Cluster-tilted algebras and slices

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Abstract

We give a criterion allowing to verify whether or not two tilted algebras have the same relation-extension (thus correspond to the same cluster-tilted algebra). This criterion is in terms of a combinatorial configuration in the Auslander-Reiten quiver of the cluster-tilted algebra, which we call local slice.

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0 Introduction

Cluster categories were introduced in [7] and, for type $\mathbb{A}$, also in [11], as a categorical model allowing to understand better the cluster algebras of Fomin and Zelevinsky [13]. Cluster-tilted algebras were defined in [11] for type $\mathbb{A}$, and in [8] for arbitrary hereditary algebras as follows: Let A be a hereditary algebra and $\tilde{T}$ be a tilting object in the associated cluster category $\mathcal{C}_A$, that is, an object such that $\mathrm{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals the rank of the Grothendieck group of A, then the algebra $B = \mathrm{End}_{\mathcal{C}_A}\tilde{T}$ is called cluster-tilted. These algebras have been studied by several authors, see, for instance, [1,8–10,12,17]. In particular, they were shown in [1] to be closely related to the tilted algebras introduced by Happel and Ringel in the early eighties [15]. Indeed, let C be a tilted algebra, then the trivial extension $\tilde{C} = C \ltimes \mathrm{Ext}^2_C(DC, C)$ of C by the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$ is cluster-tilted, and every cluster-tilted algebra is of this form. Thus, we have a surjective map $C \mapsto \tilde{C}$ from tilted to cluster-tilted algebras. However, easy examples show that this map is not injective. Our objective in this paper is to give a criterion allowing to verify whether for two tilted algebras C_1 and C_2, the corresponding cluster-tilted algebras $\tilde{C}_1$ and $\tilde{C}_2$ are isomorphic or not.

Since tilted algebras are characterised by the existence of complete slices in their Auslander-Reiten quiver (see, for instance, [15,20,19,21] or [3]), it is natural to study the corresponding concept for cluster-tilted algebras. For this purpose, we introduce what we call a *local slice*, by weakening the axioms of complete slice (thus, in a tilted algebra, complete slices are local slices). We show that a complete slice in a tilted algebra C embeds as a local slice in $\tilde{C}$ (and, in fact, any local slice in $\tilde{C}$ is of this form).

Our main theorem is the following:

Theorem 1 *Let B be a cluster-tilted algebra. Then a tilted algebra C is such that $B = C \ltimes \mathrm{Ext}^2_C(DC, C)$ if and only if there exists a local slice Σ in* $\mathrm{mod}\, B$ *such that*

$$C = B/\mathrm{Ann}_B\Sigma.$$

We also show that cluster-tilted algebras have many local slices. In fact, all but at most finitely many indecomposable modules lying in the transjective component of the Auslander-Reiten quiver belong to a local slice. If the cluster-tilted algebra is of tree type (which is the case, for instance, if it is of Dynkin type or of Euclidean type distinct from $\tilde{\mathbb{A}}$), then this is the case for all indecomposable modules in this component.

We now describe the contents of our paper. In the first section, we introduce the notion of local section in a translation quiver and in the second section we study sections and local sections in the derived category. In the third section, we introduce the concept of a local slice and prove our main result, and in section four, we prove that cluster-tilted algebras of tree type have sufficiently many local slices.

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1 Preliminaries on translation quivers

1.1 Notation

Throughout this paper, all algebras are connected finite dimensional algebras over an algebraically closed field k. For an algebra C, we denote by $\operatorname{mod} C$ the category of finitely generated right C-modules and by $\operatorname{ind} C$ a full subcategory of $\operatorname{mod} C$ consisting of exactly one representative from each isomorphism class of indecomposable modules. When we speak about a C-module (or an indecomposable C-module), we always mean implicitly that it belongs to $\operatorname{mod} C$ (or to $\operatorname{ind} C$, respectively). Also, all subcategories of $\operatorname{mod} C$ are full and so are identified with their object classes. Given a subcategory $\mathcal{C}$ of $\operatorname{mod} C$, we sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. We denote by $\operatorname{add} \mathcal{C}$ the full subcategory of $\operatorname{mod} C$ with objects the finite direct sums of modules in $\mathcal{C}$ and, if M is a module, we abbreviate $\operatorname{add}\{M\}$ as $\operatorname{add} M$. We denote the projective (or injective) dimension of a module M as $\operatorname{pd} M$ (or $\operatorname{id} M$, respectively). The global dimension of C is denoted by gl.dim.C and its Grothendieck group by $K_0(C)$. Finally, we denote by $\Gamma(\operatorname{mod} C)$ the Auslander-Reiten quiver of an algebra C, and by $\tau_C = D\,Tr$, $\tau_C^{-1} = Tr\,D$ its Auslander-Reiten translations. For further definitions and facts needed on $\operatorname{mod} C$ or $\Gamma(\operatorname{mod} C)$, we refer the reader to [3]. We also need facts on the bounded derived category $\mathcal{D}^b(\operatorname{mod} C)$ of $\operatorname{mod} C$, for which we refer to [14].

1.2 Sections

For translation quivers, we refer to [3,20]. Let (Γ, τ) be a connected translation quiver. We recall that a path $x = x_0 \to x_1 \to \ldots \to x_t = y$ in Γ is called *sectional* if, for each i with $0 < i < t$, we have $\tau\, x_{i+1} \neq x_{i-1}$. A full connected subquiver Σ of Γ is said to be *convex* in Γ if, for any path $x = x_0 \to x_1 \to \ldots \to x_t = y$ in Γ with $x, y \in \Sigma_0$, we have $x_i \in \Sigma_0$ for all i. It is called *acyclic* if there is no cycle $x = x_0 \to x_1 \to \ldots \to x_t = x$ (with $t > 0$) which is entirely contained in Σ. The following definition is due to Liu and Skowroński (see

[18,21] or else [3]).

Definition 2 *Let (Γ, τ) be a connected translation quiver. A connected full subquiver Σ of Γ is a* section *in Γ if:*

(S1) Σ is acyclic.
(S2) For each $x \in \Gamma_0$, there exists a unique $n \in \mathbf{Z}$ such that $\tau^n x \in \Sigma_0$.
(S3) Σ is convex in Γ.

This definition is motivated by the study of tilted algebras. The well-known criterion of Liu and Skowroński asserts that, if C is an algebra, and Σ is a faithful section in a component of its Auslander-Reiten quiver such that $\mathrm{Hom}_C(X, \tau_C Y) = 0$ for all $X, Y \in \Sigma_0$, then C is tilted having Σ as complete slice (see [18,21,3]).

We note that, if a translation quiver Γ contains a section, then Γ is acyclic.

1.3 Presections

We need some weaker notions. The first one is the following.

Definition 3 *Let (Γ, τ) be a connected translation quiver. A connected full subquiver Σ of Γ is called a* presection *in Γ if it satisfies the following two conditions:*

(P1) If $x \in \Sigma_0$ and $x \to y$ is an arrow, then either $y \in \Sigma_0$ or $\tau y \in \Sigma_0$.
(P2) If $y \in \Sigma_0$ and $x \to y$ is an arrow, then either $x \in \Sigma_0$ or $\tau^{-1} x \in \Sigma_0$.

The following Lemma collects the elementary properties of presections. Recall that a translation quiver is called *stable* if there are neither projective, nor injective points in Γ.

Lemma 4 *Let (Γ, τ) be a connected translation quiver.*

(a) If Σ is a section in Γ, then Σ is a presection.
(b) Any path entirely contained in a presection Σ is a sectional path.
(c) If the translation quiver Γ is stable, then conditions (P1) and (P2) are equivalent.
(d) If Σ is a presection in a stable translation quiver Γ, then Σ intersects every τ-orbit of Γ at least once.

