \leq j \leq t, \tau'(\tau_A^{-1} X_i) = X_i'$ if $i \geq t$ provided X_i is not an injective A-module, otherwise X_i' is injective in Γ'. For the remaining points of Γ', the translation τ' coincides with τ_A. We note that X_{t-1}' is injective.

Under a reasonable assumption (that will always be statisfied in the sequel), the above operation does not affect standardness.

LEMMA[5](2.4). *With the above notation, the component of* $\Gamma(\text{mod } A')$ *containing* X*, considered as an* A'*-module, is equal to* Γ'*. Further, if any walk in* Γ *from* X *to some* $\tau_A^{-1}Y_{i-1}$ *factors through one of the arrows* $Y_j \to \tau_A^{-1}Y_{j-1}$ *(where* $2 \le j \le t$*), then* Γ' *is standard.* □

Intuitively, this operation amounts to "opening" the component Γ along the arrows $X_i \to \tau_A^{-1}X_{i-1}$, "plugging" a new projective P and inserting the infinite rectangle (indicated by the dotted lines in the figure above) consisting of the points Z_{ij} and X_i'. On the other hand, those modules M in Γ such that there is a walk from M to $\tau_A^{-1}Y_{i-1}$ (for some $i, 2 \le i \le t$) not factoring through one of the arrows $Y_j \to \tau_A^{-1}Y_{j-1}$, are "removed" from the component. The reason for the appearance of two cases depending on the parity of t follows from easy combinatorial considerations involving the length functions [5](4.4). The inserted rectangle is equal to the support $\mathcal{S}(P)$ in Γ' of the functor $\text{Hom}_{A'}(P,-)|_{\Gamma'}$, where P is the new projective. We say that Γ' is obtained from Γ by inserting the rectangle determined by P.

The integer $t \ge 2$ is such that the number of infinite sectional paths parallel to $X_0 \to X_1 \to X_2 \to \dots$ in the inserted rectangle equals $t+1$. We call t the **parameter** of the operation.

Finally, together with each of the admissible operations (ad1), (ad2) and (ad3), we consider its dual, denoted by (ad1*)(ad2*) and (ad3*), respectively. These six operations are called **the admissible operations**.

Clearly, the admissible operations can be defined as operations on translation quivers rather than on Auslander-Reiten components. The definitions are done in the obvious manner (see [4] or [22] for the details).

DEFINITION. A translation quiver Γ is called a **coil** if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \dots, \Gamma_m = \Gamma$ such that Γ_0 is a stable tube and, for each i $(0 \le i < m), \Gamma_{i+1}$ is obtained from Γ_i by an admissible operation.

Observe that this use of the term coil deviates from its use in [3]. The present notion of coil is clearly a natural generalisation of the notion of coherent tube: indeed, any stable tube is (trivially) a coil, and a tube can be characterised as being a coil having the property that each admissible operation in the sequence defining it is of the form (ad1) or (ad1*). Also, a coil without injectives (or without projectives) is a tube. A **quasi−tube** is a coil having the property that each admissible operation in the sequence defining it is of the form (ad1), (ad1*), (ad2) or (ad2*).

It follows from the definition that coils share many properties with tubes. For instance, all but at most finitely many points in a coil belong to a cyclical path. A point x in a coil Γ is said to belong the the **mouth** of Γ if x is the starting, or ending, point of a mesh in Γ with a unique middle term. Also, Γ contains a (maximal) tube as a cofinite full translation subquiver. Arrows of this tube either point to the mouth or point to infinity. An infinite sectional path in Γ

$$x = x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \longrightarrow x_i \xrightarrow{\alpha_i} x_{i+1} \longrightarrow \cdots$$

is called a **ray** if there exists $i_0 \ge 1$ such that, for all $i \ge i_0$, the arrow α_i points to infinity. **Corays** are defined dually. Thus the parameter of the operation (ad1)(ad2) or (ad3) (or (ad1*), (ad2*) or (ad3*)) is used to measure the number of rays (or corays, respectively) inserted in the coil by the operation.

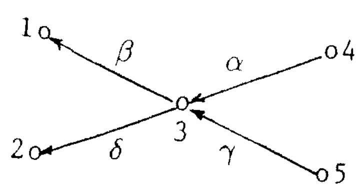
2.2. Another approach would consist in defining coils as translation quivers satisfying a set of axioms (see [5]). The following proposition shows that the approach we have chosen entails no loss of generality. Furthermore, its proof provides a method for constructing an algebra having a given coil as an Auslander-Reiten component.

PROPOSITION[5](3.2). *Let Γ be a coil. There exists a triangular algebra A such that Γ is a standard component of* $\Gamma(\text{mod } A)$.

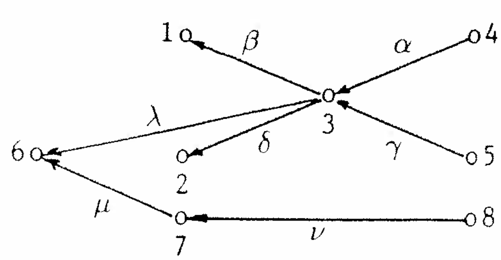
PROOF. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \Gamma$ be a sequence of translation quivers, such that Γ_0 is a stable tube and, for each i $(0 \leq i < m), \Gamma_{i+1}$ is obtained from Γ_i by an admissible operation. Clearly, there exists a tame concealed (even a hereditary) algebra C having the stable tube Γ_0 as a standard component. Inductively, we construct a sequence of algebras $C = A_0, A_1, \ldots, A_m = A$ such that A_{i+1} is obtained from A_i by the admissible operation with pivot in Γ_i such that the modified component is Γ_{i+1}. Clearly, the conditions for standardness are satisfied at each step. Then Γ is a standard component of $\Gamma(\text{mod } A)$. The triangularity of A follows from the fact that A is obtained from the triangular algebra C by a sequence of one-point extensions and coextensions. □

2.3. We illustrate the admissible operations in the following example

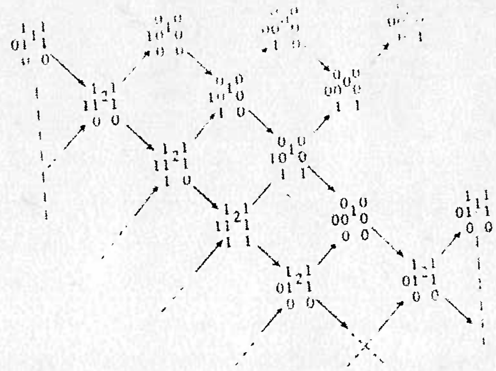
EXAMPLE. Consider the tame hereditary algebra A_0 given by the quiver



