Algebras determined by their left and right parts

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Abstract. We discuss the known generalisations of the classes of tilted and quasi-tilted algebras such as the left and right supported algebras, the laura algebras, the left and right glued algebras, the weakly shod and the shod algebras. We present their characterisations and main properties, underlining the role of the left part $\mathcal{L}_A$ and the right part $\mathcal{R}_A$ of the category of finitely generated modules over an artin algebra A.

Introduction

The objective of the representation theory of artin algebras is to characterise an algebra by properties of its module category. For this purpose, numerical invariants, such as the projective and the injective dimensions of a module, are especially useful. As a first step, one wishes to study modules of projective dimension at most one (and dually, those of injective dimension at most one). However, it is easily seen in elementary examples that such modules occur in a scattered fashion inside the module category. To overcome this difficulty, Happel, Reiten and Smalø have considered in **[42]** the so-called left and right parts of the module category. Let A be an artin algebra, and mod$A$ denote the category of finitely generated right A-modules, then the left part $\mathcal{L}_A$ is the full subcategory of mod$A$ consisting of all indecomposable A-modules whose predecessors have projective dimension at most one. The right part $\mathcal{R}_A$ is defined dually. These classes were used successfully in **[42]** to study the quasi-tilted algebras. Since then, many generalisations of the quasi-tilted algebras were introduced and studied over the years, such as the shod, weakly shod, laura or left (and right) supported algebras. In the study of all these classes, the left and the right parts of the module category have played an essential role.

The object of these notes is to present these classes of algebras, their existing characterisations and main properties, underlining the use of the left and right parts. In order to convey the flavour of the subject, we have tried to present proofs or sketches of proofs for several of the results stated here. We have also inserted many examples.

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This paper is organised as follows. After a preliminary section devoted to fixing the notations and recalling the basic facts needed in this survey, we recall the main facts on the left part in our section two : their elementary properties, the Ext-projective and Ext-injective modules, the case where its additive closure is an abelian exact subcategory and the left (and right) supported algebras. Section 3 is devoted to the Auslander-Reiten components which occur in these classes : it is seen there that properties of the left and the right parts are reflected in properties of certain paths of the module category, an observation that proves crucial in the sequel. Sections 4, 5 and 6 are respectively devoted to laura algebras of which the left (and right) glued algebras are special cases, weakly shod and shod algebras. In our last section 7, we describe three constructions which preserve these properties, namely passing to full subcategories, to subalgebras such that the original algebra becomes a split extension of the latter, and finally to skew group algebras.

1. Notations and preliminaries

1.1. Algebras. Throughout this survey, all algebras are basic and connected artin algebras. We sometimes consider an algebra A as a category, of which the object class A_0 is a complete set of primitive orthogonal idempotents, and the group of morphisms from e_i to e_j is e_iAe_j. An algebra B is a **full subcategory** of another algebra A if there exists an idempotent $e \in A$ such that $B = eAe$. It is **convex** in A if, whenever there exists a sequence $e_{i_0}, e_{i_1}, \dots, e_{i_t}$ of primitive orthogonal idempotents such that $e_{i_{l+1}}Ae_{i_l} \neq 0$ for $0 \leq l < t$ and $ee_{i_0} = e_{i_0}$, $ee_{i_t} = e_{i_t}$, then $ee_{i_l} = e_{i_l}$ for each l. Finally, A is **triangular** if there exists no sequence of primitive orthogonal idempotents $e_{i_0}, e_{i_1}, \dots, e_{i_t} = e_{i_0}$ such that $e_{i_{l+1}}Ae_{i_l} \neq 0$ for $0 \leq l < t$.

1.2. Modules. For an algebra A, we denote by mod$A$ the category of all finitely generated right A-modules, and by ind$A$ a full subcategory of mod$A$ containing exactly one representative from each isomorphism class of indecomposable A-modules. We say that a full subcategory $\mathcal{C} \subseteq$ ind$A$ is **finite** if it has only finitely many indecomposable objects and that it is **cofinite** if ind$A \setminus \mathcal{C}$ is finite. We sometimes writes $M \in \mathcal{C}$ to express that M is an object in $\mathcal{C}$. Further, we denote by add$\mathcal{C}$ the full subcategory of mod$A$ having as objects the direct sums of indecomposable summands of objects in $\mathcal{C}$, and if M is a module, we abbreviate add$\{M\}$ as add$M$. Given an algebra A, we denote by $K_0(A)$ the Grothendieck group of A. We denote the projective (or injective) dimension of a module M as pd$M$ (or id$M$, respectively). The global dimension of an algebra A is denoted by gl.dim$A$.

Given $M, N \in$ ind$A$, we write $M \rightsquigarrow N$ in case there exists a path

$$(*) \qquad M = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{t-1}} X_{t-1} \xrightarrow{f_t} X_t = N$$

$(t \geq 0)$, from M to N in ind$A$, that is, the f_i are non-zero morphisms and the X_i are indecomposable modules. In this case, we say that M is a **predecessor** of N and N is a **successor** of M. Thus each indecomposable module is a predecessor and a successor of itself. A path in ind$A$ starting and ending at the same module and involving at least one non-isomorphism is a **cycle**. An indecomposable module M which lies on no cycle in ind$A$ is a **directed** module. When each f_i in the path $(*)$ is an irreducible morphism, we say that $(*)$ is a **path of irreducible morphisms**.

For an algebra A, we denote by $\Gamma(\text{mod}A)$ its Auslander-Reiten quiver, and by τ_A the Auslander-Reiten translation DTr. Given a module M, we say that M is **left** (or **right**) **stable** if $\tau_A^n M \neq 0$, for any integer $n \geq 0$ (or $n \leq 0$, respectively). Given a connected component Γ of $\Gamma(\text{mod}A)$, we say that Γ is **convex** if, for any path $(*) : M \rightsquigarrow N$ in ind$A$ as above, where $M, N \in \Gamma$, all the X_i in $(*)$ belong to Γ. A path $(*)$ of irreducible morphisms is **sectional** if $\tau_A(X_{i+1}) \neq X_{i-1}$, for all i such that $1 < i \leq t$. In case there exists i_0 such that $\tau_A(X_{i_0+1}) = X_{i_0-1}$, the triple $(X_{i_0-1}, X_{i_0}, X_{i_0+1})$ is a **hook**. A **refinement** of $(*)$ is a path

$$(*) \qquad M = X'_0 \xrightarrow{f'_1} X'_1 \xrightarrow{f'_2} \cdots \xrightarrow{f'_{s-1}} X'_{s-1} \xrightarrow{f'_s} X'_s = N$$

with $s \geq t$ such that there exists an order-preserving function $\sigma : \{1, \cdots, t-1\} \to \{1, \cdots, s-1\}$ such that $X_i \cong X'_{\sigma(i)}$, for all i with $1 \leq i \leq t-1$.

For further definitions or facts needed on the module category, we refer the reader to [**19, 58**].

1.3. Tilted and quasi-tilted algebras. For tilting theory, we refer to [**3, 43, 58**]. We recall that, if T is a tilting module over an hereditary algebra H, then its endomorphism algebra $A = \text{End}\ T_H$ is said to be **tilted**. Tilted algebras play a crucial role in the representation theory of artin algebras. It is especially significant for us to recall that, if A is tilted, then either it is concealed, or else its Auslander-Reiten quiver $\Gamma(\text{mod}A)$ has a unique so-called **connecting component** Γ which is faithful and such that every other component either maps to Γ or receives maps from Γ (see [**47, 51**]). The quasi-tilted algebras [**42**] generalise the tilted algebras. We recall that, if T is a tilting object in a connected locally finite hereditary abelian category $\mathcal{H}$, then its endomorphism algebra $A = \text{End}\ T_{\mathcal{H}}$ is said to be **quasi-tilted**. An algebra A is quasi-tilted if and only if gl.dim$A \leq 2$ and, for every $M \in$ ind$A$, we have pd$M \leq 1$ or id$M \leq 1$ (see [**42**, (II.2.3)]). One important property of quasi-tilted algebras is the existence of a trisection of the module category, see [**42**, (II.1.7)].

2. The left and the right parts of a module category

2.1. General properties of $\mathcal{L}_A$ and $\mathcal{R}_A$. Let A be an algebra. Following [**42**], we let $\mathcal{L}_A$ denote the full subcategory of ind$A$ having as objects those modules X such that, for any predecessor Y of X, the projective dimension pd$_A Y$ of Y does not exceed one. The class $\mathcal{L}_A$ is called the **left part** of mod$A$. Dually, the **right part** $\mathcal{R}_A$ is the full subcategory of ind$A$ having as objects those modules X such that, for any successor Z of X, the injective dimension id$_A Z$ of Z does not exceed one.

For the sake of brevity, we refrain from now on stating the dual of each statement and leave the primal-dual translation to the reader.

Remark. Since, clearly, $\mathcal{L}_A$ is closed under predecessors, then $\mathcal{L}_A$ is the torsion-free class of a split torsion pair (add(ind$A \setminus \mathcal{L}_A$),add$\mathcal{L}_A$).

The definition of the left part does not say how to verify in practice whether a given indecomposable module X belongs to $\mathcal{L}_A$ or not. The following result says that instead of checking *all* predecessors of X, it suffices to look at the "immediate" predecessors.

Theorem 2.1.1. [**10**] *Let A be an artin algebra, and X be an indecomposable A-module. Then $X \in \mathcal{L}_A$ if and only if, for every indecomposable module M such that $pd_A M \geq 2$, we have* $\mathrm{Hom}_A(M, X) = 0$. □

The proof is heavily inspired from the proof of [**42**, (II.1.5)] (done under the assumption that A is quasi-tilted : this assumption turns out to be unnecessary).

The following results give a finer description of the left part and some of its properties.

Proposition 2.1.2. [**6**, (1.6)] *Let A be an artin algebra. Then $\mathcal{L}_A$ consists of the modules $N \in \mathrm{ind}A$, such that, if there exists a path from an indecomposable injective module to N, then this path can be refined to a path of irreducible morphisms, and any such refinement is sectional.*

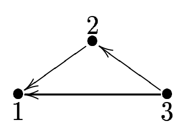
Proof (Sketch). Assume M, N are modules such that $M \rightsquigarrow N$ and $\mathrm{pd}_A M \geq 2$, then it is easy to construct a path $I \to \tau_A M \to * \to M \rightsquigarrow N$ not refinable to a sectional path. For the converse, we show that, if I is indecomposable injective, then there are only finitely many $N \in \mathcal{L}_A$ with a path $I \rightsquigarrow N$. Moreover, any such path is refinable to a path of irreducible morphisms, and any such refinement is sectional. □

Proposition 2.1.3. [**8**, (1.5)] *Let A be an artin algebra, and Γ be a connected component of $\Gamma(\mathrm{mod}A)$. If Γ contains injective modules, then $\mathcal{L}_A \cap \Gamma$ contains no module lying on a cycle of Γ.*

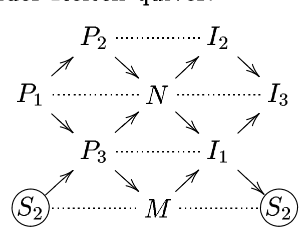
Proof (Sketch). If there exists a module $M \in \mathcal{L}_A \cap \Gamma$, lying on a cycle $(*)$ of Γ, then using [**6**, (1.4)], we construct a path of irreducible morphisms from an injective in Γ to M and, by using (2.1.2) above, we construct a sectional path containing the cycle $(*)$ as a subpath, a contradiction to [**22, 24**]. □

Corollary 2.1.4. [**8**, (1.6)] *Let A be a representation-finite artin algebra. Then $\mathcal{L}_A$ is directed.* □

Example 2.1.5. Let k be a field and A be the the radical square zero k-algebra given by the quiver



We get the following Auslander-Reiten quiver:



where we identify the two copies of S_2. Here (and in the sequel), we denote by P_x (or I_x, or S_x, or e_x) the indecomposable projective (or injective, or simple, or primitive idempotent, respectively) corresponding to the point x of the quiver. It is easy to check that $\mathcal{L}_A = \{P_1, P_2\}$, that $\mathcal{R}_A = \{I_2, I_3\}$ and that the directed modules are precisely those of $\mathcal{L}_A \cup \mathcal{R}_A$.

2.2. Ext-projective and Ext-injective modules. We recall that, if $\mathcal{C}$ is an additive full subcategory of mod$A$, closed under extensions, then an indecomposable $M \in \mathcal{C}$ is called **Ext-projective** (or **Ext-injective**) in $\mathcal{C}$ if $\mathrm{Ext}^1_A(M,-)|_{\mathcal{C}} = 0$ (or $\mathrm{Ext}^1_A(-,M)|_{\mathcal{C}} = 0$, respectively). It is shown in [**21**, (3.4)] that if $\mathcal{C}$ is a torsion-free class, then M is Ext-injective in $\mathcal{C}$ if and only if $\tau_A^{-1}M$ is a torsion object.

Roughly speaking, the Ext-projectives lie at the beginning of the subcategory $\mathcal{C}$, and the Ext-injectives at the end. Therefore, characterising them gives "bounds" for the subcategory.