PROOF. (a) This is well-known (see, for instance, [3, VIII.1.4 p.304]).

(b) Assume that Σ is a presection in Γ, and that $x = x_0 \to x_1 \to \ldots \to x_t = y$ is a path lying entirely in Σ. If this path is not sectional, then there exists a least i with $0 < i < t$ and $\tau\, x_{i+1} = x_{i-1}$. But, in this case, we have arrows $x_{i-1} \to x_i$ and $x_i \to \tau^{-1}\, x_{i-1} = x_{i+1}$ with $x_i, x_{i-1}, x_{i+1} \in \Sigma_0$, a contradiction.

(c) Assume (P1) holds and that $x \to y$ is an arrow with $y \in \Sigma_0$. Since x is not injective, there exists an arrow $y \to \tau^{-1}x$. Applying (P1) to the latter yields that $\tau^{-1}x \in \Sigma_0$ or $x = \tau(\tau^{-1}x) \in \Sigma_0$. Thus (P2) holds. The converse is shown in the same way.

(d) It suffices to prove that, if $x \in \Sigma_0$ and $y \in \Gamma_0$ lie in two neighbouring τ-orbits, then Σ intersects the τ-orbit of y. Since Γ is stable, there exists $m \in \mathbf{Z}$ such that there is an arrow $x \to \tau^m y$. But then $\tau^m y \in \Sigma_0$ or $\tau^{m+1} y \in \Sigma_0$. □

For instance, let Γ be a stable tube. A ray in Γ is a presection but clearly not a section. On the other hand, if Γ is a component of type $\mathbf{Z}A_\infty$, then a ray is a section.

1.4 Local sections

We now define an intermediate notion between those of presection and section.

Definition 5 *Let (Γ, τ) be a connected translation quiver. A presection Σ in Γ is called a* local section *if it satisfies moreover the following additional condition:*

> *Σ is sectionally convex, that is, if $x = x_0 \to x_1 \to \ldots \to x_t = y$ is a sectional path in Γ such that $x, y \in \Sigma_0$, then $x_i \in \Sigma_0$ for all i.*

The following Lemma is immediate.

Lemma 6 *Any section is a local section.*

PROOF. Indeed, any section is convex, and hence sectionally convex. We then apply Lemma 4 (b). □

The main result of this section is the following.

Proposition 7 *Let Q be a finite acyclic quiver and $\Gamma = \mathbf{Z}Q$. The following conditions are equivalent for a connected full subquiver Σ of Γ such that $|\Sigma_0| = |Q_0|$:*

(a) Σ *is a section.*
(b) Σ *is a local section.*
(c) Σ *is a presection.*

PROOF. Because of Lemma 6 (and the definition of local section), it suffices to prove that (c) implies (a). Since Γ is a stable translation quiver, then, because of Lemma 4 (d), Σ intersects every τ-orbit of Γ at least once. Then, it follows from the hypothesis that $|\Sigma_0| = |Q_0|$ that Σ intersects each τ-orbit of Γ exactly once. Since Σ is clearly acyclic (because Γ is), there only remains to prove convexity. Suppose that there exists a path $x = x_0 \to x_1 \to \ldots \to x_t = y$ in Γ with $x, y \in \Sigma_0$ but $x_1, \ldots, x_{t-1} \notin \Sigma_0$ $(t \geq 2)$. Since Σ is a presection, we have $\tau\, x_1 \in \Sigma_0$. From the arrow $\tau\, x_1 \to \tau\, x_2$, we deduce that either $\tau\, x_2 \in \Sigma_0$ or $\tau^2 x_2 \in \Sigma_0$. Repeating this argument t times, we get that one of $\tau\, y, \tau^2 y, \ldots, \tau^t y$ lies in Σ. This contradicts the fact that Σ cuts each τ-orbit exactly once, because $y \in \Sigma_0$. $\square$

2 Sections and local sections in the derived category.

Let A denote a hereditary algebra, and $\mathcal{D} = \mathcal{D}^b(\operatorname{mod} A)$ denote the bounded derived category of $\operatorname{mod} A$. We denote by $\tau_{\mathcal{D}}, \tau_{\mathcal{D}}^{-1}$ the Auslander-Reiten translations in $\mathcal{D}$. We recall that the Auslander-Reiten quiver of $\mathcal{D}$ consists of two types of components: the regular (which correspond to the regular components of $\Gamma(\operatorname{mod} A)$ and their shifts) and the transjective (which are the form $\mathbf{Z}Q$, where Q denotes the ordinary quiver of A), see [14].

We need the following result, known as Skowroński's Lemma [22,3].

Lemma 8 *Let C be an artin algebra. Assume that a C-module M is the direct sum of m pairwise non-isomorphic indecomposable modules and is such that* $\operatorname{Hom}_C(M, \tau_C M) = 0$. *Then* $m \leq \operatorname{rk} \mathrm{K}_0(C)$.

The following Lemma is motivated by [21,19].

Lemma 9 *Let Σ be a section in a connected component Γ of $\Gamma(\mathcal{D})$ then the following conditions are equivalent:*

(a) Σ *is convex in* $\operatorname{ind} \mathcal{D}$*, that is, if $X = X_0 \to X_1 \to \ldots \to Y$ is a sequence of non-zero morphisms between indecomposable objects in $\mathcal{D}$ such that $X, Y \in \Sigma_0$, then $X_i \in \Sigma_0$ for all i.*
(b) $\operatorname{Hom}(X, \tau_{\mathcal{D}} Y) = 0$ *for all* $X, Y \in \Sigma_0$.
(c) $\operatorname{Hom}(\tau_{\mathcal{D}}^{-1} X, Y) = 0$ *for all* $X, Y \in \Sigma_0$.
(d) Σ_0 *is finite.*
(e) $|\Sigma_0| = \operatorname{rk} \mathrm{K}_0(A)$.

(f) Γ *is a transjective component.*

PROOF. The equivalence of (b) and (c) follows trivially from the fact that τ is an automorphism of $\mathcal{D}$.

(a) implies (b). A non-zero morphism $X \to \tau_{\mathcal{D}} Y$ induces a path $X \to \tau_{\mathcal{D}} Y \to * \to Y$ in $\operatorname{ind} \mathcal{D}$. The hypothesis implies that both Y and $\tau_{\mathcal{D}} Y$ lie in Σ, a contradiction.

(b) implies (f). If Γ is not a transjective component, then we may assume without loss of generality that Γ is concentrated in degree zero. By Skowroński's Lemma 8, Σ is finite. Now this contradicts the fact that Σ intersects each $\tau_{\mathcal{D}}$-orbit exactly once.

(f) implies (e). The number of $\tau_{\mathcal{D}}$-orbits in a transjective component is exactly $\operatorname{rk} \mathrm{K}_0(A)$.

(e) implies (d). This is trivial.

(d) implies (a). Suppose that $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \ldots \xrightarrow{f_t} X_t = Y$ is a path, where the f_i are non-zero morphisms, the X_i lie in Γ and $X, Y \in \Sigma_0$. Now, since Σ_0 is finite, then Γ has finitely many $\tau_{\mathcal{D}}$-orbits. Therefore, Γ is transjective. This implies that the f_i lie in a finite power of the radical of $\mathcal{D}$. Therefore the path above can be refined to a path of irreducible morphisms. Convexity of Σ in Γ then implies that $X_i \in \Sigma_0$ for all i. $\square$

Corollary 10 *Let* Σ *be a connected full subquiver in a connected component* Γ *of* $\Gamma(\mathcal{D})$ *such that* $|\Sigma_0| = \operatorname{rk} \mathrm{K}_0(A)$. *The following are equivalent:*

(a) Σ *is a section.*
(b) Σ *is a local section.*
(c) Σ *is a presection.*

PROOF. Again, it suffices to prove that (c) implies (a). Since Γ is a stable translation quiver, then the presection Σ intersects each $\tau_{\mathcal{D}}$-orbit of Γ at least once. Since, by hypothesis, $|\Sigma_0| = \operatorname{rk} \mathrm{K}_0(A) < \infty$, then Γ is a transjective component. The statement then follows from Proposition 7. $\square$

3 Local slices

3.1 Definition and examples

We now define the main concept of this paper.