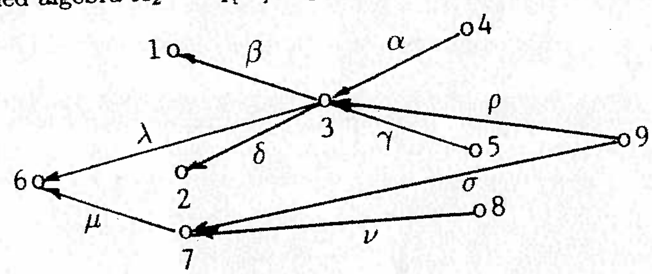
Any simple regular A_0-module can be a pivot for one of the operations (ad1) or (ad1*). We choose to apply (ad1*) with pivot the simple regular A_0-module $S(3)$, and with parameter $t = 2$ (thus, using the lower triangular matrix aglebra $D = T_2(k)$). The modified algebra A_1 is given by the quiver



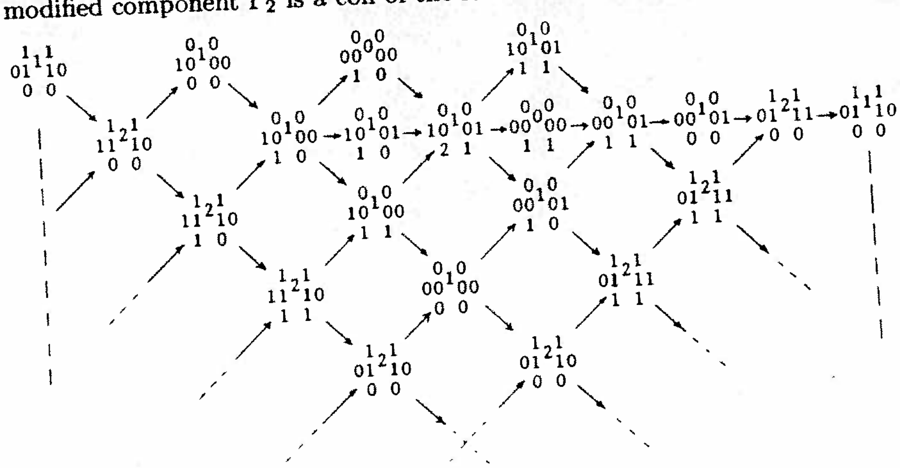
bound by $\alpha\lambda = 0, \gamma\lambda = 0$. The Auslander-Reiten quiver $\Gamma(\text{mod } A_1)$ has as standard component the modified component Γ_1 of the stable tube Γ_0 of $\Gamma(\text{mod } A_0)$ containing $S(3)_{A_0}$. This modified component Γ_1 is a tube of the form



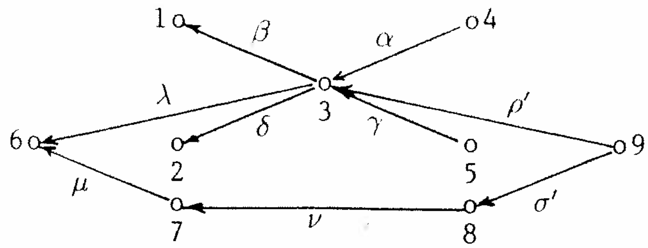
where indecomposables are represented by their dimension-vectors and one identifies along the vertical dotted lines to form the tube. At this stage, besides the admissible operations of types (ad1) or (ad1*) with pivots in ray or coray modules, respectively, we can either apply (ad3) with pivot the indecomposable A_1-module X with dimension-vector $\begin{smallmatrix} 0 & 1 & 0 \\ 1\,0 & & 0 \\ & 1\;0 & \end{smallmatrix}$, or (ad2) with pivot the indecomposable A_1-module X' with dimension-vector $\begin{smallmatrix} 0 & 1 & 0 \\ 1\,0 & & 0 \\ & 1\;1 & \end{smallmatrix}$. In each case, the parameter is $t = 2$. In the first case, the modified algebra $A_2 = A_1[X]$ is given by the quiver



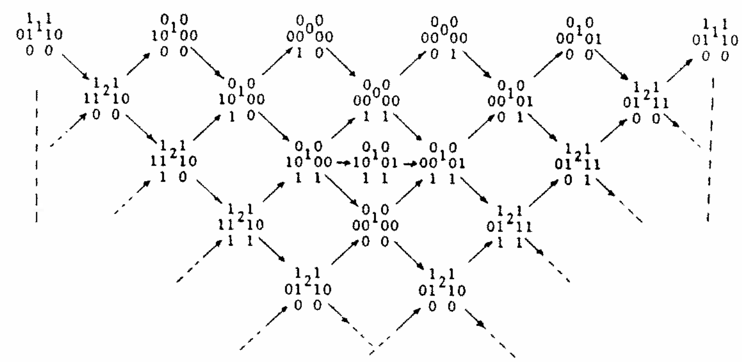
bound by $\alpha\lambda = 0, \gamma\lambda = 0, \rho\beta = 0, \rho\delta = 0$ and $\rho\lambda = \sigma\mu$. The Auslander-Reiten quiver $\Gamma(\text{mod } A_2)$ has as standard component the modified component Γ_2 of Γ_1. This modified component Γ_2 is a coil of the form



In the second case, the modified algebra $A_2' = A_1[X']$ is given by the quiver



bound by $\alpha\lambda = 0, \gamma\lambda = 0, \rho'\beta = 0, \rho'\delta = 0$ and $\rho'\lambda = \sigma'\nu\mu$. The Auslander-Reiten quiver $\Gamma(\text{mod } A_2')$ has as standard component the modified component Γ_2' of Γ_1. This modified component Γ_2' is a coil (and, in this case, even a quasi-tube) of the form



As the reader observes, each operation performed influences the following ones. We must first perform (ad1*) (or (ad1)) in order to have indecomposable modules which may serve as pivots for one of the operations (ad2), (ad3)(or (ad2*), (ad3*), respectively). Similarly, if one applies (ad1*) with a parameter equal to 1 (rather than 2, as we did above), then we may at the following step perform (ad2), but not (ad3).

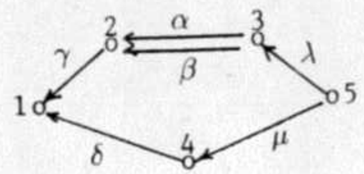
3. The structure of coil enlargements

3.1 As we have seen, the construction of a coil requires starting from a stable tube. Now, stable tubes occur, for instance, among separating tubular families, as defined by Ringel in [20](3.1). Well-known examples of classes of algebras having separating tubular families are the tame concealed algebras, and the canonical algebras. It turns out, however, that the separating property of such tubular families is not preserved by the admissible operations. We thus need to generalise the notion of separating tubular families as follows.

