Contemporary Mathematics

A note on sequential walks

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Dedicated to José Antonio de la Peña on the occasion of his sixtieth birthday

Abstract. This short note is devoted to motivate and clarify the notion of sequential walk that has been previously introduced by the authors. We also give some applications of this concept.

1. Introduction

The class of tilted algebras, introduced by Happel and Ringel [**16**], is among the most ubiquitous classes in the representation theory of algebras. For instance, any cluster-tilted algebra is the trivial extension of a tilted algebra by a particular bimodule [**2**]. Surprisingly enough, it is difficult to check whether a given algebra is tilted or not without a good knowledge of its module category. Indeed, most known criteria revolve around the existence of a combinatorial configuration called a complete slice, see, for instance, [**6**].

It was thus needed to have a handy criterion depending only on the bound quiver of the algebra. The most powerful criterion so far is the existence of so-called sequential walks. Sequential walks have a long history; they first appeared in [**1**], where it was shown that an iterated tilted algebra of type $\mathbb{A}$ is tilted if and only if it has no sequential walk. They surfaced again in [**17**] under the name of "sequential pairs" in the classification of quasi-tilted string algebras, then in [**8**] in the classification of shod string algebras and in [**12**] in the classification of laura string algebras under the name of "double zeros". Their present guise was introduced in [**4**] in the context of non-necessarily monomial algebras. It was proved there that if an algebra contains a sequential walk, then it is not tilted.

This shows that this notion is very natural. Indeed, as we prove here, it follows from simple considerations on the comparison between the shape of the bound quiver and the homological dimensions of some modules.

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It was pointed out to the authors that the definition of sequential walk given in [**4**] contained an ambiguity. It is the first purpose of this note to clarify this ambiguity. While doing so, we slightly generalize the definition of sequential walk, and try to illustrate its usefulness for computing homological dimensions.

Our main result is the following.

Theorem 1.1. *Let A be a finite dimensional algebra over an algebraically closed field. If the quiver of A contains a sequential walk, then A is not shod. In particular, A is not quasi-tilted.*

This note is organized as follows. In Section 2, we recall the definitions and known results which are necessary for the proof of our theorem. In Section 3, we give our definition of sequential walk, try to motivate it, then we prove our theorem. Section 4 is then devoted to applications and examples.

The authors thank Claire Amiot for useful discussions on the topic.

2. Preliminaries

2.1. Notation. Throughout, we let $\Bbbk$ denote an algebraically closed field and A a finite-dimensional basic $\Bbbk$-algebra. We recall that any basic and connected finite dimensional $\Bbbk$-algebra A can be written in the form $A \cong \Bbbk Q_A/I$ where $\Bbbk Q_A$ is the path algebra of the quiver Q_A of A and I is an ideal generated by finitely many relations. The pair (Q_A, I) is then called a *bound quiver*. A *relation* on a quiver is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i u_i$, where the λ_i are nonzero scalars and the u_i are paths of length at least two having all the same source and the same target, called respectively the *source* and the *target* of the relation ρ. The relation ρ is called *quadratic* if each of the u_i has length two. It is called a *monomial relation* if it is a path, and a *minimal relation* if $m \geq 2$ and, for any nonempty proper subset $J \subset \{1, 2, \ldots, m\}$ one has $\sum_{j\in J} \lambda_j u_j \notin I$.

Relations in a bound quiver (Q, I) are generated by top relations. Indeed, let $\Bbbk Q^+$ be the two-sided ideal of $\Bbbk Q$ consisting of the linear combinations of paths of length at least one and e_x be the primitive idempotent of A corresponding to a point $x \in (Q_A)_0$. Then a relation $\rho \in e_x I \, e_y$ is called a *top relation* if its residual class in $e_x \left(\frac{I}{\Bbbk Q^+ I + I \Bbbk Q^+}\right) e_y$ is nonzero. They are called top relations because they correspond to nonzero elements of the top of I, considered as a $\Bbbk Q$-$\Bbbk Q$-bimodule. Intuitively one may think of the top relations as being the shortest ones.

For a point x in the ordinary quiver of A, we denote by P_x, I_x, S_x respectively, the indecomposable projective, injective and simple A-modules corresponding to x. The *support* of an A-module M is the full subquiver $\operatorname{Supp} M$ generated by the points $x \in (Q_A)_0$ such that $Me_x \neq 0$. For further definitions and facts, we refer the reader to [**6, 20**].

2.2. Classes of algebras. We need the following classes of algebras.

Definition 2.1. (a) [**11**] An algebra A is *shod* (for small homological dimensions) if every indecomposable A-module is of projective or injective dimension at most one.

(b) [**15**] An algebra A is *quasi-tilted* if it is shod of global dimension at most two, or, equivalently, if it is the endomorphism algebra of a tilting object in a hereditary, locally finite abelian $\Bbbk$-category.

(c) [**16**] An algebra is *tilted* if it is the endomorphism algebra of a tilting module over a hereditary algebra.

