# A note on the fundamental group of a one-point extension

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## 1. Introduction

In the representation theory of finite dimensional algebras over algebraically closed fields, the use of covering techniques, initiated by Gabriel, Riedtmann and others (see, for instance [**3**]) has shown the importance of the fundamental groups of bound quivers (see [**6**] for the definition). In this note, we consider an algebra $A$ which is a one-point extension of another algebra $B$ and the morphism of fundamental groups induced by the inclusion of (the bound quiver of) $B$ into (that of)$A$. Our main result says that the cokernel of this morphism is a free group. We then deduce various consequences of this.

## 2. Preliminaries

**2.1. Algebras and quivers.** By an algebra is always meant a basic and connected finite dimensional algebra over an algebraically closed field $k$, and by module is meant a finitely generated right module. For every algebra $A$ there exists a (unique) quiver $Q_A$ and (at least) a surjective morphism of algebras $\nu : kQ_A \longrightarrow A$ so that, setting $I_\nu = \mathrm{Ker}\nu$, we have $A \simeq kQ_A/I_\nu$ (see [**3**]). The bound quiver $(Q_A, I_\nu)$ is then called a *presentation* of $A$. An algebra $A$ is said to be *triangular* whenever $Q_A$ has no oriented cycles. If $x$ is a point in $Q_A$, we denote by $e_x$ the corresponding idempotent and by $P_x = e_x A$ the corresponding indecomposable projective module.

**2.2. Fundamental groups.** Let $(Q, I)$ be a connected bound quiver. For the definitions and properties of the fundamental group $\pi_1(Q, I)$, we refer to [**6, 2**]. A triangular algebra is called *simply connected* if, for every presentation $(Q, I)$ of $A$, we have $\pi_1(Q, I) = 1$. On the other hand, to a given connected bound quiver $(Q, I)$, one can associate a CW complex $\mathcal{B} = \mathcal{B}(Q, I)$ called the *classifying space* of $(Q, I)$, and this construction behaves well with respect to homotopy, in particular one has $\pi_1(Q, I) \simeq \pi_1(\mathcal{B})$ (see [**5**]). The fundamental group of a bound quiver $(Q, I)$ affords the following description: let $T$ be a maximal tree in $Q$, $F$ be the free group with basis the set of all arrows of $Q$, and $K$ be the normal subgroup of $F$ generated by

1. all the arrows in $T$; and

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2. all expressions of the form $(\beta_1\beta_2\cdots\beta_p)(\gamma_1\gamma_2\cdots\gamma_q)^{-1}$ when $\beta_1\beta_2\cdots\beta_p$ and $\gamma_1\gamma_2\cdots\gamma_q$ are two paths appearing in a minimal relation (in the sense of **[6]**).

Then, $\pi_1(Q,I)\simeq F/K$, see **[5]**(3.2).

**2.3. One-point extensions.** Let $A$ be an algebra, $x$ be a source in $Q_A$, and $B=A/Ae_xA$. Letting $M=\mathrm{rad}P_x$, the algebra $A$ can be written in matrix form

$$A=\begin{bmatrix} B & 0\\ M & k\end{bmatrix}$$

with the usual matrix addition, and the multiplication induced from the $B-$module structure of $M$. We say that $A$ is a *one-point extension* of $B$, and write $A=B[M]$. Let $(Q_A,I_{\nu_A})$ be a presentation of $A$, and $(Q_B,I_{\nu_B})$ be the induced presentation of $B$. Let $Q^{(1)},\dots Q^{(c)}$ be the connected components of $Q_B$, and $I_\nu^{(j)}=I_{\nu_B}\cap kQ^{(j)}$, for $1\le j\le c$. For each $j$, the embedding of $Q^{(j)}$ in $Q_A$ induces a group morphism $\eta_j:\pi_1(Q^{(j)},I_\nu^{(j)})\longrightarrow\pi_1(Q_A,I_{\nu_A})$. Our aim is to compute the cokernel of the induced morphism

$$\eta=(\eta_j)_j:\prod_{j=1}^{c}\pi_1(Q^{(j)},I_\nu^{(j)})\longrightarrow\pi_1(Q_A,I_{\nu_A})\ .$$