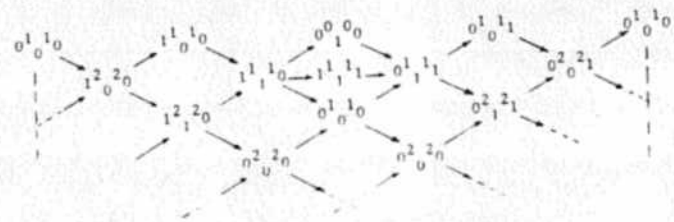
DEFINITION. Let A be an algebra. A family $\mathcal{R} = (\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ of components of $\Gamma(\text{mod } A)$ is called is a **weakly separating family** in mod A if the indecomposable A-modules not in $\mathcal{R}$ split into two classes $\mathcal{P}$ and $\mathcal{Q}$ such that:
(WS1) The components $(\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ are standard and pairwise orthogonal.
(WS2) $\text{Hom}_A(\mathcal{Q},\mathcal{P}) = \text{Hom}_(\mathcal{Q},\mathcal{R}) = \text{Hom}_A(\mathcal{R},\mathcal{P}) = 0$.
(WS3) Any morphism from $\mathcal{P}$ to $\mathcal{Q}$ must factor through add $\mathcal{R}$.

Thus, a separating tubular family is a weakly separating family of tubes in which (WS3) is strengthened to say that any morphism from $\mathcal{P}$ to $\mathcal{Q}$ factors through each of the components $\mathcal{R}_\lambda$. In particular, any separating family is weakly separating. The following example shows that the converse is not true.

EXAMPLE. Let A be given by the quiver



bound by $\lambda\alpha = 0, \alpha\gamma = 0$ and $\lambda\beta\gamma = \mu\delta$. Then A is obtained by applying to the Kronecker algebra first (ad1*) with parameter $t = 1$, and pivot a simple homogeneous module, then (ad2) with pivot the unique non-simple indecomposable injective lying in the unique non-stable tube. As we shall see below (in (3.4)), $\Gamma(\text{mod } A)$ contains a weakly separating family $\mathcal{R}$ of coils, containing the following non-trivial quasi-tube, in which lies the projective-injective $P(5) = I(1)$.



The other coils in $\mathcal{R}$ are homogeneous tubes (actually, they are the images of homogeneous tubes of the Kronecker algebra, when the latter is idencified with the obvious full convex subcategory of A). Also, $\Gamma(\text{mod } A)$ has a unique postprojective component containing all projectives except $P(5)$ and a unique preinjective component containing all injectives except $I(1)$. Since, in the notation of the above definition, the postprojective component lies in $\mathcal{P}$ and the preinjective in $\mathcal{Q}$, we have in particular $P(4) \in \mathcal{P}$ and $I(4) \in \mathcal{Q}$. Now, the canonical morphism $f : P(4) \to I(4)$ has as image the simple module $S(4)$ which belongs to the quasi-tube above. In particular, f factors through no other coil in $\mathcal{R}$.

3.2. We note that, if $\mathcal{P}, \mathcal{R}$ and $\mathcal{Q}$ are as in the definition, then $\mathcal{P}$ is closed under predecessors and $\mathcal{Q}$ is closed under successors. If $\mathcal{R}$ is a weakly separating family in mod A, and $\mathcal{P}, \mathcal{Q}$ are as in the definition, we say that $\mathcal{R}$ separates (weakly) $\mathcal{P}$ from $\mathcal{Q}$ and write ind $A = \mathcal{P} \vee \mathcal{R} \vee \mathcal{Q}$ (in the notation of [20]). This terminology is justified by the following lemma, whose proof is the same as the proof of [20](3.1)(4) p. 120.

LEMMA[6](2.1). *Let A be an algebra, and $\mathcal{R}$ be a weakly separating family in mod A. separating $\mathcal{P}$ from $\mathcal{Q}$. Then $\mathcal{P}$ and $\mathcal{Q}$ are uniquely determined by $\mathcal{R}$.* □

3.3. Let A be an algebra, and $\mathcal{R}$ be a weakly separating family in mod A, consisting of stable tubes. We wish to consider the changes in the structure of mod A which occur as a result of applying a sequence of admissible operations to $\mathcal{R}$. This leads us to the following definition.

DEFINITION. Let A be an algebra, and $\mathcal{R}$ be a weakly separating family in mod A, consisting of stable tubes. An algebra B is called a **coil enlargement of** A using modules from $\mathcal{R}$ if there exists a finite sequence of algebras $A = A_0, A_1, \dots, A_m = B$ such that, for each i $(0 \le i < m)$, A_{i+1} is obtained from A_i by an admissible operation with pivot either on a stable tube of $\mathcal{R}$, or on a coil of $\Gamma(\text{mod } A_i)$, obtained from a stable tube of $\mathcal{R}$ by means of the sequence of admissible operations done so far.

For instance, the representation-infinite tilted algebras of euclidean type and the tubular algebras are, by [20](4.9) and (5.2), coil enlargements of a tame concealed algebra using only operations (ad1) and (ad1*). In this example, the size of the coils (which are tubes) is measured by a numerical invariant, called the extension or coextension type (see [20](4.7)) whose definition can be generalised as follows.

DEFINITION. Let B be a coil enlargement of A using modules from the weakly separating family $\mathcal{R} = (\mathcal{R}_\lambda)_{\lambda\in\Lambda}$ of stable tubes. The **coil type** $c_B = (c_B^-, c_B^+)$ of B is a pair of functions $c_B^-, c_B^+ : \Lambda \to \mathbb{N}$ defined by induction on $i, 0 \le i < m$, where $A = A_0, A_1, \dots, A_m = B$ is a sequence of algebras as in the definition above.

a) $c_A = (c_0^-, c_0^+)$ is the pair of functions $c_0^- = c_0^+$ such that, for each $\lambda \in \Lambda$, the common value of $c_0^-(\lambda)$ and $c_0^+(\lambda)$ is the rank of the stable tube $\mathcal{R}_\lambda$.

b) Assume $c_{A_{i-1}} = (c_{i-1}^-, c_{i-1}^+)$ is known, and let t_i be the parameter of the admissible operation modifying A_{i-1} to A_{i+1}, then $c_{A_i} = (c_i^-, c_i^+)$ is defined by

$$c_i^-(\lambda) = \begin{cases} c_{i-1}^-(\lambda) + t_i + 1 & \text{if the operation is (ad1*) (ad2*) or (ad3*) with} \\ & \text{pivot in the coil of } \Gamma(\text{mod} A_{i-1}) \text{ arising from } \mathcal{R}_\lambda, \\ c_{i-1}^-(\lambda) & \text{otherwise} \end{cases}$$

and

$$c_{i-1}^+(\lambda) = \begin{cases} c_{i-1}^+(\lambda) + t_i + 1 & \text{if the operation is(ad1) (ad2) or (ad3) with} \\ & \text{pivot in the coil of } \Gamma(\text{mod} A_{i-1}) \text{ arising from } \mathcal{R}_\lambda, \\ c_{i-1}^+(\lambda) & \text{otherwise.} \end{cases}$$

Clearly, the coil type of a coil enlargement B of A does not depend on the sequence of admissible operations leading from A to B since, for each $\lambda \in \Lambda$, the integers $c_B^+(\lambda)$ and $c_B^-(\lambda)$ measure the rank of $\mathcal{R}_\lambda$ plus, respectively, the total number of rays and corays inserted in $\mathcal{R}_\lambda$ by the sequence of admissible operations.