In particular, every tilted algebra is quasi-tilted, and every quasi-tilted algebra is shod.

2.3. Full subcategories. There is a reduction procedure which we shall use in the proof of our main result, called taking full subcategories.

Let $e \in A$ be an idempotent. The finite dimensional $\Bbbk$-algebra eAe is called the *full subcategory* determined by e; indeed, if one considers A as a category whose objects are the elements of a complete set of primitive orthogonal idempotents and the morphism space from e_x to e_y is $e_xA\,e_y$, then a full subcategory is always of the aforementioned form with e equal to a sum of primitive idempotents.

We need the following lemma.

LEMMA 2.2. *Let A be an algebra and $e \in A$ an idempotent, then*

(a) *If A is shod, then so is eAe.*
(b) *If A is quasi-tilted, then so is eAe.*
(c) *If A is tilted, then so is eAe.*

PROOF. (a) is proved in [**18**, 1.2], (b) in [**15**, II.1.15] and (c) in [**14**, III.6.5]. □

2.4. Split-by-nilpotent extensions. Let A be an algebra and E an A-A-bimodule which is finite dimensional as a $\Bbbk$-vector space. We say that E is equipped with an *associative product* if there exists an A-A-bimodule morphism $E \otimes_A E \to E, e \otimes e' \mapsto ee'$, such that $e(e'e'') = (ee')e''$ for all $e, e', e'' \in E$.

DEFINITION 2.3. Let A, E be as before. An algebra R is called a *split extension of A by E* if

$$R = \{(a, e) \mid a \in A, e \in E\}$$

is equipped with the componentwise addition and the multiplication defined by

$$(a, e)(a', e') = (aa', ae' + ea' + ee'\}$$

for $(a, e), (a', e') \in R$. If E is nilpotent with respect to its product, then R is called a *split-by-nilpotent extension.*

It is clear that, if R is a split extension of A by E, then there exists an exact sequence

$$0 \longrightarrow E \longrightarrow R \xrightarrow{\pi} A \longrightarrow 0\ ,$$

where the projection π is an algebra morphism having a section $\sigma\colon A \to R$. Also, E is nilpotent if and only if it is contained in $\operatorname{rad} R$. If an exact sequence as above and a section σ to π are given, then we say that this sequence *realizes* R as a split extension of A by E. If $E^2 = 0$, then the split extension is called a *trivial extension.*

We need the following lemma.

LEMMA 2.4. *Let R be a split extension of an algebra A by a nilpotent bimodule E.*

(a) *If R is shod, then so is A.*
(b) *If R is quasi-tilted, then so is A.*
(c) *If R is tilted, then so is A.*

PROOF. (a) and (b) are proved in [**7**, 2.5] and (c) in [**21**]. □

2.5. Cutting arrows. Let R be a split extension of an algebra A by a nilpotent bimodule E. We recall how one can pass from R to A by dropping arrows from the quiver of R. Let w be a path in the quiver Q_A of A and α an arrow such that there exist subpaths w_1, w_2 of w such that $w = w_1 \alpha w_2$, then we write $\alpha | w$. Also, when we speak of a relation, then we can assume without loss of generality that it is monomial or minimal. Let thus $\rho = \sum_i \lambda_i u_i$ be a relation on Q_A from x to y, say, with the λ_i nonzero scalars and the u_i paths of length at least two from x to y. We say that ρ is *consistently cut* by a set S of arrows (or that S is a *consistent cut*), if, whenever there exist i and $\alpha_i | u_i$ such that $\alpha_i \in S$ then, for any $j \neq i$, there exists $\alpha_j | u_j$ satisfying $\alpha_j \in S$.

THEOREM 2.5. [**3**, 2.5] *Let $\eta_R \colon \Bbbk Q_R \to R$ be a presentation of R, let E be an ideal of R generated by the classes modulo $I_R = \ker \eta_R$ of a set S of arrows, and let $\pi \colon R \to R/E = A$ be the projection. Assume moreover that every relation in I_R is consistently cut by S, then the exact sequence*

$$0 \longrightarrow E \longrightarrow R \xrightarrow{\ \pi\ } A \longrightarrow 0$$

realizes R as a split extension of A by E.

2.6. Reduction of an algebra. We now define a notion which we call reduction of an algebra. We say that an algebra B is a *reduction* of an algebra A if there exist an idempotent $e \in A$ and a two-sided ideal E of eAe contained in its radical such that the sequence

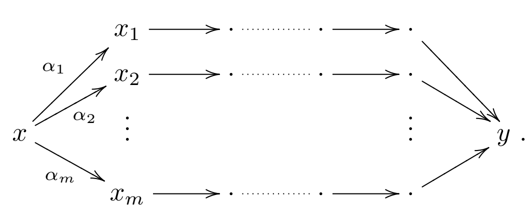
$$0 \longrightarrow E \longrightarrow eAe \longrightarrow B \longrightarrow 0$$

realizes eAe as a split extension of B by the nilpotent bimodule E. Thus B is obtained from A by the combination of two consecutive processes: one first drops points of Q_A by passing to the full subcategory eAe, then one drops arrows as explained in Theorem 2.5. One then has the following obvious corollary of lemmata 2.2 and 2.4.

