CLUSTER-TILTED AND QUASI-TILTED ALGEBRAS

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Abstract. In this paper, we prove that relation-extensions of quasi-tilted algebras are 2-Calabi-Yau tilted. With the objective of describing the module category of a cluster-tilted algebra of euclidean type, we define the notion of reflection so that any two local slices can be reached one from the other by a sequence of reflections and coreflections. We then give an algorithmic procedure for constructing the tubes of a cluster-tilted algebra of euclidean type. Our main result characterizes quasi-tilted algebras whose relation-extensions are cluster-tilted of euclidean type.

1. Introduction

Cluster-tilted algebras were introduced by Buan, Marsh and Reiten [BMR] and, independently in [CCS] for type $\mathbb{A}$ as a byproduct of the now extensive theory of cluster algebras of Fomin and Zelevinsky [FZ]. Since then, cluster-tilted algebras have been the subject of several investigations, see, for instance, [ABCP, ABS, BFPPT, BT, BOW, BMR2, KR, OS, ScSe, ScSe2].

In particular, in [ABS] is given a construction procedure for cluster-tilted algebras: let C be a triangular algebra of global dimension two over an algebraically closed field k, and consider the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$, where $D = \mathrm{Hom}_k(-, k)$ is the standard duality, with its natural left and right C-actions. The trivial extension of C by this bimodule is called the *relation-extension* $\widetilde{C}$ of C. It is shown there that, if C is tilted, then its relation-extension is cluster-tilted, and every cluster-tilted algebra occurs in this way.

Our purpose in this paper is to study the relation-extensions of a wider class of triangular algebras of global dimension two, namely the class of quasi-tilted algebras, introduced by Happel, Reiten and Smalø in [HRS]. In general, the relation-extension of a quasi-tilted algebra is not cluster-tilted, however it is 2-Calabi-Yau tilted, see Theorem 3.1 below. We then look more closely at those cluster-tilted algebras which are tame and representation-infinite. According to [BMR], these coincide exactly with the cluster-tilted algebras of euclidean type. We ask then the following question: Given a cluster-tilted algebra B of euclidean type, find all quasi-tilted algebras C

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such that $B = \widetilde{C}$. A similar question has been asked (and answered) in [ABS2], where, however, C was assumed to be tilted.

For this purpose, we generalize the notion of reflections of [ABS4]. We prove that this operation allows to produce all tilted algebras C such that $B = \widetilde{C}$, see Theorem 4.11. In [ABS4] this result was shown only for cluster-tilted algebras of tree type. We also prove that, unlike those of [ABS4], reflections in the sense of the present paper are always defined, that the reflection of a tilted algebra is also tilted of the same type, and that they have the same relation-extension, see Theorem 4.4 and Proposition 4.8 below. Because all tilted algebras having a given cluster-tilted algebra as relation-extension are given by iterated reflections, this gives an algorithmic answer to our question above.

After that, we look at the tubes of a cluster-tilted algebra of euclidean type and give a procedure for constructing those tubes which contain a projective, see Proposition 5.6.

We then return to quasi-tilted algebras in our last section, namely we define a particular two-sided ideal of a cluster-tilted algebra, which we call the partition ideal. Our first result (Theorem 6.1) shows that the quasi-tilted algebras which are not tilted but have a given cluster-tilted algebra B of euclidean type as relation-extension are the quotients of B by a partition ideal. We end the paper with the proof of our main result (Theorem 6.3) which says that if C is quasi-tilted and such that $B = \widetilde{C}$, then either C is the quotient of B by the annihilator of a local slice (and then C is tilted) or it is the quotient of B by a partition ideal (and then C is not tilted except in two cases easy to characterize).

2. Preliminaries

2.1. **Notation.** Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field k. For an algebra C, we denote by $\operatorname{mod} C$ the category of finitely generated right C-modules. All subcategories are full, and identified with their object classes. Given a category $\mathcal{C}$, we sometimes write $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. If $\mathcal{C}$ is a full subcategory of $\operatorname{mod} C$, we denote by $\operatorname{add} \mathcal{C}$ the full subcategory of $\operatorname{mod} C$ having as objects the finite direct sums of summands of modules in $\mathcal{C}$.

For a point x in the ordinary quiver of a given algebra C, we denote by $P(x)$, $I(x)$, $S(x)$ respectively, the indecomposable projective, injective and simple C-modules corresponding to x. We denote by $\Gamma(\operatorname{mod} C)$ the Auslander-Reiten quiver of C and by $\tau = D\mathrm{Tr}, \tau^{-1} = \mathrm{Tr}D$ the Auslander-Reiten translations. For further definitions and facts, we refer the reader to [ARS, ASS, S].

2.2. **Tilting.** Let Q be a finite connected and acyclic quiver. A module T over the path algebra kQ of Q is called *tilting* if $\mathrm{Ext}^1_{kQ}(T, T) = 0$ and the number of isoclasses (isomorphism classes) of indecomposable summands of

T equals $|Q_0|$, see [ASS]. An algebra C is called *tilted of type Q* if there exists a tilting kQ-module T such that $C = \mathrm{End}_{kQ}T$. It is shown in [Ri] that an algebra C is tilted if and only if it contains a *complete slice* Σ, that is, a finite set of indecomposable modules such that

1) $\bigoplus_{U\in\Sigma} U$ is a sincere C-module.
2) If $U_0 \to U_1 \to \cdots \to U_t$ is a sequence of nonzero morphisms between indecomposable modules with $U_0, U_t \in \Sigma$ then $U_i \in \Sigma$ for all i (*convexity*).
3) If $0 \to L \to M \to N \to 0$ is an almost split sequence in $\mathrm{mod}\, C$ and at least one indecomposable summand of M lies in Σ, then exactly one of L, N belongs to Σ.

For more on tilting and tilted algebras, we refer the reader to [ASS].

Tilting can also be done within the framework of a hereditary category. Let $\mathcal{H}$ be an abelian k-category which is Hom-finite, that is, such that, for all $X, Y \in \mathcal{H}$, the vector space $\mathrm{Hom}_{\mathcal{H}}(X, Y)$ is finite dimensional. We say that $\mathcal{H}$ is *hereditary* if $\mathrm{Ext}^2_{\mathcal{H}}(-, ?) = 0$. An object $T \in \mathcal{H}$ is called a *tilting object* if $\mathrm{Ext}^1_{\mathcal{H}}(T, T) = 0$ and the number of isoclasses of indecomposable objects of T is the rank of the Grothendieck group $K_0(\mathcal{H})$.

The endomorphism algebras of tilting objects in hereditary categories are called *quasi-tilted algebras*. For instance, tilted algebras but also canonical algebras (see [Ri]) are quasi-tilted. Quasi-tilted algebras have attracted a lot of attention and played an important role in representation theory, see for instance [HRS, Sk].

2.3. **Cluster-tilted algebras.** Let Q be a finite, connected and acyclic quiver. The *cluster category* $\mathcal{C}_Q$ of Q is defined as follows, see [BMRRT]. Let F denote the composition $\tau_{\mathcal{D}}^{-1}[1]$, where $\tau_{\mathcal{D}}^{-1}$ denotes the inverse Auslander-Reiten translation in the bounded derived category $\mathcal{D} = \mathcal{D}^b(\mathrm{mod}\, kQ)$, and $[1]$ denotes the shift of $\mathcal{D}$. Then $\mathcal{C}_Q$ is the orbit category $\mathcal{D}/F$: its objects are the F-orbits $\widetilde{X} = (F^iX)_{i\in\mathbb{Z}}$ of the objects $X \in \mathcal{D}$, and the space of morphisms from $\widetilde{X} = (F^iX)_{i\in\mathbb{Z}}$ to $\widetilde{Y} = (F^iY)_{i\in\mathbb{Z}}$ is

$$\mathrm{Hom}_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}}(X, F^iY).$$

Then $\mathcal{C}_Q$ is a triangulated category with almost split triangles and, moreover, for $\widetilde{X}, \widetilde{Y} \in \mathcal{C}_Q$ we have a bifunctorial isomorphism $\mathrm{Ext}^1_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) \cong D\mathrm{Ext}^1_{\mathcal{C}_Q}(\widetilde{Y}, \widetilde{X})$. This is expressed by saying that the category $\mathcal{C}_Q$ is *2-Calabi-Yau.*

An object $\widetilde{T} \in \mathcal{C}_Q$ is called *tilting* if $\operatorname{Ext}^1_{\mathcal{C}_Q}(\widetilde{T}, \widetilde{T}) = 0$ and the number of isoclasses of indecomposable summands of $\widetilde{T}$ equals $|Q_0|$. The endomorphism algebra $B = \operatorname{End}_{\mathcal{C}_Q}\widetilde{T}$ is then called *cluster-tilted* of type Q. More generally, the endomorphism algebra $\operatorname{End}_{\mathcal{C}}\widetilde{T}$ of a tilting object $\widetilde{T}$ in a 2-Calabi-Yau category with finite dimensional Hom-spaces is called a *2-Calabi-Yau tilted algebra*, see [Re].

Let now T be a tilting kQ-module, and $C = \operatorname{End}_{kQ}T$ the corresponding tilted algebra. Then it is shown in [ABS] that the trivial extension $\widetilde{C}$ of C by the C-C-bimodule $\operatorname{Ext}^2_C(DC, C)$ with the two natural actions of C, the so-called *relation-extension* of C, is cluster-tilted. Conversely, if B is cluster-tilted, then there exists a tilted algebra C such that $B = \widetilde{C}$.

Let now B be a cluster-tilted algebra, then a full subquiver Σ of $\Gamma(\operatorname{mod} B)$ is a *local slice*, see [ABS2], if:

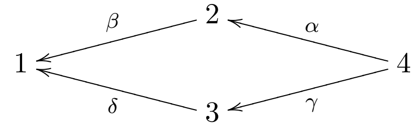
1) Σ is a *presection*, that is, if $X \to Y$ is an arrow then:
   (a) $X \in \Sigma$ implies that either $Y \in \Sigma$ or $\tau Y \in \Sigma$
   (b) $Y \in \Sigma$ implies that either $X \in \Sigma$ or $\tau^{-1}X \in \Sigma$.
2) Σ is *sectionally convex*, that is, if $X = X_0 \to X \to \cdots \to X_t = Y$ is a sectional path in $\Gamma(\operatorname{mod} B)$ then $X, Y \in \Sigma$ implies that $X_i \in \Sigma$ for all i.
3) $|\Sigma_0| = \operatorname{rk} K_0(B)$.

