Cluster categories and duplicated algebras

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Abstract

Let A be a hereditary algebra. We construct a fundamental domain for the cluster category $\mathcal{C}_A$ inside the category of modules over the duplicated algebra $\overline{A}$ of A. We then prove that there exists a bijection between the tilting objects in $\mathcal{C}_A$ and the tilting $\overline{A}$-modules all of whose non projective-injective indecomposable summands lie in the left part of the module category of $\overline{A}$.

Key words: cluster category, tilting, duplicated algebra

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0 Introduction

Cluster categories were introduced in [8], and for type A_n also in [9], as a means for a better understanding of the cluster algebras of Fomin and Zelevinsky [13,14]. The indecomposable objects (without self-extensions) in the cluster category correspond to the cluster variables in the cluster algebra and the tilting objects in the cluster category to the clusters in the cluster algebra. Our objective in this note is to give an interpretation of the cluster category and its tilting objects in terms of modules over a finite dimensional algebra. Indeed, let A be a hereditary algebra over an algebraically closed field then, by Happel's theorem [11], the derived category of bounded complexes over the category $\operatorname{mod} A$ of finitely generated right A-modules is equivalent to the stable module category over the repetitive algebra $\hat{A}$ of A (in the sense of Hughes and Waschbüsch [10]). The algebra $\hat{A}$ is infinite dimensional but, in order to study the cluster category, it suffices to look at a finite dimensional quotient of $\hat{A}$, namely the duplicated algebra $\overline{A}$ of A defined and studied in [1,5]. The resulting embedding of $\operatorname{mod} \overline{A}$ into $\operatorname{mod} \hat{A}$ induces a functor $\bar{\pi}$ from $\operatorname{mod} \overline{A}$ to the cluster category $\mathcal{C}_A$ of A. We prove that the functor $\bar{\pi}$ induces a one-to-one correspondence between the indecomposable objects in the cluster category and the non projective-injective $\overline{A}$-modules lying in the left part $\mathcal{L}_{\overline{A}}$ of $\operatorname{mod} \overline{A}$, in the sense of Happel, Reiten and Smalø [12] (we then say that $\mathcal{L}_{\overline{A}}$ is an *exact fundamental domain* for the functor $\bar{\pi}$). This opens the way to our main result.

Theorem 1 *Let A be a hereditary algebra. There exists a one-to-one correspondence between the multiplicity-free tilting objects in the cluster category $\mathcal{C}_A$ of A and the multiplicity-free tilting $\overline{A}$-modules such that all non projective-injective indecomposable summands of T lie in $\mathcal{L}_{\overline{A}}$.*

This correspondence is given explicitly as follows. Since any indecomposable projective-injective $\overline{A}$-module is necessarily a summand of T, then $T = T_0 \oplus \overline{P}$, where $\overline{P}$ is a uniquely determined projective-injective $\overline{A}$-module and T_0 has no projective-injective summands. If all the indecomposable summands of T_0 lie in $\mathcal{L}_{\overline{A}}$, then $\bar{\pi}(T_0)$ is a tilting object in $\mathcal{C}_A$ and conversely, any tilting object in $\mathcal{C}_A$ is of this form.

Since duplicated algebras appear as a perfect context to view (cluster-)tilting objects as actual tilting *modules*, we investigate these algebras further. In particular we show that the simply-laced Dynkin case corresponds to representation-finite duplicated algebras, which, in addition, are simply connected. In this case several techniques are known for computing the tilting modules, allowing us to find the clusters in the corresponding cluster algebra.

We now describe the contents of our paper. After a brief preliminary section, devoted to fixing the notation and recalling the main facts we shall be using, the second section contains a detailed description of the left part $\mathcal{L}_{\overline{A}}$. In the third section, we prove that $\mathcal{L}_{\overline{A}}$ is an exact fundamental domain for the natural functor and we prove our main result in section four. Our final section is devoted to deduce related properties of the duplicated algebra.

1 Preliminaries

1.1 Notation.

Throughout this paper, we let A denote a hereditary algebra over an algebraically closed field k. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ a full subcategory whose objects are representatives of the isomorphism classes of indecomposable objects in $\operatorname{mod} A$. The derived category of bounded complexes over $\operatorname{mod} A$ will be denoted by $\mathcal{D}^b(\operatorname{mod} A)$. For a vertex x in the quiver $\mathcal{Q}_A$ of A, we write e_x for the corresponding primitive idempotent and S_x, P_x, I_x, respectively, for the corresponding simple, indecomposable projective and indecomposable injective A-module. The functor $D = \operatorname{Hom}_k(-, k)$ is the standard duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\mathrm{op}}$, and $\tau_A = D\,Tr$, $\tau_A^{-1} = Tr\,D$ are the Auslander-Reiten translations in $\operatorname{mod} A$. We refer to [7] for further facts about $\operatorname{mod} A$, and to [16] for the tilting theory of $\operatorname{mod} A$.

1.2 The cluster category $\mathcal{C}_A$.

The cluster category $\mathcal{C}_A$ of A is defined as follows. Let F denote the endofunctor of $\mathcal{D}^b(\operatorname{mod} A)$ defined as the composition $\tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $\mathcal{D}^b(\operatorname{mod} A)$ and [1] is the shift functor. Then $\mathcal{C}_A$ is the quotient category $\mathcal{D}^b(\operatorname{mod} A)/F$. Its objects are the F-orbits of objects in $\mathcal{D}^b(\operatorname{mod} A)$ and the morphisms are given by

$$\operatorname{Hom}_{\mathcal{C}_A}(\tilde{X}, \tilde{Y}) = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(F^i X, Y) \qquad (1)$$

where X and Y are objects in $\mathcal{D}^b(\operatorname{mod} A)$ and $\tilde{X}, \tilde{Y}$ are their respective F-orbits. It is shown in [15] that $\mathcal{C}_A$ is a triangulated category. Furthermore, the canonical functor $\mathcal{D}^b(\operatorname{mod} A) \to \mathcal{C}_A$ is a functor of triangulated categories. We refer to [8] for facts about the cluster category.