If all but at most finitely many values of each of the functions c_B^- and c_B^+ equal 1, we replace each by a finite sequence, containing at least two terms and including all those which exceed 1. To enable us to compare the number of rays and corays inserted in each tube, we use the following conventions.

1. The finite sequences for c_B^- and c_B^+ contain exactly the same number of terms, where we agree to add to either sequence as many 1 as necessary.

2. c_B^- is a non-decreasing sequence, that is, if $c_B^- = (c_B^-(\lambda_1), \ldots, c_B^-(\lambda_s))$, then $c_B^-(\lambda_1) \leq \cdots \leq c_B^-(\lambda_s)$.
3. c_B^+ is the sequence consisting of the values of c_B^+ corresponding to the values of c_B^-, that is, if c_B^- is as in 2, then $c_B^+ = (c_B^+(\lambda_1), \ldots, c_B^+(\lambda_s))$.

EXAMPLES. a) The algebras A_2 and A_2' of the example (2.3) are coil enlargements of the tame hereditary algebra A_0 of type $\tilde{\mathbb{D}}_4$ whose coil type is $c_{A_0} = ((2,2,2),(2,2,2))$. We have first constructed A_1 whose coil type is $c_{A_1} = ((2,2,5),(2,2,2))$. Actually, A_1 is a tilted algebra of type $\tilde{\mathbb{D}}_7$ having a complete slice in its postprojective component and a unique non-stable tube. The algebra $A_2 = A_1[X]$ has coil type $c_{A_2} = ((2,2,5),(2,2,5))$, and the algebra $A_2' = A_1[X']$ has the same coil type $c_{A_2'} = ((2,2,5),(2,2,5))$.

b) The algebra A of the example (3.1) is a coil enlargement of the Kronecker algebra. Its coil type is $c_A = ((1,4),(1,4))$.

3.4. The following theorem describes the structure of the module category of a coil enlargement. For the notions of branch (tubular) extension and coextension, we refer the reader to [20].

THEOREM [6]. *Let A be an algebra with a weakly separating family $\mathcal{R}$ of stable tubes and B be a coil enlargement of A using modules from $\mathcal{R}$. Then:*

(a) *There is a unique maximal branch coextension B^- of A which is a full convex subcategory of B, and c_B^- is the coextension type of B^-.*

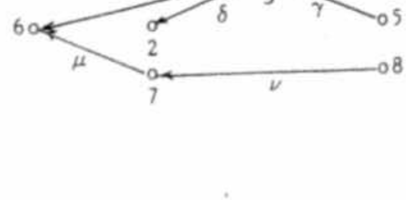
(b) *There is a unique maximal branch extension B^+ of A which is a full convex subcategory of B, and c_B^+ is the extension type of B^+.*

(c) *ind $B = \mathcal{P}' \vee \mathcal{R}' \vee \mathcal{Q}'$, where $\mathcal{R}'$ is a weakly separating family of coils of mod B, obtained from the stable tubes of $\mathcal{R}$ by the sequence of admissible operations and separating $\mathcal{P}'$ from $\mathcal{Q}'$, where $\mathcal{P}'$ consists of indecomposable B^--modules, while $\mathcal{Q}'$ consists of indecomposable B^+-modules.* □

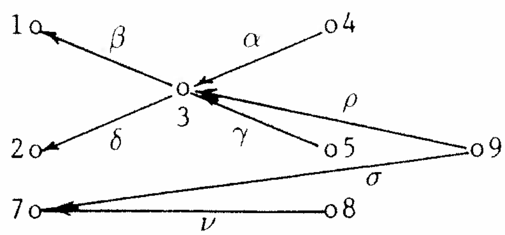
We actually obtain more, namely, we obtain a complete description of the indecomposable B-modules lying in $\mathcal{P}'$ and $\mathcal{Q}'$, in the spirit of [20](4.7) (1) p. 230. We refer the reader to [6](4.1) for the precise statement.

It is worthwhile to observe that, in the notation of the theorem, since $\mathcal{R}'$ is obtained from $\mathcal{R}$ by a sequence of admissible operations, only finitely many of the stable tubes of $\mathcal{R}$ are affected by these operations. The remaining stable tubes of $\mathcal{R}$, when considered as stable tubes in $\mathcal{R}'$, consist of A-modules. On the other hand, the non-stable tubes in $\mathcal{R}'$ may contain infinitely many non-isomorphic indecomposable modules which are neither B^+-modules, nor B^--modules: these correspond exactly to the points of intersection of the inserted rays and corays. In particular, for each $d \in \mathbb{N}$, all but at most finitely many non-isomorphic indecomposable modules in $\mathcal{R}'$ of dimension d are either B^+-modules or B^--modules.

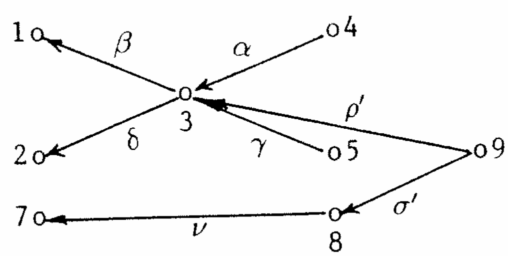
EXAMPLE. In the example (2.3), we first consider the algebra A_2. The algebras A_2^- and A_2^+ are respectively given by the quiver



bound by $\alpha\lambda = 0, \gamma\lambda = 0$ and by the quiver



bound by $\rho\beta = 0, \rho\delta = 0$. We notice that both A_2^- and A_2^+ are tilted algebras of type $\tilde{\mathbb{D}}_7$. The first has a complete slice in its postprojective component, whereas the second has a complete slice in its preinjective component. Similarly, if one considers the algebra A_2', then $A_2'^- = A_2^-$, while $A_2'^+$ is given by the quiver



bound by $\rho'\beta = 0, \rho'\delta = 0$. Thus $A_2'^+$ is a tilted algebra of type $\tilde{\mathbb{D}}_7$ having a complete slice in its preinjective component.