Let C be tilted, then, under the standard embedding $\operatorname{mod} C \to \operatorname{mod} \widetilde{C}$, any complete slice in the tilted algebra C embeds as a local slice in $\operatorname{mod} \widetilde{C}$, and any local slice in $\operatorname{mod} \widetilde{C}$ occurs in this way. If B is a cluster-tilted algebra, then a tilted algebra C is such that $B = \widetilde{C}$ if and only if there exists a local slice Σ in $\Gamma(\operatorname{mod} B)$ such that $C = B/\operatorname{Ann}_B\Sigma$, where $\operatorname{Ann}_B\Sigma = \bigcap_{X\in\Sigma} \operatorname{Ann}_B X$, see [ABS2].

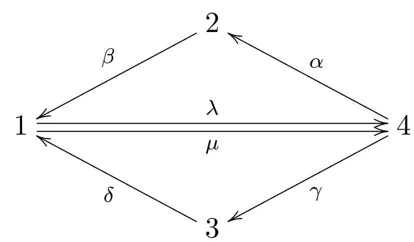
Let Σ be a local slice in the transjective component of $\Gamma(\operatorname{mod} B)$ having the property that all the sources in Σ are injective B-modules. Then Σ is called a *rightmost* slice of B. Let x be a point in the quiver of B such that $I(x)$ is an injective source of the rightmost slice Σ. Then x is called a *strong sink*. *Leftmost slices* and *strong sources* are defined dually.

3. From quasi-tilted to cluster-tilted algebras

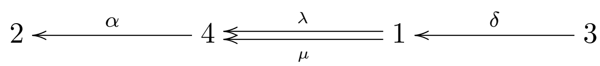
We start with a motivating example. Let C be the tilted algebra of type $\widetilde{\mathbb{A}}$ given by the quiver



bound by $\alpha\beta = 0$, $\gamma\delta = 0$. Its relation-extension is the cluster-tilted algebra B given by the quiver



bound by $\alpha\beta = 0$, $\beta\lambda = 0$, $\lambda\alpha = 0$, $\gamma\delta = 0$, $\delta\mu = 0$, $\mu\gamma = 0$. However, B is also the relation-extension of the algebra C' given by the quiver



bound by $\lambda\alpha = 0$, $\delta\mu = 0$. This latter algebra C' is not tilted, but it is quasi-tilted. In particular, it is triangular of global dimension two. Therefore, the question arises natrually whether the relation-extension of a quasi-tilted algebra is always cluster-tilted. This is certainly not true in general, for the relation-extension of a tubular algebra is not cluster-tilted. However, it is 2-Calabi-Yau tilted. In this section, we prove that the relation-extension of a quasi-tilted algebra is always 2-Calabi-Yau tilted.

Let $\mathcal{H}$ be a hereditary category with tilting object T. Because of [H], there exist an algebra A, which is hereditary or canonical, and a triangle equivalence $\Phi : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\operatorname{mod} A)$. Let T' denote the image of T under this equivalence. Because Φ preserves the shift and the Auslander-Reiten translation, it induces an equivalence between the cluster categories $\mathcal{C}_{\mathcal{H}}$ and $\mathcal{C}_A$, see [Am, Section 4.1]. Indeed, because A is canonical or hereditary, it follows that $\mathcal{C}_A \cong \mathcal{D}^b(\operatorname{mod} A)/F$, where $F = \tau^{-1}[1]$. Therefore, we have $\operatorname{End}_{\mathcal{C}_{\mathcal{H}}} T \cong \operatorname{End}_{\mathcal{C}_A} T'$.

We say that a 2-Calabi-Yau tilted algebra $\operatorname{End}_{\mathcal{C}} T$ is of *canonical type* if the 2-Calabi-Yau category $\mathcal{C}$ is the cluster category of a canonical algebra. The proof of the next theorem follows closely [ABS].

Theorem 3.1. *Let C be a quasi-tilted algebra. Then its relation-extension $\widetilde{C}$ is cluster-tilted or it is 2-Calabi-Yau titled of canonical type.*

Proof. Because C is quasi-tilted, there exist a hereditary category $\mathcal{H}$ and a tilting object T in $\mathcal{H}$ such that $C = \operatorname{End}_{\mathcal{H}} T$. As observed above, there exist an algebra A, which is hereditary or canonical, and a triangle equivalence $\Phi : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\operatorname{mod} A)$. Let $T' = \Phi(T)$.We have $\mathcal{D}^b(\operatorname{mod} C) \cong \mathcal{D}^b(\operatorname{mod} A) \cong \mathcal{D}^b(\mathcal{H})$, and therefore

$$\begin{aligned}
\operatorname{Ext}^2_C(DC, C) &\cong \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} C)}(\tau C[1], C[2]) \\
&\cong \operatorname{Hom}_{\mathcal{D}^b(\mathcal{H})}(\tau T[1], T[2]) \\
&\cong \operatorname{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, \tau^{-1} T[1]) \\
&\cong \operatorname{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, FT).
\end{aligned}$$

Thus the additive structure of $C \ltimes \mathrm{Ext}^2_C(DC, C)$ is that of

$$\begin{aligned} C \oplus \mathrm{Ext}^2_C(DC,C) &\cong \mathrm{End}_{\mathcal{H}}(T) \oplus \mathrm{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, FT) \\ &\cong \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, FT) \\ &\cong \mathrm{Hom}_{\mathcal{C}_{\mathcal{H}}}(T,T) \\ &\cong \mathrm{End}_{\mathcal{C}_{\mathcal{H}}} T. \end{aligned}$$

Then, we check exactly as in [ABS, Section 3.3] that the multiplicative structure is preserved. This completes the proof. □

Let C be a representation-infinite quasi-tilted algebra. Then C is derived equivalent to a hereditary or a canonical algebra A. Let n_A denote the tubular type of A. We then say that C has canonical type $n_C = n_A$.

Lemma 3.2. *Let C be a representation-infinite quasi-tilted. Then its relation-extension $\widetilde{C}$ is cluster-tilted of euclidean type if and only if n_C is one of*

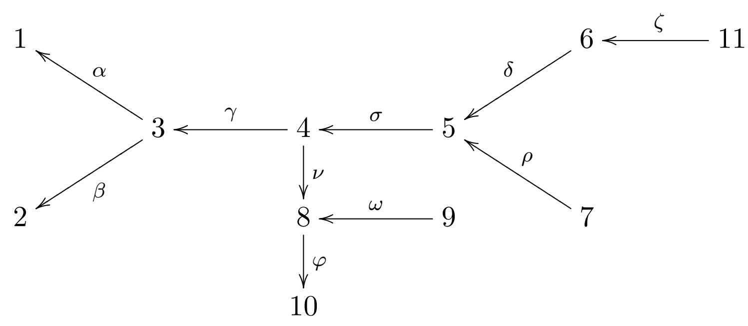
$$(p, q), (2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5), \text{ with } p \le q,\ 2 \le r.$$

Proof. Indeed, $\widetilde{C}$ is cluster-tilted of euclidean type if and only if C is derived equivalent to a tilted algebra of euclidean type, and this is the case if and only if n_C belongs to the above list. □

Remark 3.3. It is possible that C is domestic, but yet $\widetilde{C}$ is wild. Indeed, we modify the example after Corollary D in [Sk]. Recall from [Sk] that there exists a tame concealed full convex subcategory K such that C is a semiregular branch enlargement of K

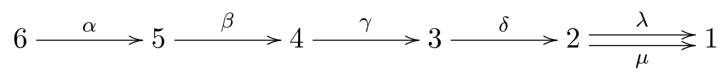
$$C = [E_i]K[F_j],$$

where E_i, F_j are (truncated) branches. Then the representation theory of C is determined by those of $C^- = [E_i]K$ and $C^+ = K[F_j]$. Let C be given by the quiver

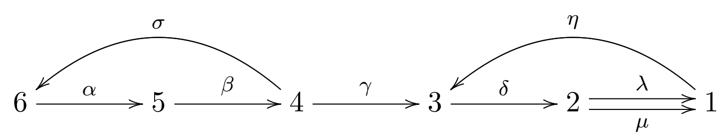


bound by the relations $\sigma\nu = 0$, $\omega\varphi = 0$, $\zeta\delta\sigma\gamma\beta = 0$. Here C^- is the full subcategory generated by $C_0 \setminus \{11\}$ and C^+ the one generated by $C_0 \setminus \{8, 9, 10\}$. Then C^- has domestic tubular type $(2, 2, 7)$ and C^+ has domestic tubular type $(2, 3, 4)$. Therefore C is domestic. On the other hand, the canonical type of C is $(2, 3, 7)$, which is wild. In this example, the 2-Calabi-Yau tilted algebra $\widetilde{C}$ is not cluster-tilted, because it is not of euclidean type, but the derived category of mod C contains tubes, see [R].

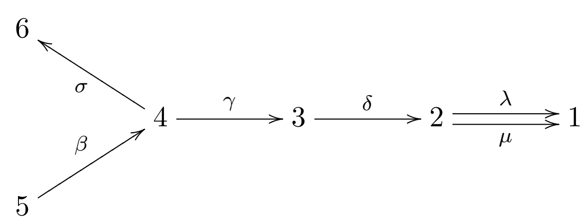
Remark 3.4. There clearly exist algebras which are not quasi-tilted but whose relation-extension is cluster-tilted of euclidean type. Indeed, let C be given by the quiver



bound by $\alpha\beta = 0, \delta\lambda = 0$. Then C is iterated tilted of type $\widetilde{\mathbb{A}}$ of global dimension 2, see [FPT]. Its relation-extension is given by



bound by $\alpha\beta = 0, \beta\sigma = 0, \sigma\alpha = 0, \delta\lambda = 0, \lambda\eta = 0, \eta\delta = 0$. This algebra is isomorphic to the relation-extension of the tilted algebra of type $\widetilde{\mathbb{A}}$ given by the quiver



bound by $\beta\sigma = 0$, $\delta\lambda = 0$. Therefore $\widetilde{C}$ is cluster-tilted of euclidean type. On the other hand, C is not quasi-tilted, because the uniserial module $\begin{smallmatrix}4\\3\end{smallmatrix}$ has both projective and injective dimension 2.

4. Reflections

Let C be a tilted algebra. Let Σ be a rightmost slice, and let $I(x)$ be an injective source of Σ. Thus x is a strong sink in C.

Definition 4.1. *We define the* completion H_x of x *by the following three conditions.*

(a) $I(x) \in H_x$.
(b) H_x *is closed under predecessors in* Σ.
(c) *If* $L \to M$ *is an arrow in* Σ *with* $L \in H_x$ *having an injective successor in* H_x *then* $M \in H_x$.

