# Abelian exact subcategories closed under predecessors

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**Abstract**

**In the category of finitely generated modules over an artinian ring, we classify all the abelian exact subcategories closed under predecessors or, equivalently, all the split torsion pairs with torsion-free class closed under quotients. In the context of Artin algebras, the result is then applied to the left part of the module category and to local extensions of hereditary algebras**

## 1 Introduction

Let $A$ be an Artin algebra, $mod_A$ be its category of right $A$-modules, and $ind_A$ be a full subcategory of $mod_A$ consisting of a complete set of representatives of the isomorphism classes of indecomposable $A$-modules. The left part $\mathcal{L}_A$ and the right part $\mathcal{R}_A$ of $mod_A$ were introduced by Happel, Reiten and Smal$\emptyset$ in their study of quasi-tilted algebras ([5]). These have repeatedly proved their usefulness in the study of homological properties of the algebra. Our initial motivation for the present paper was the following question: when is the additive

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closure $add(\mathcal{L}_A)$ of $\mathcal{L}_A$ an abelian exact subcategory of $mod_A$? (see definition below). As our study advanced, we noticed that the particular consideration of $\mathcal{L}_A$ was not essential, and our goal then shifted to classify all the full subcategories $\mathcal{C} \subseteq ind_A$, closed under predecessors, such that $add(\mathcal{C})$ is an abelian exact subcategory of $mod_A$. This is easily seen to be equivalent to the classification of all split torsion pairs in $mod_A$, with torsion-free class closed under quotients. In addition, we realized that the restriction to Artin algebras was not necessary and that our classification held in the more general context of (right) artinian rings. The desired classification is given in corollary 2.6 as a direct consequence of our main result, theorem 2.5.

This theorem states that, for a basic and connected right artinian ring $A$, the existence of such a subcategory $\mathcal{C}$ of $ind_A$ is equivalent to the existence of an isomorphism $A \cong \begin{pmatrix} C & 0 \\ M & B \end{pmatrix}$, where $M$ is a $B-C-$bimodule which is hereditary injective over $C$, and such that $\mathcal{C}$ gets identified with $ind_C$. In case $A$ is an Artin algebra or, more generally, an artinian ring with selfduality, our methods can be dualized to yield a classification of those subcategories $\mathcal{C} \subseteq ind_A$ closed under successors and such that $add(\mathcal{C})$ is an abelian exact subcategory of $mod_A$. We leave the primal-dual translation to the reader.

The paper is organized as follows. Section 2 is devoted to proving the main theorem, for which we need several equivalent characterizations of the desired subcategories (see Proposition 2.2 below). Section 3 contains applications of the theorem to Artin algebras, in the case where $\mathcal{C} = \mathcal{L}_A$. Thus, we prove that if the quiver of $A$ has no oriented cycles, then $add(\mathcal{L}_A)$ is an abelian exact subcategory of $mod_A$ if and only if $A$ is hereditary (see Corollary 3.2 below). We also prove that if $A$ is a local extension of a hereditary algebra $H$ (by a bimodule ${}_RM_H$), then $add(\mathcal{L}_A)$ is an abelian exact subcategory of $mod_A$ if, and only if, $M_H$ is injective (see Proposition 3.5).

## 2 The main theorem

Throughout this section, $A$ is a basic right artinian ring, which we assume connected (that is, indecomposable as a ring). Modules are finitely generated right modules. All subcategories of $mod_A$ or $ind_A$ are assumed closed under isomorphic images. For a full subcategory $\mathcal{C}$ of $mod_A$, we denote by $add(\mathcal{C})$ the full subcategory of $mod_A$ having as objects the direct summands of finite direct sums of modules in $\mathcal{C}$. We also write briefly $X \in \mathcal{C}$ to express that $X$ is an object of $\mathcal{C}$ . For an $A$-module $M$, $Gen(M)$ stands for the full subcategory consisting of those modules which are generated by $M$, that is, which are quotients of modules in $add(M)$. We refer the reader to [1] and [3][Chapter I] for concepts about artinian rings not specifically defined here.