Definition 11 *Let C be a finite dimensional algebra. A* local slice Σ *in* $\mathrm{mod}\, C$ *is a local section in a component Γ of $\Gamma(\mathrm{mod}\, C)$ such that $|\Sigma| = \mathrm{rk}\ \mathrm{K}_0(C)$.*

Remark 12 *Let C be an algebra, and Σ be a local slice in $\mathrm{mod}\, C$.*

(a) Since local sections are presections, every path entirely contained in Σ is sectional (because of Lemma 4 (b)). This implies that Σ is acyclic.
(b) If Γ is a stable component of $\Gamma(\mathrm{mod}\, C)$ and $\Sigma \subset \Gamma$ then, by Lemma 4 (d), Σ intersects any τ_C-orbit of Γ at least once. Since $|\Sigma_0| < \infty$, this implies that Γ has only finitely many τ_C-orbits.

Example 13 *Let C be a tilted algebra, and Γ be a connecting component of $\Gamma(\mathrm{mod}\, C)$. Then any complete slice in Γ is also a local slice. We shall prove below that any cluster-tilted algebra has (many) local slices in its Auslander-Reiten quiver.*

Example 14 *The following is an example of an algebra which is neither tilted, nor cluster-tilted, but whose Auslander-Reiten quiver contains a local slice. Let B be given by the quiver*

$$\begin{tikzcd}
1 & & 4 \arrow[dl, "\alpha"'] \\
& 3 \arrow[ul, "\beta"'] \arrow[dl, "\delta"] & \\
2 \arrow[rr, "\epsilon"'] & & 5 \arrow[ul, "\gamma"']
\end{tikzcd}$$

bound by $\alpha\,\beta = 0$, $\delta\,\epsilon = 0$, $\epsilon\,\gamma = 0$ and $\gamma\,\delta = 0$. The Auslander-Reiten quiver $\Gamma(\mathrm{mod}\, B)$ of B is shown in Figure 1, where modules are represented by their Loewy series and we identify the two copies of the underlined module $\underline{2}$. Here $\Sigma = \left\{ \begin{smallmatrix}3\\2\,1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}4\\3\\2\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix}, \begin{smallmatrix}5\\3\\1\end{smallmatrix} \right\}$ *and* $\Sigma' = \left\{ \begin{smallmatrix}2\\5\end{smallmatrix}, 5, \begin{smallmatrix}4\,5\\3\end{smallmatrix}, \begin{smallmatrix}5\\3\end{smallmatrix}, \begin{smallmatrix}5\\3\\1\end{smallmatrix} \right\}$ *are local slices. Note that neither Σ nor Σ' is a section, because both intersect twice the τ_B-orbit of $\underline{2}$. It is an interesting question to identify the algebras which have local slices.*

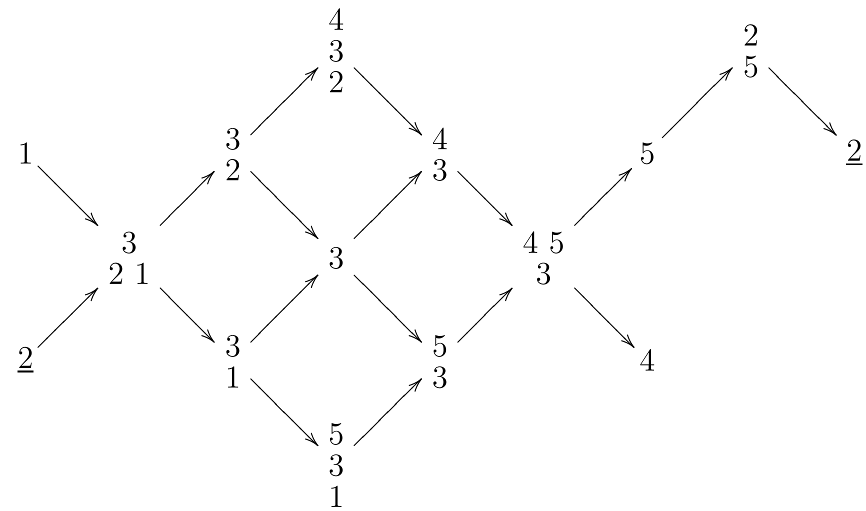


Fig. 1. Auslander-Reiten quiver of Example 13

3.2 Cluster-tilted algebras

Let A be a hereditary algebra. The *cluster category* $\mathcal{C}_A$ of A is defined as follows. Let F be the automorphism of $\mathcal{D} = \mathcal{D}^b(\mathrm{mod}\, A)$ defined as the composition $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}^{-1}$ is the Auslander-Reiten translation in $\mathcal{D}$ and [1] is the shift functor. Then $\mathcal{C}_A$ is the orbit category $\mathcal{D}/F$, that is, the objects of $\mathcal{C}_A$ are the F-orbits $\tilde{X} = (F^i X)_{i\in\mathbf{Z}}$, where $X \in \mathcal{D}$, and the set of morphisms from $\tilde{X} = (F^i X)_{i\in\mathbf{Z}}$ to $\tilde{Y} = (F^i Y)_{i\in\mathbf{Z}}$ is

$$\mathrm{Hom}_{\,\mathcal{C}_A}(\tilde{X}, \tilde{Y}) = \bigoplus_{i\in\mathbf{Z}} \mathrm{Hom}_{\,\mathcal{D}}(X, F^i Y).$$

It is shown in [7,16] that $\mathcal{C}_A$ is a triangulated category with almost split triangles. Furthermore, the projection functor $\pi_0 : \mathcal{D} \to \mathcal{C}_A$ is a functor of triangulated categories and commutes with the Auslander-Reiten translations in both categories. We refer to [7] for facts about the cluster category.

An object $\tilde{T}$ in $\mathcal{C}_A$ is called a *tilting object* provided $\mathrm{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals $\mathrm{rk}\ \mathrm{K}_0(A)$. The endomorphism algebra $B = \mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is then called a *cluster-tilted algebra* [8]. Of particular interest to us is the fact that the functor $\mathrm{Hom}_{\,\mathcal{C}_A}(\tilde{T}, -) : \mathcal{C}_A \to \mathrm{mod}\, B$ induces an equivalence

$$\mathcal{C}_A/\mathrm{add}\,(\tau\tilde{T}) \cong \mathrm{mod}\, B,$$

where τ denotes the Auslander-Reiten translation in $\mathcal{C}_A$, see [8]. This result entails several interesting consequences. For instance, it is shown it [17] that any cluster-tilted algebra is 1-Gorenstein and hence of global dimension 1 or ∞. For the convenience of the reader, we give here a short proof of this fact.

Proposition 15 *Let B be a cluster-tilted algebra.*

(a) For every injective B-module I, we have $\operatorname{pd} I \leq 1$ *(and for every projective B-module P, we have* $\operatorname{id} P \leq 1$*).*
(b) $\operatorname{gl.dim.} B \in \{1, \infty\}$.