Observe that H_x may be constructed inductively in the following way. We let $H_1 = I(x)$, and H'_2 be the closure of H_1 with respect to (c) (that is, we simply add the direct successors of $I(x)$ in Σ, and if a direct successor of $I(x)$ is injective, we also take its direct successor, etc.) We then let H_2 be the closure of H'_2 with respect to predecessors in Σ. Then we repeat the procedure; given H_i, we let H'_{i+1} be the closure of H_i with respect to (c) and H_{i+1} be the closure of H'_{i+1} with respect to predecessors. This procedure must stabilize, because the slice Σ is finite. If $H_j = H_k$ with $k > j$, we let $H_x = H_j$.

We can decompose H_x as the disjoint union of three sets as follows. Let $\mathcal{J}$ denote the set of injectives in H_x, let $\mathcal{J}^-$ be the set of non-injectives in H_x which have an injective successor in H_x, and let $\mathcal{E} = H_x \setminus (\mathcal{J} \cup \mathcal{J}^-)$ denote the complement of $(\mathcal{J} \cup \mathcal{J}^-)$ in H_x. Thus

$$H_x = \mathcal{J} \sqcup \mathcal{J}^- \sqcup \mathcal{E}$$

is a disjoint union.

Remark 4.2. If $\mathcal{J}^- = \emptyset$ then H_x reduces to the completion G_x as defined in [ABS4]. Recall that G_x does not always exist, but, as seen above, H_x does. Conversely, if G_x exists, then it follows from its construction in [ABS4] that $\mathcal{J}^- = \emptyset$.

Thus $\mathcal{J}^- = \emptyset$ if and only if G_x exists, and, in this case $G_x = H_x$.

For every module M over a cluster-tilted algebra B, we can consider a lift $\widetilde{M}$ in the cluster category $\mathcal{C}$. Abusing notation, we sometimes write $\tau^i M$ to denote the image of $\tau^i_{\mathcal{C}}\widetilde{M}$ in $\operatorname{mod} B$, and say that the Auslander-Reiten translation is computed in the cluster category.

Definition 4.3. *Let x be a strong sink in C and let Σ be a rightmost local slice with injective source $I(x)$. Recall that Σ is also a local slice in* $\operatorname{mod} B$*. Then the reflection of the slice Σ in x is*

$$\sigma_x^+\Sigma = \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1}\mathcal{E} \cup (\Sigma \setminus H_x),$$

where τ is computed in the cluster category. In a similar way, one defines the coreflection σ_y^- of leftmost slices with projective sink $P_C(y)$.

Theorem 4.4. *Let x be a strong sink in C and let Σ be a rightmost local slice in* $\operatorname{mod} B$ *with injective source $I(x)$. Then the reflection $\sigma_x^+\Sigma$ is a local slice as well.*

Proof. Set $\Sigma' = \sigma_x^+\Sigma$ and

$$\Sigma'' = \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1}\mathcal{E} \cup (\Sigma \setminus H_x) = \tau^{-1}H_x \cup (\Sigma \setminus H_x),$$

where again, Σ'' and τ are computed in the cluster category $\mathcal{C}$. We claim that Σ'' is a local slice in $\mathcal{C}$. Notice that since H_x is closed under predecessors in Σ, then, if $X \in \Sigma \setminus H_x$ is a neighbor of $Y \in H_x$, we must have an arrow $Y \to X$ in Σ. This observation being made, Σ'' is clearly obtained from Σ by applying a sequence of APR-tilts. Thus Σ'' is a local slice in $\mathcal{C}$.

We now claim that $\tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$ is closed under predecessors in Σ''. Indeed, let $X \in \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$ and $Y \in \Sigma''$ be such that we have an arrow $Y \to X$. Then, there exists an arrow $\tau X \to Y$ in the cluster category. Because $X \in \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$, we have $\tau X \in \mathcal{J} \cup \mathcal{J}^-$. Now if $Y \in \Sigma$, then the arrow $\tau X \to Y$ would imply that $Y \in H_x$, which is impossible, because $Y \in \Sigma''$ and $\Sigma'' \cap H_x = \emptyset$. Thus $Y \notin \Sigma$, and therefore $Y \in (\Sigma'' \setminus \Sigma) = \tau^{-1}H_x$. Hence $\tau Y \in H_x$. Moreover, there is an arrow $\tau Y \to \tau X$. Using that

$\tau X \in \mathcal{J} \cup \mathcal{J}^-$, this implies that τY has an injective successor in H_x and thus $Y \in \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$. This establishes our claim that $\tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$ is closed under predecessors in Σ''.

Thus applying the same reasoning as before, we get that

$$\Sigma' = (\Sigma'' \setminus \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)) \cup \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-)$$

is a local slice in $\mathcal{C}$. Now we claim that

$$\Sigma' \cap \operatorname{add}(\tau T) = \emptyset.$$

First, because $\Sigma \cap \operatorname{add}(\tau T) = \emptyset$, we have $(\Sigma \setminus H_x) \cap \operatorname{add}(\tau T) = \emptyset$. Next, $\mathcal{E}$ contains no injectives, by definition. Thus $\tau^{-1}\mathcal{E} \cap \operatorname{add}(\tau T) = \emptyset$. Assume now that $X \in \operatorname{add}(\tau T)$ belongs to $\tau^{-2}\mathcal{J}^-$. Then $\tau^2 X \in H_x$ and there exists an injective predecessor $I(j)$ of $\tau^2 X$ in H_x, and since H_x is part of the local slice Σ, there exists a sectional path from $I(j)$ to $\tau^2 X$. Applying τ^{-2}, we get a sectional path from T_j to X in the cluster category. But this means $\operatorname{Hom}_{\mathcal{C}}(T_j, X) \neq 0$, which is a contradiction to the hypothesis that $X \in \operatorname{add}(\tau T)$. Finally, if $X \in \tau^{-2}\mathcal{J}$ then X is a summand of T, which, again, is contradicting the hypothesis that $X \in \operatorname{add}(\tau T)$. □

Following [ABS4], let $\mathcal{S}_x$ be the full subcategory of C consisting of those y such that $I(y) \in H_x$.

Lemma 4.5. (a) *$\mathcal{S}_x$ is hereditary.*
(b) *$\mathcal{S}_x$ is closed under successors in C.*
(c) *C can be written in the form*

$$C = \begin{bmatrix} H & 0 \\ M & C' \end{bmatrix},$$

where H is hereditary, C' is tilted and M is a C'-H-bimodule.

Proof. (a) Let $H = \operatorname{End}(\oplus_{y \in \mathcal{S}_x} I(y))$. Then H is a full subcategory of the hereditary endomorphism algebra of Σ. Therefore H is also hereditary, and so $\mathcal{S}_x$ is hereditary.

(b) Let $y \in \mathcal{S}_x$ and $y \to z$ in C. Then there exists a morphism $I(z) \to I(y)$. Because $I(z)$ is an injective C-module and Σ is sincere, there exist a module $N \in \Sigma$ and a non-zero morphism $N \to I(z)$. Then we have a path $N \to I(z) \to I(y)$, and since $N, I(y) \in \Sigma$, we get that $I(z) \in \Sigma$ by convexity of the slice Σ in $\operatorname{mod} C$. Moreover, since $I(y) \in H_x$ and H_x is closed under predecessors in Σ, it follows that $I(z) \in H_x$. Thus $z \in \mathcal{S}_x$ and this shows (b).

(c) This follows from (a) and (b). □

We recall that the cluster duplicated algebra was introduced in [ABS3].

Corollary 4.6. *The cluster duplicated algebra $\overline{C}$ of C is of the form*

$$\overline{C} = \begin{bmatrix} H & 0 & 0 & 0 \\ M & C' & 0 & 0 \\ 0 & E_0 & H & 0 \\ 0 & E_1 & M & C' \end{bmatrix}$$

where $E_0 = \mathrm{Ext}^2_C(DC', H)$ and $E_1 = \mathrm{Ext}^2_C(DC', C')$.

Proof. We start by writing C in the matrix form of the lemma. By definition, H consists of those $y \in C_0$ such that the corresponding injective $I(y)$ lies in H_x inside the slice Σ. In particular, the projective dimension of these injectives is at most 1, hence $\mathrm{Ext}^2_C(DC, C) = \mathrm{Ext}^2_C(DC', C)$. The result now follows upon multiplying by idempotents. □

Definition 4.7. *Let x be a strong sink in C. The reflection at x of the algebra C is*

$$\sigma_x^+ C = \begin{bmatrix} C' & 0 \\ E_0 & H \end{bmatrix}$$

where $E_0 = \mathrm{Ext}^2_C(DC', H)$.

Proposition 4.8. *The reflection $\sigma_x^+ C$ of C is a tilted algebra having $\sigma_x^+\Sigma$ as a complete slice. Moreover the relation-extensions of C and $\sigma_x^+\Sigma$ are isomorphic.*

Proof. We first claim that the support $\mathrm{supp}(\sigma_x^+\Sigma)$ of $\sigma_x^+\Sigma$ is contained in $\sigma_x^+ C$. Let $X \in \sigma_x^+\Sigma$. Recall that $\sigma_x^+\Sigma = \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1}\mathcal{E} \cup (\Sigma \setminus H_x)$. If $X \in \tau^{-2}\mathcal{J}$, then $X = P(y')$ is projective corresponding to a point $y' \in H$. Thus $I(y) \in H_x$ and the radical of $P(y)$ has no non-zero morphism into $I(y)$. Therefore $\mathrm{supp}(X) \subset \sigma_X^+ C$.

Assume next that $X \in \tau^{-2}\mathcal{J}^-$, that is, $X = \tau^{-2}Y$, where $Y \in \mathcal{J}^-$ has an injective successor $I(z)$ in H_x. Because all sources in Σ are injective, there is an injective $I(y') \in \Sigma$ and a sectional path $I(y') \to \ldots \to Y \to \ldots \to I(z)$. Applying τ^{-2}, we obtain a sectional path $P(y') \to \ldots \to X \to \ldots \to P(z)$. In particular the point y' belongs to the support of X. Assume that there is a point h in H that is in the support of X. Then there exists a nonzero morphism $X \to I(h)$. But $I(h) \in \Sigma$ and there is no morphism from $X \in \tau^{-2}\Sigma$ to Σ. Therefore $\mathrm{supp}(X) \subset \sigma_x^+ C$.

By the same argument, we show that if $X \in \tau^{-1}\mathcal{E}$, then $\mathrm{supp}(X) \subset \sigma_x^+ C$.

Finally, all modules of $\Sigma \setminus H_x$ are supported in C'. This establishes our claim.