We define the following subclasses of $\mathcal{L}_A$, see [**8**, (3.1)].

DEFINITION 2.2.1. *Let*

(a) $\mathcal{E}_1 = \{\ M \in \mathcal{L}_A \mid$ *there exists an injective* $I \in \mathrm{ind}A$*, which is a predecessor of* $M\}$.
(b) $\mathcal{E}_2 = \{\ M \in \mathcal{L}_A \setminus \mathcal{E}_1 \mid$ *there exists a projective* $P \in \mathrm{ind}A \setminus \mathcal{L}_A$ *and a sectional path from* P *to* $\tau_A^{-1}M\}$.
(c) $\mathcal{E}_A = \mathcal{E}_1 \cup \mathcal{E}_2$.

We write $\mathcal{E} = \mathcal{E}_A$ when there is no place for ambiguity.

The next theorem gives a complete characterisation of the Ext-projective and the Ext-injective modules in $\mathcal{L}_A$.

THEOREM 2.2.2. [**8**, (3.1)] *Let A be an artin algebra, and M be an indecomposable A-module. Then:*

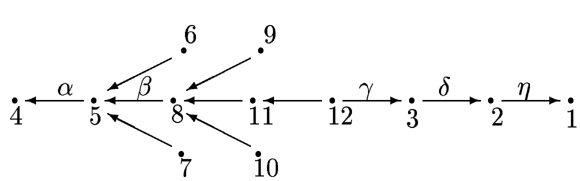
(a) *M is Ext-injective in* add$\mathcal{L}_A$ *if and only if* $M \in \mathcal{E}$;
(b) *M is Ext-projective in* add$\mathcal{L}_A$ *if and only if M is projective and lies in* $\mathcal{L}_A$. □

Since $\mathcal{L}_A$ is closed under predecessors, the Ext-injectives are evidently more useful than the Ext-projectives. In fact, their description can be made more precise with the following result.

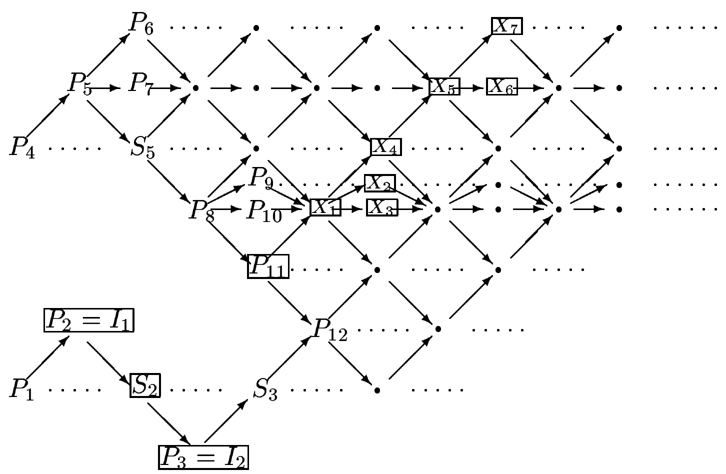
PROPOSITION 2.2.3. [**8**, (3.4)] *Assume that $M \in \mathcal{E}$ and that there exists a path $M \rightsquigarrow N$, with $N \in \mathcal{L}_A$. Then this path can be refined to a sectional path of irreducible morphisms and $N \in \mathcal{E}$. In particular, $\mathcal{E}$ is convex in* mod$A$. □

EXAMPLE 2.2.4.

Let $A = kQ/I$ be given by the quiver:



bound by $\beta\alpha = 0$, $\gamma\delta = 0$ and $\delta\eta = 0$. Then A has a unique postprojective component :



and $\mathcal{E}$ consists of the indicated modules.

2.3. The left support algebra. In order to study the objects in $\mathcal{L}_A$, it is useful to consider the following full subcategory of A.

Definition 2.3.1. [**8, 64**] *Let A be an artin algebra, and P denote the direct sum of a full set of indecomposable projectives in $\mathcal{L}_A$. The algebra A_λ =End$P$ is called the* **left support of** *A.*

It is clear that any indecomposable in $\mathcal{L}_A$ has a canonical A_λ-module structure: if $X \in \mathcal{L}_A$ and P_x is an indecomposable projective such that $\mathrm{Hom}_A(P_x, X) \neq 0$, then $P_x \in \mathcal{L}_A$.

Proposition 2.3.2. [**8**, (2.3)], [**64**, (3.1)] *Let A be an artin algebra. Then A_λ is a direct product of connected quasi-tilted algebras.*

Proof (Sketch). It is easily seen that A can be written in the form

$$A = \begin{bmatrix} A_\lambda & 0 \\ M & B \end{bmatrix}$$

and, for every $x \in B_0$, we have $P_x \notin \mathcal{L}_A$. Now, one can show that, in this situation, $\mathcal{L}_A \subseteq \mathcal{L}_{A_\lambda}$. The result then follows from [**42**, (II.1.14)]. □

As an easy consequence of this proposition, and of an obvious induction, one can show that, if A is a triangular artin algebra, then there exists a filtration of A by full subcategories $A = A_t \supset \ldots \supset A_1 \supset A_0 = A_\lambda$ and A_i-modules M_i such that $A_{i+1} = A_i[M_i]$ and $M_i \notin \mathrm{add}\mathcal{L}_{A_i+1}$. Moreover, $\mathcal{L}_A \subseteq \mathcal{L}_{A_{t-1}} \subseteq \ldots \subseteq \mathcal{L}_{A_0} = \mathcal{L}_{A_\lambda}$.

2.4. The case where add$\mathcal{L}_A$ is abelian. We consider more generally a full subcategory $\mathcal{C}$ of ind$A$, closed under predecessors, and ask when is add$\mathcal{C}$ an abelian subcategory of mod$A$ such that the embedding add$\mathcal{C} \hookrightarrow$ mod$A$ is exact (we then say simply that add$\mathcal{C}$ is an abelian exact subcategory of mod$A$).

It is easy to see that add$\mathcal{C}$ is an abelian exact subcategory of mod$A$ if and only if it is closed under cokernels, or if and only if it is closed under composition factors. We have the following theorem.

Theorem 2.4.1. **[15]** *Let A be an artin algebra, and $\mathcal{C}$ be a full subcategory of* ind$A$. *The following statements are equivalent:*

(a) *$\mathcal{C}$ is closed under predecessors and* add$\mathcal{C}$ *is an abelian exact subcategory of* mod$A$;
(b) *There exists an isomorphism* $A \cong \begin{bmatrix} C & 0 \\ M & B \end{bmatrix}$ *such that M_C is an hereditary injective C-module, and* add$\mathcal{C} \cong$ mod$C$.

If this is the case, then $\text{gl.dim.}C = \sup\{\text{pd}_A X \mid X \in \mathcal{C}\}$. □

One can show that, if $A = \begin{bmatrix} C & 0 \\ M & B \end{bmatrix}$ is as in (b), and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then any indecomposable A-module not in $\mathcal{C}$ is generated by eA.

Taking $\mathcal{C} = \mathcal{L}_A$, it follows directly from the above theorem that, if its equivalent conditions are satisfied, then the left support A_λ is an hereditary algebra.

Corollary 2.4.2. *Let A be a triangular algebra such that* add$\mathcal{L}_A$ *is an abelian exact subcategory of* mod$A$, *then $A = A_\lambda$ (and, in particular, is hereditary).*

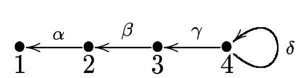
Proof. If this is not the case, there exists an indecomposable projective P_x which is not an A_λ-module. Therefore, radP_x has at least one indecomposable summand not lying in $\mathcal{L}_A$. Consequently, there exists $x_1 \notin (Q_{A_\lambda})_0$ such that $\text{Hom}_A(P_{x_1}, P_x) \neq 0$. As before, rad$P_{x_1}$ has at least one indecomposable summand not lying in $\mathcal{L}_A$. Inductively, we get a sequence of non-zero non-isomorphisms between projectives $\cdots \to P_{x_2} \to P_{x_1} \to P_x$. Since there are only finitely many indecomposable projectives, this yields a contradiction to the triangularity of A. □

Another application of this theorem is in the case of local extensions of hereditary algebras, see **[53]**. An algebra A is called a **local extension of an hereditary algebra** H if $A \cong \begin{bmatrix} H & 0 \\ M & R \end{bmatrix}$ where R is a local algebra and ${}_R M_H$ is a bimodule.

Corollary 2.4.3. *Let* $A = \begin{bmatrix} H & 0 \\ M & R \end{bmatrix}$ *be a local extension of an hereditary algebra H, where R is not a skew field, then* add$\mathcal{L}_A$ *is an abelian exact subcategory of* mod$A$ *if and only if M_H is injective.* □

Examples 2.4.4.

(a) Let k be a field, and A be the k-algebra given by the quiver



bound by $\delta^2 = 0$ and $\delta\gamma\beta = 0$. Then A is the local extension of the hereditary algebra H given by the quiver

$$\underset{1}{\bullet} \xleftarrow{\ \alpha\ } \underset{2}{\bullet} \xleftarrow{\ \beta\ } \underset{3}{\bullet}$$

by the injective module $I_1 \oplus I_2$. In this case, $\mathrm{add}\mathcal{L}_A \cong \mathrm{mod}H$ is an abelian exact subcategory of $\mathrm{mod}A$.

(b) Let A be as in (2.1.5). Then $\mathrm{add}\mathcal{L}_A$ is not an abelian exact subcategory of $\mathrm{mod}A$, because it does not contain the cokernel S_2 of the monomorphism $P_1 \to P_2$.

2.5. Left supported algebras. We now study the situation where an arbitrary A-module can be (left or right) approximated by a module in $\mathrm{add}\mathcal{L}_A$, in the sense of [**18**]. We recall from [**20**] that a full subcategory $\mathcal{C}$ of $\mathrm{mod}A$ is called **contravariantly finite** if, for any A-module M, there exists a morphism $f_\mathcal{C} : M_\mathcal{C} \to M$ such that $M_\mathcal{C} \in \mathcal{C}$ and, if $f : N \to M$ is any morphism with $N \in \mathcal{C}$, then there exists $g : N \to M_\mathcal{C}$ such that $f = f_\mathcal{C} g$. The dual notion is that of a **covariantly finite** subcategory. Since $\mathcal{L}_A$ is closed under predecessors, then $\mathrm{add}\mathcal{L}_A$ is trivially covariantly finite. This leads us to the following definition:

DEFINITION 2.5.1. [**8**] *An artin algebra A is called* **left supported** *provided the class* $\mathrm{add}\mathcal{L}_A$ *is contravariantly finite in* $\mathrm{mod}A$.

We define dually **right supported algebras**.

We need some notation. We denote by E the direct sum of all indecomposable A-modules lying in $\mathcal{E}$ (see (2.2)) and by F the direct sum of a full set of representatives of the isomorphism classes of indecomposable projective A-modules not lying in $\mathcal{L}_A$. Finally, set $T = E \oplus F$. It follows easily from (2.2.2) that T is a partial tilting module. It turns out that it is a tilting module if and only if the algebra is left supported.

THEOREM 2.5.2. [**8**, (4.2)(5.1)] *The following are equivalent for an algebra A:*

(a) *A is left supported;*
(b) $\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}E$*;*
(c) *$T = E \oplus F$ is a tilting module;*
(d) *Each connected component of A_λ is a tilted algebra, and the restriction to this component of E is a slice module.*

Proof (Sketch). (a) implies (d). This is done by checking that the Liu-Skowroński criterion is satisfied (see [**52, 63**]).
(d) implies (b). We may, without loss of generality, assume that A_λ is connected. First observe that, since A_λ admits the restriction of E as slice module, then any indecomposable A_λ-module is either a predecessor or a successor of $\mathcal{E}$. Moreover, the class of predecessors of $\mathcal{E}$ in $\mathrm{mod}A_\lambda$ coincides with the class of A_λ-modules cogenerated by E. This, together with (2.2.3), imply the statement.
(b) implies (c). Since T is a partial tilting module, it remains to show that the number of isomorphism classes of summands of E equals the number of isomorphism classes of indecomposable projective A-module lying in $\mathcal{L}_A$ that is, by (2.2.2), the number of isomorphism classes of of indecomposable Ext-projectives in $\mathcal{L}_A$. But, as $\mathrm{add}\mathcal{L}_A = \mathrm{Cogen}E$ by assumption, it follows from [**21**, (A.6)] that the latter equals

the number of isomorphism classes of indecomposable Ext-injectives in $\mathrm{add}\mathcal{L}_A$. The result then follows from (2.2.2).
(c) implies (a). It is not hard to prove that the assumption implies that the objects in $\mathcal{L}_A$ either lie in $\mathcal{E}$, or in the torsion-free class $\mathcal{F}(T)$ induced by the tilting module T. Moreover, $\mathcal{F}(T)$ is contravariantly finite by [**68**]. Therefore so is $\mathrm{add}\mathcal{L}_A$. □

The above theorem generalises the results of [**30**]. For this reason, the module T is called the *canonical tilting module*.