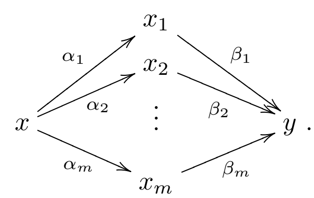
COROLLARY 2.6. *Let B be a reduction of A.*

(a) *If A is shod, then so is B.*
(b) *If A is quasi-tilted, then so is B.*
(c) *If A is tilted, then so is B.*

EXAMPLE 2.7. Taking full subcategories can be used to make a relation quadratic. Indeed, assume that the bound quiver of an algebra A contains a relation $\rho = \sum_{i=1}^{m} \lambda_i u_i$ with source x and target y



Denote by $x_1, \ldots, x_m$ the immediate successors of x on the paths $u_1, \ldots, u_m$ respectively, and let $e = e_x + \sum_{i=1}^{m} e_{x_i} + e_y + e'$, where e' is the sum of all primitive idempotents corresponding to points in Q_A not lying on the paths u_i. Then the bound quiver of eAe contains a relation $\rho' = \sum_{i=1}^{m} \lambda_i(\alpha_i\beta_i)$ with source x and target y



Clearly, ρ' is quadratic. This procedure will be used in the proof of our key Lemma 3.9 below.

3. Sequential walks

3.1. The definition. Given an arrow α in a quiver, its formal inverse is denoted by α^{-1}, where we agree that the source of α^{-1} is the target of α and the target of α^{-1} is the source of α. A *walk* w is an expression of the form $w = \alpha_1^{\epsilon_1} \cdots \alpha_t^{\epsilon_t}$, where the α_i are arrows and the $\epsilon_i \in \{+1, -1\}$ are such that the target of $\alpha_i^{\epsilon_i}$ equals the source of $\alpha_{i+1}^{\epsilon_{i+1}}$, for all i. Such a walk is called *reduced* if it contains no expression of one of the forms $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$, with α an arrow. It is called a *zigzag walk* if it is of the form $\alpha_1^{\epsilon_1} \cdots \alpha_t^{\epsilon_t}$ with $\epsilon_i \neq \epsilon_{i+1}$ for all i.

We start by recalling the definition of sequential walk as stated in [**4**]. Let w be a nontrivial walk in a bound quiver (Q, I). Assume that one writes $w = uw'v$ where each of u, w', v is a subwalk of w. We say that u, v *point to the same direction* in w if both u and v, or both u^{-1} and v^{-1} are paths in Q.

A reduced walk $w = uw'v$ having u, v pointing to the same direction was called a sequential walk in [**4**] if there is a relation $\rho = \sum_i \lambda_i u_i$ such that $u = u_1$, or $u = u_1^{-1}$, there is a relation $\sigma = \sum_i \mu_i v_i$ such that $v = v_1$, or $v = v_1^{-1}$ and no subpath w_1 of w', or of $(w')^{-1}$ is involved in a relation of the form $\sum \nu_i w_i$.

As mentioned in the introduction, there is an ambiguity in this definition arising from the undefined word "involved". Indeed, the word "involved" can be understood as meaning "is a branch of a relation". But this is not correct as shown in the following example.

Example 3.1. Let A be given by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightrightarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightrightarrows}} 3$$

bound by the relation $\alpha_1\alpha_2 + \beta_1\beta_2 = 0$. Then A is the one-point extension of the Kronecker algebra with quiver

$$2 \underset{\beta_2}{\overset{\alpha_2}{\rightrightarrows}} 3$$

by the indecomposable postprojective module ${}_{3}{}^{2}{}_{3}{}^{2}{}_{3}$. In particular, A is tilted [**19**, 3.5].

Consider the reduced walk $w = (\alpha_1\alpha_2)(\beta_2^{-1}\alpha_1^{-1})(\beta_1\beta_2)$ in the quiver of A, with $u = \alpha_1\alpha_2$, $v = \beta_1\beta_2$ and $w' = \beta_2^{-1}\alpha_1^{-1}$. Then neither w' nor $(w')^{-1} = \alpha_1\beta_2$ is a branch of any relation while u and v satisfy the conditions of the definition of [**4**]. If one understands "involved" as meaning "is a branch of a relation", then w would be a sequential walk and we would get a counterexample to [**4**, 2.4].

We propose the following definition.

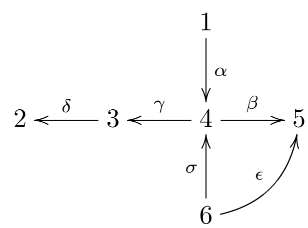
Definition 3.2. Let w be a reduced walk in a bound quiver (Q,I), then w is called a *sequential walk* if the following hold.