Now, by Theorem 4.4, $\sigma_x^+\Sigma$ is a local slice in $\mathrm{mod}\,\widetilde{C}$. Therefore $\widetilde{C}/\mathrm{Ann}\,\sigma_x^+\Sigma$ is a tilted algebra in which $\sigma_x^+\Sigma$ is a complete slice. Since the support of $\sigma_x^+\Sigma$ is the same as the support of $\sigma_x^+ C$, we are done. □

We now come to the main result of this section, which states that any two tilted algebras that have the same relation-extension are linked to each other by a sequence of reflections and coreflections.

Definition 4.9. *Let B be a cluster-tilted algebra and let Σ and Σ' be two local slices in $\operatorname{mod} B$. We write $\Sigma \sim \Sigma'$ whenever $B/\operatorname{Ann}\Sigma = B/\operatorname{Ann}\Sigma'$.*

Lemma 4.10. *Let B be a cluster-tilted algebra, and Σ_1, Σ_2 be two local slices in $\operatorname{mod} B$. Then there exists a sequence of reflections and coreflections σ such that*

$$\sigma\Sigma_1 \sim \Sigma_2.$$

Proof. Given a local slice Σ in $\operatorname{mod} B$ such that Σ has injective successors in the transjective component $\mathcal{T}$ of $\Gamma(\operatorname{mod} B)$, let Σ^+ be the rightmost local slice such that $\Sigma \sim \Sigma^+$. Then Σ^+ contains a strong sink x, thus reflecting in x we obtain a local slice $\sigma_x^+\Sigma^+$ that has fewer injective successors in $\mathcal{T}$ than Σ. To simplify the notation we define $\sigma_x^+\Sigma = \sigma_x^+\Sigma^+$. Similarly, we define $\sigma_y^-\Sigma = \sigma_y^-\Sigma^-$, where Σ^- is the leftmost local slice containing a strong source y and $\Sigma \sim \Sigma^-$.

Since we can always reflect in a strong sink, there exist sequences of reflections such that

$$\sigma_{x_r}^+ \cdots \sigma_{x_2}^+\sigma_{x_1}^+\Sigma_1 = \Sigma_\infty^1$$

$$\sigma_{y_s}^+ \cdots \sigma_{y_2}^+\sigma_{y_1}^+\Sigma_2 = \Sigma_\infty^2$$

and $\Sigma_\infty^1, \Sigma_\infty^2$ have no injective successors in $\mathcal{T}$. This implies that $\Sigma_\infty^1 \sim \Sigma_\infty^2$. Let

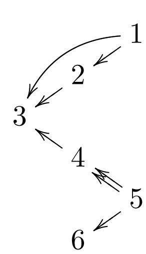
$$\sigma = \sigma_{y_1}^-\sigma_{y_2}^- \cdots \sigma_{y_s}^-\sigma_{x_r}^+ \cdots \sigma_{x_2}^+\sigma_{x_1}^+$$

thus $\sigma\Sigma_1 \sim \Sigma_2$. □

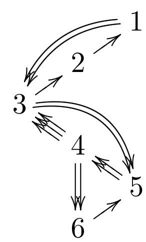
Theorem 4.11. *Let C_1 and C_2 be two tilted algebras that have the same relation-extension. Then there exists a sequence of reflections and coreflections σ such that $\sigma C_1 \cong C_2$.*

Proof. Let B be the common relation-extension of the tilted algebras C_1 and C_2. By [ABS2], there exist local slices Σ_i in $\operatorname{mod} B$ such that $C_i = B/\operatorname{Ann}\Sigma_i$, for $i = 1, 2$. Now the result follows from Lemma 4.10 and Theorem 4.4. □

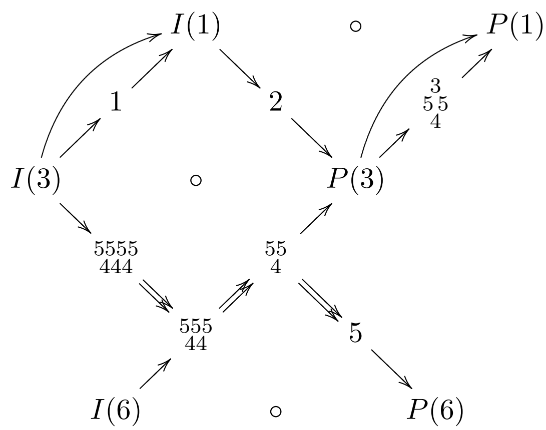
Example 4.12. *Let A be the path algebra of the quiver*



Mutating at the vertices 4,5, and 2 yields the cluster-tilted algebra B with quiver



In the Auslander-Reiten quiver of mod B *we have the following local configuration.*



where

$$I(1) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \quad I(3) = \begin{smallmatrix} 2 & 5555 \\ 11 & 444 \\ & 3 \end{smallmatrix} \quad I(6) = \begin{smallmatrix} 555 \\ 44 \\ 6 \end{smallmatrix}$$

The 6 modules on the left form a rightmost local slice Σ *in which both* $I(3)$ *and* $I(6)$ *are sources, so 3 and 6 are strong sinks. For both strong sinks the subset* $\mathcal{J}^-$ *of the completion consists of the simple module* 1. *The simple module* $2 = \tau^{-1}1$ *does not lie on a local slice.*

The completion H_6 *is the whole local slice* Σ *and therefore the reflection* $\sigma_6^+\Sigma$ *is the local slice consisting of the 6 modules on the right containing both* $P(1)$ *and* $P(6)$.

On the other hand, the completion H_3 *consists of the four modules* $I(3)$, $S(1)$, $I(1)$ *and* $\begin{smallmatrix} 5555 \\ 444 \end{smallmatrix}$, *and therefore the reflection* $\Sigma' = \sigma_3^+\Sigma$ *is the local slice consisting of the 6 modules on the straight line from* $I(6)$ *to* $P(1)$. *This local slice admits the strong sink* 6 *and the completion* H_6' *in* Σ' *consists of the two modules* $I(6)$ *and* $\begin{smallmatrix} 555 \\ 44 \end{smallmatrix}$. *Therefore the reflection* $\sigma_6^+\Sigma'$ *is equal to* $\sigma_6^+\Sigma$. *Thus*

$$\sigma_6^+\Sigma = \sigma_6^+(\sigma_3^+\Sigma).$$

This example raises the question which indecomposable modules over a cluster-tilted algebra do not lie on a local slice. We answer this question in a forthcoming publication [AsScSe].

5. Tubes

The objective of this section is to show how to construct those tubes of a tame cluster-tilted algebra which contain projectives. Let B be a cluster-tilted algebra of euclidean type, and let $\mathcal{T}$ be a tube in $\Gamma(\operatorname{mod} B)$ containing at least one projective. First, consider the transjective component of $\Gamma(\operatorname{mod} B)$. Denote by Σ_L a local slice in the transjective component that precedes all indecomposable injective B-modules lying in the transjective component. Then $B/\mathrm{Ann}_B\Sigma_L = C_1$ is a tilted algebra having a complete slice in the preinjective component. Define Σ_R to be a local slice which is a successor of all indecomposable projectives lying in the transjective component. Then $B/\mathrm{Ann}_B\Sigma_R = C_2$ is a tilted algebra having a complete slice in the postprojective component. Also, C_1 (respectively, C_2) has a tube $\mathcal{T}_1$ (respectively, $\mathcal{T}_2$) containing the indecomposable projective C_1-modules (respectively, injective C_2-modules) corresponding to the projective B-modules in $\mathcal{T}$ (respectively, injective B-modules in $\mathcal{T}$).

An indecomposable projective $P(x)$ (respectively, injective $I(x)$) B-module that lies in a tube, is said to be a *root projective* (respectively, a *root injective*) if there exists an arrow in B between x and y, where the corresponding indecomposable projective $P(y)$ lies in the transjective component of $\Gamma(\operatorname{mod} B)$.

Let $\mathcal{S}_1$ be the coray in $\mathcal{T}_1$ passing through the projective C_1-module that corresponds to the root projective $P_B(i)$ in $\mathcal{T}$. Similarly, let $\mathcal{S}_2$ be the ray in $\mathcal{T}_2$ passing through the injective that corresponds to the root injective $I_B(i)$ in $\mathcal{T}$.

Recall that if A is hereditary and $T \in \operatorname{mod} A$ is a tilting module, then there exists an associated torsion pair $(\mathscr{T}(T), \mathscr{F}(T))$ in $\operatorname{mod} A$, where

$$\mathscr{T}(T) = \{M \in \operatorname{mod} A \mid \mathrm{Ext}^1_A(T, M) = 0\}$$

$$\mathscr{F}(T) = \{M \in \operatorname{mod} A \mid \mathrm{Hom}_A(T, M) = 0\}.$$

Lemma 5.1. *With the above notation*

(a) *$\mathcal{S}_1 \otimes_{C_1} B$ is a coray in $\mathcal{T}$ passing through $P_B(i)$.*
(b) *$\mathrm{Hom}_{C_2}(B, \mathcal{S}_2)$ is a ray in $\mathcal{T}$ passing through $I_B(i)$.*

Proof. Since C_1 is tilted, we have $C_1 = \mathrm{End}_A T$ where T is a tilting module over a hereditary algebra A. As seen in the proof of Theorem 5.1 in [ScSe], we have a commutative diagram

$$\begin{array}{ccc} \mathscr{T}(T) & \xrightarrow{\mathrm{Hom}_A(T,-)} & \mathcal{Y}(T) \\ \big\downarrow & & \big\downarrow {\scriptstyle -\otimes_{C_1} B} \\ \mathcal{C}_A & \xrightarrow{\mathrm{Hom}_{\mathcal{C}_A}(T,-)} & \operatorname{mod} B \end{array}$$

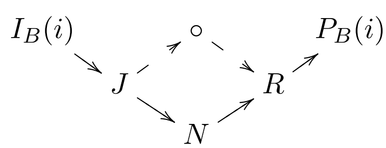
where $\mathcal{Y}(T) = \{N \in \operatorname{mod} C \mid \mathrm{Tor}^C_1(N, T) = 0\}$.