One word of caution is necessary: one could imagine that the sets $\mathcal{E}_A$ of A and $\mathcal{E}_{A_\lambda}$ of A_λ coincide and then conclude that the above conditions are equivalent to the fact that each connected component of A_λ is left supported. This is not the case as is shown in (2.5.5)(d) below.

Corollary 2.5.3. [**8**, (5.3)] *Let A be a left supported algebra, and $M \in \mathrm{add}\mathcal{L}_A$ be such that $\mathrm{Ext}^1_A(M,M)=0$. Then $C=\mathrm{End}M_A$ is a tilted algebra.*

Proof. Since $M \in \mathrm{add}\mathcal{L}_A$, then M is an A_λ-module. Furthermore, $\mathrm{Ext}^1_{A_\lambda}(M,M)=0$ and $C=\mathrm{End}M_{A_\lambda}$. By (2.5.2), there exists an hereditary algebra H and a tilting module U_H such that $A_\lambda=\mathrm{End}U_H$. Then there exists a module $V \in \mathcal{T}(U)$ such that $M=\mathrm{Hom}_H(U,V)$. Furthermore, $\mathrm{Ext}^1_H(V,V)=0$, so that V is a partial tilting module. By [**41**, (III.6.5)], $\mathrm{End}V_H$ is a tilted algebra. But now, $C=\mathrm{End}V_H$. □

Corollary 2.5.4. *Let A be a quasi-tilted algebra. The following are equivalent:*

(a) *A is left supported;*
(b) *$\mathcal{L}_A$ consists of predecessors of $\mathcal{E}$;*
(c) *$\mathcal{E} \neq \emptyset$;*
(d) *$\mathcal{L}_A$ contains an injective;*
(e) *$\Gamma(\mathrm{mod}A)$ has a connecting component containing an injective.*

In this case A is tilted, having $\mathcal{E}$ as complete slice.

Proof. Since A is quasi-tilted, then $\mathcal{L}_A$ contains all indecomposable projective A-modules by [**42**, (II.1.14)]. Hence, $\mathcal{L}_A \neq \emptyset$ and $\mathcal{E}_2=\emptyset$.
(a) implies (b). If A is left supported, then $\mathrm{add}\mathcal{L}_A=\mathrm{Cogen}E$ by (2.5.2), and so each module in $\mathcal{L}_A$ is a predecessor of $\mathcal{E}$. Now, since $\mathcal{E}\subseteq\mathcal{L}_A$ by definition, (b) follows from the fact that $\mathcal{L}_A$ is closed under predecessors.
(b) implies (c). This clearly follows from $\mathcal{L}_A \neq \emptyset$.
(c) implies (d). If $\mathcal{E}\neq\emptyset$, then $\mathcal{E}_1\neq\emptyset$ because $\mathcal{E}=\mathcal{E}_1\cup\mathcal{E}_2$ and $\mathcal{E}_2=\emptyset$. The claim follows from the definition of $\mathcal{E}_1$ and the fact that $\mathcal{L}_A$ is closed under predecessors.
(d) implies (e). Let I be an indecomposable injective A-module such that $I\in\mathcal{L}_A$. By [**37**, (6.5)], the connected component of $\Gamma(\mathrm{mod}A)$ containing I is connecting.
(e) implies (a). Let Γ be a connecting component in $\Gamma(\mathrm{mod}A)$ containing an indecomposable injective A-module and let Σ be a complete slice in Γ. Then there exists a complete slice Σ' induced from Σ containing only injective modules as sources. Indeed, if Σ satisfies this condition, then $\Sigma=\Sigma'$. Otherwise, if M is a non-injective source of Σ, replace M by $\tau_A^{-1}M$ in Σ and all the morphisms $M\to N$ by the corresponding morphisms $N\to\tau_A^{-1}M$. This procedure must stop because Γ contains injective modules by assumption. In particular, every element in Σ' is a successor of an injective. Now, since each complete slice belongs to $\mathcal{L}_A$, we infer

that $\Sigma' \subseteq \mathcal{E}_1$. But then, since Γ contains injectives and, clearly, $\mathcal{E} \subseteq \Gamma$, it follows from [**8**, (3.5)] that $|\mathcal{E}_1| = |\mathcal{E}| = |\Sigma'|$. Therefore $\mathcal{E} = \Sigma'$ and A is left supported by (2.5.2)(d) above. □

Examples 2.5.5.

(a) If A is representation-finite, then A is obviously left (and right) supported.

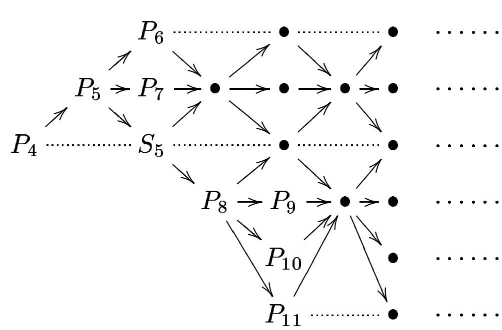
(b) For any $n \geq 2$, let $A = A(n)$ be the radical square zero algebra given by the quiver

$$\underset{1}{\bullet} \leftleftarrows \underset{2}{\bullet} \leftleftarrows \cdots\cdots \leftleftarrows \underset{n-1}{\bullet} \leftleftarrows \underset{n}{\bullet}$$

Here, $\mathcal{L}_A$ is given by the full subcategory generated by the Kronecker algebra $\underset{1}{\bullet} \leftleftarrows \underset{2}{\bullet}$. Each morphism $M \to N$ in ind$A$, with $M \in \mathcal{L}_A$ and $N \notin \mathcal{L}_A$ factors through I_2, and hence A is left supported. Since n is as large as we want, this example underlines the basic intuition about left supported algebras : A is left supported whenever the left part $\mathcal{L}_A$ is well-behaved, but the rest of the module category may have any possible shape.

(c) Let A be such that add$\mathcal{L}_A$ is an abelian exact subcategory of mod$A$. It follows from 2.4.1 that add$\mathcal{L}_A \cong$ modA_λ. Since modA_λ is cogenerated by DA_λ, it follows from [**68**] that add$\mathcal{L}_A$ is contravariantly finite in mod$A$, that is, A is left supported.

(d) Consider the algebra A of (2.2.4). Here, $F = P_{12}$ and it is easy to check that $T = E \oplus F$ is a tilting module. So, A is left supported by (2.5.2). Further, A_λ is the direct product of tilted algebras A_1 and A_2 given by the full subcategories generated by the sets of vertices $\{1, 2, 3\}$ and $\{4, 5, 6, 7, 8, 9, 10, 11\}$, respectively. Obviously, A_1 is representation-finite, and hence left supported. On the other side, the tilted algebra A_2 has a unique connecting component of the form



In particular, this component contains no injective module, and so A_2 is not left supported by (2.5.4). Moreover, $\mathcal{E}_{A_2}$ is empty while the restriction of $\mathcal{E}_A$ to A_2 is not.

Theorem 2.5.2 also gives information on the structure of the Auslander-Reiten quivers of representation-infinite left supported algebras. Indeed, if A is such an algebra, it is easy to show (see [**8**, (5.5)]) that $\mathcal{L}_A$ is infinite if and only if there exists a component Γ of $\Gamma(\text{mod}A)$ lying entirely in $\mathcal{L}_A$, and this is the case if and only if $\Gamma(\text{mod}A)$ has a postprojective component without injectives. In fact, applying

[**51**], we see that the components of $\Gamma(\text{mod}A)$ lying entirely in $\mathcal{L}_A$ are of one of the following forms : postprojective component(s), semiregular tube(s) without injectives, component(s) of the form $\mathbb{Z}\mathbb{A}_\infty$ or ray extension(s) of $\mathbb{Z}\mathbb{A}_\infty$.

3. Paths and quasi-directed components

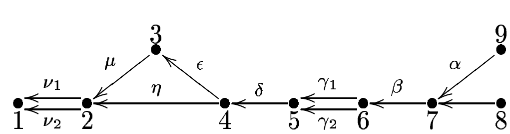
3.1. Paths from injective modules to projective modules. As we see in further sections, many families of left supported algebras may be characterised in terms of paths from indecomposable injective modules to indecomposable projective modules (see Sections 4, 5 and 6). For instance, it is well-known (see [**42**, (II.1.14)]) that an artin algebra A is quasi-tilted if and only if any path in ind$A$ from an injective module to a projective module can be refined to a sectional path.

In order to have a larger view of this process, it is convenient to study the case where the paths in ind$A$ from the injective modules to the projective modules contain a small number of hooks, that is, the case where there exists a fixed integer m_0 for which any such path contains at most m_0 distinct hooks. This leads to the (seemingly more general) case where there exists an integer n_0 such that any path in ind$A$ from an injective module to a projective module contains at most n_0 distinct modules. In fact, we show in this section that these approaches are equivalent.

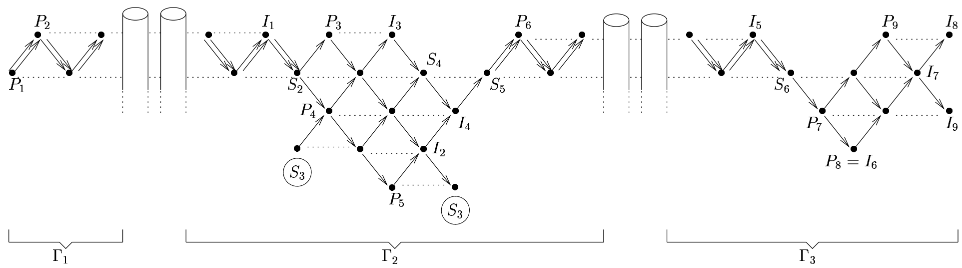
3.2. Quasi-directed components. We start with some definitions. We recall that an Auslander-Reiten component Γ is **generalised standard** if $\text{rad}^\infty(X,Y)=0$ for all X, Y in Γ, see [**62**]. A generalised standard Auslander-Reiten component is **quasi-directed** if it contains only finitely many non-directed modules, that is, modules lying on a cycle in ind$A$, see [**6**, (3.1)]. Finally, an Auslander-Reiten component Γ is **directed** if it contains only directed modules, and is **convex** if any path in ind$A$ with end-points in Γ lies entirely in Γ.

EXAMPLES 3.2.1.

(a) Let A be a representation-finite algebra. Then the unique component of $\Gamma(\text{mod}A)$ is trivially quasi-directed.
(b) Let A be a tilted algebra. It follows from the well-known description of its Auslander-Reiten quiver [**51**] that the only quasi-directed components are the postprojective component(s), the preinjective component(s) and the connecting component(s), all of which are actually directed.
(c) Let A be an artin algebra. A component Γ of $\Gamma(\text{mod}A)$ is called a π-**component** if all but at most finitely many modules in Γ lie in the τ_A-orbit of a projective module and Γ contains only finitely many non-directed modules (the dual notion is that of ι-**component**), see [**28**]. It follows from [**28**, (4.2)] and [**69**, (1.1)] that, if Γ is a π-component, then Γ is convex and quasi-directed.
(d) Let k be a field, and A be the k-algebra given by the quiver



bound by $\alpha\beta = 0$, $\beta\gamma_i = 0$, $\gamma_i\delta = 0$, $\delta\epsilon = 0$, $\epsilon\mu = 0$, $\eta\nu_i = 0$ and $\mu\nu_i = 0$ (for $i \in \{1, 2\}$). Then $\Gamma(\text{mod}A)$ is of the form:



where we identify the two copies of S_3. Then it is easy to see that the components Γ_1, Γ_2 and Γ_3 are convex and quasi-directed. In particular, the tilted algebra $B = A/\text{Ann}(\Gamma_1)$ is the Kronecker algebra generated by the set of vertices $\{1, 2\}$, and we observe that Γ_1 is a connecting component of $\Gamma(\text{mod}B)$. We recall that the **annihilator** of an Auslander-Reiten component Γ is $\text{Ann}(\Gamma) = \bigcap_{X \in \Gamma} \text{Ann}X$.

The purpose of this section is to give a complete classification of the quasi-directed components of the Auslander-Reiten quiver of an artin algebra as well as some of their basic properties. The first important property to mention at this stage is that since each generalised standard component admits at most finitely many non-periodic τ_A-orbits by [**62**, (2.3)], then any quasi-directed component contains only finitely many τ_A-orbits.

3.3. Semiregular quasi-directed components. The first theorem deals with the quasi-directed components which are semiregular (that is, do not simultaneously contain a projective module and an injective module). For the convenience of the reader, we sketch the proof.