(a) $w = uw'v$, where u, v point to the same direction, there is a top relation $\rho = \sum_i \lambda_i u_i$ such that $u = u_1$, or $u = u_1^{-1}$, and there is a top relation $\sigma = \sum_i \mu_i v_i$ such that $v = v_1$, or $v = v_1^{-1}$;
(b) no subpath of w', or of $(w')^{-1}$ lies in I, nor is a branch of a relation having a branch which has a point in common with one of the u_i or v_i;
(c) w' itself has no arrows in common with one of the u_i or v_i.

Remark 3.3. Condition (b) holds for example if no subpath of w' or $(w')^{-1}$ is the branch of a relation.

Remark 3.4. Our definition of sequential walk is clearly inspired by the definition of sequential pair in [**17**] but it is not identical to it.

Let $A = \Bbbk Q/I$ be a string algebra. In particular A is monomial. A sequential pair of monomial relations is a reduced walk w that contains exactly two zero-relations and these two zero-relations point to the same direction in w. Our definition differs from [**17**] in the case of string algebras. For example, the algebra given by the quiver



bound by the relations $\alpha\beta = \gamma\delta = \sigma\beta = 0$ contains the sequential pair $\alpha\beta\epsilon^{-1}\sigma\gamma\delta$, where $u = \alpha\beta$, $w' = \epsilon^{-1}\sigma$ and $v = \gamma\delta$, but this is not a sequential walk because the target of σ is a point on u but not an endpoint of u.

Our definition differs from [**17**] also because sequential walks do not detect overlapping relations. For example, the algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

bound by the relations $\alpha\beta = 0 = \beta\gamma$ does not admit a sequential walk.

Remark 3.5. As we shall see below, if A is a tree algebra of global dimension two, then the two notions of sequential walk and sequential pair coincide.

3.2. Why this notion is natural. It is well-known that if $A = \Bbbk Q/I$ is an algebra, and S_x a simple module, then $\operatorname{pd} S_x > 1$ if and only if x is the source of a top relation in (Q,I). This is easily seen for instance by looking at the radical of P_x. The following considerations go in this direction.

Lemma 3.6. *Let $A = kQ/I$ be an algebra and M an A-module of projective dimension $d > 1$. Then there exists $x \in (\operatorname{Supp} M)_0$ and, for each $i \leq d$, there exists $y_i \in Q_0$ such that $\operatorname{Ext}^i_A(S_x, S_{y_i}) \neq 0$.*

Proof. We first claim that M has a composition factor S_x such that $\operatorname{pd} S_x \geq d$. Consider the socle series

$$0 \subsetneq \operatorname{soc} M \subsetneq \operatorname{soc}^2 M \subsetneq \cdots \subsetneq \operatorname{soc}^\ell M = M,$$

where ℓ is the Loewy length of M, see for example [**4**, V.I]. If there exists a simple summand of $\operatorname{soc} M$ of projective dimension d, then we are done. Otherwise, there exists $i < \ell$ such that $\operatorname{pd}\operatorname{soc}^i M < d$ and $\operatorname{pd}\operatorname{soc}^{i+1} M = d$. The short exact sequence

$$0 \longrightarrow \operatorname{soc}^i M \longrightarrow \operatorname{soc}^{i+1} M \longrightarrow \frac{\operatorname{soc}^{i+1} M}{\operatorname{soc}^i M} \longrightarrow 0$$

yields an exact sequence of functors

$$\operatorname{Ext}_A^d\left(\tfrac{\operatorname{soc}^{i+1} M}{\operatorname{soc}^i M}, -\right) \longrightarrow \operatorname{Ext}_A^d(\operatorname{soc}^{i+1} M, -) \longrightarrow \operatorname{Ext}_A^d(\operatorname{soc}^i M, -) = 0.$$

Now $\operatorname{Ext}_A^d(\operatorname{soc}^{i+1} M, -) \neq 0$ implies $\operatorname{Ext}_A^d(\frac{\operatorname{soc}^{i+1} M}{\operatorname{soc}^i M}, -) \neq 0$. Hence, because the module $\frac{\operatorname{soc}^{i+1} M}{\operatorname{soc}^i M}$ is semisimple, there exists a simple composition factor S_x of M such that $\operatorname{pd} S_x \geq d$.