3.5. We now wish to give a criterion for the tameness of a coil enlargement B of a tame concealed algebra A using modules from its family of stable tubes. Since, in this case, B^- and B^+ are respectively a branch coextension and a branch extension of a tame concealed algebra, we know by [20] how to verify their tameness. We shall need the following definitions. An algebra B is called **cycle–finite** if, for any cycle in mod B

$$M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \longrightarrow M_{t-1} \xrightarrow{f_t} M_t = M_0$$

where the M_i are indecomposable B-modules and the f_i are non-zero non-isomorphisms, none of the f_i lies in the infinite power of the radical of mod B (see [3] or [22]). For the notions of tame, domestic, linear growth, polynomial growth and the Tits form of an algebra, we refer the reader to [21]. Let B be a coil enlargement of an algebra A having a weakly separating family of stable tubes. Its coil type $c_B = (c_B^-, c_B^+)$ is called **tame** if each of the sequences c_B^- and c_B^+ equals one of the following: (p,q), where $1 \le p \le q$, $(2,2,r)$, where $2 \le r$, $(2,3,3), (2,3,4), (2,3,5)$ or $(3,3,3), (2,4,4), (2,3,6), (2,2,2,2)$.

COROLLARY [6](4.3). *Let A be a tame concealed algebra and $\mathcal{R}$ be its separating tubular family. Let B be a coil enlargement of A using modules from $\mathcal{R}$. The following conditions are equivalent:*

a) *B is tame.*

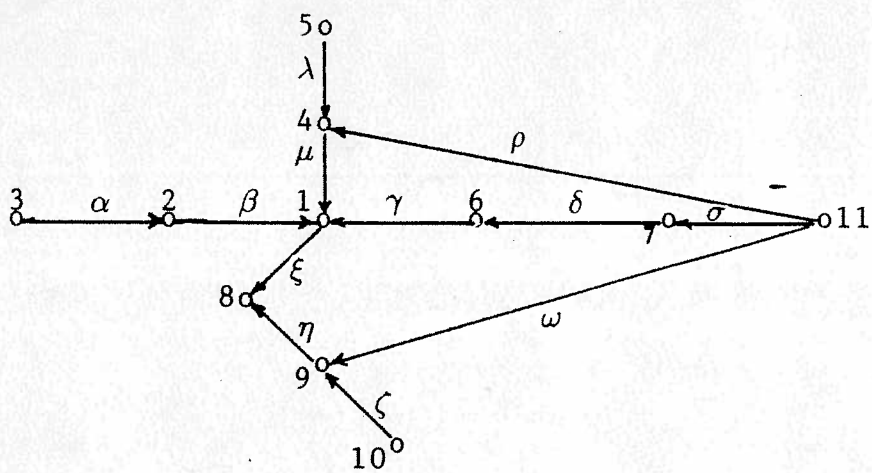
b) *B^- and B^+ are tame.*

c) *For every cycle* $M_0 \to M_1 \to \cdots \to M_t = M_0$ *in* mod B, *all the* M_i *belong to one standard coil of* Γ(mod B).
d) B *is of polynomial growth.*
e) B *is (domestic or) of linear growth.*
f) B *is cycle-finite.*
g) c_B *is tame.*
h) *The Tits form of* B *is weakly non-negative.*

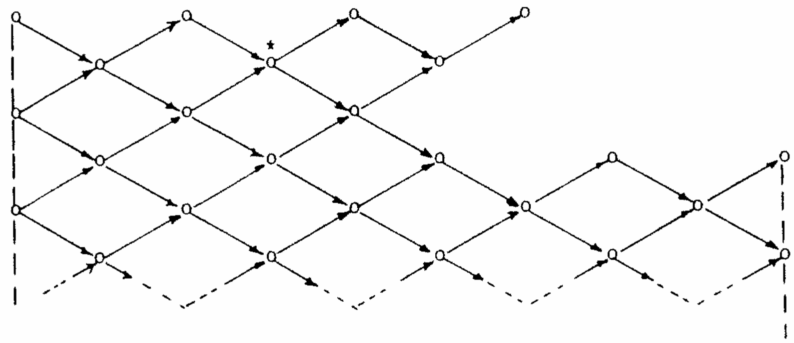
Moreover, B *is domestic if and only if each of* B^- *and* B^+ *is a tilted algebra of euclidean type.* □

For instance, the algebras A_2 and A'_2 of example (2.3) are domestic, as is also the algebra A of example (3.1). By [20], any tilted algebra of euclidean type, and any tubular algebra satisfies the conditions of our corollary. The following example shows a non-domestic tame coil enlargement having a non-trivial coil as an Auslander-Reiten component.

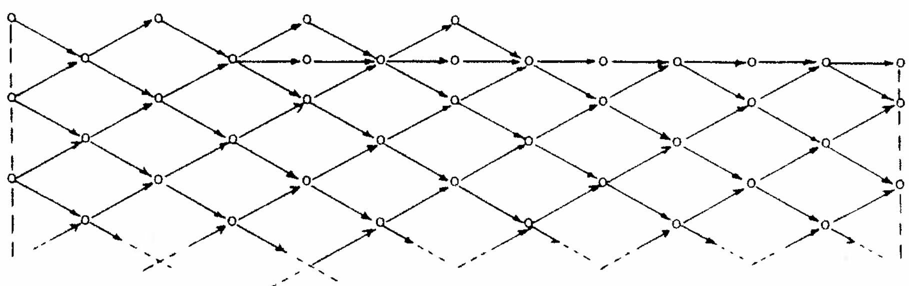
EXAMPLE. Let B be given by the quiver



bound by $\beta\xi = 0, \lambda\mu\xi = 0, \rho\mu = \sigma\delta\gamma, \omega\eta = \rho\mu\xi$. Then B is a non-domestic tame coil enlargement of a hereditary algebra of type $\tilde{\mathbb{E}}_6$ given by the full convex subcategory of B consisting of the points 1 to 7. We apply the admissible operation (ad1*) with parameter $t = 2$, taking as pivot the simple regular A-module of dimension-vector $\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$. This yields the tubular algebra C of coil type ((2, 3, 6), (2, 3, 3)) given by the full convex subcategory of B consisting of all the points except 11. This tubular algebra C has as Auslander-Reiten component a non-stable tube Γ of the form



We then apply (ad3), taking as pivot the indecomposable in Γ with dimension vector $\begin{smallmatrix} & & 0 & & \\ & & 1 & & \\ 0&0&1&1&1 \\ &1& & & \\ & &1& & \\ & &0& & \end{smallmatrix}$ (and indicated by an asterisk in the above figure). This gives B, whose coil type is ((2, 3, 6), (2, 3, 6)). Hence B is non-domestic of linear growth. The modified component of Γ in $\Gamma(\text{mod } B)$ is the coil



3.6. Recently, J.A. de la Peña and the second author established in [17] the following homological properties of coil enlargements of tame concealed algebras.