Theorem 3.3.1. [**69**] *Let A be an artin algebra. If Γ is a semiregular quasi-directed component of $\Gamma(\text{mod}A)$, then Γ is directed and convex. Furthermore, $B = A/\text{Ann}(\Gamma)$ is a tilted algebra, and Γ is the connecting component of $\Gamma(\text{mod}B)$.*

Proof (Sketch). We only prove the first statement. Suppose that Γ is quasi-directed. Since Γ is semiregular, it contains no cycles by [**50**, (2.6)]. Thus, Γ has only finitely many τ_A-orbits by [**62**, (2.3)], so it follows from [**50**, (3.6)] that Γ is isomorphic to a full subquiver of $\mathbb{Z}\Delta$, where Δ is a finite and acyclic quiver. Then Γ is a convex and directed component of $\Gamma(\text{mod}A)$. Indeed, suppose that there exists a path $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} X_t = Y$ in ind$A$, where $X, Y \in \Gamma$ but $X_i \notin \Gamma$ for some i. In particular, there is a j such that $X_{j-1} \notin \Gamma$ but $X_j \in \Gamma$. Then, for each $s \geq 0$, it follows, for instance from [**69**, (1.1)] and its dual, that there is a path of distinct modules in ind$A$ of the form $X_{j-1} \xrightarrow{g_s} Z_s \longrightarrow Z_{s-1} \longrightarrow \cdots \longrightarrow Z_0 = X_j$, where $Z_i \in \Gamma$ for each i. Since Γ is acyclic and has only finitely many τ_A-orbits, it follows from [**32**, (1.1)] that there is an $s \geq 0$ for which Z_s is a predecessor of X. Therefore, we have a cycle $X \rightsquigarrow X_{j-1} \xrightarrow{g_s} Z_s \rightsquigarrow X$ with $g_s \in \text{rad}^\infty(X_{j-1}, Z_s)$. Applying [**69**, (1.1)] to g_s yields arbitrarily many non-directed modules in Γ, a contradiction. So Γ is convex, and therefore directed. □

3.4. Non-semiregular quasi-directed components. As we saw above, the semiregular quasi-directed components are actually directed. For the non-semiregular case, the situation changes since as seen in (3.2.1)(d), there exist non-semiregular quasi-directed components with cycles. The following theorem characterises non-semiregular quasi-directed and convex Auslander-Reiten components. We refer the reader to [**69**] for the proof.

THEOREM 3.4.1. [**69**, Theorem 2] *Let A be an artin algebra. If Γ is a non-semiregular component of* $\Gamma(\text{mod}A)$*, then the following are equivalent:*

(a) *Γ is quasi-directed and convex;*
(b) *Given $X, Y \in \Gamma$, there exists an integer $n_{X,Y}$ such that any path in* ind$A$ *from X to Y contains at most $n_{X,Y}$ distinct modules;*
(c) *Given $X, Y \in \Gamma$, there exists an integer $m_{X,Y}$ such that any path in* ind$A$ *from X to Y contains at most $m_{X,Y}$ distinct hooks;*
(d) *There exists an integer n_0 such that any path in* ind$A$ *from an injective in Γ to a projective in Γ contains at most n_0 distinct module;*
(e) *There exists an integer m_0 such that any path in* ind$A$ *from an injective in Γ to a projective in Γ contains at most m_0 distinct hooks.* □

As we shall see in Section 4, the components satisfying the equivalent conditions of the theorem appears naturally as faithful Auslander-Reiten components of a certain class of algebras.

An important example of non-semiregular quasi-directed and convex component is given by the pip-bounded components. A non-semiregular Auslander-Reiten component Γ is a **pip-bounded component** if there exists an integer n_0 such that any path of non-isomorphisms in ind$A$ from an injective module in Γ to a projective module in Γ has length at most n_0, see [**32, 33**]. Similarly, a non-semiregular Auslander-Reiten component Γ is a **hip-bounded component** if there exists an integer m_0 such that any path of non-isomorphisms in ind$A$ from an injective module in Γ to a projective module in Γ passes through at most m_0 hooks.

Combining the results obtained in [**33**, (1.6)] and [**69**, (3.12)], we get easily the following theorem:

THEOREM 3.4.2. *Let A be an artin algebra. If Γ is a non-semiregular component of* $\Gamma(\text{mod}A)$*, then the following are equivalent:*

(a) *Γ is a pip-bounded component;*
(b) *Γ is a hip-bounded component;*
(c) *Given $X, Y \in \Gamma$, there exists an integer $n_{X,Y}$ such that any path of non-ismorphisms in* ind$A$ *from X to Y has length at most $n_{X,Y}$;*
(d) *Given $X, Y \in \Gamma$, there exists an integer $m_{X,Y}$ such that any path of non-ismorphisms in* ind$A$ *from X to Y passes through at most $m_{X,Y}$ hooks;*
(e) *Γ is generalised standard, convex and directed.* □

We shall study in Section 5 a class of algebras having faithful Auslander-Reiten components satisfying the equivalent conditions of the above theorem.

Examples 3.4.3.

(a) Let A be a representation-finite algebra. The unique component of $\Gamma(\text{mod}A)$ is a pip-bounded component if and only if it contains no cycles, that is, if and only if A is representation-directed.
(b) Consider the bound quiver algebra A of (3.2.1)(d) and its Auslander-Reiten quiver. The component Γ_3 is a pip-bounded component. However, since the injective I_2 and the projective P_4 lie on a common cycle, the component Γ_2 is not a pip-bounded component because we can find paths of arbitrary length from I_2 to P_4.

3.5. The case of representation-finite algebras. We conclude this section with some properties of representation-finite algebras proved in [**69**].

Proposition 3.5.1. [**69**, (3.14)] *Let A be an artin algebra. If Γ is a quasi-directed component of $\Gamma(\text{mod}A)$, then, either A is representation-finite, or $\Gamma(\text{mod}A)$ contains infinitely many non-directed modules.*

Proof (Sketch). Assume that A is representation-infinite and, consequently, that Γ is infinite. Since Γ contains only finitely many τ_A-orbits by [**62**, (2.3)], we may suppose that the right stable part ${}_r\Gamma$ of Γ is infinite. Let $\mathcal{C}$ be a component of ${}_r\Gamma$ and Σ be a maximal subsection of $\mathcal{C}$. Then, if M is the direct sum of all modules lying in Σ, we get $\text{Hom}_A(M, \tau_A M) = 0$ since Γ is generalised standard. In addition, if $B = A/\text{Ann}(\Sigma)$, then M is a faithful B-module and $\text{Hom}_B(M, \tau_B M) = 0$ since $\tau_B M$ is a submodule of $\tau_A M$. Hence, M is a slice B-module and B is tilted by the Liu-Skowroński criterion [**52, 63**]. Consequently, B admits infinitely many non-directed indecomposable modules, and so does A. □

Proposition 3.5.2. [**69**, (3.15)] *An artin algebra A is representation-finite if and only if each component of $\Gamma(\text{mod}A)$ is quasi-directed.*

Proof. Since the necessity is obvious, we assume that A is representation-infinite and that each component of $\Gamma(\text{mod}A)$ is quasi-directed and hence generalised standard. Since this implies that $\text{rad}^\infty(M, M) = 0$ for each $M \in \text{ind}A$, then, by [**60**, (8.6)], $\Gamma(\text{mod}A)$ admits infinitely many stable tubes of rank one, which are clearly not quasi-directed. □

4. Laura algebras

4.1. Gluings. The motivation for defining laura algebras comes from two sources. The first is the consideration of the non-semiregular quasi-directed components studied in Section 3. As will be seen in this section, the left (and right) stable parts of such components resemble those of the connecting component of a (non concealed) tilted algebra (see [**47**]). Moreover, the left and the right stable parts, together, comprise all of the component, except for at most a finite part. The other motivation comes from the observation that a left (or right) supported algebra has a well-behaved left (or right, respectively) part, but the rest of the module category can assume any form. In order to have a predictable module category, we not only have to require the algebra to be both left and right supported, but also that the middle part, consisting of the indecomposables lying neither in the left nor in the right part, be relatively small. In fact, a simple assumption on this middle part will prove sufficient.

4.2. Left and right glued algebras. The left glued algebras were introduced in [**4**] when studying algebras whose Auslander-Reiten quivers have components consisting of postprojective modules in the sense of Auslander and Smalø[**20**]. Such a component is a π-component [**28**] and it is quasi-directed (see (3.2.1)(c)). The next result gives equivalent definitions of left glued algebras.

THEOREM 4.2.1. [**28, 4, 6**] *The following are equivalent for an algebra A:*

(a) *$\mathcal{R}_A$ is cofinite in* $\mathrm{ind}A$*;*
(b) $\mathrm{id}_A X \leq 1$ *for all but at most finitely many X in* $\mathrm{ind}A$*;*
(c) *The support of* $\mathrm{Hom}_A(-, A)$ *is finite;*
(d) *All indecomposable projective modules lie in a unique π-component.* □

An algebra A is called **left glued** provided it satisfies one of the equivalent conditions of the theorem above. We refer to [**4**] for further characterisations. Dually, one can define **right glued algebras**. The following consequence of the above considerations characterises concealed algebras.

THEOREM 4.2.2. [**4**] *The following are equivalent for a representation-infinite algebra A:*

(a) *A is concealed;*
(b) *A is a left and right glued algebra;*
(c) *$\mathcal{L}_A \cap \mathcal{R}_A$ is cofinite in* $\mathrm{ind}A$*;*
(d) $\mathrm{pd}_A X \leq 1$ *and* $\mathrm{id}_A X \leq 1$ *for all but finitely many X in* $\mathrm{ind}A$. □

4.3. Laura algebras. Following the strategy of imposing some finiteness assumption in the middle part of the category $\mathrm{ind}A$, the class of laura algebras was introduced in [**6**] and, independently, in [**57, 64**].

DEFINITION 4.3.1. *An artin algebra is* **laura** *if the union $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in* $\mathrm{ind}A$*. A laura algebra is* **strict** *if it is not quasi-tilted.*

The next result gives several characterisations of laura algebras. The equivalence of conditions (a), (b) and (c) is shown in [**6**] (2.4). and their equivalence with (d), (e), (f) and (g) is shown in [**49**]. The equivalence of (a) and (b) is also shown in [**64**]. We denote by $\mu(M)$ the Gabriel-Roiter measure of a module M, see [**59**].

THEOREM 4.3.2. [**6**, (2.4)], [**49, 64**] *The following are equivalent for an algebra A:*

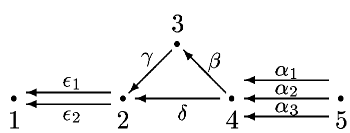
(a) *A is laura;*
(b) *There are only finitely many $M \in \mathrm{ind}A$ with a path $I \rightsquigarrow M \rightsquigarrow P$ in $\mathrm{ind}A$, with I injective and P projective;*
(c) *There are only finitely many $M \in \mathrm{ind}A$ with a path $L \rightsquigarrow M \rightsquigarrow N$ in $\mathrm{ind}A$, with $L \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$;*
(d) *The number of distinct hooks on a path $I \rightsquigarrow P$ in $\mathrm{ind}A$, with I injective and P projective, is bounded;*

(e) *The number of distinct hooks on a path* $L \rightsquigarrow M$ *in* ind$A$, *with* $L \notin \mathcal{L}_A$ *and* $M \notin \mathcal{R}_A$, *is bounded;*
(f) *The set of all* $\mu(M)$, *with* M *lying on a path* $I \rightsquigarrow M \rightsquigarrow P$ *in* ind$A$, *with* I *injective and* P *projective, is finite;*
(g) *The set of all* $\mu(M)$, *with* M *lying on a path* $L \rightsquigarrow M \rightsquigarrow N$ *in* ind$A$, *with* $L \notin \mathcal{L}_A$ *and* $N \notin \mathcal{R}_A$, *is finite.*

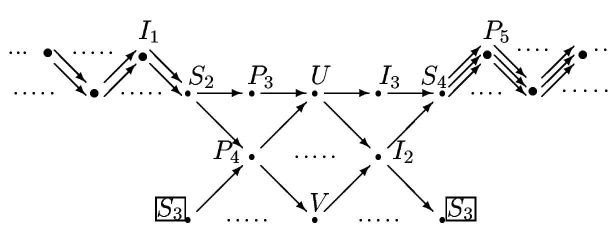
Proof. We only prove the equivalence of (a), (b) and (c).
(a) implies (b) : follows at once from (2.1.2) and the cofiniteness of $\mathcal{L}_A \cup \mathcal{R}_A$.
(b) implies (c). Let $M \in$ ind$A$ be such that there exists a path $L \rightsquigarrow M \rightsquigarrow N$ with $L \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$. So, there exists a path $X \rightsquigarrow L$ in ind$A$ with $\text{pd}_A X \geq 2$, and hence an injective I and a path $I \to \tau_A X \to * \to L \rightsquigarrow M$. Dually, we construct a path $N \rightsquigarrow P$, with P projective, which proves this implication.
(c) implies (a). If $\mathcal{L}_A \cup \mathcal{R}_A$ is not cofinite, there exists an infinite family $(M_\lambda)_\lambda$ of indecomposable modules not in $\mathcal{L}_A \cup \mathcal{R}_A$. Considering, for each λ, the path $M_\lambda \xrightarrow{1} M_\lambda \xrightarrow{1} M_\lambda$ gives a contradiction to (c). □

Examples 4.3.3.

