Cluster-tilted algebras as trivial extensions

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Abstract

Given a finite dimensional algebra C (over an algebraically closed field) of global dimension at most two, we define its relation-extension algebra to be the trivial extension $C \ltimes \mathrm{Ext}^2_C(DC, C)$ of C by the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$. We give a construction for the quiver of the relation-extension algebra in case the quiver of C has no oriented cycles. Our main result says that an algebra $\tilde{C}$ is cluster-tilted if and only if there exists a tilted algebra C such that $\tilde{C}$ is isomorphic to the relation-extension of C.

1 Introduction

Cluster categories were introduced in [6], and, for type A_n also in [12], as a means for a better understanding of the cluster algebras of Fomin and Zelevinsky [14, 15]. They are defined as follows: let A be a hereditary algebra, and $\mathcal{D}^b(\mathrm{mod}\, A)$ be the derived category of bounded complexes of finitely generated A-modules, then the cluster category $\mathcal{C}_A$ is the orbit category of $\mathcal{D}^b(\mathrm{mod}\, A)$ under the action of the functor $F = \tau^{-1}[1]$, where τ is the Auslander-Reiten translation in $\mathcal{D}^b(\mathrm{mod}\, A)$ and $[1]$ is the shift.

In [7], Buan, Marsh and Reiten defined the cluster-tilted algebras as follows. Let A be a hereditary algebra, and $\tilde{T}$ be a tilting object in $\mathcal{C}_A$, that is, an object such that $\mathrm{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals the number of isomorphism classes of simple A-modules. Then the endomorphism algebra $\mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is called cluster-tilted. Since then, these algebras have been the subject of many investigations, see, for instance, [7, 8, 9, 10, 11, 12, 13, 21]. In several particular cases, it was shown that the quiver of a cluster-tilted algebra was obtained from that of a tilted algebra by replacing relations by arrows, see, for instance [10, 11]. Our objective in this paper is to prove this statement in a more general context (not depending on the representation type). This is achieved by looking at cluster-tilted algebras as trivial extensions of tilted algebras by a bimodule which we explicitely describe (compare [3]).

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For this purpose, we let C be a finite dimensional algebra of global dimension two (over an algebraically closed field), and consider the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$ with the natural action. The trivial extension $C \ltimes \mathrm{Ext}^2_C(DC, C)$ is called the relation-extension algebra of C. Our first main result (Theorem 2.6) describes the quiver of the relation-extension of C in the case where the quiver of C has no oriented cycles: we prove that indeed this quiver is given by replacing each element in a (minimal) system of relations by an arrow (going in the opposite direction to the relation). We then prove the main result of this paper.

Theorem 1.1 *An algebra $\tilde{C}$ is cluster-tilted if and only if there exists a tilted algebra C such that $\tilde{C}$ is the relation-extension of C.*

We note that several tilted algebras may correspond to the same cluster-tilted algebra, so this mapping is not bijective. On the other hand, there clearly exist relation-extension algebras which are not cluster-tilted.

Combining the above theorem with Theorem 2.6 we deduce the construction of the quiver of a cluster-tilted algebra. This allows, for instance, as done in [11], to relate the list of tame concealed algebras of Happel and Vossieck [18] with Seven's list of minimal infinite cluster quivers [23].

This paper consists of two sections. The first one describes relation-extension algebras and their quivers, and the second is devoted to the cluster-tilted algebras. Moreover, we give several examples.

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2 Relation-extension algebras

2.1 The definition

Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field k. For an algebra C, we denote by $\mathrm{mod}\, C$ the category of finitely generated right C-modules and by $\mathcal{D}^b(\mathrm{mod}\, C)$ the derived category of bounded complexes over $\mathrm{mod}\, C$. The functor $D = \mathrm{Hom}_k(-, k)$ is the standard duality between $\mathrm{mod}\, C$ and $\mathrm{mod}\, C^{op}$. For facts about $\mathrm{mod}\, C$ or $\mathcal{D}^b(\mathrm{mod}\, C)$, we refer to [2, 22, 17].

Let C be an algebra. We recall that the *trivial extension* of C by a C-C-bimodule M is the algebra $C \ltimes M$ with underlying k-vector space

$$C \oplus M = \{(c, m) \mid c \in C, m \in M\}$$

and the multiplication defined by

$$(c, m)(c', m') = (cc', cm' + mc')$$

for $c, c' \in C$ and $m, m' \in M$. For trivial extension algebras, we refer to [16, 1].