Given $X, Y \in ind_A$, a **path** from $X$ to $Y$ is a sequence $X = X_0 \xrightarrow{f_1} X_1 \to \ldots \xrightarrow{f_t} X_t = Y$ of non-zero morphisms $f_i$ between indecomposable $A$-modules. In this case, we say that $X$ is a **predecessor** of $Y$ (and that $Y$ is a **successor** of $X$). A full subcategory $\mathcal{C} \subseteq ind_A$ is called **closed under predecessors** when every predecessor of a module in $\mathcal{C}$ lies in $\mathcal{C}$. When $\mathcal{C}$ is closed under predecessors, the direct sum $P = P_{\mathcal{C}}$ of all (indecomposable) projective modules in $\mathcal{C}$ is called the **supporting projective** module of $\mathcal{C}$.

We recall that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $mod_A$ is called a **torsion pair**, when it satifies the following two conditions: i) a module $X_A$ is in $\mathcal{T}$ if, and only if, $Hom_A(X, F) = 0$ for all $F \in \mathcal{F}$, and ii) a module $X_A$ is in $\mathcal{F}$ if, and only if, $Hom_A(T, X) = 0$ for all $T \in \mathcal{T}$. In this case, we have an idempotent subfunctor of the identity $t : mod_A \longrightarrow mod_A$, called the **torsion radical**, such that $X \in \mathcal{T}$ if and only if $t(X) = X$. The class $\mathcal{T}$ is called the **torsion class**, and the class $\mathcal{F}$ is called the **torsion-free class** of the pair. The pair $(\mathcal{T}, \mathcal{F})$ is called **split** when $t(X)$ is a direct summand of $X$, for all $X \in mod_A$, or, equivalently, when every indecomposable $A$-module $X$ either belongs to $\mathcal{T}$ or to $\mathcal{F}$.

The following lemma is well-known.

**Lemma 2.1.** *Let $\mathcal{C}$ be a full subcategory of $ind_A$. The following assertions are equivalent:*

*(1) $\mathcal{C}$ is closed under predecessors.*

*(2) If $X \in ind_A$ then either $X \in \mathcal{C}$ or $Hom_A(X, Y) = 0$, for all $Y \in \mathcal{C}$.*

*(3) $add(\mathcal{C})$ is the torsion-free class of a split torsion pair in $mod_A$.*

In this paper we use the following terminology.

**Definition 1.** *A full subcategory $\mathcal{A}$ of $mod_A$ is said to be an* **abelian exact subcategory***, when it is abelian as a category and the inclusion functor $\mathcal{A} \hookrightarrow mod_A$ is exact*

It is easily seen that a full subcategory $\mathcal{A}$ is an abelian exact subcategory of $mod_A$ if, and only if, it is closed under kernels and cokernels. In general, a full subcategory can be abelian as a category without being an abelian exact subcategory of $mod_A$.

**Proposition 2.2.** *Let $\mathcal{C}$ be a full subcategory of $ind_A$ closed under predecessors. The following statements are equivalent:*

*(1) $add(\mathcal{C})$ is an abelian exact subcategory of $mod_A$.*

*(2) $add(\mathcal{C})$ is closed under cokernels.*

*(3) $add(\mathcal{C})$ is closed under quotients.*

*(4) For every (indecomposable) projective $P_0 \in \mathcal{C}$, we have $top(P_0) \in \mathcal{C}$.*

*(5) $\mathcal{C}$ is closed under composition factors.*

*(6) $add(\mathcal{C}) = Gen(P)$, where $P$ is the supporting projective module of $\mathcal{C}$.*

*Proof.* Since $\mathcal{C}$ is closed under predecessors, $add(\mathcal{C})$ is closed under submodules and, in particular, under kernels and images. Thus (1) and (2) are clearly equivalent.

(2) is equivalent to (3): Since $add(\mathcal{C})$ is closed under submodules, every quotient of a module in $add(\mathcal{C})$ is the cokernel of a morphism in $add(\mathcal{C})$. Thus (2) implies (3). The reverse implication is trivial.

(3) implies (4): This is clear

(4) implies (5): If $X \in \mathcal{C}$, then in the radical filtration $X \supset XJ(A) \supset XJ(A)^2 \supset ....$ all the terms are direct sums of predecessors of $X$. Hence, all belong to $add(\mathcal{C})$. Since $\mathcal{C}$ is closed under predecessors, then, for every $k \geq 0$, the projective cover $P_k$ of $XJ(A)^k$ belongs to $add(\mathcal{C})$. The hypothesis 4 implies that $\frac{XJ(A)^k}{XJ(A)^{k+1}} \cong top(P_k)$ belongs to $add(\mathcal{C})$. Since every composition factor of $X$ is direct summand of some $\frac{XJ(A)^k}{XJ(A)^{k+1}}$, the statement 5 follows.