Let $\mathcal{T}_A$ be the tube in $\operatorname{mod} A$ corresponding to the tube $\mathcal{T}$ in $\operatorname{mod} B$. By what has been seen above, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_A \cap \mathscr{T}(T) & \xrightarrow{\operatorname{Hom}_A(T,-)} & \mathcal{T}_1 \\ & \searrow^{\operatorname{Hom}_{\mathcal{C}_A}(T,-)} & \downarrow{\scriptstyle -\otimes_{C_1} B} \\ & & \mathcal{T}_1 \otimes_{C_1} B \subset \mathcal{T} \end{array} .$$

Let $\mathcal{S}$ be any coray in $\mathcal{T}_1$, so it can be lifted to a coray $\mathcal{S}_A$ in $\mathcal{T}_A \cap \mathscr{T}(T)$ via the functor $\operatorname{Hom}_A(T,-)$. If we apply $\operatorname{Hom}_{\mathcal{C}_A}(T,-)$ to this lift, we obtain a coray in $\mathcal{T}_1 \otimes_{C_1} B$. Thus, any coray in $\mathcal{T}_1$ induces a coray in $\mathcal{T}$. Let $\mathcal{S}_1$ be the coray passing through the root projective $P_{C_1}(i)$. Then $\mathcal{S}_1 \otimes_{C_1} B$ is the coray passing through $P_{C_1}(i) \otimes_{C_1} B = P_B(i)$. This proves (a) and part (b) is proved dually.

However, we must still justify that the ray $\mathcal{S}_1 \otimes_{C_1} B$ and the coray $\operatorname{Hom}_{C_2}(B, \mathcal{S}_2)$ actually intersect (and thus lie in the same tube of $\Gamma(\operatorname{mod} B)$). Because $P_{C_1}(i) \in \mathcal{S}_1$, we have $P_{C_1}(i) \otimes B \cong P_B(i) \in \mathcal{S}_1 \otimes_{C_1} B$, and $P_B(i)$ lies in a tube $\mathcal{T}$. It is well-known that the injective $I_B(i)$ also lies in $\mathcal{T}$. In particular, we have the following local configuration in $\mathcal{T}$, where R is an indecomposable summand of the radical of $P_B(i)$ and J an indecomposable summand of the quotient of $I_B(i)$ by its socle.



Now $I_B(i) = \operatorname{Hom}_{C_2}(B, I_C(i))$ is coinduced, and we have shown above that the ray containing it is also coinduced. Because $I_C(i) \in \mathcal{S}_2$, this is the ray $\operatorname{Hom}_{C_2}(B, \mathcal{S}_2)$. Therefore, this ray and this coray lie in the same tube, so must intersect in a module N, where there exists an almost split sequence

$$0 \longrightarrow J \longrightarrow N \longrightarrow R \longrightarrow 0.$$

□

Remark 5.2. Knowing the ray $\operatorname{Hom}_{C_2}(B, \mathcal{S}_2)$ and the coray $\mathcal{S}_1 \otimes_{C_1} B$ for every root projective $P_B(i)$ in $\mathcal{T}$, one may apply the knitting procedure to construct the whole of $\mathcal{T}$. In this way, $\mathcal{T}$ can be determined completely.

Next we show that all modules over a tilted algebra lying on the same coray change in the same way under the induction functor.

Lemma 5.3. *Let A be a hereditary algebra of euclidean type, T a tilting A-module without preinjective summands and let $C = \operatorname{End}_A T$ be the corresponding tilted algebra. Let $\mathcal{T}_A$ be a tube in* $\operatorname{mod} A$ *and $T_i \in \mathcal{T}_A$ an indecomposable summand of T, such that* $\operatorname{pd} I_C(i) = 2$.

Then there exists an A-module M on the mouth of $\mathcal{T}_A$ such that we have

$$\tau_C \Omega_C I_C(i) = \mathrm{Hom}_A(T, M)$$

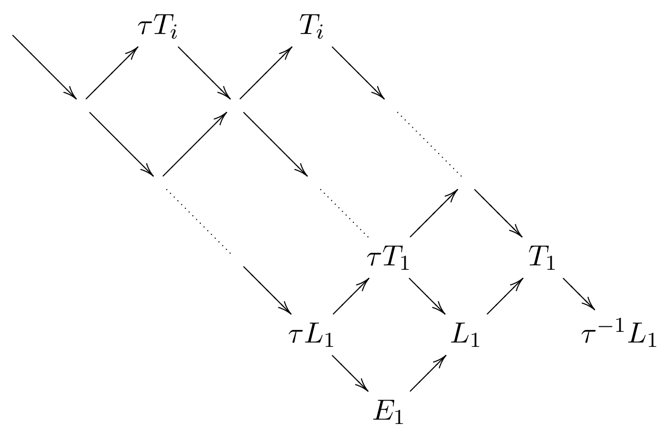
in $\mathrm{mod}\, C$*. In particular, the module $\tau_C \Omega_C I_C(i)$ lies on the mouth of the tube* $\mathrm{Hom}_A(T, \mathcal{T}_A \cap \mathscr{T}(T))$ *in* $\mathrm{mod}\, C$*.*

Proof. The injective C-module $I_C(i)$ is given by

$$I_C(i) \cong \mathrm{Ext}^1_A(T, \tau T_i) \cong D\mathrm{Hom}_A(T_i, T),$$

where the first identity holds by [ASS, Proposition VI 5.8] and the second identity is the Auslander-Reiten formula. Moreover, since T_i lies in the tube $\mathcal{T}_A$ and T has no preinjective summands, we have $\mathrm{Hom}(T_i, T_j) \neq 0$ only if T_j lies in the hammock starting at T_i. Furthermore, if T_j is a summand of T then it must lie on a sectional path starting from T_i because $\mathrm{Ext}^1(T_j, T_i) = 0$. This shows that a point j is in the support of $I_C(i)$ if and only if there is a sectional path $T_i \to \cdots \to T_j$ in $\mathcal{T}_A$. We shall distinguish two cases.

Case 1. If T_i lies on the mouth of $\mathcal{T}_A$ then let ω be the ray starting at T_i and denote by T_1 the last summand of T on this ray. Let L_1 be the direct predecessor of T_1 not on the ray ω. Thus we have the following local configuration in $\mathcal{T}_A$.



Then $I_C(i)$ is uniserial with simple top $S(1)$. Moreover there is a short exact sequence

$$0 \longrightarrow \tau T_i \longrightarrow L_1 \longrightarrow T_1 \longrightarrow 0$$

and applying $\mathrm{Hom}_A(T, -)$ yields

(5.1)

$$0 \longrightarrow \mathrm{Hom}_A(T, L_1) \longrightarrow P_C(1) \xrightarrow{f} I_C(i) \longrightarrow \mathrm{Ext}^1(T, L_1) \longrightarrow 0$$

By the Auslander-Reiten formula, we have $\mathrm{Ext}^1(T, L_1) \cong D\mathrm{Hom}(\tau^{-1} L_1, T)$ and this is zero because T_1 is the last summand of T on the ray ω. Thus the sequence (5.1) is short exact, the morphism f is a projective cover, because $I_C(i)$ is uniserial, and hence

$$\Omega_C I_C(i) \cong \mathrm{Hom}_A(T, L_1).$$

Applying τ_C yields

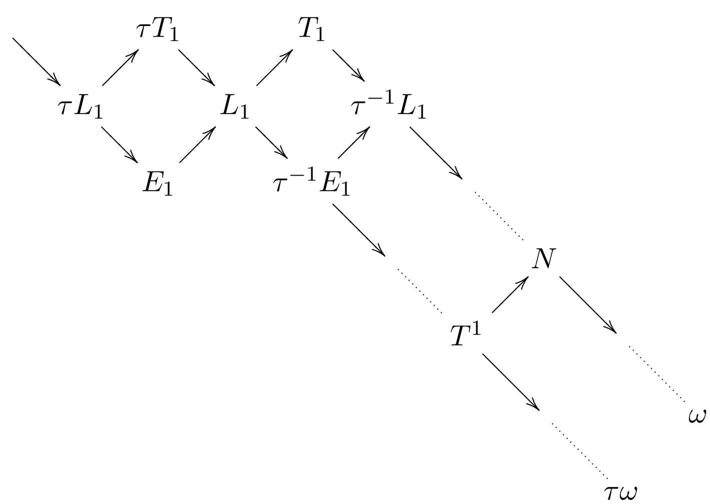
$$\tau_C \Omega_C I_C(i) \cong \tau_C \mathrm{Hom}_A(T, L_1).$$

Let E_1 be the indecomposable direct predecessor of L_1 such that the almost split sequence ending at L_1 is of the form

$$0 \longrightarrow \tau L_1 \longrightarrow E_1 \oplus \tau T_1 \longrightarrow L_1 \longrightarrow 0 \tag{5.2}$$

We claim that $E_1 \in \mathscr{T}(T)$.

Recall that L_1 is not a summand of T because $\Omega_C I_C(i) = \mathrm{Hom}_A(T, L_1)$ is non projective. Also, recall that T_1 is the last summand of T on the ray ω. Suppose $E_1 \notin \mathscr{T}(T)$, thus $0 \neq \mathrm{Ext}^1_A(T, E_1) = D\mathrm{Hom}(\tau^{-1}E_1, T)$. Then it follows that there is a summand of T on the ray $\tau\omega$ that is a successor of $\tau^{-1}E_1$. Let T^1 denote the first such indecomposable summand.



Then we have a short exact sequence

$$0 \longrightarrow L_1 \xrightarrow{\ h\ } T_1 \oplus T^1 \longrightarrow N \longrightarrow 0$$

with h an $\mathrm{add}\, T$-approximation. Applying $\mathrm{Hom}_A(-, T)$ yields

$$0 \longrightarrow \mathrm{Hom}_A(N, T) \longrightarrow \mathrm{Hom}_A(T_1 \oplus T^1, T) \xrightarrow{\ h^*\ } \mathrm{Hom}_A(L_1, T)$$
$$\longrightarrow \mathrm{Ext}^1_A(N, T) \longrightarrow 0$$

and since h is an add T-approximation, the morphism h^* is surjective. Thus $\mathrm{Ext}^1_A(N,T)=0$.

On the other hand, $T_1 \oplus T^1$ generates N, so $N \in \mathrm{Gen}\, T = \mathscr{T}(T)$, and thus $\mathrm{Ext}^1_A(T,N)=0$. But then both $\mathrm{Ext}^1_A(T,N)=\mathrm{Ext}^1_A(N,T)=0$ and we see that N is a summand of T. This is a contradiction to the assumption that T_1 is the last summand of T on the ray ω. Thus $E_1 \in \mathscr{T}(T)$.