In this section, we introduce a particular class of trivial extension algebras which are useful for studying the cluster-tilted algebras.

Definition 2.1 *Let C be a finite dimensional algebra of global dimension at most two, and consider the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$ (with the natural action). The trivial extension*

$$C \ltimes \mathrm{Ext}^2_C(DC, C)$$

is called the relation-extension *of C.*

Clearly, any hereditary algebra is (trivially) the relation-extension of itself. On the other hand, if C is of global dimension equal to two (thus not hereditary) there exist two simple C-modules S and S' such that $\mathrm{Ext}^2_C(S, S') \neq 0$. Denoting by I the injective envelope of S and by P' the projective cover of S', the short exact sequences

$$0 \longrightarrow S \longrightarrow I \longrightarrow I/S \longrightarrow 0$$

and $$0 \longrightarrow \mathrm{rad}\, P' \longrightarrow P' \longrightarrow S' \longrightarrow 0$$

induce an epimorphism $\mathrm{Ext}^2_C(I, P') \to \mathrm{Ext}^2_C(S, S')$. Thus $\mathrm{Ext}^2_C(I, P') \neq 0$ and consequently $\mathrm{Ext}^2_C(DC, C) \neq 0$.

2.2 A system of relations

We wish to describe the bound quiver of a relation-extension algebra. Let C be an algebra. It is well-known that there exists a (uniquely determined) quiver Q_C and an admissible ideal I of the path algebra kQ_C of Q_C such that $C \cong kQ_C/I$, see, for instance, [5]. We denote by $(Q_C)_0$ the set of points of Q_C and by $(Q_C)_1$ its set of arrows. For each point $x \in (Q_C)_0$, we let e_x denote the corresponding primitive idempotent of C, and by S_x, P_x, I_x respectively, the corresponding simple, indecomposable projective and indecomposable injective C-module.

Following [4], we define a *system of relations* for $C \cong kQ_C/I$ to be a subset R of $\bigcup_{x,y \in (Q_C)_0} e_x I e_y$ such that R, but no proper subset of R, generates I as a two-sided ideal of kQ_C. Thus, for any $x, y \in (Q_C)_0$, the elements of $R \cap (e_x I e_y)$ are linear combinations of paths (of length at least two) from x to y. We need the following result.

Lemma 2.2 (([4, 1.2])) *Let $C \cong kQ_C/I$ be such that Q_C has no oriented cycles and R be a system of relations for C. Then, for each $x, y \in (Q_C)_0$, the cardinality of the set $R \cap (e_x I e_y)$ is independent of the chosen system of relations for C, and equals $\dim_k \mathrm{Ext}^2_C(S_x, S_y)$.*

2.3 The quiver of a trivial extension

We start with the following easy lemma.

Lemma 2.3 *Let C be an algebra, and M be a C-C-bimodule. The quiver $Q_{C \ltimes M}$ of the trivial extension of C by M is constructed as follows:*

1. $(Q_{C \ltimes M})_0 = (Q_C)_0$

2. *For $x, y \in (Q_C)_0$, the set of arrows in $Q_{C \ltimes M}$ from x to y equals the set of arrows in Q_C from x to y plus*

$$\dim_k \frac{e_x\, M\, e_y}{e_x\, M\,(\operatorname{rad} C)\, e_y + e_x\,(\operatorname{rad} C)\, M\, e_y}$$

additional arrows from x to y.

Proof. Since $M \subset \operatorname{rad}(C \ltimes M)$, the quivers of $C \ltimes M$ and of C have the same points. The arrows in the quiver of $C \ltimes M$ correspond to a k-basis of the vector space

$$\operatorname{rad}(C \ltimes M) \;/\; \operatorname{rad}^2(C \ltimes M).$$

Now, as a vector space

$$\operatorname{rad}(C \ltimes M) = \operatorname{rad} C \oplus M$$

and since $M^2 = 0$ in $C \ltimes M$,

$$\operatorname{rad}^2(C \ltimes M) = \operatorname{rad}^2 C \oplus [M\,(\operatorname{rad} C) + (\operatorname{rad} C)\, M]\,.$$

Since $\operatorname{rad}^2 C \subset \operatorname{rad} C$ and $M\,(\operatorname{rad} C) + (\operatorname{rad} C)\, M \subset M$ and since the arrows of Q_C correspond to a basis of $\operatorname{rad} C / \operatorname{rad}^2 C$, the additional arrows of $Q_{C \ltimes M}$ correspond to a k-basis of $M/[M\,(\operatorname{rad} C) + (\operatorname{rad} C)\, M]$. The arrows from x to y are obtained upon multiplying by e_x on the left and by e_y on the right.