PROOF. (a) By [3, (IV.2.7) p.115], we need to prove that $\operatorname{Hom}_B(DB, \tau_B I) = 0$. Now $\operatorname{mod} B \cong \mathcal{C}_A/\operatorname{add}(\tau\tilde{T})$ (where A and T are as above) and, under this equivalence, every injective $\tilde{C}$-module is the image of an object of the form $\tau^2\tilde{T}_0 \in \mathcal{C}_A$, where $\tilde{T}_0 \in \operatorname{add}\tilde{T}$. It thus suffices to show that, for every $\tilde{T}_0 \in \operatorname{add}\tilde{T}$, we have $\operatorname{Hom}_{\mathcal{C}_A}(\tau^2\tilde{T}, \tau^3\tilde{T}_0) = 0$. But τ is an equivalence in $\mathcal{C}_A$, hence the result follows from $\operatorname{Hom}_{\mathcal{C}_A}(\tilde{T}, \tau\tilde{T}_0) \cong \operatorname{Ext}^1_{\mathcal{C}_A}(\tilde{T}, \tilde{T}_0) = 0$.

(b) This is the proof of [17], but we include it for completeness. It suffices to prove that, for every B-module M, $\operatorname{id} M = d < \infty$ implies $\operatorname{pd} M \leq 1$. Thus, let

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{f^1} I^1 \longrightarrow \cdots \xrightarrow{f^d} I^d \longrightarrow 0$$

be a minimal injective coresolution. Let $K^i = \operatorname{Im} f^i$ for every i. Then the exact sequence $0 \to K^{d-1} \to I^{d-1} \to I^d \to 0$ and (a) give $\operatorname{pd} K^{d-1} \leq 1$. An easy induction yields $\operatorname{pd} M \leq 1$. $\square$

It is also shown in [8] that the equivalence $\mathcal{C}_A/\operatorname{add}(\tau\tilde{T}) \cong \operatorname{mod} B$ commutes with the Auslander-Reiten translations in both categories. Let π denote the composition of the functors

$$\mathcal{D} \xrightarrow{\quad\pi_0\quad} \mathcal{C}_A \xrightarrow{\operatorname{Hom}_{\mathcal{C}_A}(\tilde{T},-)} \operatorname{mod} B,$$

where π_0 is, as above, the canonical projection. We notice that π commutes with the Auslander-Reiten translations in both categories and also that, if $X \in \mathcal{D}$, then $\pi(X) = 0$ if and only if $X \in \operatorname{add}(\tau_{\mathcal{D}}\tilde{T})$.

3.3 Auslander-Reiten quivers of cluster-tilted algebras

With the above notations, we deduce the shape of the Auslander-Reiten quiver of $\mathcal{C}_A$ and $\operatorname{mod} B$. Let Q be the ordinary quiver of A. If A is representation-finite, then the Auslander-Reiten quiver $\Gamma(\mathcal{C}_A)$ is of the form $\mathbf{Z}Q/ < \varphi >$, where φ is the automorphism of $\mathbf{Z}Q$ induced by the functor F. Since $\Gamma(\mathcal{C}_A)$ is stable and has sections isomorphic to Q, we say that it is transjective. If, on the other hand, A is representation-infinite, then $\Gamma(\mathcal{C}_A)$ consists of a unique component of the form $\mathbf{Z}Q$, which we call transjective because it is the image under π_0 of the transjective component of $\Gamma(\mathcal{D})$, and also of components which we call regular because they are the image under π_0 of the regular components of $\Gamma(\mathcal{D})$. In both cases, we deduce $\Gamma(\mathrm{mod}\, B)$ from $\Gamma(\mathcal{C}_A)$ by deleting the $|Q_0|$ points corresponding to the summands of $\tau\tilde{T}$. In particular, $\Gamma(\mathrm{mod}\, \tilde{C})$ always has a unique transjective component, deduced from that of $\Gamma(\mathcal{C}_A)$ upon applying the functor $\mathrm{Hom}_{\mathcal{C}_A}(\tilde{T}, -)$.

Lemma 16 *Let Γ be a component of the Auslander-Reiten quiver of a cluster-tilted algebra B. If Γ contains a local slice, then Γ is the transjective component.*

PROOF. Assume that Γ is not transjective. Then Γ is either a stable tube or a component of type $\mathbb{Z}\mathbb{A}_\infty$, or is obtained from one of these by deleting finitely many points. Also, since the functor $\mathrm{Hom}_{\mathcal{C}_A}(\tilde{T}, -) : \mathcal{C}_A \to \mathrm{mod}\, B$ commutes with the Auslander-Reiten translations, deleting these points will not change the τ-orbits. Consequently, a local slice Σ in Γ lifts to a unique finite local section $\tilde{\Sigma}$ in a regular component $\tilde{\Gamma}$ of $\Gamma(\mathcal{C}_A)$. In particular, $\tilde{\Gamma}$ is stable. Let thus $\tilde{X} \in \tilde{\Sigma}$ and $\tilde{X} = \tilde{X}_0 \to \tilde{X}_1 \to \ldots \to \tilde{X}_i \to \ldots$ be a sectional path of irreducible morphisms (a ray) starting at X. Then $\tilde{X}_1$ or $\tau\tilde{X}_1$ belongs to $\tilde{\Sigma}$. By induction, for each $i \geq 0$, one of the objects $\tau^i\tilde{X}_i, \tau^{i-1}\tilde{X}_i, \ldots, \tilde{X}_i$ belongs to $\tilde{\Sigma}$. Therefore $\tilde{\Sigma}$ is infinite, a contradiction. $\square$

3.4 Lifting to the derived category

Lemma 17 *Let Σ be a connected full subquiver of the transjective component of the Auslander-Reiten quiver of a cluster-tilted algebra B arising from a hereditary algebra A and $\overline{\Sigma}$ be a connected full subquiver of $D = \mathcal{D}^b(\mathrm{mod}\, A)$ such that $\pi|_{\overline{\Sigma}} : \overline{\Sigma} \to \Sigma$ is bijective. Then Σ is a local slice in $\mathrm{mod}\, B$ if and only if $\overline{\Sigma}$ is a section in $\mathcal{D}$ such that $|\Sigma_0| = \mathrm{rk}\ \mathrm{K}_0(A)$.*

PROOF. Since both subquivers are full, then the bijection $\pi|_{\overline{\Sigma}}$ induces an isomorphism of quivers. Assume that Σ is a local slice in $\mathrm{mod}\, B$. We claim that $\overline{\Sigma}$ is a presection in $\mathcal{D}$. Assume that $\overline{X} \to \overline{Y}$ is an irreducible morphism in $\mathcal{D}$ with $\overline{X} \in \overline{\Sigma_0}$. Then we have two cases to consider:

(1) If $\pi(\overline{Y}) \neq 0$, then either $\pi(\overline{Y}) \in \Sigma_0$ or $\tau_B\pi(\overline{Y}) = \pi(\tau_{\mathcal{D}}\overline{Y}) \in \Sigma_0$. Therefore $\overline{Y} \in \overline{\Sigma_0}$ or $\tau_{\mathcal{D}}\overline{Y} \in \overline{\Sigma_0}$.
(2) If $\pi(\overline{Y}) = 0$, then $\pi(\tau_{\mathcal{D}}\overline{Y}) \neq 0$ because $\mathrm{Hom}_{\mathcal{D}}(\overline{Y}, \tau_{\mathcal{D}}\overline{Y}[1]) \neq 0$. But we have an arrow $\overline{X} \to \overline{Y}$ which gives an arrow $\tau_{\mathcal{D}}\overline{Y} \to \overline{X}$, hence an arrow $\pi(\tau_{\mathcal{D}}\overline{Y}) \to \pi(\overline{X})$. Since $\pi(\overline{X}) \in \Sigma_0$, then $\pi(\tau_{\mathcal{D}}\overline{Y}) \in \Sigma_0$ and so $\tau_{\mathcal{D}}\overline{Y} \in \overline{\Sigma}_0$.