We deduce the result. Because of our claim, there exists a minimal projective resolution

$$P_{d+1} \xrightarrow{f_{d+1}} P_d \xrightarrow{f_d} P_{d-1} \xrightarrow{f_{d-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} S_x \longrightarrow 0$$

with $P_i \neq 0$ for all $i \leq d$. Consider the projection $p_i \colon P_i \to \operatorname{top} P_i$. Then $p_i f_{i+1} = 0$ because of the minimality of the resolution. Moreover, p_i does not factor through f_i because P_i is the projective cover of $\operatorname{Ker} f_{i-1}$. Therefore $\operatorname{Ext}_A^i(S_x, \operatorname{top} P_i) \neq 0$ and there exists a simple composition factor S_{y_i} of $\operatorname{top} P_i$ such that $\operatorname{Ext}_A^i(S_x, S_{y_i}) \neq 0$. □

COROLLARY 3.7. *Let M be a module of projective dimension $d > 1$. Then there exists $x \in (\operatorname{Supp} M)_0$ which is the source of a top relation in (Q, I).*

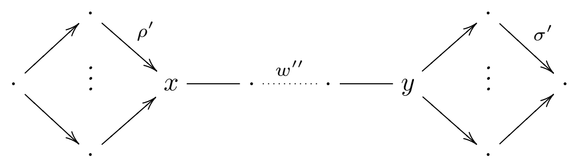
PROOF. Because $d \geq 2$, there exists y_2 such that $\operatorname{Ext}_A^2(S_x, S_{y_2}) \neq 0$. We then apply [**9**]. □

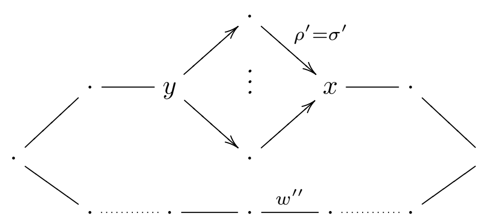
Let thus M be a module with both projective and injective dimension larger than one. Because of Corollary 3.7 and its dual, there exist two points x, y in $\operatorname{Supp} M$ which are respectively the source of a top relation and the target of a top relation. The notion of sequential walk (and all other similar notions like sequential pairs, double zeros etc.) arose from the attempt to connect y to x by a walk.

3.3. A necessary condition for shod. In this subsection, we prove our main result. We start with the following remark.

REMARK 3.8. If the quiver of an algebra A contains a nonzero walk $w = \alpha_1^{\epsilon_1} \cdots \alpha_t^{\epsilon_t}$, then there is a reduction B of A in which w is replaced by a zigzag walk which is moreover a full subcategory of B. Let $x_1, \ldots, x_r$ be all the sources on w and $y_1, \ldots, y_s$ all the sinks, then set $e = \sum_{i=1}^r e_{x_i} + \sum_{j=1}^s e_{y_j} + e'$, where e' is the sum of the primitive idempotents corresponding to points in Q_A not lying on w. In eAe, w is replaced by a new walk w' which is a zigzag walk. However, we may have in eAe new arrows between the x_i and the y_j not corresponding to subpaths of w. Let S be the set of all these arrows, namely those arrows β in Q_{eAe} whose source and target lie on w' but neither β nor β^{-1} lies on w'. Since w' is a zigzag walk, it contains no (branches of) relations, so S is a consistent cut, as defined in 2.5. Let E be the two-sided ideal of eAe generated by S and $B = eAe/E$. Then it follows from Theorem 2.5 that eAe is a split extension of B by E. Moreover, w' is a full subcategory of B.

LEMMA 3.9. *Let A have a sequential walk. Then there exists a reduction B of A containing one of the following (perhaps not full) subquivers.*





where ρ', σ' are quadratic relations, w'' is a zigzag walk having no point in common with ρ', σ' except x and y.

Moreover, w'' generates a full subcategory of B.

PROOF. Let $w = uw'v$ be a sequential walk in A. Then u, v are branches of top relations $\rho = \sum \lambda_i u_i$ from a to b, say, and $\sigma = \sum \mu_j v_j$ from c to d, say, while w' is a walk from b to c satisfying the conditions of Definition 3.2. Clearly, one may have $\rho = \sigma$. We construct B in the following steps.

(a) We make ρ, σ quadratic, using the recipe of Example 2.7 above. Let e_1 be the sum of the primitive idempotents of A corresponding to:

(1) The sources and targets a, b, c, d of ρ and σ.
(2) All immediate successors of a, c lying on one of the paths u_i, v_j respectively.
(3) All points of Q_A not lying on any of the paths u_i, v_j.

Then $A_1 = e_1 A e_1$ is a full subcategory of A whose quiver contains a sequential walk $w = u'w'v'$ where u', v' are branches of quadratic relations and w' is the same walk as before.

(b) We make w' a zigzag walk, as in Remark 3.8 above. Let $x_1, \ldots, x_r$ and $y_1, \ldots, y_s$ be respectively all the sources and all the sinks on w', and e_2 be the sum of the primitive idempotents corresponding to the x_i, y_j and all the points of Q_{A_1} not on w'. Then $A_2 = e_2 A_1 e_2$ contains a zigzag walk w'' which replaces w', such that there are perhaps additional arrows in Q_{A_2} between points of w'' not corresponding to subpaths of w'.

(c) We eliminate excessive arrows in A_2 making w'' a full subcategory of the resulting algebra. Consider the set S of all arrows α in A_2 whose source and target lie in w'', but such that neither α nor α^{-1} belongs to w''. Since there are no relations on the zigzag walk w'', the set S is a consistent cut. Let E be the two-sided ideal of A_2 generated by S, and let $B = A_2/E$. Then we get an exact sequence

$$0 \longrightarrow E \longrightarrow A_2 \longrightarrow B \longrightarrow 0$$

realizing A_2 as a split extension of B by E, as seen in Theorem 2.5.