2.4 The top of $\operatorname{Ext}^2_C(DC, C)$

In the situation of section 2.3, the C-C-bimodule $M\,(\operatorname{rad} C) + (\operatorname{rad} C)\, M$ is the radical of M, and the quotient $M/[M\,(\operatorname{rad} C) + (\operatorname{rad} C)\, M]$ is its top. In the case of relation-extension algebras, we are interested in the top of $\operatorname{Ext}^2_C(DC, C)$.

Lemma 2.4 *Let C be an algebra of global dimension two. The top of the C-C-bimodule $\operatorname{Ext}^2_C(DC, C)$ is isomorphic to $\operatorname{Ext}^2_C(\operatorname{soc} DC, \operatorname{top} C)$.*

Proof. The short exact sequences

$$0 \longrightarrow \operatorname{rad} C \xrightarrow{\;i\;} C \longrightarrow \operatorname{top} C \longrightarrow 0$$

$$0 \longrightarrow \operatorname{soc} DC \xrightarrow{\;j\;} DC \longrightarrow DC/\operatorname{soc} DC \longrightarrow 0$$

where i, j are the inclusions, induce a commutative diagram with exact rows and columns (the zeros are obtained from the condition that the global dimension of C is two).

$$
\begin{array}{ccccccc}
\operatorname{Ext}^2_C(DC/\operatorname{soc} DC, \operatorname{rad} C) & \longrightarrow & \operatorname{Ext}^2_C(DC/\operatorname{soc} DC, C) & \longrightarrow & \operatorname{Ext}^2_C(DC/\operatorname{soc} DC, \operatorname{top} C) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle j^*} & & \downarrow & & \\
\operatorname{Ext}^2_C(DC, \operatorname{rad} C) & \xrightarrow{i_*} & \operatorname{Ext}^2_C(DC, C) & \longrightarrow & \operatorname{Ext}^2_C(DC, \operatorname{top} C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & \overset{p}{\searrow} & \downarrow & & \\
\operatorname{Ext}^2_C(\operatorname{soc} DC, \operatorname{rad} C) & \longrightarrow & \operatorname{Ext}^2_C(\operatorname{soc} DC, C) & \longrightarrow & \operatorname{Ext}^2_C(\operatorname{soc} DC, \operatorname{top} C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

By the commutativity of the lower-right square, there exists an epimorphism $p : \operatorname{Ext}^2_C(DC, C) \to \operatorname{Ext}^2_C(\operatorname{soc} DC, \operatorname{top} C)$. We thus only need to show that the kernel of p is isomorphic to the radical

$$\operatorname{Ext}^2_C(DC, C)\,(\operatorname{rad} C) + (\operatorname{rad} C)\ \operatorname{Ext}^2_C(DC, C)$$

of the C-C-bimodule $\operatorname{Ext}^2_C(DC, C)$. Now an easy diagram chasing yields

$$\operatorname{Ker} p = \operatorname{Im} j^* + \operatorname{Im} i_*.$$

Thus, it suffices to prove that

$$\operatorname{Im} i_* = (\operatorname{rad} C)\ \operatorname{Ext}^2_C(DC, C) \qquad \text{and} \qquad \operatorname{Im} j^* = \operatorname{Ext}^2_C(DC, C)\,(\operatorname{rad} C).$$

We only show the first equality, the second is shown similarly. Let

$$0 \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} DC \longrightarrow 0$$

be a projective resolution of DC. By definition

$$\operatorname{Ext}^2_C(DC, C) = \operatorname{Hom}_C(P_2, C)/\operatorname{Im}\operatorname{Hom}_C(d_2, C)\ .$$