Because of Lemma 4 (c), this shows that $\overline{\Sigma}$ is a presection. Since $\pi|_{\overline{\Sigma}}$ is a bijection, we have $|\overline{\Sigma}_0| = |\Sigma_0| = \mathrm{rk}\ \mathrm{K}_0(A)$. By Corollary 10, $\overline{\Sigma}$ is a section.

Conversely, assume that $\overline{\Sigma}$ is a section in $\mathcal{D}$ and that $|\overline{\Sigma}_0| = \text{rk } \text{K}_0(A)$. By Lemma 9, $\overline{\Sigma}$ lies in a transjective component of $\Gamma(\mathcal{D})$. Note that $|\overline{\Sigma}_0| = |\Sigma_0|$ and $\text{rk } \text{K}_0(A) = \text{rk } \text{K}_0(B)$. Hence $|\Sigma_0| = \text{rk } \text{K}_0(B)$.

We show that Σ is sectionally convex. Let $X = X_0 \to X_1 \to \ldots \to X_t = Y$ be a sectional path with $X, Y \in \Sigma_0$. It lifts to a unique path $\overline{X} = \overline{X}_0 \to \overline{X}_1 \to \ldots \to \overline{X}_t = \overline{Y}$ in $\mathcal{D}$ with $\overline{X}, \overline{Y} \in \overline{\Sigma}_0$. Since $\overline{\Sigma}$ is a section, then this path is sectional and all $\overline{X}_i \in \overline{\Sigma}_0$. Applying π, we get that all X_i lie in Σ.

Finally, we show that Σ is a presection. Assume that $X \to Y$ is an irreducible morphism in $\text{mod}\, B$ with $X \in \Sigma_0$. Let $\overline{X} = \pi|_{\overline{\Sigma}}^{-1}(X)$ and choose $\overline{Y}$ in $\pi^{-1}(Y)$ such that we have an irreducible morphism $\overline{X} \to \overline{Y}$ in $\mathcal{D}$. Then $\overline{Y} \in \overline{\Sigma}_0$ or $\tau_{\mathcal{D}}\overline{Y} \in \overline{\Sigma}_0$. Hence $Y \in \Sigma_0$ or $\tau_B Y \in \Sigma_0$. Note that if $\tau_{\mathcal{D}}\overline{Y} \in \overline{\Sigma}_0$ then $\pi(\tau_{\mathcal{D}}\overline{Y}) \neq 0$ because $\pi|_{\overline{\Sigma}}$ is a bijection. We treat similarly the case of an irreducible morphism $X \to Y$ with $Y \in \Sigma_0$. $\square$

3.5 Construction of local slices

There is a close relation between tilted and cluster-tilted algebras. First, if B is a cluster-tilted algebra, then there exist a hereditary algebra A and a tilting A-module T such that $B \cong \text{End}_{\mathcal{C}_A}(\tilde{T})$, see [7, 3.3]. Also, if A is a hereditary algebra and T is a tilting A-module so that the algebra $C = \text{End}_A(T)$ is tilted, the trivial extension $\tilde{C} = C \ltimes \text{Ext}^2_C(DC, C)$ (called the *relation-extension* of C) is cluster-tilted, and conversely, every cluster-tilted algebra is of this form, see [1]. Now, since $\tilde{C} = C \ltimes \text{Ext}^2_C(DC, C)$, then any C-module can be considered as a $\tilde{C}$-module under the standard embedding $i : \text{mod}\, C \to \text{mod}\, \tilde{C}$. Note that, in general, i does not preserve irreducible morphisms. We consider the complete slice $\Sigma = \text{add}\, \text{Hom}_A(T, DA)$ in $\text{mod}\, C$, where $C = \text{End}\, T_A$ (see, for instance, [3] or [20]) and denote its image as $i(\Sigma) = \Sigma'$ in $\text{mod}\, \tilde{C}$. The following Lemma collects the important properties of Σ'.

Lemma 18 *Let* $\Sigma = \text{Hom}_A(T, DA)$ *and* $\Sigma' = i(\Sigma)$, *then*

(a) The image Σ' *is a local slice in* $\text{mod}\, \tilde{C}$.
(b) i *induces an isomorphism of quivers between* Σ *and* Σ'.
(c) $\text{Ann}_{\tilde{C}}\, \Sigma' \cong \text{Ext}^2_C(DC, C)$ *as* C-C-*bimodules.*

PROOF. (a) We have $\text{Hom}_{\mathcal{C}_A}(\tilde{T}, \widetilde{DA}) = \text{Hom}_A(T, DA)$ because T is an A-module. Thus the image $i(\Sigma) = \Sigma'$ of Σ is equal to the image of Σ under the composition

$$\text{mod}\, C \xrightarrow{j} \mathcal{D}^b(\text{mod}\, C) \xrightarrow[\cong]{\phi} \mathcal{D}^b(\text{mod}\, A) \xrightarrow{\pi} \text{mod}\, \tilde{C},$$

where j is the inclusion in degree zero and ϕ the equivalence $-\otimes^L_C T$. Let $\overline{\Sigma}$ be a connected full subquiver of $\mathcal{D}^b(\mathrm{mod}\, A)$ such that the restriction $\pi|_{\overline{\Sigma}} : \overline{\Sigma} \to \Sigma'$ is bijective, that is, $\pi(\overline{\Sigma}) = \Sigma' = i(\Sigma)$. Using the fact that ϕ is a quasi-inverse of the derived functor $R\mathrm{Hom}_A(T, -) : \mathcal{D}^b(\mathrm{mod}\, A) \to \mathcal{D}^b(\mathrm{mod}\, C)$, we see that $\overline{\Sigma} = \phi \circ j(\Sigma)$ is equal to DA which is a connected section in a transjective component of $\mathcal{D}^b(\mathrm{mod}\, A)$. Since $\mathrm{add}\,(\tau\, T) \cap \mathrm{add}\,(DA) = \emptyset$, then $\pi(I) \neq 0$ for each $I \in \mathrm{add}\, DA$, so that $\pi|_{\mathrm{add}\, DA} : DA \to \Sigma'$ is bijective. By Lemma 17, Σ' is a local slice in $\mathrm{mod}\, \tilde{C}$.

(*b*) This follows from the fact that $\Sigma' = \pi\phi j(\Sigma)$ and each one of $j|_\Sigma, \phi$ and $\pi|_{\phi j(\Sigma)}$ preserves irreducible morphisms (in the case of π, this is because $\mathrm{add}\, DA \cap \mathrm{add}\, \tau\, T = \emptyset$).

(*c*) By [1], we have an isomorphism $\mathrm{Ext}^2_C(DC, C) \cong \mathrm{Hom}_{\mathcal{D}}(T, FT)$ of C-C-bimodules. Now

$$\begin{aligned} \mathrm{Hom}_A(T, DA) \cdot \mathrm{Hom}_{\mathcal{D}}(T, FT) &= \mathrm{Hom}_{\mathcal{D}}(T, DA) \cdot \mathrm{Hom}_{\mathcal{D}}(T, FT) \\ &\subset \mathrm{Hom}_{\mathcal{D}}(T, F(DA)) \\ &= \mathrm{Hom}_{\mathcal{D}}(T, A[2]) \;=\; 0, \end{aligned}$$

because T is an A-module. Therefore $\mathrm{Hom}_{\mathcal{D}}(T, FT) \subset \mathrm{Ann}_{\tilde{C}}\, \Sigma'$.