It is then clear that the bound quiver of B contains one of the subquivers in the statement, the second one occurring if $\rho = \sigma$. $\square$

We call the reduction B as in the lemma the *standard reduction* corresponding to a given sequential walk.

LEMMA 3.10. *Let A have a sequential walk and B the corresponding standard reduction of A. Then the string module of the zigzag walk w'' in B has projective and injective dimension larger than one.*

PROOF. Let $M = M(w'')$ be the string module corresponding to w'', that is, M is the B-module defined as a representation by

$$M(x) = \begin{cases} \Bbbk & \text{if } x \text{ is a point of } w''; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$M(\alpha) = \begin{cases} \mathrm{id} & \text{if } \alpha \text{ is an arrow of } w''; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that in B, there may be arrows between points in w'' and points on one of the relations ρ', σ'. But because of the definition of the string module, for any such arrow β, we have $M(\beta) = 0$. Because no subpath of w'' is a branch of a relation in B, then M is indeed a B-module. Moreover, M is a string module, see [**10**], and in particular, it is indecomposable.

We now prove that $\mathrm{pd}\, M_B > 1$. The support of M contains the source y of σ', Therefore the projective cover of M admits a direct summand P_z such that z lies on w'', and either $z = y$ or there is an arrow $z \to y$. It is easily seen that a nonprojective summand of $\Omega^1 S_y$ is a direct summand of $\Omega^1 M$. Because $\Omega^1 S_y$ is not projective, neither is $\Omega^1 M$. This establishes the claim.

Dually, we also have $\mathrm{id}\, M_B > 1$. □

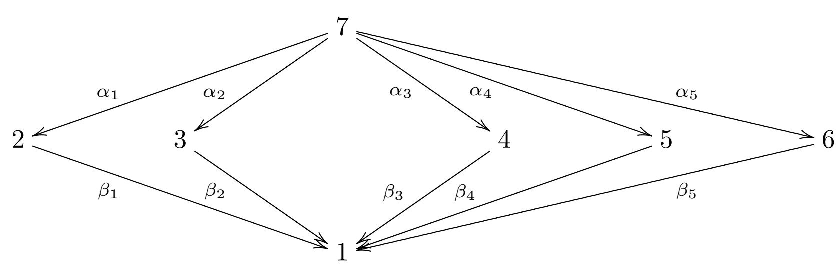
We are now ready to prove our main result. It generalizes [**4**, 2.4].

THEOREM 3.11. *Let $A = \Bbbk Q/I$ be an algebra having a sequential walk. Then A is not shod. In particular, A is not quasi-tilted.*

PROOF. Let B be the standard reduction of A corresponding to the sequential walk. By Lemma 3.10, there exists an indecomposable B-module that has both injective and projective dimension larger than one. Therefore B is not shod.

Because of Corollary 2.6, neither is A. □

EXAMPLE 3.12. Consider the quiver

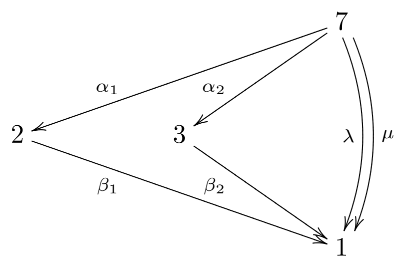


bound by the relations $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$, $\alpha_3\beta_3 + \alpha_4\beta_4 + \alpha_5\beta_5 = 0$. We show how to perform the reduction procedure of Theorem 3.11. Let here

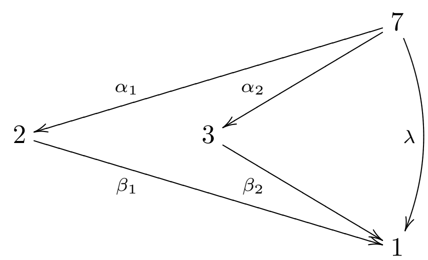
$$w = (\alpha_1\beta_1)\beta_3^{-1}\alpha_3^{-1}(\alpha_2\beta_2),$$

with $u = \alpha_1\beta_1$, $v = \alpha_2\beta_2$ and $w' = \beta_3^{-1}\alpha_3^{-1}$. It is clear that u, v and w' satisfy the conditions of Definition 3.2.

We first eliminate points by taking $e = e_1 + e_2 + e_3 + e_7$. Then eAe is given by the quiver



bound by the relation $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$. This is a split extension of the algebra B given by the quiver



bound by $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$, by the two-sided ideal generated by the arrow μ.