1.3 The duplicated algebra $\overline{A}$

The duplicated algebra of a hereditary algebra A is the matrix algebra

$$\overline{A} = \begin{bmatrix} A & 0 \\ DA & A \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ q & b \end{bmatrix} \;\middle|\; \begin{array}{c} a, b \in A, \\ q \in DA \end{array} \right\} \tag{2}$$

with the ordinary matrix addition and the multiplication induced by the bimodule structure of DA. Writing 1 for the identity of A, and setting

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad e' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3}$$

we see that $\overline{A}$ contains two copies of A given respectively by $e\overline{A}e$ and by $e'\overline{A}e'$. In order to distinguish between these we denote the first one by A and the second one by A'. Accordingly, $\mathcal{Q}'_A$ denotes the quiver of A', x' the vertex of $\mathcal{Q}'_A$ corresponding to $x \in (\mathcal{Q}_A)_0$, and e'_x the corresponding idempotent. Let $\overline{S}_x, \overline{P}_x, \overline{I}_x$ denote respectively the simple, indecomposable projective and indecomposable injective module in $\operatorname{mod} \overline{A}$ corresponding to $x \in (\mathcal{Q}_A \cup \mathcal{Q}'_A)_0$.

The ordinary quiver $\mathcal{Q}_{\overline{A}}$ of $\overline{A}$ is constructed as follows. It contains $\mathcal{Q}_A$ and $\mathcal{Q}'_A$ as full convex connected subquivers and every vertex of $\mathcal{Q}_{\overline{A}}$ lies in either $\mathcal{Q}_A$ or $\mathcal{Q}'_A$. There is an arrow $x' \to y$ whenever $\operatorname{rad}(e'_x\overline{A}e_y)/\operatorname{rad}^2(e'_x\overline{A}e_y) \neq 0$. Observe that $e'_x\overline{A}e_y = D(e_yAe_x)$ and therefore, if $e_yAe_x \neq 0$ then there is a non-zero path in $\mathcal{Q}_{\overline{A}}$ from x' to y. Also, since $e'_x\overline{A} \cong D(\overline{A}e_x)$, each $\overline{I}_x = \overline{P}_{x'}$ is projective-injective having S_x as a socle and $S_{x'}$ as a top. On the other hand, each $\overline{P}_x$ has its support lying in $\mathcal{Q}_A$ and is therefore equal to the projective A-module P_x. Dually, $\overline{I}_{x'}$ has its support lying completely in $\mathcal{Q}'_A$ and equals the injective A'-module $I_{x'}$. For facts about the duplicated algebra, we refer to [1,5].

1.4 The repetitive algebra $\hat{A}$

For our purposes, another description of $\overline{A}$ is needed. The repetitive algebra $\hat{A}$ of the hereditary algebra A, is the infinite matrix algebra

$$\hat{A} \quad = \quad \begin{bmatrix} \ddots & & & 0 & \\ & A_{m-1} & & & \\ & Q_m & A_m & & \\ & & Q_{m+1} & A_{m+1} & \\ & 0 & & & \ddots \end{bmatrix} \tag{4}$$

where matrices have only finitely many non-zero coefficients, $A_m = A$ and $Q_m = {}_A DA_A$ for all $m \in \mathbb{Z}$, all the remaining coefficients are zero and multiplication is induced from the canonical isomorphisms $A \otimes_A DA \cong_A DA_A \cong DA \otimes_A A$ and the zero morphism $DA \otimes_A DA \to 0$, see [10]. Then $\overline{A}$ is identified to the quotient algebra of $\hat{A}$ defined by the surjection

$$\hat{A} \quad \to \quad \begin{bmatrix} A_0 & 0 \\ Q_1 & A_1 \end{bmatrix}. \tag{5}$$

This identification yields an embedding functor $\operatorname{mod} \overline{A} \hookrightarrow \operatorname{mod} \hat{A}$. Similarly, the canonical surjection $\overline{A} \to e\overline{A}e = A$ yields an embedding functor $\operatorname{mod} A \hookrightarrow \operatorname{mod} \overline{A}$. Our first objective will be to look more closely at these embeddings.

2 The left part $\mathcal{L}_{\overline{A}}$ of the duplicated algebra $\overline{A}$

2.1 Definitions and a preparatory lemma

Let C be any finite dimensional k-algebra, and M, N be two indecomposable C-modules. A *path* from M to N in $\operatorname{ind} C$ is a sequence of non-zero morphisms

$$M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N \tag{6}$$

with all M_i in $\operatorname{ind} C$. We denote such a path by $M \rightsquigarrow N$ and say that M is a *predecessor* of N (or that N is a *successor* of M). When each f_i in (6) is irreducible, we say that (6) is a *path of irreducible morphisms.* A path (6) of irreducible morphisms is *sectional* if $\tau_C\ M_{i+1} \neq M_{i-1}$ for all i with $1 \leq i \leq t$.

A *refinement* of (6) is a path in $\operatorname{ind} C$:

$$M = M_0' \xrightarrow{f_1'} M_1' \xrightarrow{f_2'} \cdots \xrightarrow{f_s'} M_s' = N \tag{7}$$

with $s \geq t$ such that there exists an order-preserving injection $\sigma : \{1, \dots, t-1\} \to \{1, \dots, s-1\}$ verifying $M_i = M'_{\sigma(i)}$ for all i with $1 \leq i \leq t$.

A full subcategory $\mathcal{C}$ of $\operatorname{ind} C$ is called *convex* in $\operatorname{ind} C$ if, for any path (6) from M to N in $\operatorname{ind} C$, with M, N lying in $\mathcal{C}$, all the M_i lie in $\mathcal{C}$.

Useful examples of convex subcategories arise from the standard embeddings $\operatorname{mod} A \hookrightarrow \operatorname{mod} \overline{A}$ and $\operatorname{mod} \overline{A} \hookrightarrow \operatorname{mod} \hat{A}$, as seen in 1.4 above. We have the following lemma (see [1, 2.5], [17, 3.4, 3.5] or [18, 4.1]), which will be used quite often when considering A-modules as $\overline{A}$-modules or $\hat{A}$-modules.