(a) The following classes of algebras are, clearly, laura: (i) representation-finite algebras; (ii) quasi-tilted algebras; (iii) left and right glued algebras.
(b) Let k be a field, and A be the radical square zero k-algebra given by the quiver



The Auslander-Reiten quiver $\Gamma(\text{mod}A)$ of A consists of:
(i) The postprojective component and the family of orthogonal homogeneous tubes corresponding to the Kronecker algebra given by the full subcategory of A generated by 1 and 2;
(ii) The preinjective component and the family of regular components corresponding to the generalised Kronecker algebra given by the full subcategory of A generated by 4 and 5;
(iii) A non-semiregular component Γ of the form:



where we identify the two copies of S_3.
There are no morphisms from injective modules to one of the components in (i), and therefore, those components lie in $\mathcal{L}_A$. Also, all modules lying in the components in (i) are predecessors of S_2. Since $\text{id}_A S_2 > 1$, these components lie in $\mathcal{L}_A \setminus \mathcal{R}_A$. Dually, the components in (ii) lie in $\mathcal{R}_A \setminus \mathcal{L}_A$. Concerning the remaining component Γ, it is easy to see that $\Gamma \cap \mathcal{L}_A$

consists of the predecessors of P_3, while $\Gamma \cap \mathcal{R}_A$ consists of the successors of I_3. Therefore, $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite and so A is laura.

Before going on, we mention the following problem, formulated in [**64**]. Let A be an artin algebra such that the subcategory $\{X \in \mathrm{ind}A \mid \mathrm{pd}_A X \leq 1 \text{ or } \mathrm{id}_A X \leq 1\}$ is cofinite. Is it true that A is a laura algebra?

The next result shows that laura algebras have well-behaved left and right parts.

Proposition 4.3.4. [**8**, (4.4)], [**64**] *Let A be a strict laura algebra. Then A is left and right supported.* □

4.4. The Auslander-Reiten quiver of a laura algebra. The structure of the Auslander-Reiten quiver of a strict laura algebra resembles much that of a tilted algebra in the following way : any semiregular component is a component of a tilted algebra and the unique non-semiregular component plays the role of *connecting* the left part of the module category to its right part.

Theorem 4.4.1. [**6**, (3.4)(4.6)] *Let A be a laura algebra. Then,*

(a) *Any non-semiregular component of $\Gamma(\mathrm{mod}A)$ is quasi-directed;*

(b) *If A is not quasi-tilted then $\Gamma(\mathrm{mod}A)$ has a unique faithful non-semiregular component Γ. Moreover, if $\Gamma' \neq \Gamma$, then Γ' is semiregular and either*
  - (i) $\mathrm{Hom}_A(\Gamma', \Gamma) \neq 0$ *and Γ' lies in $\mathcal{L}_A \setminus \mathcal{R}_A$; or*
  - (ii) $\mathrm{Hom}_A(\Gamma, \Gamma') \neq 0$ *and Γ' lies in $\mathcal{R}_A \setminus \mathcal{L}_A$.* □

It follows directly from this statement that the distinguished non-semiregular component Γ is convex. Conversely, we have the following proposition.

Proposition 4.4.2. [**69**, Theorem 2] *Let A be an artin algebra and Γ be a non-semiregular quasi-directed and convex component of $\Gamma(\mathrm{mod}A)$. Then $B = A/\mathrm{Ann}(\Gamma)$ is a laura algebra, and Γ is a faithful non-semiregular component of $\Gamma(\mathrm{mod}B)$.* □

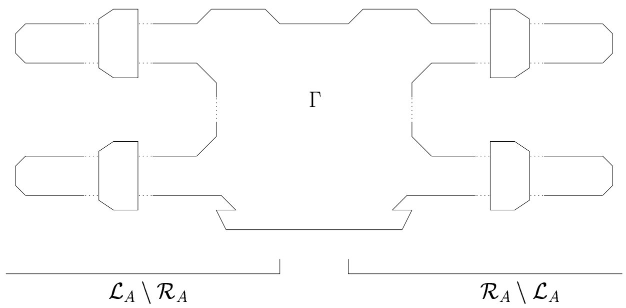
Assume now A to be a representation-infinite strict laura algebra. We have just seen that the unique faithful non-semiregular component Γ of $\Gamma(\mathrm{mod}A)$ plays a role similar to that of the connecting component of a tilted algebra. Thus, in order to describe the remaining components of $\Gamma(\mathrm{mod}A)$, it is useful to define a left and a right end algebras generalising those introduced by Kerner for tilted algebras [**47**]. Since Γ is infinite, then so is the left stable part ${}_l\Gamma$ or the right stable part ${}_r\Gamma$ of Γ. Suppose ${}_l\Gamma$ is infinite. Since Γ has finitely many τ-orbits, ${}_l\Gamma$ has finitely many non-trivial components. We choose, for each component of ${}_l\Gamma$, a maximal subsection, and denote these by ${}_1\Sigma, \cdots {}_s\Sigma$. For each i, let ${}_\infty A_i$ be the full subcategory of A generated by the support of ${}_i\Sigma$. The **left end algebra** ${}_\infty A$ of A is ${}_\infty A = {}_\infty A_1 \times {}_\infty A_2 \cdots \times {}_\infty A_s$. The **right end algebra** A_∞ is defined dually. Observe that $A_\infty = 0$ (or ${}_\infty A = 0$) if and only if A is right glued (or left glued, respectively).

Clearly, the left (or the right) end algebra does not coincide with the left (or right, respectively) support algebra (see example 4.3.3 (b) above).

Lemma 4.4.3. [**6**, (4.3)] *With the above notation, each ${}_\infty A_i$ is a tilted algebra having ${}_i\Sigma$ as a complete slice.*

Proof. This is done by checking the Liu-Skowroński criterion [**52, 63**]. □

Let now be Γ' be an Auslander-Reiten component of A, distinct from Γ. If $\mathrm{Hom}_A(\Gamma', \Gamma) \neq 0$, it is easy to show that Γ' is in fact a component of $\Gamma(\mathrm{mod}_\infty A)$. Dually if $\mathrm{Hom}_A(\Gamma, \Gamma') \neq 0$, then Γ' is in fact a component of $\Gamma(\mathrm{mod} A_\infty)$. The Auslander-Reiten quiver of A then has the following form:



It turns out that the existence of a faithful quasi-directed component actually characterises strict laura algebras.

Theorem 4.4.4. [**57**, (3.1)] *Let A be an artin algebra which is not quasi-tilted. Then A is laura if and only if* $\Gamma(\mathrm{mod} A)$ *admits a faithful quasi-directed component.* □

Remark. In [**57**] Reiten and Skowroński introduced a concept of multisection and obtain a criterion (generalising the one of Liu-Skowroński [**52, 63**]) allowing to verify whether an artin algebra which is not quasi-tilted is laura or not.

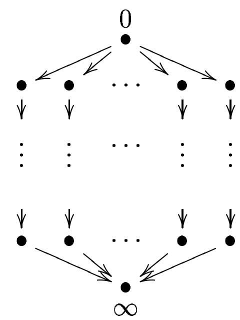
From the above description of the laura algebras, we deduce information on the infinite radical. Let A be an artin algebra. If there exists $\mu_A \in \mathbb{N}$ such that $(\mathrm{rad}^\infty(\mathrm{mod}\ A))^{\mu_A - 1} \neq 0$ but $(\mathrm{rad}^\infty(\mathrm{mod}\ A))^{\mu_A} = 0$, we say that $\mathrm{rad}^\infty(\mathrm{mod}\ A)$ is **nilpotent of nilpotency index** μ_A. We recall that A is representation-infinite if and only if $\mu_A \geq 3$, see [**35**]. The nilpotency index of quasi-tilted algebras was studied in [**67**]

Theorem 4.4.5. [**6**, (6.3)] *Let A be a representation-infinite laura algebra. The following conditions are equivalent:*

(a) *A is domestic;*
(b) *A is tame and no full convex subcategory of A is a tubular algebra;*
(c) $\mathrm{rad}^\infty(\mathrm{mod} A)$ *is nilpotent. If this is the case, and μ_A is the nilpotency index of* $\mathrm{rad}^\infty(\mathrm{mod} A)$*, then $\mu_A \in \{3, 4, 5\}$. Moreover $\mu_A = 3$ if ${}_\infty A$ or A_∞ is zero, or if A is quasi-tilted, then it is tilted.* □

Laura algebras can have arbitrary global dimension, even infinite. We refer the reader to [**6**] for examples. We end with two other classes of laura algebras.

Examples 4.4.6. (a) A quiver Q is called a **toupie** if it has a unique source 0 and a unique sink ∞, and, for any other point x there is exactly one arrow having x as source and exactly one arrow having x as target:



The distinct paths from 0 to ∞, are the branches of Q. Let now k be a field, Q be a toupie and $A = kQ/I$, where I is an admissible ideal of kQ. Set $m = \dim_k(e_0 A e_\infty)$. It is shown in [**27**] that A is laura if and only if:

(T1) $m = 0$, or
(T2) $m = 1$, and only one branch lies in I, or
(T3) $m \in \{1, 2\}$, no branch lies in I and A is simply connected, or
(T4) no branch lies in I, $\dim_k I = 1$, A is simply connected and there is at most one branch of length at least three.

(b) Let k be an algebraically closed field, and B, C be two non-simple finite dimensional k-algebras, considered as locally bounded k-categories [**25**]. An algebra A is called an **articulation of** B **and** C (denoted as $A = (B, C)$) if:

a) B and C are subcategories of A such that $A_0 = B_0 \cup C_0$;
b) $B_0 \cap C_0 \neq \emptyset$ and, if $x \in B_0 \cap C_0$, then either x is a source of B and a sink of C, or is a source of C and a sink of B;
c) For every $x, y \in A_0$, the set of morphisms $A(x, y)$ in A from x to y is given by

$$A(x,y) = \begin{cases} B(x,y) & \text{if } x, y \in B_0; \\ C(x,y) & \text{if } x, y \in C_0; \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{S}_{B\cap C}$ the set of simple A-modules corresponding to the points of $B_0 \cap C_0$. It is shown in [**38**] that $A = (B, C)$ is laura if and only if both B and C are laura, and moreover $\mathcal{S}_{B\cap C} \subseteq \Gamma^{(B)} \cap \Gamma^{(C)}$ where $\Gamma^{(B)}$ and $\Gamma^{(C)}$ are non-semiregular or connecting components of $\Gamma(\text{mod}B)$ and $\Gamma(\text{mod}C)$, respectively.

5. Weakly shod algebras

5.1. Weakly shod algebras. As seen above, the Auslander-Reiten quiver of a strict laura algebra has a non-semiregular convex quasi-directed component which is extremely useful for the understanding of the module category. We now study a subclass of the laura algebras where this distinguished component is in fact directed.

Definition 5.1.1. *An algebra A is called* **weakly shod** *if there exists a bound on the length of any path of non-isomorphisms in* ind$A$ *from an injective module to a projective module. A weakly shod algebra is called* **strict** *if it is not quasi-tilted.*

Weakly shod algebras are triangular and can have any finite global dimension (see [**33**] for examples). The next result relates weakly shod algebras, informations on the Auslander-Reiten components and the union $\mathcal{L}_A \cup \mathcal{R}_A$.

Theorem 5.1.2. *The following conditions are equivalent for an algebra A:*

(a) *A is weakly shod;*
(b) *A is a laura algebra such that none of the non-semiregular components of the Auslander-Reiten quiver contains cycles;*
(c) *There exists an integer n_1 such that any path of non-isomorphisms in* ind$A$ *from an injective module to a projective module passes through at most n_1 hooks;*
(d) *There exists an integer n_2 such that any path of non-isomorphisms $M \rightsquigarrow N$ in* ind$A$ *where $M \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$ has length at most n_2;*
(e) *There exists an integer n_3 such that any path of non-isomorphisms $M \rightsquigarrow N$ in* ind$A$ *where $M \notin \mathcal{L}_A$ and $N \notin \mathcal{R}_A$ passes through at most n_3 hooks.* □

We suggest to the reader to compare the above theorem to (4.3.2) : whereas for laura algebras, the finiteness conditions are in terms of the number of *distinct* modules, in the case of weakly shod algebras, they are in terms of their *total* number, allowing repetitions.

In order to state the following results, we need some definitions. Let $\mathcal{P}(\mathcal{R}_A)$ denote the set of all projective modules lying in $\mathcal{R}_A$. The successor relation defines a partial order in the set $\mathcal{P}(\mathcal{R}_A)$. Since this set is finite, it contains maximal elements. This leads to the next definition.

Definition 5.1.3. *Let A be an artin algebra, $P = eA$ be a maximal projective in $\mathcal{P}(\mathcal{R}_A)$ for the successor relation, $B = (1-e)A(1-e)$ and $M =$*rad$P$. *Then the one point extension $A = B[M]$ is said to be a* **maximal extension**.

Our next theorem says that any strict weakly shod algebra is obtained by iterated one-point (maximal) extensions starting from a tilted algebra.