Therefore, in the almost split sequence (5.2), we have $L_1, E_1 \in \mathscr{T}(T)$ and $\tau T_1 \in \mathscr{F}(T)$. Moreover, all predecessors of τT_1 on the ray $\tau\omega$ are also in $\mathscr{F}(T)$ because the morphisms on the ray are injective. Since $\mathrm{Hom}_A(T,-):\mathscr{T}(T)\to\mathcal{Y}(T)$ is an equivalence of categories, it follows that $\mathrm{Hom}_A(T,L_1)$ has only one direct predecessor

$$\mathrm{Hom}_A(T,E_1)\to\mathrm{Hom}_A(T,L_1)$$

in mod C and this irreducible morphism is surjective. The kernel of this morphism is $\mathrm{Hom}_A(T,t(\tau_A L_1))$ where t is the torsion radical. Thus we get

$$\tau_C\Omega_C I_C(i)=\tau_C\mathrm{Hom}_A(T,L_1)=\mathrm{Hom}_A(T,t(\tau_A L_1)).$$

We will show that $t(\tau_A L_1)$ lies on the mouth of $\mathcal{T}_A$ and this will complete the proof in case 1.

Let M be the indecomposable A-module on the mouth of $\mathcal{T}_A$ such that the ray starting at M passes through $\tau_A L_1$. Thus M is the starting point of the ray $\tau^2\omega$. Then there is a short exact sequence of the form

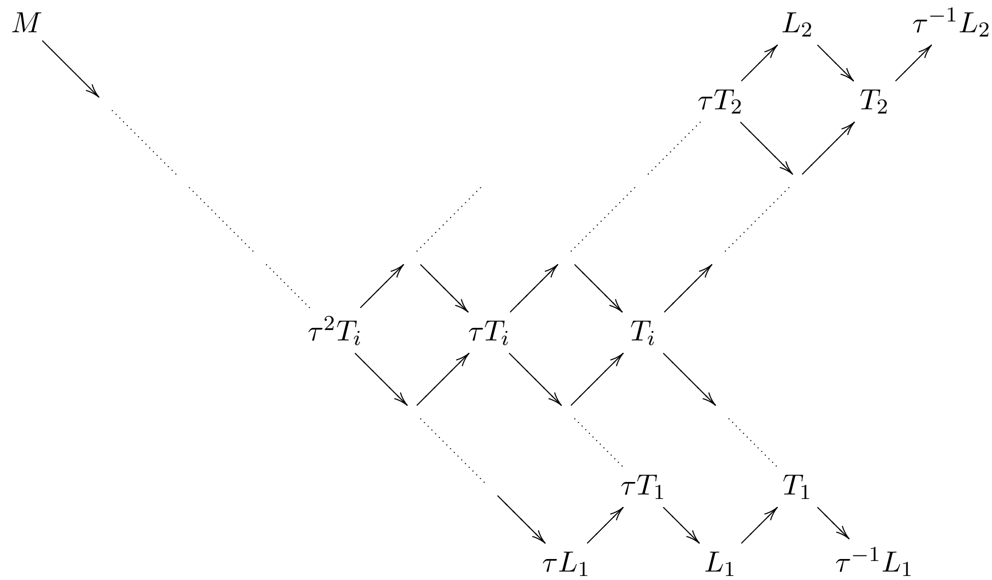
$$0\longrightarrow M\longrightarrow \tau_A L_1\longrightarrow \tau_A T_1\longrightarrow 0 \tag{5.3}$$

with $\tau_A T_1\in\mathscr{F}(T)$. We claim that $M\in\mathscr{T}(T)$.

Suppose to the contrary that $0\neq \mathrm{Ext}^1_A(T,M)=D\mathrm{Hom}_A(\tau^{-1}M,T)$. Since $\tau^{-1}M$ lies on the mouth of $\mathcal{T}_A$, this implies that there is a direct summand T^1 of T which lies on the ray $\tau\omega$ starting at $\tau^{-1}M$. Since T is tilting, T^1 cannot be a predecessor of τT_1 on this ray and since L_1 is not a summand of T, we also have $L_1\neq T^1$. Thus T^1 is a successor of L_1 on the ray $\tau\omega$. This is impossible since such a T^1 would satisfy $\mathrm{Ext}^1_A(T^1,E_1)\neq 0$ contradicting the fact that $E_1\in\mathscr{T}(T)$.

Therefore, $M\in\mathscr{T}(T)$ and the sequence (5.3) is the canonical sequence for $\tau_A L_1$ in the torsion pair $(\mathscr{T}(T),\mathscr{F}(T))$. This shows that $t(\tau_A L_1)=M$ and hence $\tau_C\Omega_C I_C(i)=\mathrm{Hom}_A(T,M)$ as desired.

Case 2. Now suppose that T_i does not lie on the mouth of $\mathcal{T}_A$. Let ω_1 denote the ray passing through T_i and ω_2 the coray passing through T_i. Denote by T_1 the last summand of T on ω_1, by T_2 the last summand of T on ω_2, and by L_j the direct predecessor of T_j which does not lie on ω_j. Note that L_2 does not exist if T_2 lies on the mouth of $\mathcal{T}_A$, and in this case we let $L_2=0$. Thus we have the following local configuration in $\mathcal{T}_A$.



The injective C-module $I_C(i) = \mathrm{Ext}^1_A(T, \tau T_i)$ is biserial with top $S(1) \oplus S(2)$. Moreover, there is a short exact sequence

$$0 \longrightarrow \tau T_i \longrightarrow L_1 \oplus L_2 \oplus T_i \longrightarrow T_1 \oplus T_2 \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(T, -)$ yields the following exact sequence.

(5.4)

$$0 \longrightarrow \mathrm{Hom}_A(T, L_1 \oplus L_2) \oplus P_C(i) \longrightarrow P_C(1) \oplus P_C(2) \xrightarrow{f} I_C(i) \longrightarrow \mathrm{Ext}^1_A(T, L_1 \oplus L_2) \longrightarrow 0.$$

By the same argument as in case 1, using that T_1 and T_2 are the last summands of T on ω_1 and ω_2 respectively, we see that $\mathrm{Ext}^1_A(T, L_1 \oplus L_2) = 0$. Therefore, the sequence (5.4) is short exact. Moreover, the morphism f is a projective cover and thus

$$\Omega_C I_C(i) = \mathrm{Hom}_A(T, L_1 \oplus L_2) \oplus P_C(i).$$

Applying τ_C yields

$$\tau_C \Omega_C I_C(i) = \tau_C \mathrm{Hom}_A(T, L_1) \oplus \tau_C \mathrm{Hom}_A(T, L_2).$$

By the same argument as in case 1 we see that

$$\tau_C \mathrm{Hom}_A(T, L_1) = \mathrm{Hom}_A(T, t(\tau_A L_1)) = \mathrm{Hom}_A(T, M)$$

where M is the indecomposable A-module on the mouth of $\mathcal{T}_A$ such that the ray starting at M passes through τL_1. In other words, M is the starting point of the ray $\tau^2 \omega$.

Therefore, it only remains to show that $\tau_C \mathrm{Hom}_A(T, L_2) = 0$. To do so, it suffices to show that L_2 is a summand of T.

We have already seen that $\mathrm{Ext}^1_A(T, L_2) = 0$. We show now that we also have $\mathrm{Ext}^1_A(L_2, T) = 0$. Suppose the contrary. Then there exists a non-zero morphism $u : T \to \tau_A L_2$. Composing it with the irreducible injective morphism $\tau_A L_2 \to \tau_A T_2$ yields a non-zero morphism in $\mathrm{Hom}_A(T, \tau_A T_2)$. But this is impossible since T is tilting.

Thus we have $\mathrm{Ext}^1_A(T, L_2) = \mathrm{Ext}^1_A(L_2, T) = 0$ and thus L_2 is a summand of T, the module $\mathrm{Hom}_A(T, L_2)$ is projective and $\tau_C \mathrm{Hom}_A(T, L_2) = 0$. This completes the proof. □

Remark 5.4. The module M in the statement of the lemma is the starting point of the ray passing through $\tau^2 T_i$.

Corollary 5.5. *Let $A, T, C, \mathcal{T}_A$ be as in Lemma 5.3, and let $B = C \ltimes E$, with $E = \mathrm{Ext}^2_C(DC, C)$. Let X, Y be two modules lying on the same coray in the tube* $\mathrm{Hom}_A(T, \mathcal{T}_A \cap \mathscr{T}(T))$ *in* $\mathrm{mod}\, C$. *Then $X \otimes_C E \cong Y \otimes_C E$ and thus the two projections $X \otimes_C B \to X \to 0$ and $Y \otimes_C B \to Y \to 0$ have isomorphic kernels.*

Proof. For all C-modules X we have

$$X \otimes_B E \cong D\mathrm{Hom}(X, DE) \cong D\mathrm{Hom}(X, \tau_C \Omega_C DC)$$

where the first isomorphism is [ScSe, Proposition 3.3] and the second is [ScSe, Proposition 4.1]. Since T has no preinjective summands, and X is regular, the only summand of $\tau\Omega DC$ for which $\mathrm{Hom}(X, \tau\Omega DC)$ can be nonzero, must lie in the same tube as X. By the lemma, the only summands of $\tau\Omega DC$ in the tube lie on the mouth of the tube. Let M denote an indecomposable C-module on the mouth of a tube. Then

$$\mathrm{Hom}_C(X, M) \cong \mathrm{Hom}_C(Y, M) \cong \begin{cases} k & \text{if } M \text{ lies on the coray passing} \\ & \quad \text{through } X \text{ and } Y, \\ 0 & \text{otherwise.} \end{cases}$$

□

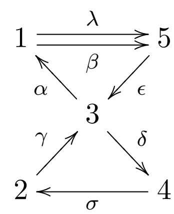
We summarize the results of this section in the following proposition.

Proposition 5.6. (a) *Let $\mathcal{S}_1$ be the coray in $\Gamma(\mathrm{mod}\, C_1)$ passing through the projective C_1-module corresponding to the root projective $P_B(i)$ Then $\mathcal{S}_1 \otimes_{C_1} B$ is a coray in $\Gamma(\mathrm{mod}\, B)$ passing through $P_B(i)$. Furthermore all modules in $\mathcal{S}_1 \otimes_{C_1} B$ are extensions of modules of $\mathcal{S}_1$ by the same module $P_{C_1}(i) \otimes E$.*

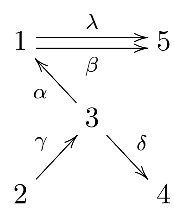
(b) *Let $\mathcal{S}_2$ be the ray in $\Gamma(\mathrm{mod}\, C_2)$ passing through the injective C_2-module corresponding to the root injective $I_B(i)$ Then* $\mathrm{Hom}_{C_2}(B, \mathcal{S}_2)$ *is a ray in $\Gamma(\mathrm{mod}\, B)$ passing through $I_B(i)$. Furthermore all modules in* $\mathrm{Hom}_{C_2}(B, \mathcal{S}_2)$ *are extensions of modules of $\mathcal{S}_2$ by the same module* $\mathrm{Hom}_{C_2}(E, I_{C_2}(i))$.