Lemma 2 *a) The embeddings* $\operatorname{mod} A \hookrightarrow \operatorname{mod} \overline{A}$ *and* $\operatorname{mod} \overline{A} \hookrightarrow \operatorname{mod} \hat{A}$ *are full, exact and preserve indecomposable modules, almost split sequences and irreducible morphisms.*
b) Under these embeddings, $\operatorname{ind} A$ *is a full convex subcategory of* $\operatorname{ind} \overline{A}$*, closed under predecessors, and* $\operatorname{ind} \overline{A}$ *is a full convex subcategory of* $\operatorname{ind} \hat{A}$.

2.2 The left part

Let again C be a finite dimensional algebra. Following Happel, Reiten and Smalø [12], we define the *left part* $\mathcal{L}_C$ of $\operatorname{mod} C$ to be the full subcategory of $\operatorname{mod} C$ consisting of all indecomposable C-modules such that if $L \rightsquigarrow M$, then the projective dimension $\operatorname{pd} L$ of L is at most one. The right part $\mathcal{R}_C$ is defined dually.

Our objective now is to compute the left part of the module category of the duplicated algebra $\overline{A}$ of a hereditary algebra A. We start by observing that, by Lemma 2, the complete slice of the Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$ of A consisting of the indecomposable injective A-modules embeds fully inside $\Gamma(\operatorname{mod} \overline{A})$. The sources in this slice are the injectives I_a with a a sink in $\mathcal{Q}_A$. For each sink a in $\mathcal{Q}_A$, the injective A-module I_a is the radical of the projective-injective $\overline{A}$-module $\overline{I}_a = \overline{P}_{a'}$.

We recall that for any algebra C and any L in $\operatorname{mod} C$, $\operatorname{pd} L \leq 1$ if and only if $\operatorname{Hom}_C(DC, \tau_C L) = 0$ (see [7, IX.1.7, p.319] or [16, p.79]).

Lemma 3 *Let M be an indecomposable $\overline{A}$-module. Then:*

a) If M belongs to $\operatorname{ind} A$*, then $M \in \mathcal{L}_{\overline{A}}$ and $\tau_{\overline{A}}^{-1} M \in \mathcal{L}_{\overline{A}}$.*

b) If M does not belong to $\operatorname{ind} A$*, then there exist a sink* $a \in (\mathcal{Q}_A)_0$ *and a path* $\overline{P}_{a'} \rightsquigarrow M$.

PROOF. a) Any A-module M admits a projective resolution in $\operatorname{mod} A$ of the form

$$0 \to P_1 \to P_0 \to M \to 0 \tag{8}$$

with P_0 and P_1 projective A-modules, hence projective $\overline{A}$-modules. Thus the projective dimension of M as an A-module and also as an $\overline{A}$-module is at most one. This shows that $\operatorname{ind} A \subset \mathcal{L}_{\overline{A}}$, because $\operatorname{ind} A$ is closed under predecessors.

To see that $\tau_{\overline{A}}^{-1} M$ is in $\mathcal{L}_{\overline{A}}$, notice that, since M is in $\operatorname{ind} A$, $\operatorname{Hom}_{\overline{A}}(\overline{I}_x, M) = 0$ for all injective $\overline{A}$-modules $\overline{I}_x$. So $\operatorname{pd}_{\overline{A}}(\tau_{\overline{A}}^{-1} M) \leq 1$ by the above remark. Furthermore, any non-projective predecessor L of $\tau_{\overline{A}}^{-1} M$ lies in $\operatorname{ind} A \cup \tau_{\overline{A}}^{-1}(\operatorname{ind} A)$, hence $\operatorname{pd} L \leq 1$.

b) Assume now that M is not in $\operatorname{ind} A$. Then there exists $b \in (\mathcal{Q}_A)_0$ such that $\operatorname{Hom}_{\overline{A}}(\overline{P}_{b'}, M) \neq 0$. If b is a sink, we are done. If not, consider the projective A-module P_b. Let S_a be a simple submodule of P_b. Note that S_a is projective since A is hereditary. Therefore $S_a = P_a$ and a is a sink. Then $\operatorname{Hom}_A(P_a, P_b) \neq 0$ implies $\operatorname{Hom}_A(I_a, I_b) \neq 0$, which induces a non-zero morphism $\overline{P}_{a'} = \overline{I}_a \to \overline{I}_b = \overline{P}_{b'}$ of $\overline{A}$-modules. This yields the required path $\overline{P}_{a'} \rightsquigarrow M$. □

2.3 A characterization of the modules in $\mathcal{L}_{\overline{A}}$

Before stating the next proposition, we recall that, by [3, 1.6], $\mathcal{L}_{\overline{A}}$ consists of all $M \in \operatorname{ind} \overline{A}$, such that, if there exists a path from an indecomposable injective module to M, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.

Proposition 4 *An indecomposable $\overline{A}$-module M is in $\mathcal{L}_{\overline{A}}$ if and only if, whenever there exists a path $\overline{P}_{a'} \rightsquigarrow M$, with a a sink in $(\mathcal{Q}_A)_0$, this path can be refined to a path of irreducible morphisms, and each such refinement is sectional.*

PROOF. Since the necessity follows directly from the above statement, we only prove the sufficiency. Assume that M satisfies the stated condition. In order to prove that $M \in \mathcal{L}_{\overline{A}}$, it suffices to show that, if there exists a path $\overline{I}_x \rightsquigarrow M$, with $\overline{I}_x$ injective in $\operatorname{mod} \overline{A}$, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional. Since $\overline{I}_x$ is not an A-module, it follows from Lemma 3 b), that there exist a sink a in $\mathcal{Q}_A$ and a path $\overline{P}_{a'} \rightsquigarrow \overline{I}_x$, giving a path $\overline{P}_{a'} \rightsquigarrow \overline{I}_x \rightsquigarrow M$. The conclusion follows at once. □