Theorem 5.1.4. [**33**, (4.9)], [**9**, (3.3)] *Let A be a strict weakly shod algebra. Then there exist a sequence of algebras $A_0, A_1, \ldots, A_m = A$ with A_0 a tilted algebra, and a sequence of A_{i-1}-modules M_i such that $A_i = A_{i-1}[M_i]$ is a maximal extension, for each i with $0 < i \leq m$.* □

We use the following notation. Let A be a strict weakly shod algebra and $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$ be a filtration of A as iterated maximal extensions with B tilted, as in the above theorem. For each i with $0 < i \leq m$, let M_i be the A_{i-1} module such that $A_i = A_{i-1}[M_i]$, P_i be the extending projective A_i-module (thus, $M_i =$rad$_A P_i$ and P_i is maximal in $\mathcal{P}(\mathcal{R}_{A_i})$) and x_i be the extension point associated to P_i. Such a filtration is called a **maximal filtration** of A. Note that, while B is a tilted algebra, all A_i, with $i > 0$, are strict weakly shod.

As well as for laura algebras, it is possible to describe the shape of the Auslander-Reiten quiver of a strict weakly shod algebra. However, we emphasise that for strict weakly shod algebras there exists a unique non-semiregular *directed* convex component, and this one is faithful. This component is a pip-bounded component, or also a hip-bounded component (see (3.4)). Conversely, we have the following result.

Proposition 5.1.5. [**69**, (3.12)] *Let A be an artin algebra, and Γ be a pip-bounded component of $\Gamma(\mathrm{mod}A)$. Then $B = A/\mathrm{Ann}(\Gamma)$ is a weakly shod algebra, and Γ is a faithful pip-bounded component of $\Gamma(\mathrm{mod}B)$.* □

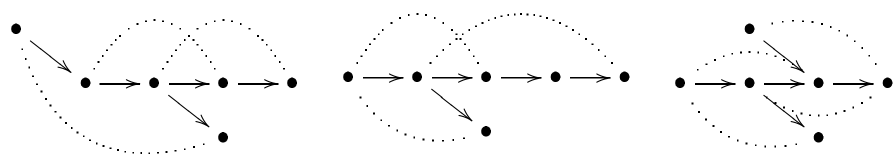
For the other components, we refer to the description of the Auslander-Reiten quiver of a laura algebra.

Examples 5.1.6.

(a) It is shown in [**27**] that a toupie algebra $A = kQ/I$ (see (4.4.6)(a) above) is weakly shod if and only if it satifies one of the conditions (T1), (T3) or (T4) of (4.4.6)(a).

(b) It is shown in [**38**] that an articulation $A = (B, C)$ (see (4.4.6)(b)) is weakly shod if and only if it is laura and moreover, every module in $\mathcal{S}_{B\cap C}$ is directed.

(c) Let k be a field, and A be the k-algebra given by one of the quivers :



(S1) (S2) (S3)

where the dotted lines indicate zero-relations. Then A is weakly shod.

5.2. Hochschild cohomology. The Hochschild cohomology groups $\mathrm{H}^i(A)$, $i \geq 1$, of a finite dimensional algebra A, introduced in [**44**], have been much investigated lately (see, for instance, [**39, 55**]). In this section we give a complete description of the Hochschild cohomology groups for weakly shod algebras.

Proposition 5.2.1. [**34**, (2.2), (2.3)], [**9**, (2.3)] *Let $A = B[M]$ be a maximal extension. Then for all $i \geq 1$, we have $\mathrm{Ext}^i_B(M, M) = 0$.*

Proof: (Sketch for $i = 1$) Assume $\mathrm{Ext}^1_B(M, M) \neq 0$. There exists an indecomposable summand N of M such that $\mathrm{Ext}^1_B(M, N) \neq 0$. Let $M = N \oplus N'$ and P be the extending projective (that is, $M = \mathrm{rad}_A P$). Then N' is a submodule of P and $L = P/N'$ is indecomposable. Since $\mathrm{Ext}^1_B(M, N) \neq 0$, it follows from [**42**, (III.2.2)(a)] that $\mathrm{id}_A L \geq 2$. Since P belongs to $\mathcal{P}(\mathcal{R}_A)$ and L is a successor of P we get a contradiction. The case $i \geq 1$ is proved using similar techniques. □

Using Happel's sequence (see [**39**]) and applying (5.1.4) and (5.2.1), we get the following result.

Corollary 5.2.2. [**9**, (2.4)] *Let $A = B[M]$ be a maximal extension. Then:*

(a) *There exists an exact sequence:*

$$0 \to \mathrm{H}^0(A) \to \mathrm{H}^0(B) \to (\mathrm{End}_A M)/k \to \mathrm{H}^1(A) \to \mathrm{H}^1(B) \to 0;$$

(b) *For all $i \geq 2$, we have* $\mathrm{H}^i(A) \cong \mathrm{H}^i(B)$. □

As a consequence of the previous results we get the main theorem of this section:

Theorem 5.2.3. [**34**, (2.4)] *Let A be a strict weakly shod algebra. Then* $\mathrm{H}^{i}(A) = 0$, *for each* $i \geq 2$. □

We recall that the Hochschild cohomology groups for quasi-tilted algebras were computed in [**40**].

5.3. Simple connectedness and the orbit graph. The previous section shows that the first Hochschild cohomology group is the only one that may not vanish for a strict weakly shod algebra. It is therefore important (for instance, for the understanding of the Hochschild cohomology ring) to describe it. A finite dimensional algebra A over an algebraically closed field k is **simply connected** if it is triangular and, for any presentation $A \cong kQ_A/I$ of A as a bound quiver algebra (see [**25**]), the fundamental group of (Q_A, I) is trivial (see, for instance, [**2, 14, 66**]). A well-known result, due to Bongartz and Gabriel [**25**, (6.5)], states that a representation-finite algebra is simply connected if and only if the orbit graph of its Auslander-Reiten quiver is a tree. On the other hand, it is shown in [**26**] that a representation-finite algebra A is simply connected if and only if $\mathrm{H}^1(A)$ vanishes. It is natural to ask whether similar results hold for a representation-infinite algebra. In this case, the Auslander-Reiten quiver is no longer connected so one should consider the orbit graph of each of its connected components. However, if one deals with a tilted algebra, then much information is contained in its connecting component(s). Indeed, it was shown in [**12**] that a tame tilted algebra A is simply connected if and only if the orbit graph of its connecting component is a tree and this, by [**39**, (1.6)] or [**11**, (1.4)], is equivalent to saying that $\mathrm{H}^1(A) = 0$. This also answered positively (for tilted algebras) Skowroński's question in [**66**](Problem 1) whether it is true that a tame triangular algebra A is simply connected if and only if $\mathrm{H}^1(A) = 0$. Now, for strict weakly shod algebras, we may use the pip-bounded component (3.4), which resembles the connecting component of a tilted algebra.

Let thus A be a strict weakly shod algebra. Using a maximal filtration $B = A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m = A$ as in (5.1.4), we may reduce the problem to the tilted algebra B.

Lemma 5.3.1. [**9**] *Let $B = A_0 \subset A_1 \subset \ldots \subset A_{m-1} \subset A_m = A$, be a maximal filtration of A. Then:*

(a) $\mathrm{H}^1(A) = 0$ *if and only if: (i)* $\mathrm{H}^1(A_0) = 0$*; and (ii) each x_i is separating;*
(b) *A is of tree type if and only if: (i) each A_i is of tree type; and (ii) each x_i is separating;*
(c) *If A is tame then A is simply connected if and only if: (i) each A_i is simply connected; and (ii) each x_i is separating.*

Proof (Sketch). (a) The pip-bounded component of a strict weakly shod algebra C contains all projectives in $\mathcal{P}(\mathcal{R}_C)$. Moreover, since it is generalised standard and directed, all its indecomposables are bricks. The statement follows from (5.2.2), [**7**, (2.2)] and an obvious descending induction.
(b) and (c) We refer to [**9**] for a proof. □

Applying the previous results, we get the main theorem of this subsection.

THEOREM 5.3.2. [**9**, Theorem A] *Let A be a strict weakly shod algebra. The following conditions are equivalent:*

(a) $\mathrm{H}^1(A) = 0$;
(b) *The orbit graph of the pip-bounded component of A is a tree.*

If, moreover, A is tame, then the above are further equivalent to:

(c) *A is simply connected.*

Proof. We first show that (a) is equivalent to (b). Let $B = A_0 \subset A_1 \subset \cdots \subset A_{m-1} \subset A_m = A$ be a maximal filtration of A. Applying (5.3.1)(a) and induction, we see that $\mathrm{H}^1(A) = 0$ if and only if $\mathrm{H}^1(B) = 0$ and each x_i is separating. Since B is a tilted algebra, $\mathrm{H}^1(B) = 0$ if and only if it is of tree type (see [**39**, (1.6)]). The statement then follows from (5.3.1)(b) and another induction. Assume now that A is tame. If A is simply connected, then, using the same notation as above, it follows from (5.3.1)(c) that each A_i is simply connected and each x_i is separating. In particular, B is simply connected. Since B is tilted and tame, this implies, by [**12**], that $\mathrm{H}^1(B) = 0$. Applying (5.3.1)(b) and induction yields $\mathrm{H}^1(A) = 0$. Conversely, if $\mathrm{H}^1(A) = 0$, then $\mathrm{H}^1(B) = 0$ and each x_i is separating. Since B is tilted and tame, it follows from [**12**] that B is simply connected. Since each x_i is separating, it follows from [**14**, (2.5)] and induction that A is simply connected. □

Since a similar result was obtained in [**7**] for the tame quasi-tilted algebras, this completely characterises the simple connectednes of a weakly shod tame algebra.

On the other hand, we conjecture that the above result holds true as well for the more general case of (not necessarily tame) strict laura algebras.

We then turn to one particular subclass, that of the strongly simply connected algebras, introduced by Skowroński in [**66**]. We recall that an algebra A is **strongly simply connected** whenever any full convex subcategory of A is simply connected. The strong simple connectedness of tame quasi-tilted algebras was characterised in [**7**]. We now consider the strict weakly shod algebras. We also recall that an algebra A is **strongly $\tilde{\mathbb{A}}$-free** if it contains no full convex subcategory which is hereditary of type $\tilde{\mathbb{A}}$. It is **separated** if, for each point x in its quiver, the number of indecomposable summands of $\mathrm{rad}P_x$ equals the number of connected components of the full subquiver generated by the non-predecessors of x. The following theorem generalises the main result of [**11**].

THEOREM 5.3.3. [**9**, Theorem B] *Let A be a strict weakly shod tame algebra. The following conditions are equivalent:*

(a) *A is strongly simply connected;*
(b) *the orbit graph of every directed component of $\Gamma(\mathrm{mod}A)$ is a tree;*
(c) $\mathrm{H}^1(A) = 0$ *and A is strongly $\tilde{\mathbb{A}}$-free;*
(d) *A is separated and strongly $\tilde{\mathbb{A}}$-free.* □

6. Shod algebras

6.1. Shod algebras. As seen above, the informations given by the subcategories $\mathcal{L}_A$ and $\mathcal{R}_A$ are useful for the understanding of the whole category $\mathrm{mod}A$ and, in particular, for the description of $\Gamma(\mathrm{mod}A)$. As stressed, this is particularly useful in case A is laura or weakly shod, both cases where the union $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in ind$A$. In this section, we look at the particular situation of when $\mathcal{L}_A \cup \mathcal{R}_A$ gives the whole of ind$A$. This is true, for instance, for quasi-tilted algebras. We start our discussion with the following result.

Theorem 6.1.1. [**31**] *The following are equivalent for an algebra A:*

(a) $\operatorname{ind}A = \mathcal{L}_A \cup \mathcal{R}_A$;
(b) *For each $X \in \operatorname{ind}A$, $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$;*
(c) *Any path in indA from an injective to a projective module can be refined to a path of irreducible maps with at most two hooks and, in case there are two, they are consecutive.*

Proof (Sketch). (a) implies (b). This is clear.
(b) implies (c) Suppose there exists a path $I \rightsquigarrow P$ in ind$A$ from an injective module I to a projective module P. Using [**31**](1.3) (see also [**29**](4.1)) this path can be refined to a path of irreducible morphisms

$$(*) \qquad I = X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{t-1} \longrightarrow X_t = P.$$

Assume that $(*)$ has two hooks. Hence, there exist j and l such that $\tau_A^{-1} X_j = X_{j+2}$, $\tau_A X_l = X_{l-2}$ and the paths $I \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_{j+1}$ and $X_{l-1} \longrightarrow \cdots \longrightarrow X_{t-1} \longrightarrow P$ are sectional. Since there are at least two hooks, we infer that $j+1 < l-1$. Observe that $\operatorname{pd}_A X_{j+2} \geq 2$ (because $\operatorname{Hom}_A(I, \tau_A X_{j+2}) \neq 0$) and $\operatorname{id}_A X_{l-2} \geq 2$ (because $\operatorname{Hom}_A(\tau_A^{-1} X_{l-2}, P) \neq 0$). If now $j+2 < l-1$, we get a path $X_{j+2} \rightsquigarrow X_{l-2}$, a contradiction to [**31**](1.2). So, $j+2 = l-1$ and in this case the above path has only two hooks and they are consecutive.
(c) implies (a) Suppose there exists an indecomposable module X which does not lie in the union $\mathcal{L}_A \cup \mathcal{R}_A$. Then X has a predecessor Y with $\operatorname{pd}_A Y \geq 2$ and a successor Z with $\operatorname{id}_A Z \geq 2$. Hence, there exists a path

$$I \longrightarrow \tau_A Y \longrightarrow * \longrightarrow Y \rightsquigarrow X \rightsquigarrow Z \longrightarrow * \longrightarrow \tau_A^{-1} Z \longrightarrow P$$

in ind$A$, where I is an injective module and P is a projective module. Clearly, such a path passes through two non-consecutive hooks, a contradiction. □

Definition 6.1.2. *An algebra satisfying the equivalent conditions of theorem (6.1.1) is called* **shod** *(for* small **ho***mological* **d***imension). A shod algebra which is not quasi-tilted is called* **strict shod**.

