ADDENDUM TO "CLUSTER AUTOMORPHISMS AND COMPATIBILITY OF CLUSTER VARIABLES"

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We prove that Conjecture 1.1 and 1.2 hold true for cluster algebras of Dynkin type. For definitions and notations we refer to [ASS2].

Theorem 0.1. *Let $\mathcal{A}$ be a cluster algebra of Dynkin type, then $\mathcal{A}$ is unistructural.*

Proof. Assume that $\mathcal{A}$ is given two cluster structures $\mathcal{X} = \cup_\alpha \mathbf{x}_\alpha = \cup_\beta \mathbf{x}'_\beta$ where $\mathbf{x}_\alpha$ and $\mathbf{x}'_\beta$ are clusters in their respective structures. Denote the two cluster structures by S and S', respectively. Let $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ be the initial cluster of S. We claim that $\mathbf{x}$ is also a cluster in S'. If not, then there exist two initial variables x_i, x_j which are not compatible in S': indeed, this can be seen by using the well-known bijections between clusters and tilting objects in the associated cluster category, and between cluster variables and rigid indecomposable objects. Because of the positivity theorem, each of x_i and x_j is a positive element in $\mathcal{A}$ (in both structures), hence so is their product $x_i x_j$. Cerulli-Irelli showed that the cluster monomials form an atomic basis for $\mathcal{A}$, see [C] Th.1.1, which implies that every positive element is a linear combination of cluster monomials with non-negative coefficients.

Therefore the product $x_i x_j$ in the structure S' can be written as a positive linear combination of cluster monomials: $x_i x_j = \sum \lambda_{M'} M'$. Each of the cluster monomials M' is a product of S'-compatible cluster variables and each of these cluster variables can be written as a positive Laurent polynomial in $x_1, \ldots, x_n$, because the latter is a cluster of S and both structures have the same set of cluster variables. Thus the cluster monomial M' can also be written as a positive Laurent polynomial $\mathcal{L}(M')$ in $\{x_1, \ldots, x_n\}$.

Replacing each M' by $\mathcal{L}(M')$ in the sum $\sum \lambda_{M'} M'$, we get

$$x_i x_j = \sum \lambda_{M'} \mathcal{L}(M')$$

and because of positivity there is no cancellation of terms in the right hand side. Therefore the sum $\sum \lambda_{M'} M'$ has only one term $M' = x_i x_j$ and $\lambda_{M'} = 1$. But this means that x_i and x_j are S'-compatible, a contradiction. This proves that $\{x_1, \ldots, x_n\}$ is a cluster in the structure S'.

In order to complete the proof it suffices to show that the quiver Q' of the cluster $\{x_1, \ldots, x_n\}$ in the structure S' is equal or opposite to the quiver Q of the same cluster in the structure S. The mutations μ_i and μ'_i in the direction i applied to the cluster $\{x_1, \ldots, x_n\}$ in both structures S and S' will produce a variable whose denominator is x_i. Namely,

$$\mu_i(x_i) = \frac{\prod\limits_{i \to j \text{ in } Q} x_j + \prod\limits_{i \leftarrow j \text{ in } Q} x_j}{x_i} \qquad \text{and} \qquad \mu'_i(x_i) = \frac{\prod\limits_{i \to j \text{ in } Q'} x_j + \prod\limits_{i \leftarrow j \text{ in } Q'} x_j}{x_i}.$$

Since both cluster structures have the same cluster variables and since in the Dynkin case the denominators determine the cluster variables, it follows that $\mu_i(x_i) =$

$\mu'_i(x_i)$ and therefore either

$$\prod_{i \to j \text{ in } Q} x_j = \prod_{i \to j \text{ in } Q'} x_j \qquad \text{and} \qquad \prod_{i \leftarrow j \text{ in } Q} x_j = \prod_{i \leftarrow j \text{ in } Q'} x_j$$

or

$$\prod_{i \to j \text{ in } Q} x_j = \prod_{i \leftarrow j \text{ in } Q'} x_j \qquad \text{and} \qquad \prod_{i \leftarrow j \text{ in } Q} x_j = \prod_{i \to j \text{ in } Q'} x_j.$$

Since i is arbitrary and Q is connected, this implies that $Q = Q'$ or $Q = Q'^{op}$. □

Corollary 0.2. *Conjecture 1.1 holds true for cluster algebras of Dynkin type.*

Proof. This follows from the above theorem and Theorem 1.4. □

References

[ASS2] I. Assem, R. Schiffler and V. Shramchenko, Cluster automorphisms and compatibility of cluster variables, Glasgow Mathematical Journal

[C] G. Cerulli Irelli, Positivity in skew-symmetric cluster algebras of finite type, arxiv: 1102.3050