THEOREM [17]*Let A be a coil enlargement of a tame concealed algebra. Then* gldim$A \leq 3$ *and, for any indecomposable A-module M,* pd $M \leq 2$ or id $M \leq 2$. □

3.7 In [29], B. Tomé shows how one can iterate the procedure described above in the spirit of [18], and calls the resulting class of algebras iterated coil enlargements. One is able to give a description of the module category of an iterated coil enlargement and to prove a generalisation of [18] (3.4). In [19], J.A. de la Peña and B. Tomé prove the following partial converse of corollary (3.5).

THEOREM [19]. *Let A be a strongly simply connected tame algebra. Assume that* mod A *has a weakly separating family of coils $\mathcal{R}$ such that, if* ind $A = \mathcal{P} \vee \mathcal{R} \vee \mathcal{Q}$, *then any indecomposable projective (or injective) module lies in $\mathcal{P}$ or $\mathcal{R}$ (or in $\mathcal{R}$ or $\mathcal{Q}$, respectively). Then A is a coil enlargement of a tame concealed algebra.* □

We recall that an algebra A is called **strongly simply connected** [24] if every full convex subcategory of A is simply connected or, equivalently, if every full convex subcategory of A satisfies the separation property.

4. Multicoil algebras.

4.1. We wish to characterise a class of tame algebras satisfying the condition(c) of corollary (3.5), that is, such that every cycle of non-zero non-isomorphisms lies in a standard coil. We first need to define the notion of multicoil. Roughly speaking, a multicoil consists of a finite set of coils glued together by a directed part. More precisely, we define it as follows.

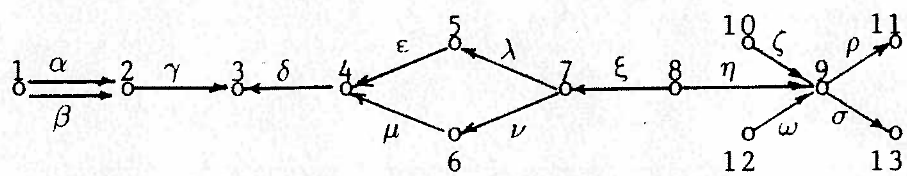
DEFINITION. A translation quiver Γ is called a **multicoil** if it contains a full translation subquiver Γ' such that:

a) Γ' is a disjoint union of coils, and

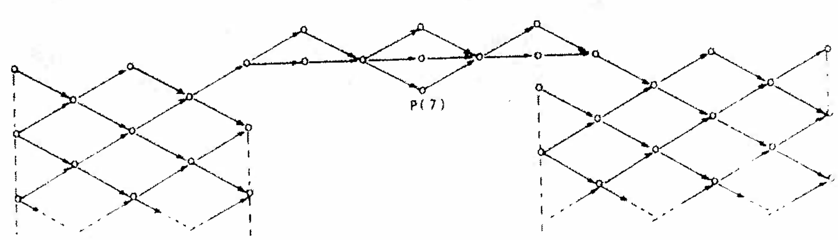
b) no point in $\Gamma \backslash \Gamma'$ belongs to a cyclical path.

This implies that any cyclical path in the multicoil Γ lies entirely in one coil in Γ'. While every coil is trivially a multicoil, the following example shows a multicoil which is not a coil.

EXAMPLES. Let A be given by the quiver



bound by $\beta\gamma = 0, \varepsilon\delta = 0, \mu\delta = 0, \lambda\varepsilon = \nu\mu, \xi\nu = 0, \xi\lambda = 0, \eta\rho = 0, \eta\sigma = 0$. Then $\Gamma(\text{mod}A)$ contains the following multicoil



4.2. We are now ready to define multicoil algebras.

DEFINITION. An algebra A is called a **multicoil algebra** if, for any cycle $M_0 \to M_1 \to \cdots \to M_t = M_0$ in mod A, all the M_i belong to one standard coil of a multicoil in $\Gamma(\text{mod } A)$.

EXAMPLES. a) A representation-finite algebra is a multicoil algebra if and only if it has a directed module category.

b) It follows from corollary (3.5)(c) that a tame coil enlargement of a tame concealed algebra is a multicoil algebra.

c) The algebra of example (4.1) above is a multicoil algebra.

d) Let A be a tilted algebra. It follows from the results of O. Kerner [13] that A is a multicoil algebra if and only if A is tame.

e) The iterated tubular algebras of [18] and the iterated coil enlargements of [29] are multicoil algebras.

f) The algebras such that every indecomposable projective module is directing were studied by the second author and M. Wenderlich in [28]. In particular, such an algebra is tame if and only if it is a multicoil algebra [28](4.1).

g) Let A be a sincere, tame, strongly simply connected algebra which contains a full convex subcategory which is either representation-infinite tilted of type $\tilde{\mathbb{E}}_p$ ($p = 6, 7, 8$) or a tubular algebra. The second author and J.A. de la Peña have proved in [16] that A is a multicoil algebra.

4.3. PROPOSITION [4](4.6). *Let A be a multicoil algebra. Then A is of polynomial growth.*

PROOF. It follows from the definition that any multicoil algebra is cycle-finite. By [26], this implies it is of polynomial growth. □

4.4. PROPOSITION [5](3.5). *Let A be a multicoil algebra. Then A is triangular (hence of finite global dimension).*

PROOF. If A is not triangular, then mod A contains a cycle of non-zero non-isomorphisms between indecomposable projective A-modules. By definition, this cycle belongs to a standard coil Γ of a multicoil in $\Gamma(\text{mod } A)$. Thus there exists a cycle of projective objects in the mesh category $k(\Gamma)$ of the standard coil Γ. By (2.2), there exists a triangular algebra B and a component Γ' in $\Gamma(\text{mod } B)$ such that $k(\Gamma) \cong k(\Gamma')$. This implies that Γ' contains a cycle of non-zero non-isomorphisms between indecomposable projective B-modules, a contradiction to the triangularity of B. □

4.5. One of the main properties of multicoil algebras is their good behaviour with respect to taking full convex subcategories.

THEOREM [5](5.6). *Let A be a multicoil algebra, and B be a full convex subcategory of A. Then B is a multicoil algebra.* □

4.6. We deduce from the above theorem the following characterisation of minimal representation-infinite multicoil algebras, which generalises [3] (2.3) (compare also with [25] (4.1)).