2.4 Ext-injectives in $\mathcal{L}_{\overline{A}}$

We now characterize the Ext-injectives in the additive full subcategory add $\mathcal{L}_{\overline{A}}$ of mod $\overline{A}$ generated by the left part. We recall from [6] that, if $\mathcal{A}$ is an additive full subcategory of mod $\overline{A}$, closed under extensions, then an indecomposable module M in $\mathcal{A}$ is called an Ext-*injective in* $\mathcal{A}$ if $\mathrm{Ext}^1_{\overline{A}}(\ \ ,M)|_{\mathcal{A}} = 0$. It is known that M is Ext-injective in add $\mathcal{L}_{\overline{A}}$ if and only if $\tau^{-1}_{\overline{A}}M$ is not in add $\mathcal{L}_{\overline{A}}$ (see [6, 3.4]). We denote by Σ the set of all indecomposable Ext-injectives in add $\mathcal{L}_{\overline{A}}$. The following corollary says that $\mathcal{L}_{\overline{A}} = \mathrm{ind}\, A \cup \Sigma$.

Corollary 5 *The following are equivalent for an $\overline{A}$-module M:*
a) M is in Σ.
b) M is in $\mathcal{L}_{\overline{A}}$ and M is not in $\mathrm{ind}\, A$.
c) M is in $\mathcal{L}_{\overline{A}}$ and there exist a sink $a \in (\mathcal{Q}_A)_0$ and a path $\overline{P}_{a'} \rightsquigarrow M$.
d) There exist a sink $a \in (\mathcal{Q}_A)_0$ and a path $\overline{P}_{a'} \rightsquigarrow M$ and any such path is refinable to a sectional path.

PROOF.
a) implies b) since $\mathrm{ind}\, A \cup \tau^{-1}(\mathrm{ind}\, A) \subset \mathcal{L}_{\overline{A}}$ by Lemma 3 a).
b) implies c) follows from Lemma 3 b).
c) implies d) follows from Proposition 4.
d) implies a) Proposition 4 implies that M is in $\mathcal{L}_{\overline{A}}$. The fact that there exist a sink $a \in (\mathcal{Q}_A)_0$ and a path $\overline{I}_a = \overline{P}_{a'} \rightsquigarrow M$ (hence a sectional path), implies that $\mathrm{Hom}_{\overline{A}}(\overline{I}_a, M) \neq 0$ by [7, III.2.4, p.239]. By the remark before Lemma 3, it follows that $\mathrm{pd}_{\overline{A}}(\tau^{-1}_{\overline{A}}M) \geq 2$ and therefore $\tau^{-1}_{\overline{A}}M$ is not in $\mathcal{L}_{\overline{A}}$. $\square$

Corollary 6 *The set Σ of all indecomposable Ext-injectives in* add $\mathcal{L}_{\overline{A}}$ *consists of all the projective-injectives lying in $\mathcal{L}_{\overline{A}}$ as well as all the modules of the form $\tau^{-1}I_x$ with $x \in (\mathcal{Q}_A)_0$, that is*

$$\Sigma = \{\tau^{-1}_{\overline{A}} I_x \mid x \in (Q_A)_0\} \cup \{\overline{P}_{x'} \mid \overline{P}_{x'} \in \mathcal{L}_{\overline{A}}\}. \tag{9}$$

PROOF. Clearly, projective-injective modules which lie in $\mathcal{L}_{\overline{A}}$ belong to Σ. Now let $x \in (\mathcal{Q}_A)_0$ and consider $\tau^{-1}_{\overline{A}}I_x$. Let a be a sink and I_a be a maximal indecomposable injective A-module such that there is an epimorphism $I_a \to I_x$. Then there is a non-zero map $\tau^{-1}_{\overline{A}}I_a \to \tau^{-1}_{\overline{A}}I_x$ and therefore a path $\overline{I}_a \to \tau^{-1}_{\overline{A}}I_a \to \tau^{-1}_{\overline{A}}I_x$. Since $\tau^{-1}_{\overline{A}}I_x$ is in $\mathcal{L}_{\overline{A}}$, it follows that $\tau^{-1}_{\overline{A}}I_x$ is in Σ by Corollary 5 d).

Conversely, suppose X belongs to Σ, but is not a projective-injective lying in $\mathcal{L}_{\overline{A}}$. By Corollary 5, there exists a sink a and a sectional path in ind $\overline{A}$

$$\overline{P}_{a'} = \overline{I}_a = M_0 \to M_1 \to \ldots \to M_t = X \tag{10}$$

with $t \geq 1$ and $M_1 = \overline{I}_a/S_a = \tau_{\overline{A}}^{-1} I_a$. We claim that no M_i (with $i \geq 1$) is a projective $\overline{A}$-module. Indeed, assume first that M_i (with $i \geq 1$) is projective-injective. By hypothesis, $i < t$. Then $M_{i-1} = \operatorname{rad} M_i$ and $M_{i+1} = M_i/\operatorname{soc} M_i = \tau_{\overline{A}}^{-1} M_{i-1}$, contradicting the sectionality of the above path. On the other hand, for any $i \leq t$, $\operatorname{Hom}_{\overline{A}}(\overline{P}_{a'}, M_i) \neq 0$ hence M_i is not an A-module, and a fortiori not projective in $\operatorname{mod} A$. This establishes our claim. We infer the existence of a sectional path in $\operatorname{ind} \overline{A}$

$$I_a = \tau_{\overline{A}} M_1 \to \tau_{\overline{A}} M_2 \ldots \to \tau_{\overline{A}} M_t = \tau_{\overline{A}} X. \tag{11}$$

Since $X \in \mathcal{L}_{\overline{A}}$, then, for any $i \leq t$, $M_i \in \mathcal{L}_{\overline{A}}$ and so $\operatorname{pd} M_i \leq 1$ implying that $\operatorname{Hom}_{\overline{A}}(\overline{P}_{x'}, \tau_{\overline{A}} M_i) = 0$ for any $x \in (\mathcal{Q}_A)_0$. This shows that the above path lies entirely in $\operatorname{mod} A$. Since I_a is injective, all the modules on it are injective. In particular, there exists $x \in (\mathcal{Q}_A)_0$ such that $\tau_{\overline{A}} X = I_x$. □

We now give another expression for the set of all indecomposable Ext-injectives in $\operatorname{add} \mathcal{L}_{\overline{A}}$. For this, we need to recall that, if M is an $\overline{A}$-module, then its first cosyzygy $\Omega_{\overline{A}}^{-1} M$ is the cokernel of an injective envelope $M \to \overline{I}$ in $\operatorname{mod} \overline{A}$.