We first claim that the image of the map

$$i_0 = \operatorname{Hom}_C(P_2, i) : \operatorname{Hom}_C(P_2, \operatorname{rad} C) \to \operatorname{Hom}_C(P_2, C)$$

is equal to $(\operatorname{rad} C)\ \operatorname{Hom}_C(P_2, C)$. Indeed, the product rf with $r \in \operatorname{rad} C$ and $f \in \operatorname{Hom}_C(P_2, C)$ is easily seen to factor through $\operatorname{rad} C$. Therfore, we have $(\operatorname{rad} C)\ \operatorname{Hom}_C(P_2, C) \subset \operatorname{Im} i_0$. On the other hand, there is an isomorphism of k-vector spaces

$$(\operatorname{rad} C)\ \operatorname{Hom}_C(P_2, C) \cong \operatorname{Hom}_C(P_2, \operatorname{rad} C).$$

Since i_0 is injective, this establishes our claim.

Now, the image of i_* is generated by the residual classes (modulo the image of $\operatorname{Hom}_C(d_2, C)$) of the products ig, with $g \in \operatorname{Hom}_C(P_2, \operatorname{rad} C)$. These are the residual classes of the elements in $\operatorname{Im} i_0$ thus, by our claim above, the residual classes of the elements of the form rf with $r \in \operatorname{rad} C$ and $f \in \operatorname{Hom}_C(P_2, C)$. We deduce that $\operatorname{Im} i_* = (\operatorname{rad} C)\ \operatorname{Ext}^2_C(DC, C)$, as required.

Remark 2.5 *The proof of this lemma can easily be generalised to show that, for an algebra C of global dimension at most m, the top of the bimodule $\operatorname{Ext}^m_C(DC, C)$ is equal to $\operatorname{Ext}^m_C(\operatorname{soc} DC, \operatorname{top} C)$.*

2.5 The quiver of a relation-extension

The following theorem states that the quiver of the relation-extension algebra is obtained from the quiver of the original algebra by adding, for each pair of points x, y, one arrow from x to y for each relation from y to x. This justifies the name “relation-extension”.

Theorem 2.6 *Let $C \cong kQ_C/I$ be an algebra of global dimension at most two, such that Q_C has no oriented cycles, and let R be a system of relations for C. The quiver of the relation-extension algebra $C \ltimes \operatorname{Ext}^2_C(DC, C)$ is constructed as follows:*

(a) $(Q_{C\ltimes \operatorname{Ext}^2_C(DC,C)})_0 = (Q_C)_0$

(b) For $x, y \in (Q_C)_0$, the set of arrows in $Q_{C\ltimes \operatorname{Ext}^2_C(DC,C)}$ from x to y equals the set of arrows in Q_C from x to y plus $\operatorname{Card}(R \cap (e_y\, I\, e_x))$ *additional arrows.*

Proof. Let $S_1, S_2, \ldots, S_n$ denote a complete set of representatives of the isomorphism classes of simple C-modules, and set $S = \oplus_{i=1}^n S_i$. Since C is basic, the module S is isomorphic to the top of C and to the socle of DC. By Lemma 2.2, the relations of R correspond to a k-basis of $\operatorname{Ext}^2_C(S, S)$. By Lemma 2.4, the C-C-bimodule $\operatorname{Ext}^2_C(S, S)$ is isomorphic to the top of $\operatorname{Ext}^2_C(DC, C)$. Lemma 2.3 then implies that the number of additional arrows from x to y equals the k-dimension of the vector space $e_x \operatorname{Ext}^2_C(S, S)e_y = \operatorname{Ext}^2_C(S_y, S_x)$, and the result follows.

In particular, the quiver of a non-hereditary relation-extension algebra always contains oriented cycles.

2.6 The indecomposable projectives

It would be useful to know a system of relations for the relation-extension algebra $C \ltimes \operatorname{Ext}^2_C(DC, C)$ starting from one for C. In actual examples, such a system is easily obtained once we know the indecomposable projective modules. In order to state the next lemma, we need a notation: for each $x \in (Q_C)_0$, we denote by $\tilde{P}_x$ the corresponding indecomposable projective $C \ltimes \operatorname{Ext}^2_C(DC, C)$-module. Also, we note that C-modules can always be considered as $C \ltimes \operatorname{Ext}^2_C(DC, C)$-modules under the standard embedding.