(5) implies (2): If $f : X \longrightarrow Y$ is a morphism between modules in $\mathcal{C}$, the hypothesis guarantees that all composition factors of $Z = coker(f)$ lie in $\mathcal{C}$. In particular, $top(Z) \in add(\mathcal{C})$. Since $\mathcal{C}$ is closed under predecessors, we have $Z \in add(\mathcal{C})$, so that $add(\mathcal{C})$ is closed under cokernels.

(3) implies (6) : Since $\mathcal{C}$ is closed under predecessors, the projective cover of a module $X \in \mathcal{C}$ belongs to $add(\mathcal{C})$ and, consequently, to $add(P)$. Hence $add(\mathcal{C}) \subseteq Gen(P)$. The reverse inclusion follows from the fact that $add(\mathcal{C})$ is closed under quotients.

Since (6) trivially implies (4), the proof is complete. □

We recall that an additive full subcategory $\mathcal{D}$ of $mod_A$ is **contravariantly finite** if, for every $X \in mod_A$, there is a morphism $f : D_X \longrightarrow X$ (called a right approximation) such that $D_X \in \mathcal{D}$ and, for any other morphism $g : D \longrightarrow X$, with $D \in \mathcal{D}$, there exists $h : D \longrightarrow D_X$ such that $f \circ h = g$. **Covariantly finite** subcategories are defined dually, and a subcategory is called **functorially finite** if it is both covariantly and contravariantly finite (see [4]).

**Corollary 2.3.** *Let $\mathcal{C}$ be a full subcategory of $ind_A$ closed under predecessors such that $add(\mathcal{C})$ is an abelian exact subcategory of $mod_A$, and let $e \in A$ be an idempotent such that $P = eA$ is (isomorphic to) the supporting projective of $\mathcal{C}$. The following assertions hold:*

*(1) $add(\mathcal{C})$ is functorially finite in $mod_A$*

*(2) $(Gen((1-e)A), add(\mathcal{C})) = (Gen((1-e)A), Gen(eA))$ is the split torsion pair in $mod_A$ having $add(\mathcal{C})$ as torsion-free class.*

*(3) The torsion radical $t$ of the above torsion pair is given by $t(X) = X(1-e)A$, for every $X \in mod_A$*

*Proof.* (1) Every torsion-free class is covariantly finite. By Proposition 2.2(6), $add(\mathcal{C}) = Gen(P)$ is contravariantly finite, the (minimal) right approximation of $X$ being the inclusion $t_P(X) \hookrightarrow X$, where $t_P(X)$ is the trace of $P$ in $X$, that is, $t_P(X) = \sum_{f \in Hom_A(P,X)} Im(f)$.

(2) Consider the split torsion pair $(\mathcal{T}, add(\mathcal{C}))$. Since $\mathcal{C}$ is closed under composition factors, an $A$-module $X$ lies in $\mathcal{T}$ if, and only if, $top(X)$ contains no simple summand from $\mathcal{C}$, that is, if and only if $top(X) \in Gen((1-e)A)$. This is equivalent to saying that $X$ is generated by $(1-e)A$.

(3) $t(X)$ is the (unique) maximal submodule of $X$ belonging to $Gen((1-e)A)$, which is the trace $t(X) = \sum_{f \in Hom_A((1-e)A,X)} Im(f) = X(1-e)A$

□

We recall that an $A$-module $I$ is called **hereditary injective** if every quotient of $I$ (or of $I^r$, with $r > 0$) is an injective $A$-module.

**Remark 2.4.** *If* $A = \begin{pmatrix} C & 0 \\ M & B \end{pmatrix}$, *where* $M$ *is a* $B-C-$*bimodule, then the right* $A$*-modules can be viewed as triples* $(X, Y, \varphi)$, *where* $X \in mod_C$, $Y \in mod_B$ *and* $\varphi : Y \otimes_B M \longrightarrow X$ *is a morphism in* $mod_C$ *(see [3][Chapter III]). In this case, we may, and shall, identify* $mod_C$ *with the full subcategory of* $mod_A$ *having as objects the triples* $(X, 0, 0)$, *with* $X \in mod_C$.