Proposition 7 *Let $x \in (Q_A)_0$. Then $\Omega_{\overline{A}}^{-1} P_x \cong \tau_{\overline{A}}^{-1} I_x$. Consequently,*

$$\Sigma = \{\Omega_{\overline{A}}^{-1} P_x \mid x \in (Q_A)_0\} \cup \{\overline{P}_{x'} \mid \overline{P}_{x'} \in \mathcal{L}_{\overline{A}}\}. \tag{12}$$

PROOF. We prove this by induction on the Loewy length of the projective module P_x. Recall that the Loewy length of a module M is the smallest integer i with $\operatorname{rad}^i M = 0$. Let $P_a = S_a$ be a simple projective module. Then $\Omega_{\overline{A}}^{-1} P_a \cong \overline{I}_a/S_a$. On the other hand, from the almost split sequence:

$$0 \to I_a \to \overline{I}_a \oplus I_a/S_a \to \overline{I}_a/S_a \to 0 \tag{13}$$

it follows that $\Omega_{\overline{A}}^{-1} P_a \cong \tau_{\overline{A}}^{-1} I_a$ for any sink a, which proves our claim in this case. For an indecomposable non-simple projective P_x let the radical be $\operatorname{rad} P_x = \oplus P_{y_i}$. Then there are the following isomorphisms of the injective envelopes: $I_0\,(P_x) = I_0\,(\operatorname{rad} P_x) \cong \oplus I_0\,(P_{y_i})$. Then $\Omega_{\overline{A}}^{-1}(P_x) = I_0\,(P_x)/P_x$ and $\Omega_{\overline{A}}^{-1}(\operatorname{rad} P_x) = \oplus \Omega_{\overline{A}}^{-1}(P_{y_i}) \cong I_0\,(P_x) / (\oplus P_{y_i})$. A simple application of the snake lemma yields $\Omega_{\overline{A}}^{-1}(P_x) \cong \Omega_{\overline{A}}^{-1}(\operatorname{rad} P_x) / S_x$. Now, it is easy to see that there is an almost split sequence

$$0 \to I_x \to \left(\oplus \left(\tau_{\overline{A}}^{-1} I_{y_i}\right)\right) \oplus I_x/S_x \to \tau_{\overline{A}}^{-1} I_x \to 0. \tag{14}$$

Since each morphism in this sequence is irreducible, it is either a monomorphism or an epimorphism. Since S_x is the kernel of the morphism $I_x \to$

I_x/S_x, another application of the snake lemma and the induction hypothesis $\tau_{\overline{A}}^{-1}(I_{y_i}) \cong \Omega_{\overline{A}}^{-1}(P_{y_i})$ yield

$$\tau_{\overline{A}}^{-1} I_x \cong \left(\oplus \left(\tau_{\overline{A}}^{-1} I_{y_i}\right)\right) \Big/ S_x \cong \left(\oplus \left(\Omega_{\overline{A}}^{-1} P_{y_i}\right)\right) \Big/ S_x \cong \Omega_{\overline{A}}^{-1}(P_x). \tag{15}$$

□

3 Fundamental domain for the cluster category.

3.1 $\mathcal{L}_{\overline{A}}$ *as a subcategory of* $\operatorname{mod} \hat{A}$

As a consequence of the above description, the left part $\mathcal{L}_{\overline{A}}$ is nicely embedded in $\operatorname{mod} \overline{A}$, and thus in $\operatorname{mod} \hat{A}$.

Corollary 8 *The embedding $\mathcal{L}_{\overline{A}} \hookrightarrow \operatorname{mod} \overline{A} \hookrightarrow \operatorname{mod} \hat{A}$ is full, exact and preserves indecomposable modules, irreducible morphisms and almost split sequences.*

3.2 Relation between $\mathcal{L}_{\overline{A}}$ *and* $\mathcal{C}_A$

We are now able to describe an exact fundamental domain for the cluster category $\mathcal{C}_A$ inside $\operatorname{mod} \overline{A}$, and actually inside $\mathcal{L}_{\overline{A}}$. Indeed, since A is hereditary, and thus of finite global dimension, we have a triangulated equivalence $\mathcal{D}^b(\operatorname{mod} A) \cong \underline{\operatorname{mod}}\, \hat{A}$ (see [11]). Let

$$\hat{\pi} : \operatorname{mod}\, \hat{A} \twoheadrightarrow \underline{\operatorname{mod}}\, \hat{A} \cong \mathcal{D}^b(\operatorname{mod} A) \twoheadrightarrow \mathcal{C}_A \tag{16}$$

be the canonical functor. We define an *exact fundamental domain* for $\hat{\pi}$ to be a full convex subcategory of $\operatorname{ind} \hat{A}$ which contains exactly one point of each fibre $\hat{\pi}^{-1}(X)$, with X an indecomposable object in $\mathcal{C}_A$.

We recall at this point that $\operatorname{ind} \overline{A}$ is a full convex subcategory of $\operatorname{ind} \hat{A}$.