Lemma 2.7 *Let C be an algebra of global dimension at most two. Then, for each $x \in (Q_C)_0$, we have a short exact sequence in* $\operatorname{mod}\big(C \ltimes \operatorname{Ext}^2_C(DC, C)\big)$

$$0 \to \operatorname{Ext}^2_C(DC, P_x) \to \tilde{P}_x \xrightarrow{p_x} P_x \to 0$$

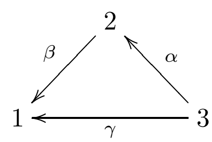
where p_x is a projective cover.

Proof. Since both P_x and $\tilde{P}_x$ admit S_x as a simple top, there indeed exists a projective cover morphism $p_x : \tilde{P}_x \to P_x$. On the other hand, $\operatorname{Ext}^2_C(DC, P_x) \cong e_x \operatorname{Ext}^2_C(DC, C)$ is clearly a submodule of the $C \ltimes \operatorname{Ext}^2_C(DC, C)$-module $\tilde{P}_x$. The result then follows from the isomorphism of k-vector spaces

$$\tilde{P}_x = e_x \left(C \ltimes \operatorname{Ext}^2_C(DC, C) \right) \cong P_x \oplus \operatorname{Ext}^2_C(DC, P_x).$$

2.7 An example

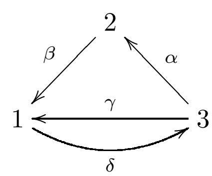
Example 2.8 *Let C be given by the quiver*



bound by the relation $\alpha\beta = 0$. Thus

$$C_C = 1 \oplus \begin{array}{c} 2 \\ 1 \end{array} \oplus \begin{array}{ccc} & 3 & \\ 1 & & 2 \end{array} \qquad \text{and} \qquad (DC)_C = \begin{array}{ccc} 2 & & 3 \\ & 1 & \end{array} \oplus \begin{array}{c} 3 \\ 2 \end{array} \oplus 3$$

where the indecomposable projectives and injectives are represented by their Loewy series. It is easily seen that the global dimension of C is two. By Theorem 2.6, the quiver of $C \ltimes \operatorname{Ext}^2_C(DC, C)$ is obtained by adding to Q_C a single arrow $\delta : 1 \to 3$.



We now compute the new indecomposable projective modules. A simple calculation yields

$$\begin{array}{rclcrcl} \operatorname{Ext}^2_C(I_3, P_1) & \cong & k & , & \operatorname{Ext}^2_C(I_3, P_3) & \cong & k \\ \operatorname{Ext}^2_C(I_1, P_1) & \cong & k & , & \operatorname{Ext}^2_C(I_1, P_3) & \cong & k. \end{array}$$

Since the projective dimension of I_2 is one and the injective dimension of P_2 is also one, this yields $\dim_k \operatorname{Ext}^2_C(DC, C) = 4$. Using Lemma 2.7, we get the new indecomposable projectives

$$\begin{array}{c} 1 \\ 3 \\ 1 \end{array} \quad , \quad \begin{array}{ccc} & 3 & \\ 1 & & 2 \\ 3 & & \\ 1 & & \end{array} \quad , \quad \begin{array}{c} 2 \\ 1 \end{array} .$$

Thus, a system of relations for the relation-extension algebra is $\alpha\beta = 0, \delta\alpha = 0, \beta\delta = 0$ and $\delta\gamma\delta = 0$.

3 Cluster-tilted algebras

3.1 Preliminaries

Let A be a hereditary algebra. The cluster category $\mathcal{C}_A$ of A is defined as follows. Let F denote the automorphism of $\mathcal{D}^b(\mathrm{mod}\,A)$ defined as the composition $\tau^{-1}_{\mathcal{D}^b(\mathrm{mod}\,A)}[1]$, where $\tau^{-1}_{\mathcal{D}^b(\mathrm{mod}\,A)}$ is the Auslander-Reiten translation in $\mathcal{D}^b(\mathrm{mod}\,A)$, and $[1]$ is the shift functor. Then $\mathcal{C}_A$ is the quotient category $\mathcal{D}^b(\mathrm{mod}\,A)/F$. Its objects are the F-orbits $\tilde{X} = (F^iX)_{i\in\mathbb{Z}}$, where X is an object in $\mathcal{D}^b(\mathrm{mod}\,A)$. The set of morphisms from $\tilde{X} = (F^iX)_{i\in\mathbb{Z}}$ to $\tilde{Y} = (F^iY)_{i\in\mathbb{Z}}$ in $\mathcal{C}_A$ is given by

$$\mathrm{Hom}_{\mathcal{C}_A}(\tilde{X},\tilde{Y}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(X, F^iY).$$