Proof. (a) The first statement is Lemma 5.1, and the second statement is a restatement of Corollary 5.5. □

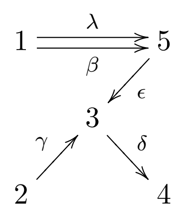
Example 5.7. *Let B be the cluster-tilted algebra given by the quiver*



bound by $\alpha\beta = 0, \beta\epsilon = 0, \epsilon\alpha = 0, \gamma\delta = 0, \sigma\gamma = 0, \delta\sigma = 0$. *The algebras* C_1 *and* C_2 *are respectively given by the quivers*

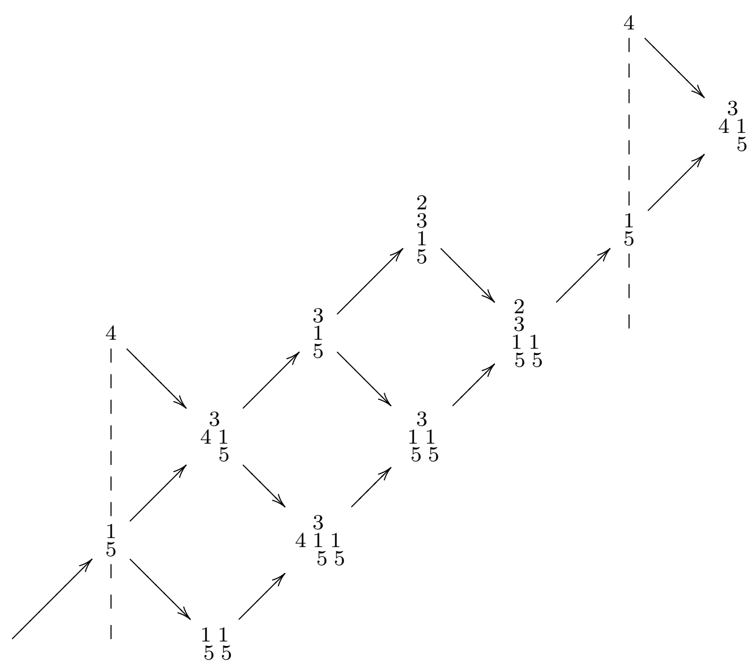


and



with the inherited relations. We can see the tube in $\Gamma(\operatorname{mod} C_1)$ *below and the coray passing through the root projective* $P_{C_1}(3) = \begin{smallmatrix} & 3 \\ 4 & 1 \\ & 5 \end{smallmatrix}$ *is given by*

$$\mathcal{S}_1 : \qquad \cdots \longrightarrow \begin{smallmatrix} 1 \\ 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} & 3 \\ 4 & 1 \\ & 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 \\ 1 \\ 5 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 5 \end{smallmatrix}.$$



Dually, the ray in $\Gamma(\operatorname{mod} C_2)$ *passing through the root injective* $I_{C_2}(3) = \begin{smallmatrix}1&&\\5&2\\&3\end{smallmatrix}$ *is given by*

$$\mathcal{S}_2 : \qquad \begin{smallmatrix}1\\5\\3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\5\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1&\\5&2\\&3\end{smallmatrix} \longrightarrow \begin{smallmatrix}1\\5\end{smallmatrix} \longrightarrow \cdots$$

The root projective $P_B(3)$ *lies on the coray*

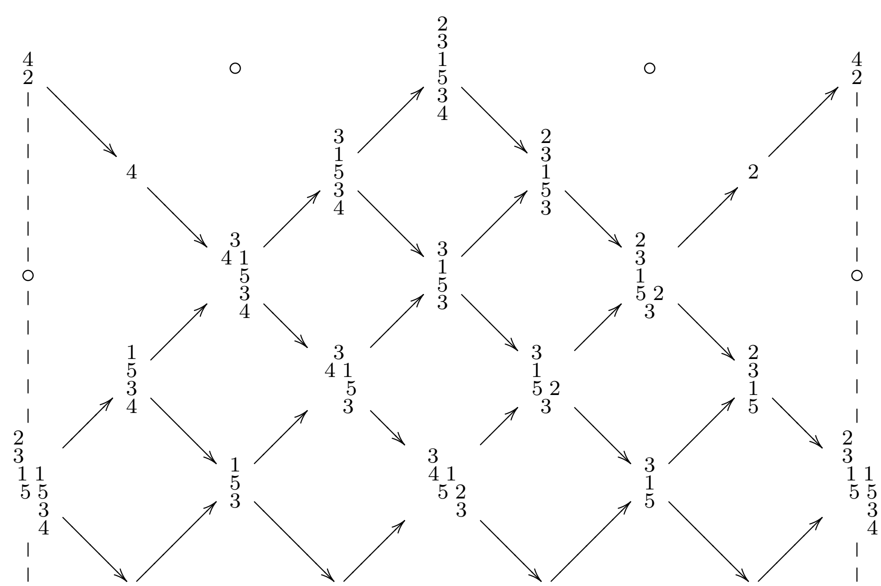
$$\mathcal{S}_1 \otimes_{C_1} B : \qquad \cdots \longrightarrow \begin{smallmatrix}1\\5\\3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}&3\\4&1\\&5\\&3\\&4\end{smallmatrix} \longrightarrow \begin{smallmatrix}3\\1\\5\\3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\5\\3\\4\end{smallmatrix}$$

and the root injective $I_B(3)$ *lies on the ray*

$$\operatorname{Hom}_{C_2}(B, \mathcal{S}_2) : \qquad \begin{smallmatrix}2\\3\\1\\5\\3\\4\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\5\\3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2&\\3&\\1&\\5&2\\&3\end{smallmatrix} \longrightarrow \begin{smallmatrix}2\\3\\1\\5\end{smallmatrix} \longrightarrow \cdots$$

Note that by Proposition 5.6, every module in $\mathcal{S}_1 \otimes_{C_1} B$ *is an extension of a module in* $\mathcal{S}_1$ *by* $\begin{smallmatrix}3\\4\end{smallmatrix}$*. Similarly, every module in* $\operatorname{Hom}_{C_2}(B, \mathcal{S}_2)$ *is an extension of a module in* $\mathcal{S}_2$ *by* $\begin{smallmatrix}2\\3\end{smallmatrix}$*.*

Applying the knitting algorithm we obtain the tube in $\Gamma(\operatorname{mod} B)$ *containing both* $\mathcal{S}_1 \otimes_{C_1} B$ *and* $\operatorname{Hom}_{C_2}(B, \mathcal{S}_2)$*.*

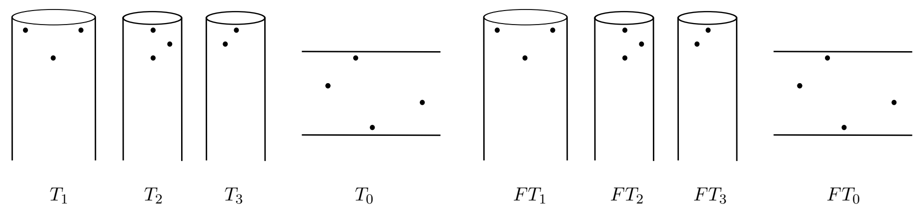


6. From cluster-tilted algebras to quasi-tilted algebras

Let B be cluster-tilted of euclidean type Q and let $A = kQ$. Then there exists $T \in \mathcal{C}_A$ tilting such that $B = \operatorname{End}_{\mathcal{C}_A} T$.

Because Q is euclidean, $\mathcal{C}_A$ contains at most 3 exceptional tubes. Denote by T_0, T_1, T_2, T_3 the direct sums of those summands of T that respectively lie in the transjective component and in the three exceptional tubes.

In the derived category $\mathcal{D}^b(\operatorname{mod} A)$, we can choose a lift of T such that we have the following local configuration.



Let $\mathcal{H}$ be a hereditary category that is derived equivalent to $\operatorname{mod} A$ and such that $\mathcal{H}$ is not the module category of a hereditary algebra. Then $\mathcal{H}$ is of the form $\mathcal{H} = \mathcal{T}^- \vee \mathcal{C} \vee \mathcal{T}^+$, where $\mathcal{T}^-, \mathcal{T}^+$ consist of tubes, and $\mathcal{C}$ is a transjective component, see [LS]. Let T_-, T_+ be the direct sum of all indecomposable summands of T lying in $\mathcal{T}^-$, $\mathcal{T}^+$ respectively. We define two subspaces L and R of B as follows.

$$L = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(F^{-1}T_+, T_0) \quad \text{and} \quad R = \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(T_0, FT_-).$$

The transjective component of $\operatorname{mod} B$ contains a left section Σ_L and a right section Σ_R, see [A]. Thus Σ_L, Σ_R are local slices, Σ_L has no projective predecessors, and Σ_R has no projective successors in the transjective component. Define K to be the two-sided ideal of B generated by $\operatorname{Ann}\Sigma_L \cap \operatorname{Ann}\Sigma_R$ and the two subspaces L and R. Thus

$$K = \langle \operatorname{Ann}\Sigma_L \cap \operatorname{Ann}\Sigma_R, L, R\rangle.$$

We call K the *partition ideal* induced by the partition $\mathcal{T}^- \vee \mathcal{C} \vee \mathcal{T}^+$.