For any right artinian ring $R$, we denote by $gl.dim(R)$ the global dimension of $R$. We are now able to state, and prove, the main result of this paper.

**Theorem 2.5.** *Let* $A$ *be a basic connected right artinian ring and* $\mathcal{C}$ *be a full subcategory of* $ind_A$. *The following assertions are equivalent:*

*(1)* $\mathcal{C}$ *is closed under predecessors and* $add(\mathcal{C})$ *is an abelian exact subcategory of* $mod_A$

*(2) There exists a ring isomorphism* $A \cong \begin{pmatrix} C & 0 \\ M & B \end{pmatrix}$ *such that* $M_C$ *is a hereditary injective* $C$*-module and* $add(\mathcal{C}) \cong mod_C$

*(3) There exists an idempotent* $e \in A$ *such that* $eA(1-e) = 0$, $\mathcal{C}$ *consists of those* $X \in ind_A$ *such that* $Xe = X$ *and every* $Y \in ind_A \setminus \mathcal{C}$ *is generated by* $(1-e)A$

*Further, if this is the case, then* $gl.dim(C) = Sup\{pd(X_A) : X \in \mathcal{C}\}$

*Proof.* (1) implies (3): Let $e \in A$ be an idempotent such that $eA = P$ is the supporting projective of $\mathcal{C}$. By proposition 2.2, $add(\mathcal{C}) = Gen(eA)$ and, by corollary 2.3, the corresponding split torsion pair is $(Gen((1-e)A, Gen(eA))$. Therefore $eA(1-e) \cong Hom_A((1-e)A, eA) = 0$ and so $XeA = Xe$, for all $X \in mod_A$. Hence $X \in add(\mathcal{C})$ if, and only if, $X = Xe$. The last statement follows from the fact that the torsion pair is split.

(3) implies (1): Since $eA(1-e) = 0$, we have $Gen(eA) = \{X \in mod_A : X = Xe\}$. The hypothesis (3) says exactly that $(Gen((1-e)A), Gen(eA)) =$

$(Gen((1-e)A), add(\mathcal{C})$ is a split torsion pair. The statement then follow from lemma 2.1 and proposition 2.2

(2) implies (3): Setting $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we have $1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ so that, clearly, $eA(1-e) = 0$. The equality $\mathcal{C} = \{X \in ind_A : Xe = X\} = \{X \in ind_A : X(1-e) = 0\}$ follows from the interpretation of $mod_C$ as a full subcategory of $mod_A$. There remains to prove that $ind_A \setminus \mathcal{C} \subset Gen(1-e)A)$. Let $X \notin \mathcal{C}$ be indecomposable. We claim that

$$X(1-e)AeA = X(1-e)A \cap XeA$$

Clearly, we have $X(1-e)AeA \subseteq X(1-e)A \cap XeA$. Conversely, if $x \in X(1-e)A \cap XeA$ then $x = xe$, due to the equality $XeA = Xe$. On the other hand, $x = \sum_{1 \leq i \leq n} y_i(1-e)a_i$, with $a_i \in A$ and $y_i \in X$. But then $x = xe = \sum_{1 \leq i \leq n} y_i(1-e)a_ie$ belongs to $X(1-e)AeA$, thus establishing our claim.

The $A$-module $X(1-e)AeA$ is generated by $(1-e)AeA = M$ which, by hypothesis, is a hereditary injective $C$-module. Hence, $X(1-e)AeA$ is injective in $mod_C$, and so we have a decomposition

$$XeA = X(1-e)AeA \oplus X'$$

in $mod_C$. Considering this decomposition in $mod_A$ via the embedding $mod_C \hookrightarrow mod_A$, we have

$$X = XeA + X(1-e)A = X(1-e)AeA + X' + X(1-e)A = X' + X(1-e)A$$

But $X' \cap X(1-e)A \subseteq XeA \cap X(1-e)A = X(1-e)AeA$, and so $X' \cap X(1-e)A \subseteq X' \cap X(1-e)AeA = 0$. We thereby get a decomposition

$$X = X' \oplus X(1-e)A$$

in $mod_A$. Since $X_A$ is indecomposable and $X(1-e) \neq 0$ (because $X \notin \mathcal{C}$), we conclude that $X' = 0$ and, hence, $X = X(1-e)A \in Gen((1-e)A)$ as desired.