THEOREM [5](5.7). *Let A be an algebra. The following conditions are equivalent:*

(a) *A is a tame concealed algebra.*

(b) *A is a representation-infinite multicoil algebra and, for every $0 \neq e^2 = e \in A$, the algebra A/AeA is representation-finite.*
(c) *A is a representation-infinite multicoil algebra and every proper full convex subcategory of A is representation-finite.* □

4.7. Corollary [5](5.8). *Let A be a representation-infinite multicoil algebra. Then A contains a tame concealed full convex subcategory.* □

4.8. We restate now the following characterisation of strongly simply connected algebras of polynomial growth, already stated in the introduction. It plays a crucial rôle in the study of the tame simply connected algebras.

Theorem [27],[22](9.4). *Let A be a strongly simply connected algebra. Then A is of polynomial growth if and only if A is a multicoil algebra.* □

4.9. The above theorem and properties of coils have been used essentially in [17] to prove the following homological and geometric characterisations of polynomial growth strongly simply connected algebras.

Theorem [17]. *Let A be a strongly simply connected algebra. The following conditions are equivalent:*
(a) *A is of polynomial growth.*
(b) *The Tits form of A is weakly non-negative and $\mathrm{Ext}^2_A(M,M) = 0$ for any indecomposable A-module M.*
(c) $\dim_k \mathrm{Ext}^1_A(M,M) \leq \dim_k \mathrm{End}_A M$ *and* $\mathrm{Ext}^i_A(M,M) = 0$ *for any* $i \geq 2$ *and any indecomposable A-module M.* □

4.10. Let A be an algebra, having n points in its ordinary quiver. For a vector z in $\mathbb{N}^n$, we denote by $\underline{\underline{\mathrm{mod}}}_A(z)$ the scheme of A-modules having z as dimension-vector. The set of rational points $\mathrm{mod}_A(z)$ in $\underline{\underline{\mathrm{mod}}}_A(z)$ is the corresponding variety of modules having z as dimension-vector. The isomorphism classes of modules in $\mathrm{mod}_A(z)$ are the orbits of the action of the corresponding affine algebraic group $G(z)$. Finally, we denote by χ_A the Euler characteristic of A. We may now state the following theorem.

Theorem [17]. *Let A be a strongly simply connected algebra. The following conditions are equivalent:*
(a) *A is of polynomial growth.*
(b) *For each $z \in \mathbb{N}^n$, we have* $\dim \mathrm{mod}_A(z) \leq \dim G(z)$ *and every indecomposable A-module M is $\mathrm{mod}_A(z)$ is a smooth point.*
(c) *For each $z \in \mathbb{N}^n$ and every indecomposable module M in $\mathrm{mod}_A(z)$, we have* $0 \leq \dim G(z) - \dim_M \mathrm{mod}_A(z) = \chi_A(\dim M)$. □

5. Indecomposable modules over multicoil algebras

5.1 In this last section, we discuss the structure of an indecomposable module M over a multicoil algebra A. We first consider the case where M is directing. It was shown in [20], Addendum to 4.2, p. 375, that the support B of M is a tilted algebra. Since, moreover, A is tame (even, of polynomial growth) then so is B. It was shown by J.A. de la Peña in [14] that a tame algebra with a sincere directing indecomposable module is domestic in at most two one-parameters. Further, he classified in [15] those tame algebras B with a sincere directing indecomposable module, having exactly two one-parameters and whose ordinary quiver has at least 20 points. We thus concentrate here on the case where M is a non-directing indecomposable A-module.

Let C be the convex hull of the support of M. It follows from (4.5) that C is itself a multicoil algebra. This remark implies that in order to study the indecomposable non-directing modules over a multicoil algebra, it suffices to consider the case where the algebra equals the convex hull of the support of the module.

Since, by definition of a multicoil algebra, any cycle of non-zero non-isomorphisms lies in a standard coil, then any non-directing indecomposable module lies in a coil. The first case to consider is the case where the coil is a stable tube. We can assume the coil to be sincere. Indeeed, it is shown in [5](5.1) that the support algebra of a coil (thus, in particular, of a stable tube) in the Auslander-Reiten quiver of a multicoil algebra is a full convex subcategory of the latter, hence is, by (4.5), itself a multicoil algebra. In this case, we have the following structure theorem.

THEOREM [4](4.1). *Let A be a multicoil algebra. The following conditions are equivalent:*

(a) *A is tame concealed or tubular.*
(b) *There exists a sincere indecomposable A-module lying in a stable tube.*
(c) *There exist infinitely many non-isomorphic sincere indecomposable A-modules of the same dimension lying in homogeneous tubes.* □

Recall that the structure of the indecomposable modules over a tame concealed or a tubular algebra is described in [20].

5.2. COROLLARY [4](4.5). *Let A be a multicoil algebra, and Γ be a stable tube of $\Gamma(\text{mod } A)$. Then the support algebra of Γ is a full convex subcategory of A which is tame concealed or tubular, and has Γ as a full component.*

PROOF. By the remarks in (5.1), the support algebra B of Γ is a multicoil algebra. Moreover, B has clearly the tube Γ as a full component, and a sincere indecomposable module in Γ. There just remains to apply the theorem. □

5.3. COROLLARY [4](4.7). *A multicoil algebra A is domestic if and only if it contains no tubular algebra as a full convex subcategory.*

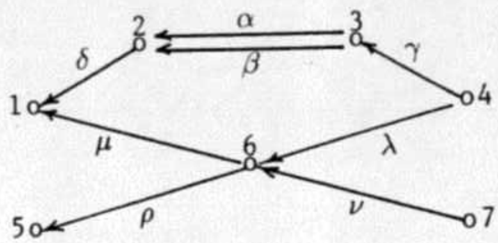
PROOF. If A contains a tubular algebra as a full convex subcategory, it is not domestic (by [20](5.2) or [21](3.6)). Conversely, if A contains no tubular algebra as a full convex subcategory, then all full convex subcategories of A satisfying condition (c) of the theorem are tame concealed and in particular domestic. Since the set of points in the ordinary quiver of A is finite, so is the set of all such full convex subcategories. Consequently, A is domestic. □