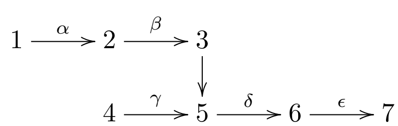
The indecomposable B-module M of the proof is the module $M = \begin{smallmatrix}7\\1\end{smallmatrix}$ supported by the arrow λ. Clearly, we have a minimal projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_2 \oplus P_3 \longrightarrow P_7 \longrightarrow M \longrightarrow 0$$

so $\operatorname{pd} M_B = 2$. Notice that $\Omega^1 M = \begin{smallmatrix}2&&3\\&1&\end{smallmatrix}$ while $\Omega^1 S_7 = \begin{smallmatrix}2&&3\\&1&\end{smallmatrix} \oplus 1$. Thus $\Omega^1 M$ and $\Omega^1 S_7$ have a common summand but are not equal. Similarly, $\operatorname{id} M_B = 2$. Thus B, and A, are not shod.

The following examples illustrate that the converse of Theorem 3.11 does not hold without additional conditions on the algebra A or the module M. In Section 4, we give examples of such additional conditions.

Example 3.13. If an indecomposable module has both projective and injective dimension larger than one, this does not necessarily imply the existence of a sequential walk. Indeed, there exist x, y in the support of the module such that x is the source of a top relation v, and y the target of a top relation u. Then there exists a walk w' from x to y inside the support of M, but this does not imply that $w = uw'v$ is a sequential walk because it may not be reduced. For instance, let A be the monomial tree algebra given by the quiver



bound by the relations $\alpha\beta = 0 = \gamma\delta\epsilon$, then there is no sequential walk but the module $M = \begin{smallmatrix}3&&4\\&5&\end{smallmatrix}$ has both projective and injective dimension 2.

EXAMPLE 3.14. There exist sincere indecomposable modules of projective and injective dimension 2. Let A be given by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightrightarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightrightarrows}} 3$$

bound by the relations $\alpha_1\alpha_2 = 0$ and $\beta_1\beta_2 = 0$. Note that A is gentle and in particular tame. The indecomposable module $M = \begin{smallmatrix}1\\2\\3\end{smallmatrix}$ is sincere and of both projective and injective dimension 2.

4. Applications and examples

4.1. The case of global dimension two. Because sequential walks do not detect overlaps, it is natural to think of them in the context of algebras of global dimension two.

PROPOSITION 4.1. *If A is a monomial algebra of global dimension 2 and M is a uniserial A-module whose injective and projective dimensions are both larger than one then there exists a sequential walk in A.*

PROOF. Since M is uniserial, the support of M is of the form

$$z_1 \longrightarrow z_2 \longrightarrow \cdots \qquad \longrightarrow z_\ell \ .$$

By Corollary 3.7 and its dual, there exist $z_i, z_j \in \operatorname{Supp} M$ such that z_j is the source of a top relation v, and z_i the target of a top relation u. Since A is monomial of global dimension 2, the two relations u, v do not overlap [**13**]. Thus $i \leq j$. Now let w' be a path $z_i \to \cdots \to z_j$ in $\operatorname{Supp} M$. Then the composition $uw'v$ is a sequential walk in A. □

COROLLARY 4.2. *Let A be a Nakayama algebra of global dimension 2. Then there exists a sequential walk in A if and only if A is not tilted.*

PROOF. Necessity follows from Theorem 3.11. To show sufficiency, suppose A is not quasi-tilted. Then there exists an indecomposable A-module M of both projective and injective dimension 2. Since A is a Nakayama algebra, A is monomial and M is uniserial. Now the result follows from Proposition 4.1 and the fact that quasi-tilted Nakayama algebras are representation-finite, and hence tilted. □

We have the following characterization of projective dimension 2.

PROPOSITION 4.3. *Let M be an indecomposable module over an algebra of global dimension 2 such that one of the sinks in the support of M is the source of a top relation. Then the projective dimension of M is two.*

PROOF. Indeed, we have $\operatorname{pd} S_x > 1$ and an exact sequence

$$0 \longrightarrow S_x \longrightarrow M \longrightarrow M/S_x \longrightarrow 0.$$

Hence we have an exact sequence of functors

$$\operatorname{Ext}^2_A(M/S_x, -) \longrightarrow \operatorname{Ext}^2_A(M, -) \longrightarrow \operatorname{Ext}^2_A(S_x, -) \longrightarrow 0,$$

because $\operatorname{Ext}^3_A = 0$. Hence $\operatorname{Ext}^2_A(S_x, -) \neq 0$ implies $\operatorname{Ext}^2_A(M, -) \neq 0$. □

For tree algebras of global dimension two, sequential walks are easy to characterize.