Theorem 9 *The functor $\hat{\pi}$ induces a one-to-one correspondence between the non projective-injective modules in $\mathcal{L}_{\overline{A}}$ and the indecomposable objects in $\mathcal{C}_A$. In particular, $\mathcal{L}_{\overline{A}}$ is an exact fundamental domain for $\hat{\pi}$.*

PROOF. Since $\mathcal{L}_{\overline{A}}$ is a full convex subcategory of $\operatorname{ind} \overline{A}$, it is also convex inside $\operatorname{ind} \hat{A}$. Furthermore, the non projective-injective modules in $\mathcal{L}_{\overline{A}}$ are just the modules in $\operatorname{ind} A$ and those of $\{\Omega_{\overline{A}}^{-1} P_x \mid x \in (\mathcal{Q}_A)_0\}$. The statement follows at once from the definition of $\mathcal{C}_A$ and from the fact that under the triangle equivalence $\mathcal{D}^b(\mathrm{mod}\, A) \cong \underline{\mathrm{mod}}\, \hat{A}$, the shift of $\mathcal{D}^b(\mathrm{mod}\, A)$ corresponds to $\Omega_{\hat{A}}^{-1}$ (see [11]). □

4 Tilting modules vs tilting objects

4.1 The main theorem

In this section, we prove our main theorem, which compares the tilting $\overline{A}$-modules with the tilting objects in $\mathcal{C}_A$. For this purpose, we assume without loss of generality that our tilting modules and our tilting objects are multiplicity-free. We start by observing that, if T is a tilting $\overline{A}$-module, then every indecomposable projective-injective $\overline{A}$-module is a direct summand of T. Hence T decomposes uniquely as $T = T_0 \oplus e'\overline{A}$, where T_0 has no projective-injective direct summands. We say that T is an *$\mathcal{L}$-tilting module* if $T_0 \in \mathrm{add}\, \mathcal{L}_{\overline{A}}$.

We denote by $\bar{\pi} : \mathrm{mod}\, \overline{A} \to \mathcal{C}_A$, the composition of the inclusion $\mathrm{mod}\, \overline{A} \hookrightarrow \mathrm{mod}\, \hat{A}$ and the functor $\hat{\pi}$. By abuse of notation, the modules will be often denoted by the same letter even when considered as objects in different categories.

Theorem 10 *There is a one-to-one correspondence*

$$\{\mathcal{L} - \textit{tilting modules}\} \longleftrightarrow \{\textit{Tilting objects in } \mathcal{C}_A\}$$

given by $$T = T_0 \oplus e'\,\overline{A} \longleftrightarrow \bar{\pi}(T_0).$$

PROOF. Let $T = T_0 \oplus e'\overline{A}$ be an $\mathcal{L}$-tilting module and let $X = \bar{\pi}(T_0)$. Say $T_0 = \oplus_{i=1}^{n} T_i$ where the T_i are pairwise non-isomorphic indecomposable $\overline{A}$-modules. Then $X = \oplus_{i=1}^{n} X_i$ with $X_i = \bar{\pi}(T_i)$. We first notice that, clearly, the number n of indecomposable summands of T_0 is equal to the rank of the Grothendieck group of A. Hence, in order to show that X is a tilting object in $\mathcal{C}_A$, it suffices to prove that $\mathrm{Ext}^1_{\mathcal{C}_A}(X, X) = 0$. Suppose to the contrary that there exist i, j such that $\mathrm{Ext}^1_{\mathcal{C}_A}(X_j, X_i) \neq 0$. Since Ext^1 is symmetric in the cluster category by [8, 1.7], we also have $\mathrm{Ext}^1_{\mathcal{C}_A}(X_i, X_j) \neq 0$. Thus there are non-zero morphisms $X_i \to \tau_{\mathcal{C}_A} X_j$ and $X_j \to \tau_{\mathcal{C}_A} X_i$ in $\mathcal{C}_A$. Let $\hat{F} = \Omega_{\hat{A}}^{-1} \tau_{\hat{A}}^{-1}$. Then there exist integers $s, t \geq 0$ such that the previous morphisms lift to non-zero morphisms in $\underline{\mathrm{mod}}\, \hat{A}$

$$T_i \to \hat{F}^s \tau_{\hat{A}} T_j \quad \text{and} \quad T_j \to \hat{F}^t \tau_{\hat{A}} T_i, \tag{17}$$

by definition of the cluster category and the triangulated structure of $\mathrm{mod}\,\hat{A}$, see [11]. Moreover $s \neq 0$ and $t \neq 0$ since by hypothesis $\mathrm{Ext}^1_{\overline{A}}(T_j, T_i) = \mathrm{Ext}^1_{\overline{A}}(T_i, T_j) = 0$. Now T_i, T_j are in $\mathcal{L}_{\overline{A}} = \mathrm{ind}\,A \cup \Sigma$. We then have 3 cases to consider.

(1) $T_i, T_j \in \Sigma$. Then X_i and X_j lie on a slice of $\mathcal{C}_A$, hence $\mathrm{Ext}^1_{\mathcal{C}_A}(X_i, X_j) = 0$, a contradiction.

(2) $T_i, T_j \in \mathrm{ind}\,A$. If $s = 1$, then there is a non-zero morphism $T_i \to \hat{F}\tau_{\hat{A}} T_j = \Omega^{-1}_{\hat{A}} T_j$ in $\underline{\mathrm{mod}}\,\hat{A}$. But this is impossible since

$$\underline{\mathrm{Hom}}_{\hat{A}}(T_i, \Omega^{-1}_{\hat{A}} T_j) = \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T_i, T_j[1]) = \mathrm{Ext}^1_A(T_i, T_j) = 0 \quad (18)$$

where we have identified the modules T_i and T_j with the corresponding stalk complexes in $\mathcal{D}^b(\mathrm{mod}\,A)$. Assume thus that $s \geq 2$. Now, either $\tau_{\hat{A}} T_j$ is an A-module, or T_j is a projective A-module, and then $\tau_{\hat{A}} T_j[1]$ is an A-module. But this fact and the structure of the morphisms in the derived category (see [11]) imply that

$$\underline{\mathrm{Hom}}_{\hat{A}}(T_i, \hat{F}^s \tau_{\hat{A}} T_j) = \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T_i, F^s \tau_{\hat{A}} T_j) = 0, \quad (19)$$

again a contradiction.