Following **[2]**(2.1), we denote by $x^{\rightarrow}$ the set of all the arrows of $Q_A$ starting in $x$. Let $\approx$ be the least equivalence on $x^{\rightarrow}$ such that $\alpha\approx\beta$ whenever there exists a minimal relation $\sum_{i=1}^{r}\lambda_iw_i$ such that $w_1=\alpha w_1'$, $w_2=\beta w_2'$. We denote by $[\alpha]_{\nu_A}$ the equivalence class of $\alpha$, and by $t(\nu_A)$ the number of equivalence classes $[\alpha]_{\nu_A}$ in $x^{\rightarrow}$. Finally, we denote by $s=s(x)$ the number of indecomposable direct summands of $M$. It is shown in **[2]** (2.2) that $c\le t(\nu_A)\le s$.

## 3. The results

**3.1. Theorem.** *Let $A=B[M]$ be given an arbitrary presentation $(Q_A,I_{\nu_A})$. Then the cokernel of the morphism $\eta:\prod_{j=1}^{c}\pi_1(Q^{(j)},I_\nu^{(j)})\longrightarrow\pi_1(Q_A,I_{\nu_A})$ induced by the inclusion is the free group $L_{t(\nu_A)-c}$ in $t(\nu_A)-c$ generators.*

**Proof:** Clearly, if $\delta_1$ is an arrow from $x$ to a point in $Q^{(1)}$, and $\delta_2$ is an arrow from $x$ to a point in $Q^{(2)}$, then $\delta_1\not\approx\delta_2$. We may then assume, without loss of generality that $B$ is connected, that is, $c=1$. We set $t=t(\nu_A)$, and let $\{\alpha_1,\alpha_2,\dots\alpha_t\}$ be a complete set of representatives of the classes $[\alpha]_\nu$. For $i\in\{1,2,\dots t\}$, we set $[\alpha_2]_\nu=a_i$. Our aim is to show that the cokernel of $\eta$ is the free group $L_{t-1}$ having as basis the set $\{a_2,\dots a_t\}$.

Let $T_B$ be a maximal tree in $Q_B$, and $F_B$, $K_B$ as in (2.2). Denote by $T_A$ the maximal tree in $Q_A$ obtained from $T_B$ by adding the arrow $\alpha_1$ and the vertex $x$. Denote again by $F_A$, $K_A$ the groups as in (2.2). We have $\pi_1(Q_B,I_{\nu_B})\simeq F_B/K_B$, $\pi_1(Q_A,I_{\nu_A})\simeq F_A/K_A$, and the morphism $\eta:F_B/K_B\longrightarrow F_A/K_A$ is given by $\alpha K_B\longmapsto\alpha K_A$ (where $\alpha$ is an arrow in $Q_B$).

We define a group morphism $\tilde{\varphi}:F_A\longrightarrow L_{t-1}$ as follows. For an arrow $\alpha$ in $Q_A$, we set

$$\tilde{\varphi}(\alpha) = \begin{cases} a_i & \text{if } \alpha \approx \alpha_i \text{ for some } i \in \{2, \dots t\}, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that $\tilde{\varphi}(K_A) = 1$. Clearly, if $\alpha$ is an arrow in $T_A$, then $\tilde{\varphi}(\alpha) = 1$. Let $\beta_1\beta_2\cdots\beta_p$ and $\gamma_1\gamma_2\cdots\gamma_q$ be two paths in $(Q_A, I_{\nu_A})$ appearing in some minimal relation. If the source of this relation is not $x$, then it lies in $Q_B$ so that

$$\tilde{\varphi}(\beta_1\beta_2\cdots\beta_p) = 1 = \tilde{\varphi}(\gamma_1\gamma_2\cdots\gamma_q)$$

If, on the other hand, the source is $x$, then $\beta_1,\ \gamma_1 \in x^{\rightarrow}$. Moreover, there exists $i_0 \in \{1, 2, \dots t\}$ such that $[\beta_1]_\nu = [\gamma_1]_\nu = a_{i_0}$. Therefore

$$\begin{aligned} \tilde{\varphi}(\beta_1\beta_2\cdots\beta_p) &= \tilde{\varphi}(\beta_1)\tilde{\varphi}(\beta_2\cdots\beta_p) \\ &= \tilde{\varphi}(\gamma_1)\tilde{\varphi}(\gamma_2\cdots\gamma_q) \\ &= \tilde{\varphi}(\gamma_1\gamma_2\cdots\gamma_q) \end{aligned}$$

This establishes our claim. We infer the existence of a group morphism (which is in fact an epimorphism) $\varphi : F_A/K_A \longrightarrow L_{t-1}$ defined by

$$\varphi(\alpha K_A) = \begin{cases} a_i & \text{if } \alpha \approx \alpha_i \text{ for some } i \in \{2, \dots t\}, \\ 1 & \text{otherwise.} \end{cases}$$

(where $\alpha$ denotes an arrow in $Q_A$).