As observed in [**42**, (II.1.1)], a shod algebra has global dimension at most 3. The original impetus for studying shod algebras was to extend the existence of the trisection $\operatorname{ind}A = (\mathcal{L}_A \setminus \mathcal{R}_A) \vee (\mathcal{L}_A \cap \mathcal{R}_A) \vee (\mathcal{R}_A \setminus \mathcal{L}_A)$ proven by Happel-Reiten-Smalø for a quasi-tilted algebra A to a broader class of algebras. It is not difficult to see that a quasi-tilted algebra is a shod algebra of global dimension 2. A shod algebra is strict if and only if it has global dimension 3.

Observe that a shod algebra is weakly shod. Indeed, suppose there are paths of non-isomorphisms from an injective module I to a projective module P of arbitrary length. Since A is shod, any such path passes through at most two hooks and so, there are paths from I to P having sectional subpaths of arbitrary length. However, if $X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_t$ is a sectional path with $t > \operatorname{rk}(\mathrm{K}_0(A))$, then $\operatorname{Hom}_A(X_i, \tau_A X_j) \neq 0$ for some i and j (by [**65**]) and so one can construct a path from X_1 to X_t with one hook $\tau_A X_j \longrightarrow * \longrightarrow X_j$. Hence, it is possible to construct paths from I to P with an arbitrary number of hooks, a contradiction, which proves the claim.

Remark 6.1.3. In [**57**], Reiten-Skowroński introduced a concept of double sections and obtained a criterion (generalising the Liu-Skowroński criterion) allowing to verify whether an artin algebra is strict shod or not. See also [**61**].

Example 6.1.4. Let k be an algebraically closed field. A finite dimensional k-algebra A is a **string algebra** if $A \cong kQ/I$, with (Q, I) a bound quiver satisfying:

(a) The ideal I is generated by a set of paths;
(b) Each point of Q is the source and the target of at most two arrows;
(c) For any arrow α, there exist at most one arrow β and at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ do not belong to I.

A reduced walk ω in (Q, I) is a **double-zero** if ω contains exactly two zero-relations which point to the same direction. It was shown by Huard and Liu in [**46**] that a string algebra $A \cong kQ/I$ is quasi-tilted if and only if (Q, I) contains no double-zero. We now consider the shod case. Let ω be a reduced walk in (Q, I) with at least two zero-relations and such that all zero-relations point to the same direction in ω, then any subwalk in ω having at least two zero-relations is a **consecutive-zero**. Finally, a path ω in Q is said to contain two **overlapping** zero-relations if $\omega = \omega_1\omega_2\omega_3$ where the ω_i are non-trivial non-zero paths such that $\omega_1\omega_2$ and $\omega_2\omega_3$ are zero-relations. If ω_1 (or ω_3) is an arrow, we say that ω is a **start-tight** (or **end-tight**, respectively).
It was shown by Bélanger and Tosar in [**23**] that a string algebra $A \cong kQ/I$ is shod if and only if A is triangular and (Q, I) satisfies the following conditions :

(i) Every consecutive zero in (Q, I) contains at most two zero-relations. In case there are two, then they are overlapping and the path containing them is either start-tight or end-tight;
(ii) (Q, I) contains no full subcategory of one of the forms (S1), (S2) and (S3) of (5.1.6)(c) or their duals.

As a consequence, it is possible to classify the shod gentle algebras as well, and hence, also, the shod algebras which are derived equivalent to a hereditary algebra of type $\mathbb{A}_n$ or $\tilde{\mathbb{A}}_n$, and the shod algebras with discrete derived category not of Dynkin type, see [**23**] for details.

6.2. The Auslander-Reiten quiver of a shod algebra. Let A be a strict shod algebra. Clearly, A is strict laura and so $\Gamma(\text{mod}A)$ contains a non-semiregular quasi-directed component Γ and any other possible component is also a component of a tilted algebra. Not much more can be said in general about the other components but one can describe Γ better. Firstly, Γ is directed (since A is weakly shod). Secondly, it can be embedded in a stable component.

Theorem 6.2.1. [**1**] *Let A be a strict shod algebra and Γ be the unique non-semiregular component of $\Gamma(\text{mod}A)$. Then Γ can be embedded in a stable component $\mathbb{Z}\Delta$, where Δ is a finite quiver without cycles.* □

We refer to [**1**] for a proof of this result. Actually, it is a bit more general than stated here. As shown in [**1**] this result cannot be extended in general to weakly shod algebras.

6.3. One-point extension shod algebras. Since shod algebras are triangular, it is convenient to look at them as one-point extensions of *smaller* algebras. Let $A = B[M]$ be the one-point extension algebra of B by M. If A is shod, so is B but for the converse, clearly, we also need that M be *well-behaved.* If M is directed, then either M is a projective B-module or $\tau_B M \in \mathrm{add}(\mathcal{L}_B)$ and we refer to [**36**] for a further discussion of these two cases. However, we mention an useful result for the case when M is a projective A-module. The equivalence of conditions (a) and (c) was first established by Huard in [**45**].

Theorem 6.3.1. [**36, 45**] *Let B be an algebra and let M be a projective B-module. The following conditions are equivalent:*

(a) $A = B[M]$ *is shod;*
(b) *For each* $(k^t, X, f) \in \mathrm{ind}A$*, either* $X \in \mathrm{add}\mathcal{L}_B$ *or* $X \in \mathrm{add}\mathcal{R}_B$*;*
(c) *For each* $(k^t, X, f) \in \mathrm{ind}A$*, either* $\mathrm{pd}_B X \leq 1$ *or* $\mathrm{id}_B X \leq 1$. □

We also refer to the survey article [**29**] for further details on shod algebras.

7. Full subcategories, split-by-nilpotent extensions and skew group algebras

7.1. Full subcategories. We consider the following problem. Let A, B be artin algebras such that $\mathrm{mod}B$ is embedded in $\mathrm{mod}A$, then which properties of $\mathrm{mod}A$ are inherited by $\mathrm{mod}B$? More specifically, we are interested in the case where A belongs to one of the classes studied before, namely, those of the left supported, laura, left (or right) glued, weakly shod and shod algebras.

We first assume that B is a full subcategory of A. Let A be an algebra, and $e \in A$ be an idempotent. We may, without loss of generality, choose e so that $B = eAe$ is connected. It is well-known (see, for instance, [**19**](II.2.5)) that, if $P = eA$, then the functor $\mathrm{Hom}_A(P, -) : \mathrm{mod}A \longrightarrow \mathrm{mod}B$ induces an equivalence between $\mathrm{mod}B$ and the full subcategory of $\mathrm{mod}A$ consisting of the P-presented modules, that is, of the A-modules M admitting a presentation

$$P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_0, P_1 \in \mathrm{add}P$.

Now, for any B-module X, the A-module $X \otimes_B P_A$ is P-presented: indeed, applying to a presentation $B_B^m \longrightarrow B_B^n \longrightarrow X_B \longrightarrow 0$ the right exact functor $- \otimes_B P_A$ yields an exact sequence $P_A^m \longrightarrow P_A^n \longrightarrow X \otimes_B P_A \longrightarrow 0$. Furthermore, applying to the second sequence the exact functor $\mathrm{Hom}_A(P, -)$ and comparing with the first yields a functorial isomorphism $\mathrm{Hom}_A(P, X \otimes_B P) \cong X_B$.

Lemma 7.1.1. [**5**, (2.2)] *Let M be a P-presented A-module.*

(a) *If* $\mathrm{pd}_A M \leq 1$*, then* $\mathrm{pd}_B\, \mathrm{Hom}_A(P, M) \leq 1$*;*
(b) *If* $M \in \mathcal{L}_A$*, then* $\mathrm{Hom}_A(P, M) \in \mathcal{L}_B$*;*
(c) *If* $M \in \mathcal{R}_A$*, but* $\mathrm{Hom}_A(P, M) \notin \mathcal{R}_B$*, then there exists a projective A-module* $P' \in \mathcal{R}_A$ *and a path* $M \rightsquigarrow P'$*;*
(d) *If* $M \in \mathcal{R}_A \setminus \mathcal{L}_A$*, then* $\mathrm{Hom}_A(P, M) \in \mathcal{R}_B$.

Proof. We only prove (a) and (b). Assume that M is a P-presented module such that $\mathrm{pd}_A M \leq 1$. There exists a presentation $P_1' \longrightarrow P_0' \longrightarrow M \longrightarrow 0$ with $P_0', P_1' \in \mathrm{add}P$. Therefore, in a minimal projective resolution $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, the projective modules P_0 and P_1 are, respectively, summands of P_0' and P_1' and thus lie in $\mathrm{add}P$. Applying $\mathrm{Hom}_A(P, -)$ shows that $\mathrm{pd}_B \mathrm{Hom}_A(P, M) \leq 1$. This proves (a).

If now $M \in \mathcal{L}_A$, let $X_0 \xrightarrow{u_1} X_1 \xrightarrow{u_2} \cdots \xrightarrow{u_t} X_t = \mathrm{Hom}_A(P, M)$ be a path in $\mathrm{ind}B$. Setting, for each i, $M_i = X_i \otimes_B P$ and $f_i = u_i \otimes_B P$, the comments above show that we have a path $M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = M$ in $\mathrm{ind}A$, where all M_i are P-presented. Since $M \in \mathcal{L}_A$, then $\mathrm{pd}_A M_0 \leq 1$. By (a), $\mathrm{pd}_B X_0 \leq 1$. Thus $\mathrm{Hom}_A(P, M) \in \mathcal{L}_B$ and (b) is proven. □

We now state the first theorem of this section, of which part (d) was first shown in [**48**](1.2) and part (e) in [**42**](II.1.5).

Theorem 7.1.2. [**5**]. *Let A be an algebra, and $e \in A$ be an idempotent such that $B = eAe$ is connected.*

- (a) *If A is a laura algebra, then so is B;*
- (b) *If A is a left (or right) glued algebra, then so is B;*
- (c) *If A is a weakly shod algebra, then so is B;*
- (d) *If A is a shod algebra, then so is B;*
- (e) *If A is a quasi-tilted algebra, then so is B.*

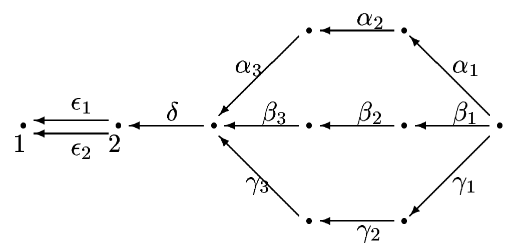
Proof. (a) Let $X \in \mathrm{ind}B$. If $X \notin \mathcal{L}_B \cup \mathcal{R}_B$, then by (7.1.1), the P-presented A-module $X \otimes_B P$ does not lie in $\mathcal{L}_A \cup \mathcal{R}_A$. The statement follows.
(b) This is shown similarly.
(d) If A is shod, and $X \in \mathrm{ind}B$, then $X \otimes_B P$ either lies in $\mathcal{L}_A$ or in $\mathcal{R}_A \setminus \mathcal{L}_A$. In the first case $X \in \mathcal{L}_B$ and, in the second, $X \in \mathcal{R}_B$.
(e) Let P' be an indecomposable projective B-module. Since A is quasi-tilted the projective A-module $P' \otimes_B P_A$ lies in $\mathcal{L}_A$. By (7.1.1), $P' \in \mathcal{L}_B$. Hence, B is quasi-tilted by [**42**, (II.1.14)].
(c) [Sketch] Assume A to be weakly shod. We may, by (e), suppose that it is not quasi-tilted. Let s_1 denote the number of P-presented A-modules $M \in \mathcal{R}_A$ such that $\mathrm{Hom}_A(P, M) \notin \mathcal{R}_B$. By (2.1.2) and (7.1.1), s_1 is finite. Let also s_2 be the cardinality of the set $\mathrm{ind}A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)$. Clearly $s_2 < \infty$. In view of (5.1.2), it suffices to show that $s_1 + s_2 - 1$ is a bound on the length of the paths from an indecomposable not in $\mathcal{L}_B$ to one not in $\mathcal{R}_B$. □

We recall from [**41**, (III.6.5)] that every full subcategory of a tilted algebra is tilted. We now prove that a tubular algebra cannot occur as a full subcategory of a strict laura algebra.