Now let $f : T \to T$ be a non-zero morphism. Since its image is an A-module, then $\mathrm{Hom}_A(\mathrm{Im} f, DA) \neq 0$. Let $g : \mathrm{Im} f \to DA$ be a non-zero morphism and denote by $p : T \to \mathrm{Im} f$ the canonical epimorphism. Since DA is an injective module, there exists $g' : T \to DA$ such that $g'f = gp \neq 0$. Therefore $\mathrm{Ann}_{\tilde{C}}\, \Sigma' \cap \mathrm{Hom}_A(T, T) = 0$. Since, as a k-vector space, $\tilde{C} = \mathrm{Hom}_A(T, T) \oplus \mathrm{Hom}_{\mathcal{D}}(T, FT)$. We have $\dim \tilde{C} = \dim \mathrm{Hom}_{\mathcal{D}}(T, T) + \dim \mathrm{Hom}_{\mathcal{D}}(T, FT) \leq \dim \mathrm{Hom}_{\mathcal{D}}(T, T) + \dim \mathrm{Ann}_{\tilde{C}} \Sigma'$, because $\dim \mathrm{Hom}_{\mathcal{D}}(T, FT) \subset \mathrm{Ann}_{\tilde{C}} \Sigma'$. Since $\mathrm{Hom}_{\mathcal{D}}(T, FT)$ and $\mathrm{Ann}_{\tilde{C}} \Sigma'$ are in direct sum, then we have $\dim \mathrm{Hom}_{\mathcal{D}}(T, T) + \dim \mathrm{Ann}_{\tilde{C}} \Sigma' = \dim(\mathrm{Hom}_{\mathcal{D}}(T, T) + \mathrm{Ann}_{\tilde{C}} \Sigma') \leq \dim \tilde{C}$. Hence $\dim \mathrm{Ann}_{\tilde{C}} \Sigma' = \dim \mathrm{Hom}_{\mathcal{D}}(T, FT)$. This shows that $\mathrm{Hom}_{\mathcal{D}}(T, FT)$ and $\mathrm{Ann}_{\tilde{C}} \Sigma'$ are equal as subspaces of $\tilde{C}$, and the statement follows. $\square$

3.6 Main result

We are now ready for the proof of our main theorem.

Theorem 19 *Let B be a cluster-tilted algebra. Then a tilted algebra C is such that $B = C \ltimes \mathrm{Ext}^2_C(DC, C)$ if and only if there exists a local slice Σ in $\mathrm{mod}\, B$ such that $C = B/\mathrm{Ann}_B\, \Sigma$.*

PROOF. Necessity. It is well-known (see, for instance, [3]) that any complete slice Σ in $\mathrm{mod}\, C$ is of the form $\Sigma = \mathrm{add}\, \mathrm{Hom}_A(T, DA)$ for some hereditary algebra A and some tilting A-module T. By Lemma 18 (a), Σ embeds as a local slice in $\mathrm{mod}\, B$ (because $B = \tilde{C}$). Moreover, by Lemma 18 (c), we have $\mathrm{Ann}_B \Sigma \cong \mathrm{Ext}^2_C(DC, C)$ as C-C-bimodules. Therefore $C = B/\mathrm{Ext}^2_C(DC, C) = B/\mathrm{Ann}_B \Sigma$.

Sufficiency. Let B be cluster-tilted, and Σ be a local slice in $\mathrm{mod}\, B$. Set $C_1 = B/\mathrm{Ann}_B \Sigma$. Because of the definition of a cluster-tilted algebra, there exist a hereditary algebra A and a tilting object $\tilde{T} \in \mathcal{C}_A$ such that $B = \mathrm{End}_{\mathcal{C}_A}(\tilde{T})$. Let $\overline{\Sigma}$ be a connected component of the preimage $\pi^{-1}(\Sigma)$ of Σ in $\mathcal{D}^b(\mathrm{mod}\, A)$. Since the local slice Σ can only occur in the transjective component of $\Gamma(\mathrm{mod}\, B)$, because of Lemma 16, then $\overline{\Sigma}$ lies in a transjective component of $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A))$. Since Σ and $\overline{\Sigma}$ have the same number of points, then $\pi|_{\overline{\Sigma}} : \overline{\Sigma} \to \Sigma$ is bijective, whence $\overline{\Sigma}$ is a section in $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A))$ such that $|\overline{\Sigma}_0| = \mathrm{rk}\ \mathrm{K}_0(A)$, by Lemma 17. We may suppose without loss of generality that $\overline{\Sigma} = \mathrm{add}\, DA$.

The fact that $\pi|_{\mathrm{add}\, DA} : \mathrm{add}\, DA \to \Sigma$ is a bijection implies that the F-orbit $\tau_{\mathcal{D}}\, \pi^{-1}(\tilde{T})$ in $\mathcal{D}^b(\mathrm{mod}\, A)$ does not intersect $\mathrm{add}\, DA$. Therefore, we have $\mathrm{add}\, \pi^{-1}(\tilde{T}) \cap \mathrm{add}\, A[1] = \emptyset$, because $\tau_{\mathcal{D}}^{-1} DA = A[1]$ in $\mathcal{D}^b(\mathrm{mod}\, A)$. Thus, we can choose a representative T in the F-orbit $\pi^{-1}(\tilde{T})$ such that T is an A-module and $\pi(T) = \tilde{T}$. Then T is a tilting A-module. Let $C_2 = \mathrm{End}_A(T)$ be the corresponding tilted algebra. By [1], we have $B = C_2 \ltimes \mathrm{Ext}^2_{C_2}(DC_2, C_2)$. By Lemma 18 (c), we have $\mathrm{Ext}^2_{C_2}(DC_2, C_2) = \mathrm{Ann}_B \Sigma$. Thus $C_2 = B/\mathrm{Ann}_B \Sigma = C_1$. In particular, C_1 is tilted and verifies $B = C_1 \ltimes \mathrm{Ext}^2_{C_1}(DC_1, C_1)$. $\square$

Corollary 20 *Let C be a tilted algebra and $\tilde{C}$ be the corresponding cluster-tilted algebra. Then any complete slice in* $\mathrm{mod}\, C$ *embeds as a local slice in* $\mathrm{mod}\, \tilde{C}$ *and any local slice in* $\mathrm{mod}\, \tilde{C}$ *arises this way.*

PROOF. This follows directly from Lemma 18 and Theorem 19. $\square$

3.7 Computing the annihilator

Our Theorem 19 actually gives a concrete way to compute the tilted algebra C starting from $\tilde{C}$. Given a local slice Σ in $\mathrm{mod}\, \tilde{C}$, one computes its annihilator using the following result.