(1) and (3) imply (2): From (3), letting $C = eAe$, $e' = 1 - e$, $B = e'Ae'$ and $M = e'Ae$, we may identify $A$ with the matrix algebra $A = \begin{pmatrix} C & 0 \\ M & B \end{pmatrix}$ and $\mathcal{C}$ with $ind_C = \{X \in ind_A : Xe = X\} = \{X \in ind_A : Xe' = 0\}$. By corollary 2.3, the torsion radical associated with the split torsion pair $(Gen(e'A), add(\mathcal{C}))$ is given by $t(X) = Xe'A$, so that $Xe'A$ is a direct summand of $X$, for every $X \in mod_A$. Let us fix a complete set of primitive orthogonal idempotents $\{e_1, ..., e_s\}$ of $B = e'Ae'$, so that $e' = e_1 + ... + e_s$. Interpreting $A$-modules as triples, as in remark 2.4(2), we have that $e_iJ(A)e'A = (e_iJ(B)M, e_iJ(B), \mu)$ is a direct summand of $e_iJ(A) = (e_iM, e_iJ(B), \mu')$ in $mod_A$ (where $\mu$, $\mu'$ are the respective multiplication maps). This clearly implies that $e_iJ(B)M$ is a direct summand of $e_iM$ in $mod_C$.

We fix, for each $i$, a decomposition $e_iM = M'_i \oplus e_iJ(B)M$ in $mod_C$, and let $M' = \oplus_{1\leq i\leq s}M'_i$. Then $e_iM' = M'_i$ for each $i$, and $M = M' \oplus J(B)M$ in $mod_C$. An easy induction shows that $J(B)M = J(B)M' + J(B)^kM$, for all $k > 0$. The nilpotency of $J(B)$ yields $J(B)M = J(B)M'$. Since the left multiplication by an element of $B$ gives an endomorphism of $M_C$, the equality $J(B)M = J(B)M'$ implies that $J(B)M$ is generated by $M'$ in $mod_C$. But then $M_C = M' \oplus J(B)M$ is also generated by $M'$ in $mod_C$. In order to prove that $M_C$ is hereditary injective, it suffices to show that each $M'_i = e_iM'_C$ is hereditary injective.

Suppose that this is not the case and consider an epimorphism $g : N \twoheadrightarrow Z$, where $N$ is an indecomposable summand of some $M'_i$ and $Z$ is a non-injective indecomposable $C$-module. Decomposing $M'_i = N \oplus N'$ in $mod_C$, we see that $e_iJ(A) = N \oplus N' \oplus e_iJ(B)A$ is a decomposition in $mod_A$, where the $C$-modules $N$, $N'$ are viewed as $A$-modules and, by definition, $e_iJ(B)A = (e_iJ(B)M, e_iJ(B), \mu)$, with $\mu$ the multiplication map. We deduce an embedding $Ker(g) \oplus N' \oplus e_iJ(B)A \hookrightarrow e_iJ(A) \hookrightarrow e_iA$. The corresponding quotient $X = \frac{e_iA}{Ker(g)\oplus N'\oplus e_iJ(B)A}$ has simple top, hence is indecomposable. We also have

$$XJ(A) = \frac{e_iJ(A)}{Ker(g)\oplus N'\oplus e_iJ(B)A} = \frac{N\oplus N'\oplus e_iJ(B)A}{Ker(g)\oplus N'\oplus e_iJ(B)A} \cong Z$$

Since $Z$ is not injective in $mod_C$, the functor $Ext^1_C(-, Z)$ is non-zero. It is easily seen that this is equivalent to the existence of some simple $C$-module $S$ such that $Ext^1_C(S, Z) \neq 0$. We fix a non-split exact sequence

$$0 \to Z \xrightarrow{j} V \xrightarrow{p} S \to 0$$

in $mod_C$ which, clearly, is also non-split in $mod_A$. By the above comments, the canonical inclusion $XJ(A) \hookrightarrow X$ induces an embedding $i : Z \longrightarrow X$. We thus have an amalgamated sum (pushout) diagram:

$$\begin{array}{ccccccccc}
0 & \to & Z & \xrightarrow{j} & V & \xrightarrow{p} & S & \to & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle r} & & \| & & \\
0 & \to & X & \xrightarrow{u} & W & \xrightarrow{w} & S & \to & 0
\end{array}$$