Clearly, we have $\varphi\eta = 1$. Now, assume there exist a group morphism $\varphi' : F_A/K_A \longrightarrow G$ such that $\varphi'\eta = 1$. We define $\bar{\varphi} : L_{t-1} \longrightarrow G$ by $a_i \longmapsto \varphi'(\alpha K_A)$, where $[\alpha]_\nu = a_i$. We must verify that $\varphi'(\alpha K_A)$ does not depend on the choice of the representative $\alpha$. Indeed, if $[\beta]_\nu = [\gamma]_\nu$, we may assume without loss of generality that there exists two paths $\beta\beta_2\cdots\beta_p$, and $\gamma\gamma_2\cdots\gamma_q$ in $Q_A$ appearing in the same minimal relation. But then

$$\begin{aligned} \varphi'(\beta K_A) &= \varphi'(\beta K_A)\varphi'\eta(\beta_2\cdots\beta_p K_B) \\ &= \varphi'(\beta K_A)\varphi'(\beta_2\cdots\beta_p K_A) \\ &= \varphi'(\beta\beta_2\cdots\beta_p K_A) \\ &= \varphi'(\gamma\gamma_2\cdots\gamma_q K_A) \\ &= \varphi'(\gamma K_A)\varphi'(\gamma_2\cdots\gamma_q K_A) \\ &= \varphi'(\gamma K_A)\varphi'\eta(\gamma_2\cdots\gamma_q K_B) \\ &= \varphi'(\gamma K_A) \end{aligned}$$

Thus, $\bar{\varphi}$ is well-defined. Clearly $\bar{\varphi}\varphi = \varphi'$, and $\bar{\varphi}$ is uniquely determined (because $\varphi$ is an epimorphism). This completes the proof.

□

**3.2. Corollary [2](2.3).** *Let $A = B[M]$ be given by an arbitrary presentation $(Q_A, I_{\nu_A})$, and $Z$ be any abelian group. There exists an exact sequence of abelian groups*

$$0 \longrightarrow Z^{t(\nu_A)-c} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\pi_1(Q_A, I_{\nu_A}), Z) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\eta, Z)} \prod_{j=1}^{c} \mathrm{Hom}_{\mathbb{Z}}(\pi_1(Q^{(j)}, I_\nu^{(j)}), Z)$$

□

**3.3. Corollary.** *Let $A = B[M]$. There exists a presentation $(Q_A, I_{\mu_A})$ such that the cokernel of the morphism $\eta : \prod_{j=1}^{c} \pi_1(Q^{(j)}, I_{\mu}^{(j)}) \longrightarrow \pi_1(Q_A, I_{\mu_A})$ is the free group $L_{s-c}$ in $s-c$ generators.*

**Proof:** By [**2**](2.2), there exists a presentation $(Q_A, I_{\mu_A})$ such that $t(\mu) = s$. □

**3.4. Corollary** [**2**]**(2.6).** *Let $A$ be simply connected. Then all sources of $Q_A$ are separating (see for instance* [**2**]*).*

**Proof:** Since $A$ is simply connected, it follows from the previous result that $s = c$. □

**3.5. Corollary.** *Let $A = B[M]$ and $(Q_A, I_{\mu_A}$ be a presentation of $A$ as in corollary 3.3. If $\pi_1(Q_A, I_{\mu_A})$ is finite, then $M$ is a separated module*

**Proof:** If this is not the case, then $L_{s-c} \neq 1$ and, since the short exact sequence $1 \longrightarrow \mathrm{Ker}\varphi \longrightarrow \pi_1(Q_A, I_{\mu_A}) \xrightarrow{\varphi} L_{s-c} \longrightarrow 1$ splits (because $L_{s-c}$ is a free group), we get $\pi_1(Q_A, I_{\mu_A}) \simeq L_{s-c} \ltimes \mathrm{Ker}\varphi$, which is infinite. □