Corollary 21 *Let B be a cluster-tilted algebra and Σ be a local slice in* $\mathrm{mod}\, B$. *Then* $\mathrm{Ann}_B\Sigma$ *is generated (as an ideal) by arrows in the quiver of B.*

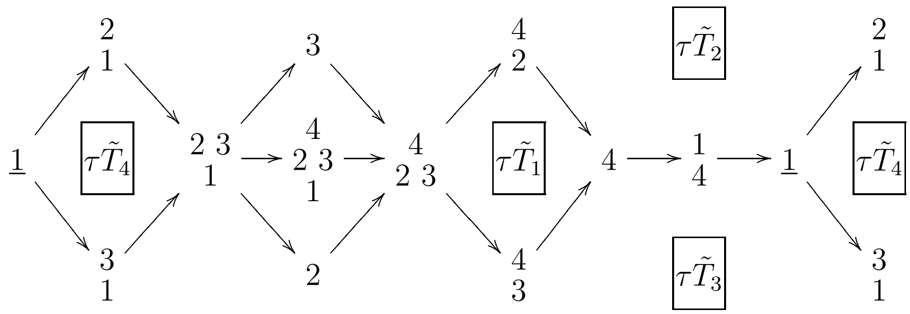


Fig. 2. Auslander-Reiten quiver of Example 3.8

PROOF. This follows from [4, 1.3] using that $\tilde{C}$ is a trivial extension (hence a split extension) of C by the C-C-bimodule $\mathrm{Ext}^2_C(DC, C) \cong \mathrm{Ann}_{\tilde{C}}\Sigma$. $\square$

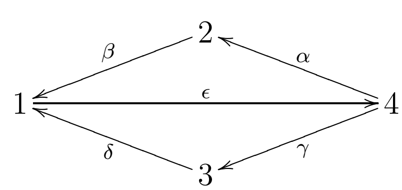
We can be a bit more precise. Let $\tilde{C} = k\tilde{Q}/\tilde{I}$. Since $\tilde{C} = \mathrm{End}_{\mathcal{C}_A}\tilde{T}$, there is a bijection between the points $x \in \tilde{Q}_0$ and the indecomposable summands $\tilde{T}_x$ of $\tilde{T}$ so that each arrow $\alpha : x \to y$ in $\tilde{Q}_1$ corresponds to a non-zero morphism $f_\alpha \in \mathrm{Hom}_{\mathcal{C}_A}(\tilde{T}_y, \tilde{T}_x) = \mathrm{Hom}_{\mathcal{D}}(T_y, T_x) \oplus \mathrm{Hom}_{\mathcal{D}}(T_y, FT_x)$. With this notation, $\mathrm{Ann}_{\tilde{C}}\Sigma$ is generated by all arrows $\alpha : x \to y$ such that $f_\alpha \in \mathrm{Hom}_{\mathcal{D}}(T_y, FT_x)$. This indeed follows immediately from the isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(T, FT) \cong \mathrm{Ext}^2_C(DC, C) \cong \mathrm{Ann}_{\tilde{C}}\Sigma.$$

Note that, as shown in [2], the arrows α which generate $\mathrm{Ann}_{\tilde{C}}\Sigma$ have to satisfy certain conditions. Moreover, if $C = kQ/I$, then $I = \tilde{I} \cap kQ$.

3.8 Example

Let $\tilde{C}$ be the cluster-tilted algebra (of type $\mathbb{D}_4$) given by the quiver



bound by $\alpha\beta = \gamma\delta$, $\beta\epsilon = 0$, $\delta\epsilon = 0$, $\epsilon\alpha = 0$, $\epsilon\gamma = 0$. The Auslander-Reiten quiver of $\tilde{C}$ is of the form shown in Figure 2 where indecomposable modules are represented by their Loewy series, and one identifies the two copies of the underlined simple module 1 (and also the two copies of the modules $\begin{smallmatrix}2\\1\end{smallmatrix}$ and $\begin{smallmatrix}3\\1\end{smallmatrix}$).

The entries $\boxed{\tau\tilde{T}_i}$ indicate the position of $\tau\tilde{T}_i$ in the cluster category. We find

$$C_1 \qquad \begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 1 & & 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & 3 \arrow[ul, "\delta"] & \end{tikzcd} \qquad \Sigma_1 = \left\{ \begin{smallmatrix} & 4 & \\ 2 & & 3 \\ & 1 & \end{smallmatrix} , \begin{smallmatrix} & 4 & \\ 2 & & 3 \end{smallmatrix} , \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} , \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \right\}$$

$$C_2 \qquad \begin{tikzcd} & 2 \arrow[dl, "\beta"'] & \\ 1 \arrow[rr, "\epsilon"] & & 4 \\ & 3 \arrow[ul, "\delta"] & \end{tikzcd} \qquad \Sigma_2 = \left\{ \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} , 1 , \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} , \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right\}$$

$$C_3 \qquad \begin{tikzcd} & 2 & \\ 1 \arrow[rr, "\epsilon"] & & 4 \arrow[ul, "\alpha"'] \arrow[dl, "\gamma"] \\ & 3 & \end{tikzcd} \qquad \Sigma_3 = \left\{ \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} , \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} , 4 , \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} \right\}$$

Fig. 3. Tilted algebras and corresponding local slices

easily all the local slices hence all the tilted subalgebras of $\tilde{C}$ which realise it as a relation-extension. There are only the three algebras C_i shown in Figure 3 each corresponding to a local slice Σ_i. with the inherited relations in each case.

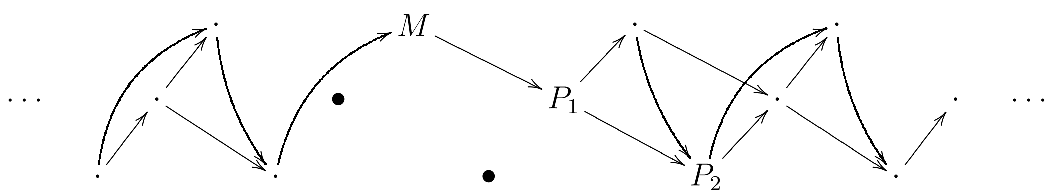
4 Cluster-tilted algebras of tree type

In view of our main result, Theorem 19, it is reasonable to ask whether there exist sufficiently many local slices in the Auslander-Reiten quiver of a cluster-tilted algebra. Since the latter is deduced from the Auslander-Reiten quiver of the cluster category by dropping finitely many points, and since local slices can only occur in the transjective component, then all but at most finitely many indecomposables lying in the transjective component of the cluster-tilted algebra belong to a local slice. That not necessarily all indecomposables in the transjective component belong to a local slice is seen in the following example.

Example 22 *Let A be the hereditary algebra given by the quiver*

$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ll] \arrow[ul] \end{tikzcd}$$

and consider the tilting A-module $T = 1 \oplus \begin{smallmatrix} & 3 & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$. Note that while $T_1 = 1$ and $T_2 = \begin{smallmatrix} & 3 & \\ 1 & & 2 \\ & 1 & \end{smallmatrix}$ are projective A-modules, the module $T_3 = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ is simple regular non-homogeneous. The transjective component of the Auslander-Reiten quiver of $\tilde{C} = \mathrm{End}_{\mathcal{C}_A}\tilde{T}$ is thus of the form



where $P_i = \mathrm{Hom}_{\mathcal{C}_A}(\tilde{T}, T_i)$, for $i \in \{1,2\}$, and the two $\bullet$ represent τT_1 and τT_2. It is easily seen that the indecomposable $\tilde{C}$-module $M = \mathrm{rad}\, P_1$ lies on no local slice.

We say that a cluster-tilted algebra $\mathrm{End}_{\mathcal{C}_A}(\tilde{T})$ is of *tree type* if the ordinary quiver of A is a tree.

Theorem 23 *Let $\tilde{C}$ be a cluster-tilted algebra of tree type. Then any indecomposable $\tilde{C}$-module lying in the transjective component lies on a local slice.*

PROOF. Let $\tilde{M}$ be an indecomposable $\tilde{C}$-module in the transjective component and M be an indecomposable object in $\pi^{-1}(\tilde{M})$; here, as in section 3, π denotes the composition of the natural functors

$$\mathcal{D}^b(\mathrm{mod}\, A) \xrightarrow{\pi_0} \mathcal{C}_A \xrightarrow{\mathrm{Hom}_{\mathcal{C}_A}(\tilde{T},-)} \mathrm{mod}\, \tilde{C} \, .$$

We claim that M lies on a section Σ in $\mathcal{D}^b(\mathrm{mod}\, A)$ such that $|\Sigma_0| = \mathrm{rk}\ \mathrm{K}_0(A)$ and $\Sigma \cap \mathrm{add}\, \tau\, \tilde{T} = \emptyset$.