Theorem 6.1. *The algebra $C = B/K$ is quasi-tilted and such that $B = \widetilde{C}$. Moreover C is tilted if and only if $L = 0$ or $R = 0$.*

Proof. We have $B = \operatorname{End}_{\mathcal{C}_A} T = \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} A)}(T, F^iT)$, where the last equality is as k-vector spaces. Using the decomposition $T = T_- \oplus T_0 \oplus T_+$, we see that B is equal to

$$\begin{array}{lccccc} & \operatorname{Hom}_{\mathcal{D}}(T_-,T_-) & \oplus & \operatorname{Hom}_{\mathcal{D}}(T_-,T_0) & \oplus & \operatorname{Hom}_{\mathcal{D}}(T_-,FT_-) \\ \oplus & \operatorname{Hom}_{\mathcal{D}}(T_0,T_0) & \oplus & \operatorname{Hom}_{\mathcal{D}}(T_0,T_+) & \oplus & \operatorname{Hom}_{\mathcal{D}}(T_0,FT_-) \\ \oplus & \operatorname{Hom}_{\mathcal{D}}(T_0,FT_0) & \oplus & \operatorname{Hom}_{\mathcal{D}}(F^{-1}T_+,FT_0) & \oplus & \operatorname{Hom}_{\mathcal{D}}(F^{-1}T_+,T_+) \\ \oplus & \operatorname{Hom}_{\mathcal{D}}(T_+,T_+), & & & & \end{array}$$

where all Hom spaces are taken in $\mathcal{D}^b(\operatorname{mod} A)$. On the other hand,

$$\begin{array}{rcl} \operatorname{End}_{\mathcal{H}} T & = & \operatorname{Hom}_{\mathcal{H}}(T_-,T_-) \oplus \operatorname{Hom}_{\mathcal{H}}(T_-,T_0) \oplus \operatorname{Hom}_{\mathcal{H}}(T_0,T_0) \\ & & \oplus \operatorname{Hom}_{\mathcal{H}}(T_0,T_+) \oplus \operatorname{Hom}_{\mathcal{H}}(T_+,T_+) \end{array}$$

is a quasi-tilted algebra. Thus in order to prove that C is quasi-tilted it suffices to show that K is the ideal generated by

$$\operatorname{Hom}_{\mathcal{D}}(T_-,FT_-) \oplus \operatorname{Hom}_{\mathcal{D}}(T_0,FT_- \oplus FT_0) \oplus \operatorname{Hom}_{\mathcal{D}}(F^{-1}T_+, T_0 \oplus T_+).$$

But this follows from the definition of L and R and the fact that the annihilators of the local slices Σ_L and Σ_R are given by the morphisms in $\mathrm{End}_{\mathcal{C}_A} T$ that factor through the lifts of the corresponding local slice in the cluster category. More precisely,

$$\begin{aligned}\mathrm{Ann}\,\Sigma_L &\cong \mathrm{Hom}_{\mathcal{D}}(F^{-1}T_0 \oplus F^{-1}T_+ \oplus T_- \ ,\ T_0 \oplus T_+ \oplus FT_-),\\ \mathrm{Ann}\,\Sigma_R &\cong \mathrm{Hom}_{\mathcal{D}}(F^{-1}T_+ \oplus T_- \oplus T_0 \ ,\ T_+ \oplus FT_- \oplus FT_0),\end{aligned}$$

and thus

$$\begin{aligned}\mathrm{Ann}\,\Sigma_L \cap \mathrm{Ann}\,\Sigma_R \ \cong\ & \mathrm{Hom}_{\mathcal{D}}(T_0, FT_0) \oplus \mathrm{Hom}_{\mathcal{D}}(T_-, FT_-)\\ & \oplus \mathrm{Hom}_{\mathcal{D}}(F^{-1}T_+, T_+),\end{aligned}$$

where we used the fact that $\mathrm{Hom}_{\mathcal{D}}(T_-, T_+) = \mathrm{Hom}_{\mathcal{D}}(T_+, T_-) = 0$. This completes the proof that C is quasi-tilted.

Since $C = \mathrm{End}_{\mathcal{H}} T$, we have $\widetilde{C} = \mathrm{End}_{\mathcal{C}_{\mathcal{H}}} T \cong \mathrm{End}_{\mathcal{C}_A} T = B$.

Now assume that $R = 0$. Then $T_- = 0$ and thus K is generated by $(\mathrm{Ann}\,\Sigma_L \cap \mathrm{Ann}\,\Sigma_R) \oplus L$, and this is equal to

$$\mathrm{Hom}_{\mathcal{D}}(T_0, FT_0) \oplus \mathrm{Hom}_{\mathcal{D}}(F^{-1}T_+, T_+) \oplus \mathrm{Hom}_{\mathcal{D}}(F^{-1}T_+, FT_0). \tag{6.1}$$

On the other hand, $T_- = 0$ implies that

$$\mathrm{Ann}\,\Sigma_L = \mathrm{Hom}_{\mathcal{D}}(F^{-1}T_0 \oplus F^{-1}T_+, T_0 \oplus T_+),$$

and since $\mathrm{Hom}_{\mathcal{D}}(F^{-1}T_0, T_+) = 0$, this implies that $K = \mathrm{Ann}\,\Sigma_L$ is the annihilator of a local slice. Therefore $C = B/K$ is tilted by [ABS2]. The case where $L = 0$ is proved in a similar way.

Conversely, assume C is tilted. Then $K = \mathrm{Ann}\,\Sigma'$ for some local slice Σ' in $\mathrm{mod}\,B$. We show that $K = \mathrm{Ann}\,\Sigma_L$ or $K = \mathrm{Ann}\,\Sigma_R$. Suppose to the contrary that Σ' has both a predecessor and a successor in $\mathrm{add}\,T_0$. Then there exists an arrow α in the quiver of B such that $\alpha \in \mathrm{Hom}_{\mathcal{D}}(T_0, T_0)$ and $\alpha \in \mathrm{Ann}\,\Sigma' = K$. But by definition of Σ_L, Σ_R, L and R, we see that this is impossible.

Thus $K = \mathrm{Ann}\,\Sigma_L$ or $K = \mathrm{Ann}\,\Sigma_R$. In the former case, we have $R = 0$, by the computation (6.1), and in the latter case, we have $L = 0$. □

Theorem 6.2. *If C is quasi-tilted of euclidean type and $B = \widetilde{C}$ then*

$$C = B/\mathrm{Ann}(\Sigma^- \oplus \Sigma^+),$$

where Σ^- is a right section in the postprojective component of C and Σ^+ is a left section in the preinjective component.

Proof. C being quasi-tilted implies that there is a hereditary category $\mathcal{H}$ with a tilting object T such that $C = \mathrm{End}_{\mathcal{H}} T$. Moreover, $B = \mathrm{End}_{\mathcal{C}_{\mathcal{H}}} T$ is the corresponding cluster-tilted algebra. As before we use the decomposition $T = T_- \oplus T_0 \oplus T_+$. Then the algebras

$$C^- = \mathrm{End}_{\mathcal{H}}(T_- \oplus T_0) \quad \text{and} \quad C^+ = \mathrm{End}_{\mathcal{H}}(T_0 \oplus T_+)$$

are tilted. Let Σ^- and Σ^+ be complete slices in $\mathrm{mod}\,C^-$ and $\mathrm{mod}\,C^+$ respectively. Note that Σ^- lies in the postprojective component and Σ^+ lies in the preinjective component of their respective module categories.

Then C is a branch extension of C^- by the module

$$M^+ = \mathrm{Hom}_{\mathcal{H}}(T_+, T_+) \oplus \mathrm{Hom}_{\mathcal{H}}(T_0, T_+).$$

Similarly C is a branch coextension of C^+ by the module

$$M^- = \mathrm{Hom}_{\mathcal{H}}(T_-, T_-) \oplus \mathrm{Hom}_{\mathcal{H}}(T_-, T_0).$$

Observe that the postprojective component of C^- does not change under the branch extension, and the preinjective component of C^+ does not change under the branch coextension. Therefore Σ^- is a right section in the postprojective component of C and Σ^+ is a left section in the preinjective component. Moreover, by construction, we have

$$\mathrm{Ann}_B \Sigma^- = M^+ \oplus \mathrm{Ext}^2_C(DC, C) \quad \text{and} \quad \mathrm{Ann}_B \Sigma^+ = M^- \oplus \mathrm{Ext}^2_C(DC, C),$$

and therefore

$$\mathrm{Ann}_B(\Sigma^- \oplus \Sigma^+) = \mathrm{Ann}_B \Sigma^- \cap \mathrm{Ann}_B \Sigma^+ = \mathrm{Ext}^2_C(DC, C).$$

This completes the proof. □

The main theorem of this section is the following.

Theorem 6.3. *Let C be a quasi-tilted algebra whose relation-extension B is cluster-tilted of euclidean type. Then C is one of the following.*

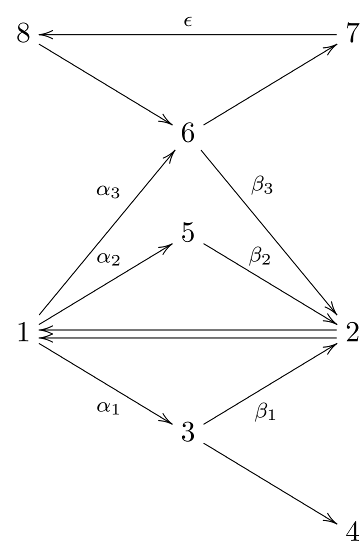
(a) *$C = B/\mathrm{Ann}\,\Sigma$ for some local slice Σ in $\Gamma(\mathrm{mod}\, B)$.*
(b) *$C = B/K$ for some partition ideal K.*

Proof. Assume first that C is tilted. Then, because of [ABS2], there exists a local slice Σ in the transjective component of $\Gamma(\mathrm{mod}\, B)$ such that $B/\mathrm{Ann}\,\Sigma = C$. Otherwise, assume that C is quasi-tilted but not tilted. Then, because of [LS], there exists a hereditary category $\mathcal{H}$ of the form

$$\mathcal{H} = \mathcal{T}^- \vee \mathcal{C} \vee \mathcal{T}^+$$

and a tilting object T in $\mathcal{H}$ such that $C = \mathrm{End}_{\mathcal{H}} T$. Because of Theorem 6.1 we get $C = B/K$ where K is the partition ideal induced by the given partition of $\mathcal{H}$. □

Example 6.4. *Let B be the cluster-tilted algebra of type $\widetilde{\mathbb{E}}_7$ given by the quiver*



As usual let T_i denote the indecomposable summand of T corresponding to the vertex i of the quiver. In this example T has two transjective summands T_1, T_2, and the other summands lie in three different tubes. T_3, T_4 lie in a tube $\mathcal{T}_1$, T_5 lies in a tube $\mathcal{T}_2$ and T_6, T_7, T_8 lie in a tube $\mathcal{T}_3$.

Choosing a partition ideal corresponds to choosing a subset of tubes to be predecessors of the transjective component. Thus there are 8 different partition ideals corresponding to the 8 subsets of $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$. If the tube $\mathcal{T}_i$ is chosen to be a predecessor of the transjective component, then the arrow β_i is in the partition ideal. And if $\mathcal{T}_i$ is not chosen to be a predecessor of the transjective component, then it is a successor and consequently the arrow α_i is in the partition ideal. The arrow ϵ is always in the partition ideal since it corresponds to a morphim from T_8 to FT_7 in the derived category.

Sumarizing, the 8 partition ideals K are the ideals generated by the following sets of arrows.

$$\{\alpha_i, \beta_j, \epsilon \mid i \notin I, j \in I, I \subset \{1, 2, 3\}\}.$$

The quiver of the corresponding quasi-tilted algebra B/K is obtained by removing the generating arrows from the quiver of B. Exactly 2 of these 8 algebras are tilted, and these correspond to cutting $\alpha_1, \alpha_2, \alpha_3, \epsilon$, respectively $\beta_1, \beta_2, \beta_3, \epsilon$.

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