Since $(Gen(e'A), add(\mathcal{C}))$ is a split torsion pair, we have $W = W_1 \oplus W_2$, with $W_1 \in add(\mathcal{C}) = Gen(eA)$ (whence it is a $C$-module) and $W_2 \in Gen(e'A)$. Since $X \in Gen(e'A)$, the composition of $u$ with the projection $W \longrightarrow W_1$ vanishes, so that $u(X) \subseteq W_2$. The obvious inequalities between composition lengths $l(X) \leq l(W_2) \leq l(W) = l(X) + 1$ lead to two cases:

1. Assume first that $l(W_2) = l(W) = l(X) + 1$. Then $W = W_2$ and $W_1 = 0$, so that $W \in Gen(e'A)$. But $w : W \longrightarrow S$ is non-zero, and $S \in \mathcal{C}$. This is a contradiction.

2. Assume $l(X) = l(W_2) = l(W) - 1$. Identifying $X$ with $u(X)$, we have $X = W_2$ so that $W_1 \cong W/X \cong S$ and the above diagram becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z & \xrightarrow{\quad j \quad} & V & \xrightarrow{\quad p \quad} & S & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle (h\ p)^t} & & \Big\| & & \\ 0 & \longrightarrow & X & \xrightarrow{\ (1\ 0)^t\ } & X \oplus S & \xrightarrow{\ (0\ 1)\ } & S & \longrightarrow & 0 \end{array}$$

for some $h : V \longrightarrow X$. In particular, $h \circ j = i$ and $p \circ j = 0$. On the other hand, since $V$ is a $C$-module, we have $Im(h) \subseteq XeA \subseteq XJ(A) \cong Z$ because $X$ has a simple top isomorphic to $S_i = \frac{e_i A}{e_i J(A)}$. We then get a morphism $h' : V \longrightarrow Z$ such that $i \circ h' = h$. But then $i = h \circ j = i \circ h' \circ j$ and, since $i$ is a monomorphism, we get $h' \circ j = 1_Z$. This contradicts the fact that the upper sequence in the above diagram is not split.

In either case we have reached a contradiction. Hence each $M'_i$ is hereditary injective. That completes the proof of the equivalence of (1), (2) and (3).

The last statement of the theorem follows from the fact that, if we identify $add(\mathcal{C})$ with the full subcategory $mod_C$ of $mod_A$, then the minimal projective resolution of any $X \in \mathcal{C}$ is the same in $mod_C$ and $mod_A$. □

Given a complete set of primitive orthogonal idempotents $\mathcal{E} = \{e_1, ..., e_n\}$ of $A$, and a subset $\Sigma = \{e_{i_1}, ..., e_{i_r}\}$ of $\mathcal{E}$, we denote by $e_\Sigma$ the sum $e_{i_1} + ... + e_{i_r}$. With this notation, the desired classification of the split torsion pairs with torsion-free class closed under quotients follows directly from our theorem.

**Corollary 2.6.** *Let $\mathcal{E} = \{e_1, ..., e_n\}$ be a complete set of primitive orthogonal idempotents of $A$. There is a one-to-one correspondence between:*

*(1) The full subcategories $\mathcal{C}$ of $ind_A$ closed under predecessors such that $add(\mathcal{C})$ is an abelian exact subcategory of $mod_A$*

*(2) The split torsion pairs in $mod_A$, with torsion-free class closed under quotients*

*(3) The subsets $\Sigma \subseteq \mathcal{E}$ such that $(1 - e_\Sigma)Ae_\Sigma$ is a hereditary injective $e_\Sigma A e_\Sigma$-module and $e_\Sigma A(1 - e_\Sigma) = 0$*

# 3 Applications to Artin algebras

Throughout this section, we assume that our algebras are basic and connected Artin algebras. We denote by $Q_A$ the (valued) quiver of $A$ and by $(Q_A)_0$ the set of points of $Q_A$. The idempotent corresponding to a point $x \in (Q_A)_0$ is denoted by $e_x$, while we denote by $P_x$ (or $S_x$) the corresponding indecomposable projective ( or simple, respectively). For general facts about the module category of $A$, we refer the reader to [3].

