A NOTE ON AISLES IN A TRIANGULATED KRULL SCHMIDT CATEGORY.

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ABSTRACT. We assume that $\mathcal{T}$ is a triangulated Hom-finite Krull-Schmidt k-category and that M is a strong generator such that $Hom_{\mathcal{T}}(M, M[j]) = 0$, for all $j \neq 0$. We show that the suspended subcategory $\mathcal{U}_M$ generated by M is an aisle. Further, if $\mathcal{T}$ has almost split triangles then the orthogonal $\mathcal{U}_M^{\perp}$ equals the co-aisle ${}_{\tau M}\mathcal{U}$ cogenerated by the Auslander-Reiten translate τM of M.

INTRODUCTION

The notion of triangulated category (see [V]) has proved very useful in the representation theory of algebras. In particular, there is a strong relationship between the study of t-structures and tilting theory (see, for instance, [KV, P, H, ST]). In [KV](1.1), Keller and Vossieck consider certain subcategories called aisles, and show that, if $\mathcal{U}$ is an aisle, then $(\mathcal{U}_M, \mathcal{U}_M^{\perp}[1])$ is a t-structure, and conversely any t-structure is of this form.

In this note, we give a construction procedure for aisles and hence for t-structures. We recall that, for instance, it was shown in [ST] that every perfect complex generates a t-structure on $\mathbf{D}^b(mod - A)$, where A is a Noether algebra (see also [KV](5.1)).

We say that an object M in a triangulated category is a strong generator if $\mathcal{T}$ equals the smallest triangulated subcategory containing M and closed under direct summands. We prove the following theorem:

Theorem: Let k be a field, $\mathcal{T}$ be a triangulated Hom-finite Krull-Schmidt k-category and M be a strong generator such that $Hom_{\mathcal{T}}(M, M[j]) = 0$, for all $j \neq 0$. Then the suspended subcategory $\mathcal{U}_M$ generated by M is an aisle in $\mathcal{T}$. Dually, the cosuspended subcategory ${}_M\mathcal{U}$ cogenerated by M is a co-aisle in $\mathcal{T}$.

We next consider the case where $\mathcal{T}$ has almost split triangules. A necessary and sufficient condition for the existence of such triangles is given in [RV]. We denote by τ the Auslander-Reiten translation in $\mathcal{T}$.

Corollary: Let $\mathcal{T}$ and M be as in the theorem, and assume that $\mathcal{T}$ has almost split triangles. Then $(\mathcal{U}_M)^{\perp} ={}_{\tau M}\,\mathcal{U}$ and $\mathcal{U}_M = {}^{\perp}({}_{\tau M}\mathcal{U})$.

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1. The theorem.

1.1. Following [KV], we say that a full subcategory $\mathcal{U}$ of a triangulated category $\mathcal{T}$ is a **suspended subcategory** if $\mathcal{U}[1] \subset \mathcal{U}$, and it is closed under extensions (that is, if $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{T}$ and $X, Z \in \mathcal{U}$, then Y belongs to $\mathcal{U}$).

A suspended subcategory $\mathcal{U}$ is called an **aisle** in $\mathcal{T}$ if the inclusion functor $\mathcal{U} \to \mathcal{T}$ has a right adjoint functor $t_{\mathcal{U}} : \mathcal{T} \to \mathcal{U}$ (see [KV](1.1)). We define dually **co-suspended** subcategories and **co-aisles**.

Given an object M in $\mathcal{T}$, we denote by $\mathcal{U}_M$ (or ${}_M\mathcal{U}$) the smallest suspended (or cosuspended, respectively) subcategory of $\mathcal{T}$ containing M.

1.2. Let M be an object in a triangulated category $\mathcal{T}$. We define a sequence of classes of objects $(\mathcal{E}_i)_{i\geq 0}$ of $\mathcal{T}$ as follows. Let $\mathcal{E}_0 = add(\oplus_{i\in\mathbb{Z}} M\,[i])$ consist of all the summands of finite sums of copies of translates of M. Assume that $i \geq 1$, and that $\mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_{i-1}$ are already known. The class $\mathcal{E}_i$ consists of all the objects X which are direct summands of objects X' such that there is a triangle $X_0 \to X' \to X_{i-1} \to X_0[1]$, where X_0 lies in $\mathcal{E}_0$ and X_{i-1} lies in $\mathcal{E}_{i-1}$. It is not hard to show that $\cup_{i\geq 0}\mathcal{E}_i$ is the smallest triangulated subcategory of $\mathcal{T}$ closed under direct summands and containing M.

We say that M is a **strong generator** of $\mathcal{T}$ if $\mathcal{T} = \cup_{i\geq 0}\mathcal{E}_i$.

For instance, let $\mathcal{T}'$ be a triangulated compactly generated category and M be a compact generator of $\mathcal{T}'$, then M is a strong generator of the full subcategory consisting of the compact objects (see [N]).