5.4. We also deduce from the above results the following structure theorem for coils in a multicoil algebra.

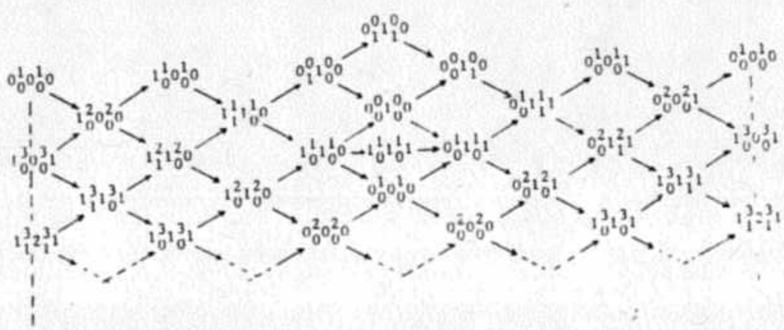
THEOREM [5] (5.9). *Let A be a multicoil algebra and Γ be a non-stable coil of $\Gamma(\text{mod } A)$. Then there exists a tame concealed full convex subcategory C of A and a stable tube Γ_0 of $\Gamma(\text{mod } C)$ such that Γ is obtained from Γ_0 by a sequence of admissible operations and the support algebra of Γ is obtained from C by the corresponding sequence of admissible operations.* □

5.5. There thus remains to describe the structure of the indecomposable modules over a multicoil algebra which lie in a non-stable coil. The classification should be completed soon [7, 8]. We try here to convey its flavour through the following example.

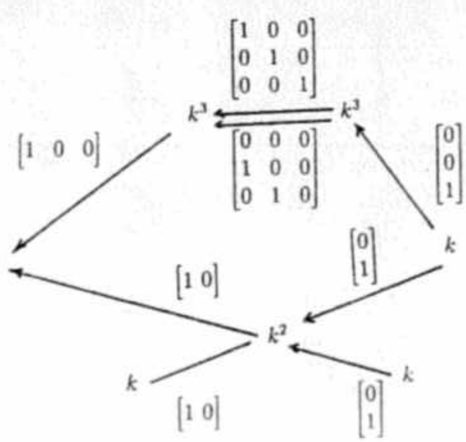
EXAMPLE. Let A be given by the quiver



bound by $\nu\mu = 0, \lambda\rho = 0, \gamma\beta = 0$ and $\beta\delta = 0$. Then $\Gamma(\text{mod } A)$ contains the following quasi-tube



One can see a sincere indecomposable module M of dimension-vector $\begin{smallmatrix} 3 & & 3 \\ 1 & 2 & 1 \\ 1 & & 1 \end{smallmatrix}$ lying at the intersection closer to the mouth of the ray starting with, and the coray ending with the projective-injective module $P(7) = I(5)$. Actually, it is easily seen that all the indecomposable modules lying at the intersections of this ray and this coray are sincere (recall the remarks following theorem (3.4) above). The structure of M is given by the following vector spaces and linear maps.



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References

[1] I. Assem and de la Peña, J.A., *The fundamental groups of a triangular algebra*, Comm. Algebra, to appear.

[2] I. Assem and A. Skowroński, Proc. London Math. Soc. **3, No. 56** (1988), 417–450.

[3] I. Assem and A. Skowroński, *Minimal representation-infinite coil algebras*, Manuscripta Math. **67** (1990), 305–331.

[4] I. Assem and A. Skowroński, *Indecomposable modules over multicoil algebras*, Math. Scand. **71** (1992), 31–61.

[5] I. Assem and A. Skowroński., *Multicoil algebras*, Can. Math. Soc. Conf. Proc. **14** (1993), 29–68.

[6] I. Assem and A. Skowroński and B. Tomé, *Coil enlargements of algebras*, Tsukuba J. Math., to appear.

[7] I. Assem, A. Skowroński and B. Tomé, *Indecomposable modules in quasi-tubes*, in preparation.

[8] I. Assem, A. Skowroński and B. Tomé, *Indecomposable modules in coils*, in preparation.

[9] M. Auslander and I. Reiten, *Representation theory of artin algebras III and IV*, Comm. Algebra **3** (1975), 239–294 and 5(1977), 443–518.

[10] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. **65** (1981/82), 331–378.

[11] P. Dräxler, *Completely separating algebras*, J. Algebra **165** (1994), 550–565.

[12] G. d'Este and C.M. Ringel, *Coherent tubes*, J. Algebra **87** (1984), 150–201.

[13] O. Kerner, *Tilting wild algebras*, J. London Math. Soc. **(2)39** (1989), no. 21, 29–47.

[14] J. A. de la Peña, *Tame algebras with sincere directing modules*, J. Algebra **161, No. 1** (1993), 171–185.

[15] J. A. de la Peña, *The families of two-parametric tame algebras with sincere directing modules*, Can. Math. Soc. Conf. Proc. **14** (1993), 361–392.

[16] J. A. de la Peña and A. Skowroński, *Forbidden subalgebras of non-polynomial growth tame simply connected algebras*, Canad. J. Math., to appear.

[17] J. A. de la Peña and A. Skowroński, *Geometric and homological characterizations of polynomial growth simply connected algebras*, preprint (1995), Universidad Nacional Autónoma de México.

[18] J. A. de la Peña and B. Tomé, *Iterated tubular algebras*, J. Pure Appl. Algebra **64** (1990), 303–314.

[19] J. A. de la Peña and B. Tomé, *Tame algebras with a weakly separating family of coils*, Proc. ICRA VII (1994), Can. Math. Soc. Conf. Proc., to appear.

[20] C.M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics 1099 (1984), Springer-Verlag, Berlin, Heidelberg, New York.

[21] A. Skowroński, *Algebras of polynomial growth, Topics in Algebra, Part 1 - Rings and Representations of Algebras*, Banach Centre Publications **26** (1990), PWN, Warsaw, 535–568.

[22] A. Skowroński, *Cycles in module categories, in: Finite dimensional algebras and related topics*, NATO ASI Series, Series C **424** (1994), Kluwer Academic Publishers, 309–346.

[23] A. Skowroński, *Criteria for polynomial growth of algebras*, Bull. Polish Acad. Sci. **42** (1994), Mathematics, 173–183.

[24] A. Skowroński, *Simply connected algebras and Hochschild cohomologies*, Can. Math. Soc. Conf. Proc. **Vol. 14** (1993), 431–448.

[25] A. Skowroński, *Minimal representation-infinite artin algebras*, Math. Proc. Camb. Phil. Soc. **116** (1994), 229–243.

[26] A. Skowroński, *Cycle-finite algebras*, J. Pure Appl. Algebra, to appear.

[27] A. Skowroński, *Tame algebras with simply connected Galois coverings* (preprint 1995), Nicholas Copernicus University, Toruń.

[28] A. Skowroński and M. Wenderlich, *Artin algebras with directing indecomposable projective modules*, J. Algebra **165** (1995), 507–530.

[29] B. Tomé, *Iterated coil enlargements of algebras*, Fund. Math. **146** (1995), 251–266.

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