It is shown in [20], that $\mathcal{C}_A$ is a triangulated category. Furthermore, the canonical functor $\mathcal{D}^b(\mathrm{mod}\,A) \to \mathcal{C}_A$ is a functor of triangulated categories. We refer to [6] for facts about the cluster category.

An object $\tilde{T}$ in $\mathcal{C}_A$ is called a *tilting object* provided $\mathrm{Ext}^1_{\mathcal{C}_A}(\tilde{T},\tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals the number of isomorphism classes of simple A-modules (that is, the number of points in the quiver of A). The algebra of endomorphisms $\tilde{C} = \mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is then called a *cluster-tilted algebra* [7].

Cluster-tilted algebras may also be expressed in terms of modules. We recall that an A-module T is called a tilting module provided $\mathrm{Ext}^1_A(T,T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals the number of isomorphism classes of simple A-modules. Denoting by $\tilde{T}$ the F-orbit of T, we have the following theorem.

Theorem 3.1 (([6, 3.3])) *Let $\tilde{C}$ be a cluster-tilted algebra, then there exist a hereditary algebra A and a tilting A-module T such that $\tilde{C} \cong \mathrm{End}_{\mathcal{C}_A}(\tilde{T})$.*

We further recall that the endomorphism algebra of a tilting module over a hereditary algebra is called a *tilted algebra*, see, for instance, [22]. We need the following result.

Theorem 3.2 (([17])) *Let A be a hereditary algebra, T be a tilting A-module and $C = \mathrm{End}_A(T)$ be the corresponding tilted algebra. Then*

(a) The derived functor $\mathrm{RHom}_A(T,-) : \mathcal{D}^b(\mathrm{mod}\,A) \to \mathcal{D}^b(\mathrm{mod}\,C)$ is an equivalence of categories which maps the A-module T to the C-module C.

(b) $\mathrm{RHom}_A(T,-)$ commutes with the Auslander-Reiten translations and the shifts in the respective categories.

3.2 Cluster-tilted algebras are trivial extensions

For any object X in $\mathcal{D}^b(\mathrm{mod}\,A)$, the k-vector space $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(X, FX)$ has a natural structure of $\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\,A)}(X)$-$\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\,A)}(X)$-bimodule under the action

$$\begin{array}{rcl}\mathrm{End}(X)\times\mathrm{Hom}(X,FX)\times\mathrm{End}(X) & \to & \mathrm{Hom}(X,FX)\\ (u,f,v) & \mapsto & Fu\circ f\circ v\end{array}$$

The following lemma is proved in [3, 3.1]. We include a simple proof for the convenience of the reader.

Lemma 3.3 *Let $\tilde{C}$ be a cluster tilted algebra. Then, for each hereditary algebra A and tilting A-module T such that $\tilde{C}=\mathrm{End}_{\mathcal{C}_A}(\tilde{T})$, we have*

$$\tilde{C}\cong\mathrm{End}_A(T)\ltimes\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,FT).$$

Proof. By definition of $\mathcal{C}_A$, we have

$$\tilde{C}=\mathrm{End}_{\mathcal{C}_A}(\tilde{T})=\oplus_{i\in\mathbb{Z}}\ \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,F^iT)$$

as k-vector spaces, and the multiplication is given by

$$(g_i)_{i\in\mathbb{Z}}(f_j)_{j\in\mathbb{Z}}=\left(\sum_{i+j=l}F^j\,g_i\circ f_j\right)_{l\in\mathbb{Z}}.$$

Since A is hereditary, then, for any two A-modules M and N, we have that $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(M,N[i])=0$ for all $i\geq 2$. Therefore, as a k-vector space

$$\tilde{C}=\mathrm{End}_{\mathcal{C}_A}(\tilde{T})\ =\ \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,T)\oplus\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,FT).$$

The multiplication of two elements $f,g\in\mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is given as follows. Assume $f=(f_0\,,f_1)$ and $g=(g_0\,,g_1)$, with $f_0,g_0\in\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,T)$ and $f_1,g_1\in\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,FT)$, then, since $Fg_1\circ f_1=0$,

$$gf=(g_0\circ f_0\ ,\ Fg_0\circ f_1+f_0\circ g_1).$$

In view of the bimodule structure of $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,FT)$ defined above, this shows indeed that $\tilde{C}=\mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is the trivial extension of $\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T)=\mathrm{End}_A(T)$ by the bimodule $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\,A)}(T,FT)$.