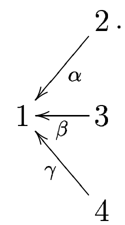
(3) $T_i \in \mathrm{ind}\,A, T_j \in \Sigma$. Then by Proposition 7, there exists an indecomposable projective A-module P_x such that $T_j = \Omega^{-1}_{\hat{A}} P_x$. Since $\hat{A}$ is self-injective, it follows from [7, IV.3.7] that $\Omega^{-2}_{\hat{A}} = \nu_{\hat{A}} \tau^{-1}_{\hat{A}}$ where $\nu_{\hat{A}}$ is the Nakayama functor in $\mathrm{mod}\,\hat{A}$. Thus $\hat{F}\,\tau_{\hat{A}}\,T_j = \Omega^{-2}_{\hat{A}}\,P_x = \nu_{\hat{A}}\,\tau^{-1}_{\hat{A}}\,P_x$ which is an A'-module (unless A is of Dynkin type A_n, linearly oriented and P_x is projective-injective, in which case $\hat{F}\,\tau_{\hat{A}}\,T_j[-1]$ is an A'-module). Therefore the modules T_i and $\hat{F}^s\,\tau_{\hat{A}}\,T_j$ have disjoint supports for any $s \geq 1$. Therefore $\underline{\mathrm{Hom}}_{\hat{A}}(T_i, \hat{F}^s \tau_{\hat{A}} T_j) = 0$ for any $s \geq 1$, contradiction.

This completes the proof that $X = \hat{\pi}(T_0)$ is a tilting object in $\mathcal{C}_A$.

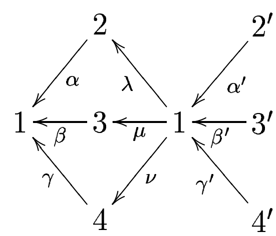
Conversely, let $X = \oplus_{i=1}^n X_i$ be any tilting object in $\mathcal{C}_A$, where we assume that the objects X_i are indecomposable and pairwise non-isomorphic. By Theorem 9, there exists, for each i with $1 \leq i \leq n$, a unique module $T_i \in \mathcal{L}_{\overline{A}}$ in the fibre $\hat{\pi}^{-1}(X_i)$. Let $T_0 = \oplus_{i=1}^n T_i$. Then, clearly $\hat{\pi}(T_0) = X$. We want to show that $T = T_0 \oplus e'\overline{A}$ is an $\mathcal{L}$-tilting $\overline{A}$-module. Since $T_0 \in \mathrm{add}\,\mathcal{L}_{\overline{A}}$ by construction and, on the other hand, the number of indecomposable summands of T_0 is equal to the rank of the Grothendieck group of A, we only have to prove that $\mathrm{Ext}^1_{\overline{A}}(T, T) = 0$. Suppose to the contrary, that there exist i, j such that $\mathrm{Ext}^1_{\overline{A}}(T_i, T_j) \neq 0$. Then $\mathrm{Hom}_{\overline{A}}(T_j, \tau_{\overline{A}} T_i) \neq 0$. In particular, T_i is not projective in $\mathrm{mod}\overline{A}$. Now, $T_i \in \mathcal{L}_{\overline{A}}$ implies that $\tau_{\overline{A}} T_i = \tau_{\hat{A}} T_i$. By Lemma 3 and Corollary 5, we also have $\tau_{\overline{A}} T_i \in \mathrm{ind}\,A$. Therefore $\mathrm{Hom}_{\overline{A}}(T_j, \tau_{\overline{A}} T_i) \neq 0$ implies that $T_j \in \mathrm{ind}\,A$ (because $\mathrm{ind}\,A$ is closed under predecessors in $\mathrm{ind}\,\overline{A}$). Thus $\mathrm{Hom}_A(T_j, \tau_{\overline{A}} T_i) \neq 0$ and then $\mathrm{Ext}^1_{\mathcal{C}_A}(X_i, X_j) \neq 0$, contradiction. □

4.2 Example

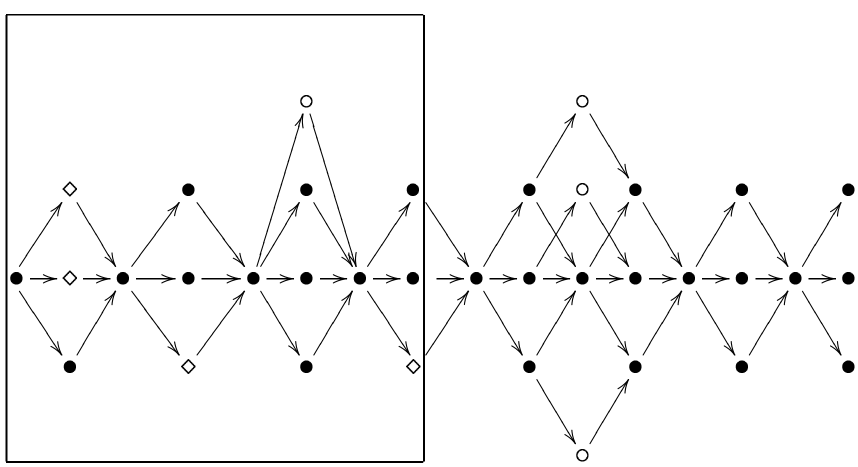
Let A be given by the quiver



Then the ordinary quiver of $\overline{A}$ is given by



bound by the relations $\lambda\alpha = \mu\beta = \nu\gamma,\ \alpha'\mu = \alpha'\nu = \beta'\nu = \beta'\lambda = \gamma'\lambda = \gamma'\mu = 0$. The Auslander-Reiten quiver of $\overline{A}$ is given by



$\mathcal{L}_{\overline{A}}$

where we have indicated the left part $\mathcal{L}_{\overline{A}}$. We have also indicated an $\mathcal{L}$-tilting module $T = T_0 \oplus e'\overline{A}$, where $T_0 \in \operatorname{add} \mathcal{L}_{\overline{A}}$. The summands of T_0 are indicated by diamonds and the (projective-injective) summands of $e'\overline{A}$ by circles.