1.3. **Proof of our theorem:** We first prove the following claim: for every $X \in \mathcal{T}$, there is a finite set $F_X \subset \mathbb{Z}$ such that $Hom_{\mathcal{T}}(M[l], X) = 0$, for all $l \notin F_X$. Indeed, assume $X = \oplus_{i\in F} M_i^{(F_i)} \in \mathcal{E}_0$ where $M_i \simeq M[i]$ with F_i and F finite subsets of $\mathbb{Z}$. Then, $Hom_{\mathcal{T}}(M[t], X) = Hom_{\mathcal{T}}(M[t], \oplus_{i\in F} M_i^{(F_i)}) = 0$ if $t \notin F$, using the hypothesis on M. This shows our claim for X (and hence for its direct summands). Assume now $j > 0$ and $X \in \mathcal{E}_j$. Then there is a triangle $X_0 \xrightarrow{r} X \xrightarrow{s} X_{j-1} \to X_0[1]$ with $X_0 \in \mathcal{E}_0$ and $X_{j-1} \in \mathcal{E}_{j-1}$.

Let $t \in \mathbb{Z}$ and $f \in Hom_{\mathcal{T}}(M\,[t]\,, X)$. If $sf = 0$, there exists a morphism $h : M[t] \to X_0$ such that $rh = f$. But we know that there are only finitely many indices l such that $Hom_{\mathcal{T}}(M[l], X_0) \neq 0$. On the other hand, the induction hypothesis says that we have only finitely many indices l such that $sf \in Hom_{\mathcal{T}}(M[l], X_{j-1})$ is non-zero. Therefore, the set $\{l \in \mathbb{Z} \mid Hom_{\mathcal{T}}(M[l], X_j) \neq 0\}$ is finite. If now Y is a direct summand of X as before, then $Hom_{\mathcal{T}}(M[l], Y)$ is a direct summand of $Hom_{\mathcal{T}}(M[l], X)$. This establishes our claim.

For an $l \in \mathbb{Z}$, let $S_l = dim_k Hom_{\mathcal{T}}(M[l], X)$ and $U_0 = \oplus_{l\in F_X \cap \mathbb{N}} M[l]^{S_l}$. Then the induced morphism

$$Hom_{\mathcal{T}}(-, U_0)_{|\mathcal{U}_M} \to Hom_{\mathcal{T}}(-, X)_{|\mathcal{U}_M}$$

is an epimorphism. Applying [KV](1.3) yields that $\mathcal{U}_M$ is an aisle in $\mathcal{T}$.

The second statement is proved dually. □

2. The corollary

2.1. We now assume that the triangulated category $\mathcal{T}$ has almost split triangles, or equivalently, that there is a triangulated equivalence $\tau : \mathcal{T} \to \mathcal{T}$ and an isomorphism, called the Auslander-Reiten formula,

$$\beta_{X,Y} : DHom_{\mathcal{T}}(X, Y[1]) \to Hom_{\mathcal{T}}(Y, \tau X),$$

functorial in both variables, X, Y in $\mathcal{T}$ (see [RV] for details.).

An example of such a situation is the case of $D^b(mod\, A)$, the derived category of bounded complexes of finitely generated (right) $A-$modules, where A is a finite dimensional $k-$algebra with finite global dimension.

For a full subcategory $\mathcal{U}$ of $\mathcal{T}$, we denote by $\mathcal{U}^{\perp}$ (or ${}^{\perp}\mathcal{U}$) the full subcategory consisting of the objects $X \in \mathcal{T}$ such that $Hom_{\mathcal{T}}(-, X)_{|_{\mathcal{U}}} = 0$ (or $Hom_{\mathcal{T}}(X, -)_{|_{\mathcal{U}}} = 0$, respectively).

2.2. The following lemma seems to be well-known. We provide its proof for the convenience of the reader.

Lemma: The aisle $\mathcal{U}_M$ coincides with the full subcategory consisting of the objects X such that $Hom_{\mathcal{T}}(M[i], X) = 0$ for all $i < 0$.

Proof: Let $\mathcal{S}$ be the full subcategory of $\mathcal{T}$ consisting of the objects X verifying the condition of the statement. Then $\mathcal{S}$ is closed under extensions, direct summands, positive translations and M lies in $\mathcal{S}$. Hence $\mathcal{U}_M \subseteq \mathcal{S}$.

Let $X \in \mathcal{S}$, and consider the triangle $N \to X \to B \to N[1]$ given by the definition of aisle. Applying the cohomological functor $Hom_{\mathcal{T}}(M, -)$ to the above triangle, yields $Hom_{\mathcal{T}}(M[j], B) = 0$ for all $j < 0$, because $N, X \in \mathcal{S}$. However, $Hom_{\mathcal{T}}(M[j], B) = 0$ for all $j \geq 0$, because $B \in \mathcal{U}_M{}^{\perp}$. Since M is a strong generator, $B = 0$. Hence $X \simeq N$ lies in $\mathcal{U}_M$. □

2.3. **Proof of our corollary:** Applying the above Lemma and the Auslander-Reiten formula, we get that X belongs to $\mathcal{U}_M$ if and only if, for all $j \leq 0$, $Hom_{\mathcal{T}}(X, (\tau M)[j]) \cong Hom_{\mathcal{T}}(M[j], X[1]) = 0$. This means, if and only if $X \in {}^{\perp}({}_{\tau M}\mathcal{U})$ (see [ST] (2.3)). The second statement is obtained dually. □

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