PROPOSITION 4.4. *Let A be a tree algebra of global dimension two. Then the two notions of sequential walk and sequential pair coincide.*

PROOF. Indeed, because A is a tree algebra (not necessarily a string algebra) it is monomial, and the global dimension two means that no two top relations overlap. Because Q is a tree, two points of Q are connected by a unique walk, and so a sequential walk as well as a sequential pair mean a walk of the form $w = uw'v$, where u, v are monomial relations pointing to the same direction while w' is any walk not containing a relation. □

4.2. An application to laura algebras. For our next corollary, we recall a few notions. Given an algebra A, we denote by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of exactly one representative from each isomorphism class of indecomposable modules. The *left part* $\mathcal{L}_A$ of $\operatorname{mod} A$ consists of all modules M in $\operatorname{ind} A$ such that, for every L for which there exists a path of nonzero morphisms from L to M, we have $\operatorname{pd} L \leq 1$. The *right part* $\mathcal{R}_A$ of $\operatorname{mod} A$ is defined dually, and an algebra A is called a *laura algebra* if and only if $\mathcal{L}_A \cup \mathcal{R}_A$ is cofinite in $\operatorname{ind} A$.

In [**12**], Dionne has shown that a string algebra is laura if and only if its bound quiver does not contain a combinatorial configuration called intertwined double zero which we now define.

DEFINITION 4.5. [**12**, 2.1.1] Let $A = \Bbbk Q/I$ be a string algebra. A reduced walk w in Q is called an *intertwined double zero* if $w = \rho_1 w_1 w_2 w_3 \rho_2$ where

(a) $\rho_1 = \alpha_1 \dots \alpha_n, \rho_2 = \beta_1 \dots \beta_m$ are monomial relations pointing in the same direction,
(b) neither $\alpha_2 \dots \alpha_n w_1 w_2 w_3 \beta_1 \dots \beta_{m-1}$ nor its inverse contains a monomial relation, and
(c) w_2 is a band.

The next corollary shows that one direction of Dionne's result follows from ours. Observe first that, if $A = \Bbbk Q/I$ is a string algebra, and w is an intertwined double zero then w is a sequential walk in our sense.

COROLLARY 4.6. *Let A be a string algebra having an intertwined double zero. Then A is not laura.*

PROOF. It follows from Definition 4.5 that if $w = \rho_1 w_1 w_2 w_3 \rho_2$ as above is an intertwined double zero, then, for any $n \geq 1$, the reduced walk $w_n = \rho_1 w_1 w_2^n w_3 \rho_2$ is also a sequential walk in the bound quiver of A. Because of Lemma 3.10, we have $\operatorname{pd} M(w_n) > 1$ and $\operatorname{id} M(w_n) > 1$. In particular, the $M(w_n)$ form an infinite family of nonisomorphic indecomposable modules lying neither in $\mathcal{L}_A$ nor in $\mathcal{R}_A$. Thus A is not laura. □

4.3. An application to 2-Calabi-Yau tilted algebras. Let $C = \Bbbk Q/I$ be a quasi-tilted algebra. In particular, its relation extension $\widetilde{C} = \Bbbk\widetilde{Q}/\widetilde{I}$, which is the trivial extension of C by the bimodule $\operatorname{Ext}^2_C(DC, C)$ is a cluster-tilted algebra or a 2-Calabi-Yau tilted algebra of canonical type, see [**5**, 3.1]. A walk $w = \alpha w' \beta$ in $(\widetilde{Q}, \widetilde{I})$ is called a *C-sequential walk* if

(i) w' consists entirely of old arrows, see [**2**] for the terminology "old" vs "new" arrows;
(ii) α, β are new arrows corresponding respectively to old relations $\rho = \sum_i \lambda_i u_i$ and $\sigma = \sum_j \mu_j v_j$;

(iii) For all i, j, the walk $w = u_i w' v_j$ is sequential in (Q, I).

Then we have

Corollary 4.7. *Let C be a quasi-tilted algebra. Then the bound quiver of its relation extension $\widetilde{C}$ contains no C-sequential walk.*

4.4. Example. We have seen in Example 3.1 that the algebra A given by the quiver

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightrightarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightrightarrows}} 3$$

bound by the relation $\alpha_1\alpha_2 + \beta_1\beta_2 = 0$ is a tilted algebra. Notice that here $w = (\alpha_1\alpha_2)\beta_2^{-1}\alpha_1^{-1}(\beta_1\beta_2)$ is not a sequential walk in the sense of Definition 3.2, since the subpath $(w')^{-1} = \alpha_1\beta_2$ has arrows in common with the branches of the relation $u = \alpha_1\alpha_2$ and $v = \beta_1\beta_2$. In fact, this bound quiver contains no sequential walks.

On the other hand, the algebra A' given by the same quiver but bound by the relation $\alpha_1\alpha_2 = 0$ contains evidently the sequential walk $w = (\alpha_1\alpha_2)\beta_2^{-1}\beta_1^{-1}(\alpha_1\alpha_2)$. In particular, Theorem 3.11 implies that this algebra is not tilted.

It is interesting to note that the relation extensions of both algebras A and A' have the same quiver $\widetilde{Q}$. Therefore the associated cluster categories $\mathcal{C}_A$ and $\mathcal{C}_{A'}$ are categorifications of the same cluster algebra $\mathcal{A}(\widetilde{Q})$.

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