A first consequence of our main theorem is the following combinatorial result:

**Corollary 3.1.** *Let $A$ be an algebra satisfying the equivalent conditions of the theorem. Then, for every arrow $y \to x$ in $Q_A$, with $y \in (Q_B)_0$ and $x \in (Q_C)_0$, the point $x$ is a source in $Q_C$.*

*Proof.* Since there exists an arrow $y \to x$ in $Q_A$, then $\frac{e_y J(A) e_x}{e_y J(A)^2 e_x} \neq 0$. Notice that $e_y J(A) e_x$ is identified with $e_y M e_x$ and $e_y J(A)^2 e_x$ with $e_y [J(B)M + MJ(C)] e_x$. Then $\frac{e_y M e_x}{e_y [J(B)M + MJ(C)] e_x} \neq 0$ and, in particular, $\frac{e_y M e_x}{e_y MJ(C) e_x} \neq 0$. This says that the simple $C$-module $S_x$ is a direct summand of the top of the $C$-module $e_y M$ and, hence, also of $top(M_C)$. Since $M_C$ is hereditary injective, we conclude that $S_x$ is a simple injective $C$-module, so that $x$ is a source in $Q_C$. □

We now consider the case where $\mathcal{C}$ is the left part $\mathcal{L}_A$ of $mod_A$, that is, the full subcategory of $ind_A$ consisting of those $X \in ind_A$ such that every predecessor of $X$ has projective dimension at most one (see [5]). Thus, $\mathcal{L}_A$ is closed under predecessors. The endomorphism algebra of the supporting projective of $\mathcal{L}_A$ is denoted by $A_\lambda$ and is called the **left support** of $A$ (see [2] and [7]).

We recall that $A$ is called **left supported** when $add(\mathcal{L}_A)$ is contravariantly finite in $mod_A$ (see [2]). Many important classes of algebras are left supported such as, for instance, the laura algebras which are not quasi-tilted (see [2], [7]).

**Corollary 3.2.** *Let $A$ be an Artin algebra such that $add(\mathcal{L}_A)$ is an abelian exact subcategory of $mod_A$. Then:*

*(1) The left support $A_\lambda$ of $A$ is hereditary*

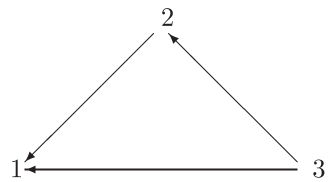
*(2) The algebra $A$ is left supported*

*(3) If, furthermore, the valued quiver of $A$ has no oriented cycles, then $A = A_\lambda$. In particular, $A$ itself is hereditary*

*Proof.* (1) follows from the last statement of the theorem, and (2) follows from corollary 2.3(1). In order to prove (3), suppose that $A \neq A_\lambda$. There exists a point $x_0 \in (Q_A)_0$ such that $P_{x_0} \notin \mathcal{L}_A$. In particular, the radical $P_{x_0} J(A)$ of $P_{x_0}$ admits an indecomposable summand $R_{x_0}$ which is not in $\mathcal{L}_A$. Hence there exists a point $x_1 \in (Q_A)_0$ such that $P_{x_1} \notin \mathcal{L}_A$ and $Hom_A(P_{x_1}, R_x) \neq 0$. This yields a non-zero non-isomorphism $f_1 : P_{x_1} \longrightarrow P_{x_0}$. Repeating the process for $x_1$ instead of $x_0$ yields a point $x_2 \in (Q_A)_0$ such that $P_{x_2} \notin \mathcal{L}_A$ and there exists a non-zero non-isomorphism $f_2 : P_{x_2} \longrightarrow P_{x_1}$. Inductively, we get a sequence of non-zero non-isomorphisms between indecomposable projective modules $...P_{x_n} \xrightarrow{f_n} P_{x_{n-1}} ... \xrightarrow{f_2} P_{x_1} \xrightarrow{f_1} P_{x_0}$. Since $(Q_A)_0$ is finite, this sequence yields necessarily an oriented cycle in $Q_A$, which is a contradiction. □

We note that, if $A_\lambda$ is hereditary, it does not follow in general that $add(\mathcal{L}_A)$ is an abelian exact subcategory of $mod_A$, as is shown by the following example.