**3.6. Corollary.** *Let $A = B[M]$ be given an arbitrary presentation $(Q_A, I_{\nu_A})$ and assume $B$ to be a direct product of simply connected algebras. Then*

*a)* $\pi_1(Q_A, I_{\nu_A}) \simeq L_{t(\nu_A)-c}$, *and*
*b) $A$ is simply connected if and only if $M$ is separated.*

**Proof:** Since *a)* is clear we only prove *b)*: By definition, $M$ is separated if and only if $s = c$. Moreover, it follows from [**2**](2.2) that $s = c$ if and only if $t(\nu_A) = c$ for every presentation $(Q_A, I_{\nu_A})$. Finally, using *a)* one can see that this holds if and only if $A$ be simply connected. □

Note that the sufficiency of *b)* was proven in [**2**](2.5), [**7**](2.3).

We now suppose $A$ to be triangular and schurian (that is, for every two vertices $x$, $y$ of $Q_A$, we have $\dim_k e_x A e_y \leq 1$). In this case, the fundamental group does not depend on the presentation, so it can unambiguously be denoted by $\pi_1(A)$. Moreover, the classifying space $\mathcal{B}$ of $(Q_A, I_{\nu_A})$ is a simplicial complex [**4**]. We denote by $\mathrm{SH}_1(A)$ the first simplicial homology group of $A$, and recall that $\mathrm{SH}_1(A)$ is the abelianisation of $\pi_1(A)$.

**3.7. Corollary.** *Let $A = B[M]$ be a triangular schurian algebra, and $B_1, \dots B_c$ the connected components of $B$. Then, there exists an exact sequence of abelian groups*

$$\prod_{j=1}^{c} \mathrm{SH}_1(B_j) \xrightarrow{\eta_*} \mathrm{SH}_1(A) \longrightarrow \mathbb{Z}^{s-c} \longrightarrow 0$$

*Furthermore, if $A$ contains no quasi-crowns (in the sense of* [**1**]*), then $\eta_*$ is a monomorphism.*

**Proof:** By 3.2 there exists, for each abelian group $Z$, an exact sequence of abelian groups

$$0 \longrightarrow Z^{s-c} \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\pi_1(A), Z) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\eta, Z)} \prod_{j=1}^{c} \mathrm{Hom}_{\mathbb{Z}}(\pi_1(B_j), Z)$$

By [**1**](3.8), the absence of quasi-crowns forces $\mathrm{Hom}_{\mathbb{Z}}(\eta, Z)$ to be an epimorphism. Since the above sequence may be rewritten as

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{s-c}, Z) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{SH}_1(A), Z) \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\eta_*, Z)} \prod_{j=1}^{c} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{SH}_1(B_j), Z)$$

where $\eta_*$ is induced from $\eta$ by abelianising, replacing $Z$ by the injective cogenerator $\mathbb{Q}/\mathbb{Z}$ yields the result.

□

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## References

[1] I. Assem, D. Castonguay, E. N. Marcos, and S. Trepode. Schurian strongly simply connected algebras an multiplicative bases. preprint, 2004.
[2] I. Assem and J.A. de la Peña. The fundamental groups of a triangular algebra. *Comm. Algebra*, 24(1):187–208, 1996.
[3] K. Bongartz and P. Gabriel. Covering spaces in representation theory. *Invent. Math*, 65(3):331–378, 1981-1982.
[4] J. C. Bustamante. On the fundamental group of a schurian algebra. *Comm. Algebra*, 30(11):5305–5327, 2002.
[5] J. C. Bustamante. The classifying space of a bound quiver. To appear in *J. Algebra*, 2004.
[6] R. Martínez-Villa and J.A. de la Peña. The universal cover of a quiver with relations. *J. Pure Appl. Algebra*, 30:873–887, 1983.
[7] A. Skowroński. Simply connected algebras and Hochschild cohomologies. In *Proceedings of the sixth international conference on representation of algebras*, number 14 in Ottawa-Carleton Math. Lecture Notes Ser., pages 431–448, Ottawa, ON, 1992.

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