5 More on duplicated algebras of hereditary algebras

It follows from our main theorem that the duplicated algebras of hereditary algebras are quite a natural class to consider, since all the tilting objects of the cluster category correspond to the actual modules over the duplicated algebras. In this section we study other properties of these algebras, which are consequences of the description of the left part $\mathcal{L}_{\overline{A}}$ and the Ext-injectives as done in the previous sections.

We recall that a finite dimensional algebra C is called *left (or right) supported* provided the class $\operatorname{add}\mathcal{L}_C$ (or $\operatorname{add}\mathcal{R}_C$) is contravariantly finite (or covariantly finite, respectively) in $\operatorname{mod} C$, see [4,2].

Corollary 11 *The duplicated algebra $\overline{A}$ of a hereditary algebra is both left and right supported.*

PROOF. By [4, 3.3], the canonical module $T = U \oplus V$ (with $U = \oplus_{X\in\Sigma}X$ and $V = \oplus_{\overline{P}_x \notin \mathcal{L}_{\overline{A}}} \overline{P}_x$) is a partial tilting module. Now the number of its indecomposable summands equals the number of isomorphism classes of indecomposable injective A-modules plus the number of isomorphism classes of indecomposable projective-injective $\overline{A}$-modules. Hence T is a tilting module and $\overline{A}$ is left supported, by [4, thm. A]. The other statement follows by symmetry. □

Remark 12 *The assumption that A is a hereditary algebra is essential. If A is a tilted algebra which is the endomorphism algebra of a regular tilting module, then it is easily seen that $\overline{A}$ is neither left nor right supported.*

Equivalent statements to duplicated algebras being representation-finite are given in the next corollary. We recall that an algebra C is said to be a *laura algebra* [3] provided the class $\operatorname{ind} C \setminus (\mathcal{L}_C \cup \mathcal{R}_C)$ contains only finitely many indecomposables.

Corollary 13 *Let A be a hereditary algebra. The following conditions are equivalent:*

(a) *$\overline{A}$ is a laura algebra.*

(b) *A is of Dynkin type.*

(c) *$\overline{A}$ is representation-finite.*

If this is the case, then $\overline{A}$ is simply connected.

PROOF. We denote by Σ' the set of all indecomposable Ext-projectives in add $\mathcal{R}_{\overline{A}}$. By Lemma 3, Corollary 5 and their duals, the duplicated algebra $\overline{A}$ is laura if and only if the class $[\tau_{\overline{A}}\Sigma, \tau_{\overline{A}}^{-1}\Sigma']$ of all the $M \in \operatorname{ind}\overline{A}$ such that there exists a path $L \rightsquigarrow M \rightsquigarrow N$, with $\tau_{\overline{A}}^{-1}L \in \Sigma$ and $\tau_{\overline{A}}N \in \Sigma'$ consists of finitely many indecomposables. Now, by [1, 2.6] this class is an exact fundamental domain for the module category over the trivial extension $T(A)$ of A by its minimal injective cogenerator DA. Therefore $\overline{A}$ is laura if and only if $T(A)$ is representation finite, or, by [17], if and only if A is of Dynkin type which, by [1, 2.6] is the case if and only if $\overline{A}$ is representation-finite. The last statement follows from [1, 2.7]. □

Remark 14 *Assume A to be representation-infinite. Then, of course, Theorems 5 and 10 still apply. In this case as well, a good description of the module category of the duplicated algebra $\overline{A}$ is known (see [1,5]) and, at least in the tame case, it is possible to compute explicitly the $\mathcal{L}$-tilting modules.*

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References

[1] I. Assem, On representations of the bimodule DA, *Tsukuba J. Math.* **9**, No.2 (1985) 217–232.

[2] I. Assem, J. Cappa, M.I. Platzeck and S. Trepode, Some characterisations of supported algebras , *Preprint No.21, Dép. de Math., U. de Sherbrooke* (2005)

[3] I. Assem, F.U. Coelho, Two-sided gluings of tilted algebras , *J. Algebra* **269**, (2) (2003) 456–470.

[4] I. Assem, F.U. Coelho and S. Trepode, The left and right parts of a module category, *J. Algebra* **281**,(2) (2004), 518–534.

[5] I. Assem, J. Nehring and W. Schewe, Fundamental domains and duplicated algebras, *CMS Conference Proceedings* **11**, (1991) 25–51.

[6] M. Auslander and S.O. Smalø, Almost split sequences in subcategories *J. Algebra* **69**, (1981) 426–454.

[7] M. Auslander, I. Reiten and S.O. Smalø, *Representation Theory of Artin Algebras* Cambridge Studies in Advanced Math. 36, (Cambridge University Press, 1995).

[8] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, to appear in *Adv. Math.*

[9] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case), to appear in *Trans. Amer. Math. Soc.*

[10] D. Hughes and J. Waschbüsch, Trivial extensions of tilted algebras *Proc. London Math. Soc.* **46** (1983) 347–364.

[11] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras* London Mathematical Society. Lecture Notes Series 119, (Cambridge University Press, 1988).

[12] D. Happel, I. Reiten and S.O. Smalø, Tilting in abelian categories and quasi tilted algebras, *Memoirs Amer. Math. Soc.* **575**, (1996).

[13] S. Fomin and A. Zelevinsky, Cluster algebras I. Foundations, *J. Amer. Math. Soc.* **15**(2), (2002), 497–529 (electronic)

[14] S. Fomin and A. Zelevinsky, Cluster algebras II. Finite type classification, *Inventiones Mathematicae* **154**(1), (2003), 63–121.

[15] B. Keller, Triangulated orbit categories, preprint (2005).

[16] C.M. Ringel, *Tame algebras and integral quadratic forms* Lecture Notes in Math. 1099, (Springer Verlag, 1984).

[17] H. Tachikawa, Representations of trivial extensions of hereditary algebras *Proc. ICRA II* , Lecture Notes in Math. 832, (Springer Verlag 1980) 579–599.

[18] K. Yamagata, Extensions over hereditary artinian rings with self-dualities I. *J. Algebra* **73** (2) (1981) 386–433.