**Example 3.3.** *Let $K$ be a field and $A$ be the radical square zero $K$-algebra given by the quiver*




*Here $\mathcal{L}_A = \{P_1, P_2\}$ and its support is the hereditary $K$-algebra with quiver*

$$1 \longleftarrow 2$$

*However, $add(\mathcal{L}_A)$ is not an abelian exact subcategory of $mod_A$ because it does not contain the cokernel $S_2$ of the inclusion $P_1 \longrightarrow P_2$*

Our final application is to local extensions of hereditary algebras. We recall that a triangular matrix algebra $A = \begin{pmatrix} H & 0 \\ M & R \end{pmatrix}$, where ${}_RM_H$ is an $R-H-$bimodule, is called a **local extension** of $H$ in case $R$ is a local algebra (see [6]). Taking $R$ a skew field, we see that this notion generalizes that of a one-point extension. However, we are interested in the case where $R$ is not a skew field, a hypothesis that we assume in the sequel. We denote by $y$ the unique point in $Q_R$. For general facts about the module category of a local extension, we refer the reader to [6]

**Lemma 3.4.** *Let $A = \begin{pmatrix} H & 0 \\ M & R \end{pmatrix}$ be a local extension of the hereditary algebra $H$. Then the left support $A_\lambda$ is equal to $H$.*

*Proof.* Let $P_x$ be any indecomposable projective $H$-modules. The predecessors of $P_x$ in $ind_A$ are (projective) $H$-modules and, hence, $P_x \in \mathcal{L}_A$. On the other hand, the only other indecomposable projective $P_y$ lies on an oriented cycle of projectives in $ind_A$. Therefore $y \notin (Q_{A_\lambda})_0$, because $A_\lambda$ is quasi-tilted by [2][2.1] and hence triangular by [5] □

It follows from the above lemma, or from [2][2.1], that we have an inclusion $\mathcal{L}_A \subseteq ind_H$. Our final result says exactly when equality holds:

**Proposition 3.5.** *Let $A = \begin{pmatrix} H & 0 \\ M & R \end{pmatrix}$ be a local extension of the hereditary algebra $H$, where $R$ is not a skew-field. The following statements are equivalent:*

*(1) $add(\mathcal{L}_A)$ is an abelian exact subcategory of $mod_A$*

*(2) $\mathcal{L}_\mathcal{A} = ind_H$*

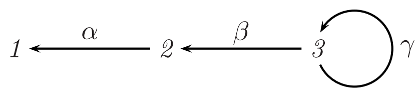
*(3) $M_H$ is injective*

*Proof.* (1) implies (3): By lemma 3.4, we have $H = A_\lambda$. Our main theorem 2.5 gives that $M_H$ is injective.

(3) implies (2): From theorem 2.5 we get $\mathcal{C} = ind_H$. Also, for any $X \in ind_H$, we have $pd_A(X) \leq gl.dim(H) = 1$. Then $ind_H \subseteq add(\mathcal{L}_A)$, so that $\mathcal{L}_A = ind_H$

(2) implies (1): The hypothesis gives $mod_H = add(\mathcal{L}_\mathcal{A})$, and the statement follows at once. □

**Example 3.6.** *Let $K$ be a field and let $A$ be the $K$-algebra given by the quiver*




*with relations $\gamma^2 = 0$ and $\gamma\beta\alpha = 0$. Denoting the indecomposables by their Loewy series, the regular module $A_A$ is given by:*

$$1 \;\oplus\; \begin{array}{c} 2 \\ 1 \end{array} \;\oplus\; \begin{array}{ccc} & 3 & \\ 2 & & 3 \\ 1 & & 2 \end{array}$$

*Here, $A$ is a local extension of the hereditary algebra $H$ given by the quiver $1 \xleftarrow{\alpha} 2$, taking $M_H = P_2 \oplus S_2$, which is an injective $H$-module. The hypothesis of proposition 3.5 is satisfied, and therefore $add(\mathcal{L}_A) = mod_H$ is an abelian exact subcategory of $mod_A$. Notice that if we put here $C = H$ and $B = R = K[\gamma]/(\gamma^2)$, then $J(B)M \cong S_2$ so that, taking $M' = P_2$, we get the decomposition $M_C = M' \oplus J(B)M$ of the proof of theorem 2.5.*

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