Since the algebra $\mathrm{End}_A(T)$ of the lemma is tilted, any cluster-tilted algebra is a trivial extension of a tilted algebra. However, the hereditary algebra A and the A-module T above are not unique. Therefore, one cannot apply directly the lemma to construct a map from cluster tilted algebras to tilted algebras.

3.3 The main result

We are now able to prove the main theorem of this section.

Theorem 3.4 *An algebra $\tilde{C}$ is cluster-tilted if and only if there exists a tilted algebra C such that $\tilde{C}$ is the relation-extension of C.*

Proof. Let C be a tilted algebra. Then there exist a hereditary algebra A and a tilting A-module T such that $C = \mathrm{End}_A(T)$. Let $\tilde{T}$ denote as usual the F-orbit of T in $\mathcal{D}^b(\mathrm{mod}\, A)$. Then $\tilde{C} = \mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is a cluster-tilted algebra. By Lemma 3.3, we have

$$\tilde{C} = \mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\, A)}(T) \ltimes \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(T, FT). \tag{1}$$

By Theorem 3.2, the derived functor $\mathrm{RHom}_A(T, -)$ induces C-C-bimodule isomorphisms

$$\mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\, A)}(T) \cong \mathrm{End}_{\mathcal{D}^b(\mathrm{mod}\, C)}(C) \cong \mathrm{End}_C(C) \cong C$$

and

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, A)}(T, FT) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, C)}(C, F'C)$$

where $F' = \tau^{-1}_{\mathcal{D}^b(\mathrm{mod}\, C)}[1]$ is the functor corresponding to F in the derived category $\mathcal{D}^b(\mathrm{mod}\, C)$. Thus we get

$$\tilde{C} \cong C \ltimes \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, C)}(C, F'C).$$

Moreover, we have the following sequence of C-C-bimodule isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, C)}(C, F'C) &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, C)}(\, \tau_{\mathcal{D}^b(\mathrm{mod}\, C)}\, C\,[1]\,,\, C\,[2]\,) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}\, C)}(\, DC\,,\, C[2]\,) \\ &\cong \mathrm{Ext}^2_C(DC, C),\end{aligned}$$

where the first is obtained by applying to both arguments the automorphism $\tau_{\mathcal{D}^b(\mathrm{mod}\, C)}[1]$, the second uses the fact that $\tau_{\mathcal{D}^b(\mathrm{mod}\, C)}C \cong DC[-1]$ and the third is a property of the derived category. This shows that the relation-extension $C \ltimes \mathrm{Ext}^2_C(DC, C)$ is a cluster-tilted algebra. Finally, by Lemma 3.3, every cluster-tilted algebra is obtained in this way.

3.4 Remarks and examples

(a) Since the quiver of a tilted algebra has no oriented cycles, it follows directly from Theorem 3.4 and Theorem 2.6 that we have a construction for the quiver of a cluster-tilted algebra $\tilde{C}$ starting from the quiver of a tilted algebra C. This construction is easily seen to generalise the one in [11, 4.1] and, thus, can be used to relate the Happel-Vossieck list of tame concealed algebras [18] with Seven's list of minimal infinite cluster quivers [23].