Corollary 7.1.3. [**5**, (3.4)] *Let A be a strict laura algebra. If B is quasi-tilted, then it is tilted.*

Proof. If all indecomposable summands of $P = eA$ lie in $\mathcal{L}_A$, then P is a module over the left support A_λ, which is tilted, by (4.3.4). Hence, $B = \mathrm{End}P$ is tilted. Otherwise, let P' be an indecomposable summand of P lying in $\mathcal{R}_A \setminus \mathcal{L}_A$. By (7.1.1)(d), $\mathrm{Hom}_A(P, P') \in \mathcal{R}_B$ and is projective, so if B is quasi-tilted, it follows from [**42**, (II.3.4)] that B is tilted. □

Example 7.1.4. If A is left supported, it does not follow that so is B. Let A be given by the quiver



bound by $\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 + \gamma_1\gamma_2\gamma_3 = 0$, $\alpha_3\delta = 0$, $\beta_3\delta = 0$, $\gamma_3\delta = 0$. $\delta\epsilon_1 = 0$, $\delta\epsilon_2 = 0$. Then A is left supported, but the full (even convex!) subcategory generated by all points except 1 and 2, is tubular, thus is not left supported.

7.2. Split extensions. We now consider another construction. Informally, if one can roughly think of taking full subcategories as *deleting points*, the construction we now outline can be thought of as *deleting arrows*.

Definition 7.2.1. **[13]** *Let A and B be artin algebras and let Q be a nilpotent ideal of A (that is, $Q \subseteq \operatorname{rad} A$). We say that A is a* **split extension of** *B* **by** *Q if there exists a split surjective algebra morphism $A \longrightarrow B$ with kernel Q.*

In particular, B is a subalgebra of A (and has the same primitive idempotents). For instance, if $Q^2 = 0$, then the above definition coincides with that of trivial extension of B by Q. Another example is the following: let $A = \begin{bmatrix} C & 0 \\ M & k \end{bmatrix}$ be the one point extension of C by M_C, and x be the extension point. Then A is a split extension of $B = C \times k$ by the bimodule Q such that $Q_B \cong M_B$ and ${}_BQ \cong S_x^t$ for some $t \geq 1$. For further examples, we refer the reader to **[13, 16, 17]**.

Clearly, if A and B are as above, and B is a connected algebra, then so is A, but the converse is generally not true.

We have the change of rings functors $- \otimes_B A : \operatorname{mod} B \longrightarrow \operatorname{mod} A$, $- \otimes_A B : \operatorname{mod} A \longrightarrow \operatorname{mod} B$, $\operatorname{Hom}_B(A_B, -) : \operatorname{mod} B \longrightarrow \operatorname{mod} A$, $\operatorname{Hom}_A(B_A, -) : \operatorname{mod} A \longrightarrow \operatorname{mod} B$, which satisfy the functorial isomorphisms $- \otimes_B A \otimes_A B \cong 1_{\operatorname{mod} B}$ and $\operatorname{Hom}_A(B_A, \operatorname{Hom}_B(A_B, -)) \cong 1_{\operatorname{mod} B}$.

We refrain from proving the following key lemma.

Lemma 7.2.2. **[17**, (2.4)**]** *Let X be an indecomposable B-module.*

(a) *If $X \otimes_B A$ belongs to $\mathcal{L}_A$, then X_B belongs to $\mathcal{L}_B$;*
(b) *$X \otimes_B A$ belongs to $\mathcal{R}_A$, then X_B belongs to $\mathcal{R}_B$;*
(c) *If $\operatorname{Hom}_B(A, X)$ belongs to $\mathcal{R}_A$, then X_B belongs to $\mathcal{R}_B$;*
(d) *If $\operatorname{Hom}_B(A, X)$ belongs to $\mathcal{L}_A$, then X_B belongs to $\mathcal{L}_B$.* □

The second main theorem of this section is the following.

Theorem 7.2.3. **[17]** *Let A be a split extension of B.*

(a) *If A is a laura algebra, then so is B;*

(b) *If A is a left (or right) glued algebra, then so is B;*
(c) *If A is a weakly shod algebra, then so is B;*
(d) *If A is a shod algebra, then so is B;*
(e) *If A is a quasi-tilted algebra, then so is B.*

Proof. (a) Let $X \in \text{ind}B$. If $X \notin \mathcal{L}_B \cup \mathcal{R}_B$, then by (7.2.2) $X \otimes_B A \notin \mathcal{L}_A \cup \mathcal{R}_A$. The statement follows.
(b) This is shown similarly.
(c) Follows directly from (7.2.2) and (5.1.2).
(d) Follows directly from (7.2.2).
(e) Let P_B be indecomposable projective. Since A is quasi-tilted, the indecomposable projective A-module $P \otimes_B A$ lies in $\mathcal{L}_A$. Hence $P_B \in \mathcal{L}_B$, by (7.2.2). Invoking [**42**, (II.1.14)] concludes the proof. □

It is not known whether, if A and B are as above, and A is tilted, then so is B. We have, however, the following partial result.

Proposition 7.2.4. [**17**, (2.6)] *Assume that B is connected. If A is tilted having an indecomposable projective (or injective) in a connecting component of its Auslander-Reiten quiver (for instance, if A is tame), then B is tilted.*

Proof. Assume that B is connected, so is A. Also, by (7.2.3), B is quasi-tilted. By [**42**, (II.3.4)], there exists, up to duality, an indecomposable projective P_B such that $P \otimes_B A \in \mathcal{R}_A$ By (7.2.2), $P \in R_B$. Another application of [**42**, (II.3.4)] finishes the proof. □

An interesting problem would be the following. Assume that B is a laura algebra (or is left or right glued, or is weakly shod, or is shod, or is quasi-tilted, or is tilted) then under which conditions on Q does one have A lying in the same class?

Example 7.2.5. Since one-point extensions are special cases of split extensions, it follows from (7.1.3) that, if A is laura not quasi-tilted, and B is quasi-tilted, then B is tilted. We show that any of the remaining cases may occur. Let A be the radical square zero algebra given by the quiver of (4.3.3)(b). Then A is a laura algebra which is not weakly shod.

(a) Let Q_1 be the ideal of A generated by α_3. Then A is a split extension of the laura, not weakly shod, algebra A/Q_1;
(b) Let Q_2 be the ideal of A generated by ϵ_1,ϵ_2. Then A is a split extension of the left glued, not weakly shod, algebra A/Q_2;
(c) Let Q_3 be the ideal generated by δ. Then A is a split extension of the weakly shod algebra, not shod, algebra A/Q_3;
(d) Let Q_4 be the ideal of A generated by β, γ. Then A is a split extension of the shod, not quasi-tilted, algebra A/Q_4;
(e) Let Q_5 be the ideal do A generated by $\alpha_1, \alpha_2, \alpha_3, \beta$. Then A is a split extension of the quasi-tilted, even tilted, algebra A/Q_5.

7.3. Skew group algebras. Another contructing preserving homological properties is that of skew group algebras. We recall the relevant definitions and refer the reader to [**19, 56, 54**] for details. A finite group G (of identity 1) is said to **act** on an artin k-algebra A if there is a function: $G \times A \to A$, $(\sigma, a) \mapsto \sigma(a)$ satisfying the following:

i) $\sigma : A \to A$, $a \mapsto \sigma(a)$ is a k-linear automorphism for all $\sigma \in G$;
ii) $(\sigma_1\sigma_2)(a) = \sigma_1(\sigma_2(a))$ for all $\sigma_1, \sigma_2 \in G$ and $a \in A$;
iii) $1(a) = a$ for all $a \in A$.

If G acts on A, then the **skew group algebra** $R = A[G]$ is defined as follows:

i) As an A-module, $A[G]$ is the free right A-module having the elements of G as a basis;
ii) The multiplication in $A[G]$ is defined by the rule:

$$(\rho b)(\sigma a) = (\rho\sigma)(\sigma(b)a) \text{ for all } a, b \in A \text{ and } \sigma, \rho \in G.$$

Throughout, we assume that the order $|G|$ of G is invertible in A. The embedding $A \hookrightarrow R$ given by $a \mapsto 1.a$ induces the functors $- \otimes_A R : \text{mod}A \to \text{mod}R$ (induction) and $\text{Hom}_R({}_AR_R, -) : \text{mod}R \to \text{mod}A$ (restriction of scalars). It is shown in [**56**] that $- \otimes_A R$ is both left and right adjoint to $\text{Hom}_R({}_AR, -)$. The interest of skew group algebras comes, among other reasons, from its connexion with finite coverings [**54**]. It was shown in [**42, 56**] that the skew group algebra $R = A[G]$ is quasi-tilted (or tilted) if and only if so is A. In fact, if one reads carefully the proof in [**42**, (III.1.6)], it is also shown that R is shod if and only if so is A.

Lemma 7.3.1. [**10**] *With the above notation, we have:*

(a) $(\text{add}\mathcal{L}_A) \otimes_A R \subseteq \text{add}\mathcal{L}_R$;
(b) $(\text{add}\mathcal{R}_A) \otimes_A R \subseteq \text{add}\mathcal{R}_R$;
(c) $\text{Hom}_R(R, \text{add}\mathcal{L}_R) \subseteq \text{add}\mathcal{L}_A$;
(d) $\text{Hom}_R(R, \text{add}\mathcal{R}_R) \subseteq \text{add}\mathcal{R}_A$. □

The proof of this lemma uses essentially (2.1.1).

Theorem 7.3.2. [**10**] *Let A be an artin algebra, G be a finite group acting on A such that $|G|$ is invertible in A, and $R = A[G]$ be the skew group algebra. Then:*

(a) *R is a laura algebra if and only if so is A;*
(b) *R is a left (or right) glued algebra if and only if so is A;*
(c) *R is weakly shod if and only if so is A;*
(d) *R is shod if and only if so is A.*

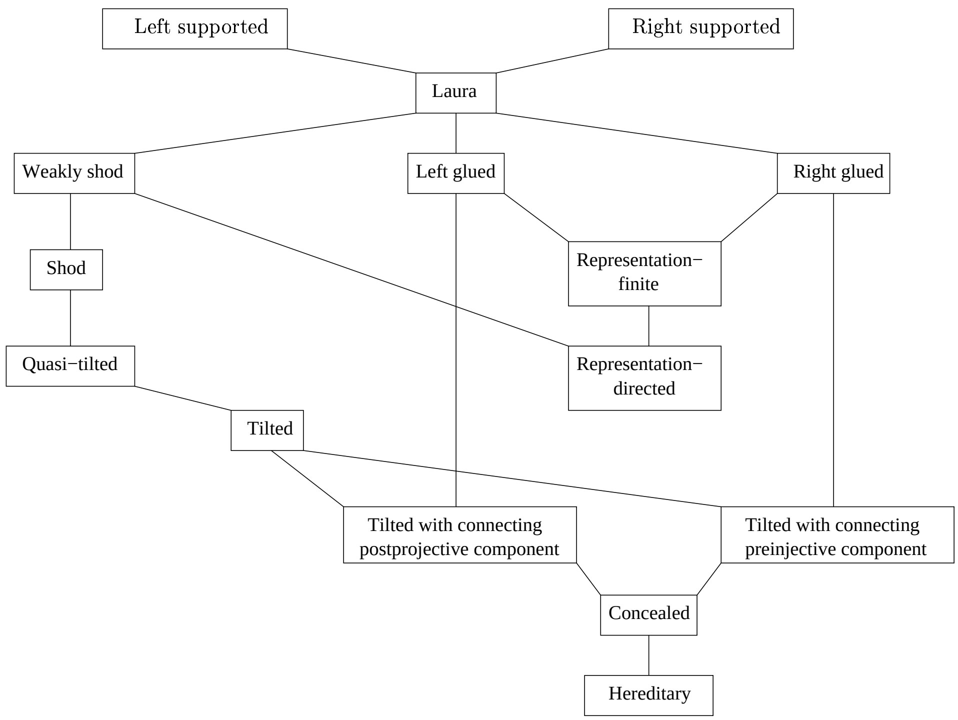
Proof. We just prove (a). Assume A to be laura, and let $\mathcal{F}$ be the subset of $\text{ind}R$ consisting of the summands of the modules in the class

$$[\text{ind}A \setminus (\mathcal{L}_A \cup \mathcal{R}_A)] \otimes_A R.$$

Since A is laura, then $\mathcal{F}$ is finite. Let $X \in \text{ind}R$ be such that $X \notin \mathcal{L}_R \cup \mathcal{R}_R$. By the above lemma, $X \notin \text{add}\mathcal{L}_A \otimes_A R$ and $X \notin \text{add}\mathcal{R}_A \otimes_A R$. Consequently $X \in \mathcal{F}$. Thus $\text{ind}R \setminus (\mathcal{L}_R \cup \mathcal{R}_R) \subseteq \mathcal{F}$ and so R is laura. The converse is proven in the same manner. □

SUMMARY

The following inclusion diagram between classes of algebras may be useful.



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