Before proving this claim, we show that the theorem follows from it. Indeed, under these conditions, $\pi|_\Sigma : \Sigma \to \pi(\Sigma)$ is bijective and $\pi(\Sigma)$ is a connected full subquiver of the transjective component of the Auslander-Reiten quiver of $\tilde{C}$. By Lemma 17, $\pi(\Sigma)$ is a local slice. Obviously $\tilde{M} = \pi(M) \in \pi(\Sigma)$.

We now prove the claim. First, we fix some terminology. For any connected full subquiver $\mathcal{T}$ of $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A))$ such that $\mathcal{T}$ is a tree and $M \in \mathcal{T}_0$, there is a unique reduced walk in $\mathcal{T}$ from M to any other point $N \in \mathcal{T}_0$. We define the distance $d_{\mathcal{T}}(M, N)$ between M and N in $\mathcal{T}$ to be the number of arrows of this walk. Moreover, we define

$$d_{\mathcal{T}} = \begin{cases} \mathrm{rk}\ \mathrm{K}_0(A) & \text{if } \mathcal{T} \cap \mathrm{add}\, \tau \tilde{T} = \emptyset \\ \min\{d_{\mathcal{T}}(M,N) \mid N \in \mathcal{T} \cap \mathrm{add}\, \tau \tilde{T}\} & \text{if } \mathcal{T} \cap \mathrm{add}\, \tau \tilde{T} \neq \emptyset. \end{cases}$$

Thus $d_{\mathcal{T}}$ measures the distance between M and the set $\mathrm{add}\, \tau \tilde{T}$ in the tree $\mathcal{T}$. Note that $d_{\mathcal{T}} \geq 1$ because $M \notin \mathrm{add}\, \tau \tilde{T}$ and $d_{\mathcal{T}} = \mathrm{rk}\ \mathrm{K}_0(A)$ if and only if $\mathcal{T} \cap \mathrm{add}\, \tau \tilde{T} = \emptyset$.

Now, let Σ_1 be any section in $\mathcal{D}^b(\mathrm{mod}\, A)$ containing M. By hypothesis, Σ_1 is a tree. If $d_{\Sigma_1} = \mathrm{rk}\ \mathrm{K}_0(A)$ we are done. Suppose that $d_{\Sigma_1} < \mathrm{rk}\ \mathrm{K}_0(A)$. Consider the set $\mathcal{N} = \{N \in \Sigma_1 \cap \mathrm{add}\, \tau \tilde{T} \mid d_{\Sigma_1} = d_{\Sigma_1}(M, N)\}$ and let $N \in \mathcal{N}$. Consider the unique reduced walk

$$M = X_0 — X_1 — \ldots — X_{(d_{\Sigma_1}-1)} = L — X_{d_{\Sigma_1}} = N$$

in Σ_1 from M to N. Deleting the edge $L — N$ cuts Σ_1 into two subtrees. Let Σ_1^M be the subtree containing M (and L) and let Σ_1^N the subtree containing N. There are two cases to consider, according to the orientation of the arrow between L and N.

(1) If $L \to N$, we define Σ_2 to be the full subquiver of $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A))$ having as points those of $\Sigma_1^M \cup \tau\, \Sigma_1^N$.
(2) If $L \leftarrow N$, we define Σ_2 to be the full subquiver of $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A))$ having as points those of $\Sigma_1^M \cup \tau^{-1}\, \Sigma_1^N$.

By construction, Σ_2 is a connected tree, it lies in the transjective component and it intersects every τ-orbit exactly once.

We now show that Σ_2 is also convex. Assume that $L \to N$ (the proof in case $L \leftarrow N$ is entirely similar). Then Σ_2 has two subtrees Σ_1^M and $\tau\, \Sigma_1^N$, connected by the arrow $\tau N \to L$. We show first that these two subtrees are convex. Suppose that $X = X_0 \to X_1 \to \ldots \to X_s = Y$ is a path in $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A)$ with $X, Y \in \Sigma_1^M$. By convexity of Σ_1, we have $X_i \in \Sigma_1$ for all i. Since there is exactly one walk from X to Y in Σ_1 we actually have X_i in Σ_1^M for all i. Thus Σ_1^M is convex. Similarly, $\tau\, \Sigma_1^N$ is also convex. Now suppose that $X = X_0 \to X_1 \to \ldots \to X_s = Y$ is a path with $X, Y \in \Sigma_2$. If $X \in \Sigma_1^M$, then $Y \in \Sigma_1^M$ because of the structure of the transjective component, and then the convexity of Σ_1^M implies the result. If $X, Y \in \tau\,\Sigma_1^N$ we are done by convexity of $\tau\,\Sigma_1^N$. Hence the only remaining case is when $X \in \tau\,\Sigma_1^N$ and $Y \in \Sigma_1^M$. Suppose that there is an $i \in \{1, \ldots, s-1\}$ such that $X_i \notin \Sigma_1$. We may suppose, without loss of generality, that $i = 1$. Now $\tau\,\Sigma_1$ is a section such that $|(\tau\,\Sigma_1)_0| = \mathrm{rk}\ \mathrm{K}_0(A)$, hence, in particular, it is a presection. Therefore $X \in \tau\,\Sigma_1$, $X \to X_1$ and $X_1 \notin \tau\,\Sigma_1$ imply $\tau\,X_1 \in \tau\,\Sigma_1$, hence $X_1 \in \Sigma_1$. Then all X_i with $1 \leq i \leq s$ lie in Σ_1 because Σ_1 is convex. Since $X_1 \notin \Sigma_2$, we have $X_1 \in \Sigma_1^N$. Then, since $Y \in \Sigma_1^M$ and the subtrees Σ_1^N, Σ_1^M are only joined by the arrow $L \to N$, there exists j such that $X_j = N$ and $X_{j+1} = L$. But then there is an arrow $L \leftarrow N$, a contradiction. This shows that Σ_2 is convex and hence is a section in $\mathcal{D}^b(\mathrm{mod}\, A)$ satisfying $|(\Sigma_2)_0| = \mathrm{rk}\ \mathrm{K}_0(A)$.

We repeat this construction for every element N of $\mathcal{N}$. Note that, since $N \in \mathrm{add}\,\tau\,\tilde{T}$, neither $\tau\,N$ nor $\tau^{-1}N$ are in $\mathrm{add}\,\tau\,\tilde{T}$. In this way, we obtain, after $k = |\mathcal{N}|$ steps, a section Σ_{k+1} in $\Gamma(\mathcal{D}^b(\mathrm{mod}\, A))$ that contains M, such that $|(\Sigma_{k+1})_0| = \mathrm{rk}\ \mathrm{K}_0(A)$ and

$$d_{\Sigma_{k+1}} > d_{\Sigma_1}.$$

If $d_{\Sigma_{k+1}} = \mathrm{rk}\ \mathrm{K}_0(A)$ then $\Sigma_{k+1} \cap \mathrm{add}\,(\tau\,\tilde{T}) = \emptyset$ and we are done. Otherwise, we repeat the construction with the set

$$\mathcal{N}_{k+1} = \{N \in \Sigma_{k+1} \mid d_{\Sigma_{k+1}}(M, N) = d_{\Sigma_{k+1}}\}.$$

This algorithm will find the required section Σ in a finite number of steps. $\square$

Corollary 24 *Let $\tilde{C}$ be a representation-finite cluster-tilted algebra. Then any indecomposable $\tilde{C}$-module lies on a local slice.*

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