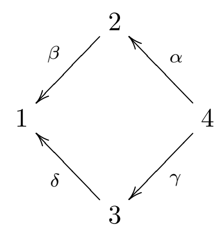
(b) A different description, inspired from [19], of the relation-extension algebra is sometimes useful. Consider the following doubly infinite matrix algebra

$$\hat{C} \quad = \quad \begin{bmatrix} \ddots & & & 0 & \\ & C_{i-1} & & & \\ & M_i & C_i & & \\ & & M_{i+1} & C_{i+1} & \\ & 0 & & & \ddots \end{bmatrix}$$

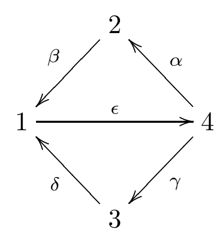
where matrices are assumed to have only finitely many non-zero coefficients, $C_i = C$ and $M_i = \mathrm{Ext}^2_C(DC, C)$ for all $i \in \mathbb{Z}$, all the remaining coefficients are zero. The addition is the usual addition of matrices while the multiplication is induced from the bimodule structure of $\mathrm{Ext}^2_C(DC, C)$ and the zero map $\mathrm{Ext}^2_C(DC, C) \otimes_C \mathrm{Ext}^2_C(DC, C) \to 0$. Clearly, $\hat{C}$ is a Galois covering of $C \ltimes \mathrm{Ext}^2_C(DC, C)$ with group $\mathbb{Z}$: the identity maps $C_i \to C_{i+1}$, $M_i \to M_{i+1}$ induce an automorphism η of $\hat{C}$ and $\hat{C}/\eta \cong C \ltimes \mathrm{Ext}^2_C(DC, C)$.

(c) As observed before, different tilted algebras C may correspond to the same cluster-tilted algebra $\tilde{C}$ (thus, the surjective map $C \mapsto \tilde{C}$ is not injective). We give an example of such an occurrence.

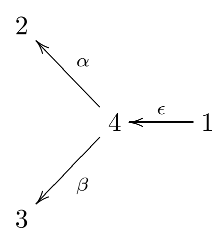
Example 3.5 *Let C_1 be given by the quiver*



bound by $\alpha\beta = \gamma\delta$. This is a tilted algebra of Dynkin type D_4, and the corresponding cluster-tilted (relation-extension) algebra $\tilde{C}_1$ is given by the quiver



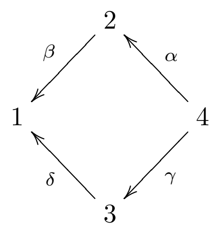
bound by $\alpha\beta = \gamma\delta$, $\beta\epsilon = 0$, $\delta\epsilon = 0$, $\epsilon\alpha = 0$, $\epsilon\gamma = 0$. Let now C_2 be the tilted algebra given by the quiver



bound by $\epsilon\alpha = 0$, $\epsilon\beta = 0$. Then it is easily seen that $\tilde{C}_1 = \tilde{C}_2$.

(d) Not surprisingly, it is possible that C is representation-finite whereas $\tilde{C}$ is representation-infinite: it suffices to have two points $x, y \in (Q_C)_0$ such that $\dim_k \mathrm{Ext}^2_C(I_y, P_x) > 1$. We give an example of such a situation.

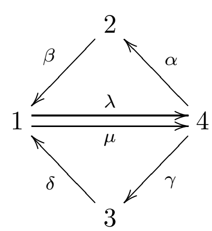
Example 3.6 *Let C be given by the quiver*



bound by $\alpha\beta = 0$, $\gamma\delta = 0$. This is a representation-finite tilted algebra of euclidean type $\tilde{A}_3$. The injective resolution

$$0 \to P_1 \to I_1 \to I_2 \oplus I_3 \to I_4 \oplus I_4 \to 0$$

shows that $\dim_k \operatorname{Ext}^2_C(I_4, P_1) = 2$. The corresponding cluster-tilted algebra $\tilde{C}$ is given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$, $\delta\lambda = 0$, $\lambda\gamma = 0$, $\beta\mu = 0$, $\mu\alpha = 0$. The indecomposable projective $\tilde{C}$-modules are given by

$$\begin{matrix} & 1 & \\ 4 & & 4 \\ 2 & & 3 \end{matrix} \quad , \quad \begin{matrix} 2 \\ 1 \\ 4 \\ 2 \end{matrix} \quad , \quad \begin{matrix} 3 \\ 1 \\ 4 \\ 3 \end{matrix} \quad , \quad \begin{matrix} & 4 & \\ 2 & & 3 \end{matrix}$$

Clearly, $\tilde{C}$ is representation-infinite.

(e) The relation-extension algebra in Example 2.8 is not a cluster-tilted algebra. This follows from the fact that cluster-tilted algebras contain no oriented cycles of length two.

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